Research article

On the well-posedness of the "Bando-follow the leader" car following model and a time-delayed version

Xiaoqian Gong[1,*]**and Alexander Keimer**[2,*]

[1] School of Mathematical and Statistical Science, Arizona State University, Tempe, AZ, 85281, USA

[2] Institute of Transportation Studies, UC Berkeley, US and Department of Mathematics, FAU Erlangen-Nürnberg, Germany

* **Correspondence:** Email: xgong14@asu.edu, keimer@berkeley.edu.

Abstract: In this contribution we study the "Bando-follow the leader" car-following model, a second order ordinary differential equation, for its well-posedness. Under suitable conditions, we provide existence and uniqueness results, and also bounds on the higher derivatives, i.e., velocity and acceleration. We then extend the result to the "reaction" delay case where the delay is instantiated in reacting on the leading vehicle's position and velocity. We prove that the solution of the delayed model converges to the undelayed when the delay converges to zero and present some numerical examples underlying the idea that it is worth looking in more details into delay as it might explain problems in traffic flow like "phantom shocks" and "stop and go" waves.

Keywords: ODE; initial value problem; well-posedness; existence; time delay; information delay; car following model; Bando-follow the leader model

1. Introduction

In approximating traffic flow dynamics by dynamical systems, we observe phenomena like "stop and go" waves [34] and "phantom shocks" [33, 16]. It is likely that these phenomena are partially caused by a reaction delay of the participating drivers. In most of the existing car-following models, no delay had been introduced, on the contrary the delay was often interpreted model intrinsically. However, dynamical systems allow in a very straight forward way to introduce a time-delay. This is what we will cover in this contribution. Using the well-known Bando-follow the leader (Bando-FtL) car following model [34] we assume that velocity and position of the leading car is only taken into account in a delayed way for determining the ego vehicle' (i.e., the follower's) acceleration and study the well-posedness of the underlying model (compare in particular Definition 4.1). This is a first step in analyzing rigorously delay in car following models although the delay in this contribution is not

realized as typically done in delay ordinary differential equations (the solutions itself are subject to delay) but only the leading vehicle's velocity and position is incorporated in the acceleration function of the ego vehicle in a delayed sense.

Another point which is addressed in this contribution is the question of existence of solutions and lower and upper bounds on specific derivatives. As it has been noticed in [1], a careful study of existence of solutions of car-following models has to the authors' knowledge not been undertaken and leaves most car following models, particularly the second order nonlinear models-which are intensively used in applied sciences-in a position that there is just "the hope" that they will behave properly also for "non-standard" initial data. This is why we prove the existence and uniqueness of solutions to the Bando-FtL second order car-following model and provide explicit bounds under the assumption that the leading vehicle does not drive backwards. These bounds will come in handy for different tasks, among these studying related optimal control problems and existence of minimizers. The well-posedness, i.e., existence and uniqueness of solutions, becomes also more and more important as numerical traffic simulators use specific classes of car following models with rather complex lane changing dynamics necessitating a careful study on the well-posedness which can also shed light on the required discretization in the numerical approximation and more.

We also prove that when delay of the delayed car following model converges to zero, the solutions converge to the solutions of the undelayed model, somewhat underlining the intuition that the considered car following model is robust (in the uniform topology) with regard to delay and that-therefore-the currently used non-delayed versions might be an acceptable approximation of the "closer" to reality delayed car-following models.

1.1. Literature

The literature on car following models is vast and we will only give a short overview, mentioning that this overview cannot be complete.

First attempts in car following modeling can be found in [11, 15]. Among all the car-following models existing in the literature (see for instance [3, 6, 8, 30, 31]) it is worth naming the Optimal Velocity Model [4], the Bando-follow-the-leader (Bando-FtL) model [34] which is the model we analyze in this contribution, and the Intelligent Driver Model (IDM) (with modifications and extensions) [35, 25, 14, 32, 26, 27]. For a comprehensive overview of the main car-following models we refer to [36, 28, 19].

For car-following models with delay, a lot of contributions can be found as well. We name [38] which proposes an ODE (ordinary differential equation) model with two delays, one in the velocity of the considered vehicle (potentially time dependent) and one in the headway difference and thus delaying also the dependency of the acceleration on its own position. Yu et al. consider in [37] again a second order car following model with delay in both relative position and relative velocity so that once more the delay is not just in the position and velocity of the leading vehicle. Bando et al. study in [5] delayed car-following models where the delay is located in position and velocity and the leading vehicle's velocity is not taken into account (first and second order). In [12], a second order car following model based on the optimal velocity (which is also part of the Bando-FtL model we use in this work Definition 2.1) with delay in the headway is investigated. Zhu et al. model car following in [39] with a first order model and delay in the headway. Keane et al. calibrate with recorded data in [22] in a general car-following model with different delays using adjoint methods to optimize the

incorporated parameters. Jiang et al. investigate in [20] the famous Nagel Schreckenberg cellular automaton (see also) [29] with delay, again both velocities (leading and ego vehicle) are delayed. None of the previous contributions perform a mathematical analysis of the well-posedness (i.e., existence and uniqueness of solutions) and clearly, for specific versions of delay the solution can break down in finite time if one for instance has a denominator in the model which incorporates headway. Also, none of the named contributions considers a delay only in the information about the leading vehicle' position and velocity which we deal with in this contribution (see particularly Definition 4.1). For macroscopic modelling with explicit time delay, we refer the reader to [23] where nonlocal conservation laws are used to model traffic and where the authors prove existence and uniqueness of solutions, and to [10] which considers similarly macroscopic conservation laws to model traffic but this time locally in space, running, due to delay, into the problem of infeasible solutions for specific initial datum.

In [24, 17], the foundations for using car-following models in mean-field modelling are laid necessitating the study of well-posedness of the used car-following models.

1.2. Outline

Section 2 defines the involved functions and parameters for the "Bando-FtL" car following model and Section 3 then presents the considered system of initial value problems together with the result on the well-posedness under rather weak conditions on the involved functions in Theorem 3.3. The key idea is that the car-following model can partially be integrated so that lower bounds can be obtained. Section 4 then introduces the delayed car-following model which is detailed in Definition 4.1. Thanks to the structure of the delay, the well-posedness can be proven analogously as in the undelayed case and one obtains almost identical lower and upper bounds on the involved derivatives. In Section 5 we then study the convergence of solutions when the delay approaches zero and can indeed prove in Theorem 5.2 the convergence in $W^{1,\infty}$, to the non-delayed solution and thus in the uniform topology not just for the position but also the velocity. The main argument is based on some compactness results together with the bounds on the solution, derived before. Section 6 generalizes the existence and convergence results described before to the many car case, the typical situation in applications. In Section 7, some numerical simulations are presented to illustrated the obtained results and in particular the effect of the delay, increasing the variation on velocity (phantom shocks, etc.) significantly. We conclude in Section 8 with some challenging open problems, particularly considering corresponding optimal control problems and more general versions of delay.

2. Problem definition

In this section, we first introduce the considered car following model, the Bando-FtL model, which had been introduced in [34]. We start with defining the "Bando-FtL" acceleration.

Definition 2.1 (Bando-FtL acceleration). Let $l \in \mathbb{R}_{>0}$ be the car length,

$$A := \{(x, x_\ell, c, d) \in \mathbb{R}^4 : \ x_\ell - x - l > 0\},$$

$(\alpha, \beta) \in \mathbb{R}_{>0}^2$ and

$$V \in C^1(\mathbb{R}_{\geq 0}; \mathbb{R}_{\geq 0}) \cap L^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}_{\geq 0}) \ \text{ satisfying } V' > 0, \tag{2.1}$$

we call the following function Acc the Bando-FtL acceleration

$$\text{Acc} : \begin{cases} A & \to \mathbb{R} \\ (x, x_\ell, v, v_\ell) & \mapsto \alpha\Big(V(x_\ell - x - l) - v\Big) + \beta\frac{v_\ell - v}{(x_\ell - x - l)^2}. \end{cases} \tag{2.2}$$

Remark 1 (The definition of Acc). If we take $(x, v), (x_\ell, v_\ell) \in \mathbb{R} \times \mathbb{R}_{\geq 0}$ as the ego vehicle' and the leading vehicle' position-velocity vectors, the definition of Acc in Definition 2.1 indicates that the acceleration is dependent on leading vehicle's velocity and position as well as ego vehicle's velocity and position. However, in the considered setup, the situation is slightly more easy, as the acceleration term only depends on the ego vehicle's space headway $x_\ell - x - l$ and not explicitly on the positions x_ℓ and x. To keep consistent and to be able to incorporate later dynamics for the ego vehicle where the position and velocity of the leader are delayed, the current notation seems reasonable (see Definition 4.1 and in particular Eq (4.4)).

The "follow the leader" term in the definition of Acc in Definition 2.1, i.e.,

$$\beta\frac{v_\ell - v}{(x_\ell - x - l)^2},$$

guarantees that the ego vehicle would accelerate if its velocity is smaller than the leading vehicle's velocity, i.e., $v < v_\ell$ and decelerate if $v > v_\ell$. In the extreme case when both the ego vehicle and its leading vehicle drive at the same velocity and small headway, the Bando term in the definition of Acc in Definition 2.1, i.e.

$$\alpha\Big(V(x_\ell - x - l) - v\Big),$$

makes the ego vehicle decelerate to increase its space headway due to the monotonicity of the optimal velocity function V in Eq (2.1). On the other hand, if the headway would be very large, $V(x_\ell - x - l)$ would be almost maximal (thanks to the properties on V in Eq (2.1)) so that the headway would decrease (compare in particular Remark 2 and Eq (2.3)).

Remark 2 (An example for the optimal velocity function V). Let $v_{\max} \in \mathbb{R}_{>0}$ be the maximal velocity and $d_s \in \mathbb{R}_{>0}$ a given parameter. One example of an optimal velocity function, see Figure 1, consists of

$$V(\cdot) = v_{\max}\frac{\tanh(\cdot - d_s) + \tanh(l + d_s)}{1 + \tanh(l + d_s)}, \tag{2.3}$$

where $l \in \mathbb{R}_{>0}$ is the car length. In particular, if the ego vehicle's space headway $x_\ell - x - l$ is large enough, the Bando term in the definition of Acc in Definition 2.1 forces the ego vehicle's velocity toward the maximal velocity $v_{\max} \in \mathbb{R}_{>0}$.
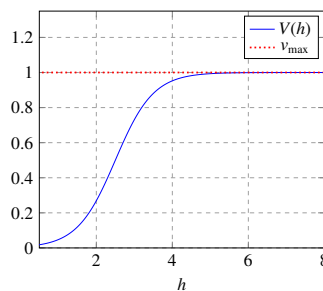

Figure 1. One choice for the optimal velocity function V, the function in Eq (2.3) with parameters $l = 4.5$ and $d_s = 2.5$ and $v_{\max} = 1$.

3. Well-posedness: Real-time information available–the undelayed case

In the following, we give the precise definition of the car-following model we will study. In the most simplistic case it is enough to have only two vehicles, the leading vehicle which has its velocity and position given, and the ego vehicle which will be subject to the car-following dynamics as follows:

Definition 3.1 (The Bando-FtL – the undelayed case). For $T, v_{\ell\max}, l \in \mathbb{R}_{>0}$, and Acc as in Definition 2.1 we consider the following initial value problem in $(x_\ell, x) \in W^{2,\infty}((0, T))^2$

$$\dot{x}_\ell(t) = v_\ell(t) \qquad\qquad t \in [0, T] \qquad (3.1)$$
$$\dot{v}_\ell(t) = u_{\text{lead}}(t) \qquad\qquad t \in [0, T] \qquad (3.2)$$
$$\dot{x}(t) = v(t) \qquad\qquad t \in [0, T] \qquad (3.3)$$
$$\dot{v}(t) = \text{Acc}(x(t), x_\ell(t), v(t), v_\ell(t)) \qquad\qquad t \in [0, T] \qquad (3.4)$$
$$(x(0), v(0)) = (x_0, v_0) \qquad\qquad (3.5)$$
$$(x_\ell(0), v_\ell(0)) = (x_{\ell 0}, v_{\ell 0}) \qquad\qquad (3.6)$$

with the leading vehicle acceleration satisfying

$$u_{\text{lead}} \in \mathcal{U}_{\text{lead}} := \left\{ u \in L^\infty((0, T)) : \ v_{\ell\max} \geq v_{\ell 0} + \int_0^t u(s)\, \mathrm{d}s \geq 0 \ \forall t \in [0, T] \right\} \qquad (3.7)$$

and the initial values $(x_0, x_{\ell 0}) \in \mathbb{R}^2$ so that

$$x_{\ell 0} - x_0 - l > 0 \text{ as well as } (v_0, v_{\ell 0}) \in \mathbb{R}_{\geq 0}^2.$$

Remark 3 (The leading vehicle trajectory). As can be seen from the previous Definition 3.1, the leading vehicle's position x_ℓ is uniquely determined by u_{lead} and its initial position and velocity $x_{\ell 0}, v_{\ell 0}$. The condition in Eq (3.7) assures that the leading vehicle's velocity is never negative and bounded from above by $v_{\ell\max}$, a reasonable assumption for microscopic vehicle dynamics. Thus, generally, we can assume that x_ℓ is just a given trajectory. However, as we are later on also interested in optimal control related questions, we describe the leading vehicle's trajectory in terms of acceleration and not in terms of position/velocity.

In the definition of $\mathcal{U}_{\text{lead}}$ it might make sense to add additional lower and upper bounds, for instance replacing $\mathcal{U}_{\text{lead}}$ by $\tilde{\mathcal{U}}_{\text{lead}}$ with

$$\tilde{\mathcal{U}}_{\text{lead}} := \{ u \in \mathcal{U}_{\text{lead}} : u_{\min} \leq u(t) \leq u_{\max} \ \forall t \in [0, T] \text{ a.e.} \}$$

with the lower bound $u_{\min} \in \mathbb{R}_{<0}$ and upper bound $u_{\max} \in \mathbb{R}_{>0}$ on the acceleration. From an application point of view, such bounds are quite natural as maximal deceleration and acceleration in ordinary traffic situations are estimated to be around $-2\frac{m}{s^2}$ and $4\frac{m}{s^2}$ (m denoting meters and s seconds), see [7, 2, 21]. However, for our analysis we only require the velocity of the leading vehicle to not become negative, and left these additional and in application for sure reasonable assumptions aside.

As described in Remark 3 in our analysis we only need to focus on Eqs (3.3)–(3.5) and can just assume that leader's position and velocity is given. We consider as solutions classical solutions in the extended case which we will state as follows:

Definition 3.2 (Extended classical solution-Non-delayed). Given $(T, v_{\ell\max}, l) \in \mathbb{R}^3_{>0}$, Acc as in Definition 2.1 and the initial values $(x_0, x_{\ell 0}, v_0, v_{\ell 0}) \in \mathbb{R}^2 \times \mathbb{R}^2_{\geq 0}$ so that

$$x_{\ell 0} - x_0 - l > 0,$$

the mapping

$$((x_\ell, x, v) \in W^{2,\infty}((0,T))^2 \times W^{1,\infty}((0,T))$$

is called the classical solution of the Eqs (3.1)–(3.6) in the extended case if it satisfies the following integral equation

$$x_\ell(t) = x_{\ell 0} + t v_{\ell 0} + \int_0^t (t-s) u_{\text{lead}}(s) \, ds \qquad \forall t \in [0, T]$$

$$x(t) = x_0 + \int_0^t v(s) \, ds \qquad \forall t \in [0, T]$$

$$v(t) = v_0 + \int_0^t \text{Acc}(x(s), x_\ell(s), v(s), \dot{x}_\ell(s)) \, ds \qquad \forall t \in [0, T].$$

Now we study the well-posedness, that is the existence and uniqueness of the solution of the Eqs (3.1)–(3.6) in the extended case.

Theorem 3.3 (Well-posedness, minimal distance for the Bando-FtL model). *Consider the Bando-FtL model in Definition 3.1 with the assumptions therein. Then, there exists a unique solution $x \in W^{2,\infty}((0,T))$ in the sense of Definition 3.2 on any finite time horizon $T \in \mathbb{R}_{>0}$, the solution satisfies with the constants*

$$A(t) := -v_0 - \alpha t \|V\|_{L^\infty(\mathbb{R}_{>0})} + \alpha(x_{\ell 0} - x_0 - l) - \beta \frac{1}{x_{\ell 0} - x_0 - l} \tag{3.8}$$

$$d_{\min}(t) := \frac{A(t) + \sqrt{A(t)^2 + 4\alpha\beta}}{2\alpha} \tag{3.9}$$

$$B(t) := \frac{\alpha d_{\min}(t)^2 + \beta}{d_{\min}(t)^2} \tag{3.10}$$

the minimal distance $\forall t \in [0, T]$

$$x_\ell(t) - x(t) - l \geq d_{\min}(t), \tag{3.11}$$

has nonnegative bounded velocity (from above and from below) $\forall t \in [0, T]$

$$v_0 \exp\left(-\int_0^t B(s) \, ds\right) \leq \dot{x}(t) \leq \max\left\{v_0, \|V\|_{L^\infty(\mathbb{R})} + \frac{\beta}{\alpha} \frac{v_\ell(t)}{d_{\min}(t)^2}\right\}, \tag{3.12}$$

and upper and lower bounded acceleration $\forall t \in [0, T]$

$$-B(t) \max\left\{v_0, \|V\|_{L^\infty(\mathbb{R})} + \frac{\beta}{\alpha} \frac{v_\ell(t)}{d_{\min}(t)^2}\right\} \leq \ddot{x}(t)$$

$$\alpha \|V\|_{L^\infty(\mathbb{R}_{\geq 0})} - \alpha v_0 e^{-\int_0^t B(s) \, ds} + \beta \frac{v_\ell(t)}{d_{\min}(t)^2} \geq \ddot{x}(t). \tag{3.13}$$

Proof. As the initial datum is assumed to satisfy $x_{\ell 0} - x_0 - l > 0$ and as $V \in C^1(\mathbb{R}_{\geq 0}; \mathbb{R}_{\geq 0}) \cap L^\infty(\mathbb{R}_{\geq 0})$ there exists a small time horizon $T^* \in \mathbb{R}_{>0}$ so that on $t \in [0, T^*]$ the right hand side of the second order ODE in Eqs (3.3) and (3.4) is Lipschitz-continuous so that we have by the Picard-Lindelöf theorem

[18, Chapter II, Theorem 1.1] a unique solution (for the initial value problem in x_ℓ there is nothing to do, it does not depend on anything except an a priori given function u_{lead}). Let us next consider-on this time horizon $[0, T^*] \ni t$ – the difference in the change of velocities and estimate:

$$\dot{v}_\ell(t) - \dot{v}(t) = u_{\text{lead}}(t) - \alpha(V(x_\ell(t) - x(t) - l) - v(t)) - \beta \frac{v_\ell(t) - v(t)}{\left(x_\ell(t) - x(t) - l\right)^2}$$

$$\overset{(2.1)}{\geq} u_{\text{lead}}(t) - \alpha(\|V\|_{L^\infty(\mathbb{R}_{\geq 0})} - v(t)) - \beta \frac{v_\ell(t) - v(t)}{\left(x_\ell(t) - x(t) - l\right)^2} \tag{3.14}$$

Integrating over $[0, t]$ and recalling that $\dot{x}_\ell - \dot{x} \equiv v_\ell - v$ we end up with

$$v_\ell(t) - v(t) \geq v_{\ell 0} - v_0 + \int_0^t u_{\text{lead}}(s)\,\mathrm{d}s - \alpha t \|V\|_{L^\infty(\mathbb{R}_{\geq 0})} + \alpha \int_0^t v(s)\,\mathrm{d}s$$

$$+ \beta \frac{1}{x_\ell(t) - x(t) - l} - \beta \frac{1}{x_{\ell 0} - x_0 - l}$$

$$\geq v_{\ell 0} - v_0 + \int_0^t u_{\text{lead}}(s)\,\mathrm{d}s - \alpha t \|V\|_{L^\infty(\mathbb{R}_{\geq 0})} + \alpha x(t) - \alpha x_0$$

$$+ \beta \frac{1}{x_\ell(t) - x(t) - l} - \beta \frac{1}{x_{\ell 0} - x_0 - l}$$

$$\overset{(4.7)}{\geq} -v_0 - \alpha t \|V\|_{L^\infty(\mathbb{R}_{\geq 0})} - \alpha(x_\ell(t) - x(t) - l) + \alpha x_\ell(t) - \alpha(x_0 + l)$$

$$+ \beta \frac{1}{x_\ell(t) - x(t) - l} - \beta \frac{1}{x_{\ell 0} - x_0 - l}$$

$$\overset{(4.7)}{\geq} -v_0 - \alpha t \|V\|_{L^\infty(\mathbb{R}_{\geq 0})} - \alpha(x_\ell(t) - x(t) - l) + \alpha(x_{\ell 0} - x_0 - l)$$

$$+ \beta \frac{1}{x_\ell(t) - x(t) - l} - \beta \frac{1}{x_{\ell 0} - x_0 - l}.$$

Substituting $h(t) := x_\ell(t) - x(t) - l$, we then have

$$h'(t) \geq -v_0 - \alpha t \|V\|_{L^\infty(\mathbb{R}_{\geq 0})} + \alpha(x_{\ell 0} - x_0 - l) - \beta \frac{1}{x_{\ell 0} - x_0 - l} - \alpha h(t) + \beta \frac{1}{h(t)}$$

$$= A(t) - \alpha h(t) + \beta \frac{1}{h(t)}$$

with A as in Eq (3.8). As can be seen, for $h(t)$ small, the right hand side becomes strictly positive so that the distance cannot decrease anymore. And indeed, we have

$$h(t) < \frac{A(t) + \sqrt{A(t)^2 + 4\alpha\beta}}{2\alpha} \implies h'(t) > 0. \tag{3.15}$$

However, from this we can conclude as lower bound for h

$$h(t) \geq \frac{A(t) + \sqrt{A(t)^2 + 4\alpha\beta}}{2\alpha}$$

as long as the initial datum $h(0) = x_{\ell 0} - x_0 - l$ also satisfies this lower bound. If it does not, the initial datum itself becomes the lower bound as by Eq (3.15) h is then increasing. Concluding, we have

$$x_\ell(t) - x(t) - l \geq \min\left\{\frac{A(t) + \sqrt{A(t)^2 + 4\alpha\beta}}{2\alpha}, x_{\ell 0} - x_0 - l\right\} > 0.$$

As we will show in the following, it holds that

$$\frac{A(t) + \sqrt{A(t)^2 + 4\alpha\beta}}{2\alpha} \leq x_{\ell 0} - x_0 - l \ \forall t \in [0, T]$$

so that the lower bound can be simplified. In fact, estimating as follows and squaring the discriminant we have for $t \in [0, T]$

$$
\begin{aligned}
(A(t)^2 + 4\alpha\beta) &= v_0^2 + t^2\alpha^2\|V\|^2_{L^\infty(\mathbb{R}_{>0})} + \alpha^2(x_{\ell 0} - x_0 - l)^2 + \frac{\beta^2}{(x_{\ell 0} - x_0 - l)^2} \\
&\quad + 2v_0 t\alpha\|V\|^2_{L^\infty(\mathbb{R}_{>0})} - 2v_0\alpha(x_{\ell 0} - x_0 - l) + \frac{2v_0\beta}{x_{\ell 0} - x_0 - l} \\
&\quad - 2t\alpha^2\|V\|^2_{L^\infty(\mathbb{R}_{>0})}(x_{\ell 0} - x_0 - l) + \frac{t\alpha\beta\|V\|_{L^\infty(\mathbb{R}_{>0})}}{x_{\ell 0} - x_0 - l} + 2\alpha\beta \\
&\leq v_0^2 + t^2\alpha^2\|V\|^2_{L^\infty(\mathbb{R}_{>0})} + \alpha^2(x_{\ell 0} - x_0 - l)^2 + \frac{\beta^2}{(x_{\ell 0} - x_0 - l)^2} \\
&\quad + 2v_0 t\alpha\|V\|^2_{L^\infty(\mathbb{R}_{>0})} + 2v_0\alpha(x_{\ell 0} - x_0 - l) + \frac{2v_0\beta}{x_{\ell 0} - x_0 - l} \\
&\quad + 2t\alpha^2\|V\|^2_{L^\infty(\mathbb{R}_{>0})}(x_{\ell 0} - x_0 - l) + \frac{t\alpha\beta\|V\|_{L^\infty(\mathbb{R}_{>0})}}{x_{\ell 0} - x_0 - l} + 2\alpha\beta \\
&= \left(v_0 + t\alpha\|V\|_{L^\infty(\mathbb{R}_{>0})} + \alpha(x_{\ell 0} - x_0 - l) + \frac{\beta}{x_{\ell 0} - x_0 - l}\right)^2 \\
&= (2\alpha(x_{\ell 0} - x_0 - l) - A(t))^2.
\end{aligned}
$$

This is exactly the term which we need to estimate against to obtain that

$$
\frac{A(t) + \sqrt{A(t)^2 + 4\alpha\beta}}{2\alpha} \leq x_{\ell 0} - x_0 - l
$$

from which it follows that

$$
x_\ell(t) - x(t) - l \geq \frac{A(t) + \sqrt{A(t)^2 + 4\alpha\beta}}{2\alpha} =: d_{\min}(t) \ \forall t \in [0, T]
$$

what we wanted to show. Furthermore, for all $t \in [0, T]$, we have

- that Eq (3.11) holds as long as the solution exists,
- the existence and uniqueness of the solution on the considered time horizon $[0, T]$ as we can lower bound the distance with the estimate from above.

To show that the velocity of the ego vehicle, $\dot{x} \equiv v$ satisfies the claimed bound we estimate as follows-taking advantage of the fact that by Eq (3.7) we have $v_\ell(t) \geq 0 \ \forall t \in [0, T]$ and by Eq (2.1) $V \geqq 0$

$$
\dot{v}(t) = \alpha\Big(V(x_\ell(t) - x(t) - l) - v(t)\Big) + \beta\frac{v_\ell(t) - v(t)}{(x_\ell(t) - x(t) - l)^2} \tag{3.16}
$$

$$
\geq -\alpha v(t) - \beta\frac{v(t)}{d_{\min}(t)^2} \tag{3.17}
$$

and integrating over $[0, t]$ gives

$$
v(t) \geq v_0 \exp\Big(-\int_0^t \frac{\alpha d_{\min}(s)^2 + \beta}{d_{\min}(s)^2}\,\mathrm{d}s\Big), \qquad \forall t \in [0, T] \tag{3.18}
$$

which is the lower bound in Eq (3.12). As both latter terms, v_0 and $t \mapsto \exp\Big(-\int_0^t \frac{\alpha d_{\min}(s)^2 + \beta}{d_{\min}(s)^2}\,\mathrm{d}s\Big)$, are nonnegative, the velocity is bounded from below by zero. For the upper bound, we estimate (using that $v \geqq 0$ as established previously)

$$
\dot{v}(t) \leq \alpha\|V\|_{L^\infty(\mathbb{R}_{>0})} - \alpha v(t) + \beta\frac{v_\ell(t)}{d_{\min}(t)^2}, \qquad \forall t \in [0, T].
$$

If $v(t) > \|V\|_{L^\infty(\mathbb{R})} + \frac{\beta}{\alpha} \frac{v_\ell(t)}{d^2_{\min}(t)}$, we have $\dot{v}(t) < 0$ so that we indeed obtain as upper bound

$$v(t) \leq \max\left\{v_0, \|V\|_{L^\infty(\mathbb{R})} + \frac{\beta}{\alpha} \frac{v_\ell(t)}{d_{\min}(t)^2}\right\}.$$

Finally, let us prove the acceleration bounds. Recall the acceleration formula in Definition 2.1, Eq (2.2) once more

$$\dot{v}(t) = \alpha\big(V(x_\ell(t) - x(t) - l) - v(t)\big) + \beta\frac{v_\ell(t)-v(t)}{(x_\ell(t)-x(t)-l)^2}, \qquad \forall t \in [0, T]$$

and the upper and lower bound of the velocity in Eq (3.12), we estimate for the upper bound on the acceleration

$$\dot{v}(t) \overset{(3.12)}{\leq} \alpha\|V\|_{L^\infty(\mathbb{R})} - \alpha v_0 \exp\Big(-\int_0^t B(s)\,\mathrm{d}s\Big) + \beta\frac{v_\ell(t)}{d_{\min}(t)^2} \qquad \forall t \in [0, T]$$

which is exactly Eq (3.13). For the lower bound, estimate as in Eq (3.17) for $t \in [0, T]$

$$\dot{v}(t) \geq -\alpha v(t) - \beta\frac{v(t)}{d_{\min}(t)^2}$$
$$\overset{(3.12)}{\geq} -B(t)\max\left\{v_0, \|V\|_{L^\infty(\mathbb{R})} + \frac{\beta}{\alpha}\frac{v_\ell(t)}{d_{\min}(t)^2}\right\}.$$

This concludes the proof.

Remark 4 (Some comments on the Bando-FtL model)**.** The results in Theorem 3.3 need to be interpreted in terms of their applicability and reasonability. It is worth mentioning that none of the previously obtained bounds depend on the behavior of the leading vehicle, except that it never drives backwards, i.e., its velocity is non-negative, and make the estimates to some extend not very sharp. On the other hand, they are strong as they do not require any additional assumptions which comes in particular handy when applying the results to some optimal control problems where uniform bounds are required but the leading vehicle's trajectory cannot be specified a priori except from knowing that the leading vehicle's velocity does not become negative.

3.1. Strictly positive velocity

The inequality in Eq (3.12) guarantees that the velocity never becomes negative (independent on what the leading vehicle trajectory does) and not even zero as long as the initial velocity v_0 was not zero. From an application point of view, one might criticize a behavior stating that the velocity never becomes zero. On the other hand, the velocity can become very small so that the modelling can still be justified. The upper bound on the velocity is clearly not sharp, as it relies on $d_{\min}$ which is the "worst case" estimate for the lower bound on the distance. Additionally, we have not postulated anything additional on V so that also V is estimated in a "worst case" way.

3.2. The lower bound and the impact of A, B

Looking at $d_{\min}$ as in Eq (3.9) one can observe that $d_{\min} > 0$ as $4\alpha\beta > 0$ and $\alpha > 0$. The constant A could either be positive or negative, however for large time horizon $T \in \mathbb{R}_{>0}$ or small initial space headway $x_{\ell 0} - x_0 - l$, the constant A will be negative. Thus the minimal distance $d_{\min}$ approaches 0 as the time horizon T gets large or the initial space headway $x_{\ell 0} - x_0 - l$ approaches 0.

3.3. The bounds on the acceleration

The derived bounds on the acceleration in Eq (3.13) are once more "worst case" estimates and are clearly not sharp. However, for theoretical studies they enable uniform bounds independent on what the leading vehicle does as long as it does not drive backwards, and prove to be of significant help.

The previous Theorem 3.3 has shown among other things how to obtain a lower bound on the distance and can be refined further. In the following remark, we detail this:

Remark 5 (Minimal distance of the Bando-FtL model revisited). In Theorem 3.3 we estimated the ego vehicle's optimal velocity on the time $t \in [0, T]$, $T \in \mathbb{R}_{>0}$, $V(x_\ell(t) - x(t) - l)$, (see in particular Eq (3.14)) from above by $\|V\|_{L^\infty(\mathbb{R}_{\geq 0})}$. One can also keep the ego vehicle's optimal velocity and obtain for $t \in [0, T]$

$$h'(t) \geq -v_0 - \alpha \int_0^t V(h(s)) \, \mathrm{d}s + \alpha(x_{\ell 0} - x_0 - l) - \beta \frac{1}{x_{\ell 0} - x_0 - l} - \alpha h(t) + \beta \frac{1}{h(t)}$$

$$= A_1 - \alpha \int_0^t V(h(s)) \, \mathrm{d}s - \alpha h(t) + \beta \frac{1}{h(t)}$$

with

$$h(t) := x_\ell(t) - x(t) - l \qquad\qquad t \in [0, T]$$

$$A_1 := -v_0 + \alpha(x_{\ell 0} - x_0 - l) - \beta \frac{1}{x_{\ell 0} - x_0 - l}.$$

As can be seen, for $h(t)$ small, the right hand side of the above inequality becomes strictly positive as long as the integral term does not increase too much so that the space headway h cannot decrease anymore. Indeed, define for $t \in [0, T]$, $F[h](t) := A_1 - \alpha \int_0^t V(h(s)) \, \mathrm{d}s - \alpha h(t) + \beta \frac{1}{h(t)}$. Then we are interested in the zeros for this operator as it will tell us when the headway increases/decreases. Thus, we study

$$F[h^*](t) = 0 \implies \int_0^t V(h^*(s)) \, \mathrm{d}s = \frac{\beta}{\alpha h^*(t)} - h^*(t) + \frac{A}{\alpha}$$

and as this needs to hold for all $t \in [0, T]$, we can take the derivative with respect to t on both sides and obtain

$$\implies V(h^*(t)) = -\frac{\beta}{\alpha} \frac{1}{(h^*(t))^2} h^{*\prime}(t) - h^{*\prime}(t)$$

$$\implies 1 = \frac{-\frac{\beta}{\alpha} \frac{1}{(h^*(t))^2} h^{*\prime}(t) - h^{*\prime}(t)}{V(h^*(t))}$$

$$\implies t = \int_0^t \frac{-\frac{\beta}{\alpha} \frac{1}{(h^*(s))^2} h^{*\prime}(s) - h^{*\prime}(s)}{V(h^*(s))} \, \mathrm{d}s$$

$$\implies t = \int_{h(0)}^{h^*(t)} \frac{-\frac{\beta}{\alpha} \frac{1}{u^2} - 1}{V(u)} \, \mathrm{d}u. \tag{3.19}$$

Let

$$G : \mathbb{R} \to \mathbb{R} \text{ with } a \mapsto \int_{h(0)}^a \frac{-\frac{\beta}{\alpha} \frac{1}{u^2} - 1}{V(u)} \, \mathrm{d}u. \tag{3.20}$$

Then, G is strictly monotonically decreasing with respect to a and thus invertible, which is why we can write

$$h^*(t) = G^{-1}(t) \ \forall t \in [0, T].$$

We claim that $h^*(t) > 0 \ \forall t \in [0, T]$. To this end, recall Eq (3.19) and estimate as follows assuming that $h^*(t) \leq h(0)$

$$t = \int_{h(0)}^{h^*(t)} \frac{-\frac{\beta}{\alpha}\frac{1}{u^2}-1}{V(u)} \, \mathrm{d}u \geq \int_{h(0)}^{h^*(t)} \frac{-\frac{\beta}{\alpha}\frac{1}{u^2}-1}{\|V\|_{L^\infty(\mathbb{R})}} \, \mathrm{d}u$$

$$= \frac{1}{\alpha \|V\|_{L^\infty(\mathbb{R})}} \left(\frac{\beta}{h^*(t)} - \alpha h^*(t) - \frac{\beta}{h(0)} + \alpha h(0) \right)$$

and solving the least inequality for $h^*(t)$ we obtain

$$t\alpha\|V\|_{L^\infty(\mathbb{R})} h^*(t) \geq \beta - \alpha(h^*(t))^2 + (\alpha h(0) - \tfrac{\beta}{h(0)}) h^*(t)$$

and thus $\forall t \in [0, T]$

$$h^*(t) \geq \frac{-\left(\frac{\beta}{h^*(0)} - \alpha h^*(0) + t\alpha\|V\|_{L^\infty(\mathbb{R})} \right) + \sqrt{\left(\frac{\beta}{h^*(0)} - \alpha h^*(0) + t\alpha\|V\|_{L^\infty(\mathbb{R})} \right)^2 + 4\alpha\beta}}{2\alpha} > 0$$

which is indeed comparable with the lower bound in Eq (3.9).

However, similarly to the reasoning in Theorem 3.3, we can conclude as lower bound for h then

$$h(t) \geq G^{-1}(t) \ \forall t \in [0, T] \tag{3.21}$$

which is generally a sharper bound. The only problem with this bound is that one needs to find the function G^{-1} which is only implicitly defined by Eq (3.20). As it only contains the computation of an integral, this is at least numerically feasible. Clearly, based on this estimate, the bounds on the derivatives of x, i.e., velocity and acceleration (compare Eqs (3.12) and (3.13) can be improved as well, however, we will not detail this.

Example 1 (Lower bounds). Now we assume that the leading vehicle's acceleration u_{lead} at time $t \in [0, \infty)$ satisfies the following

$$u_{\text{lead}}(t) := \begin{cases} 1, & t \in \left[\sum\limits_{k=0}^{n} 4k + n + 1, \ \sum\limits_{k=0}^{n} 4k + 2(n+1) \right) \\ -1, & t \in \left[\sum\limits_{k=0}^{n} 4k + 3(n+1)), \ \sum\limits_{k=0}^{n+1} 4k \right) \qquad n \in \mathbb{N}_{\geq 0} \\ 0, & \text{otherwise,} \end{cases} \tag{3.22}$$

which is illustrated in Figure 2 and the following vehicles' acceleration is determined by the Bando-FtL model as defined in Definition 2.1. Then, we obtain the following on the time horizon $T = 25$ assuming that the ego vehicle's acceleration has the constants as in Eq (7.1). Assume that the initial datum for the leading and following vehicle satisfy $(x_{\ell 0}, x_0) = (l + d_s, 0) = (4.5 + 2.5, 0)$ and $(v_{\ell 0}, v_0) = (0, 0)$. In addition, we choose the parameters $\alpha = 0.5, \beta = 20, v_{\max} = 10, l = 4.5, d_s = 2.5$ (these are according to [21] reasonable choices). According to Theorem 3.3, we have

$$A(t) := -5t + 0.5 \cdot 2.5 - 20\tfrac{1}{2.5} = -5t - 6.75 \tag{3.23}$$

$$d_{\min}(t) := \frac{A(t)+\sqrt{A(t)^2+4\alpha\beta}}{2\alpha} = -5t - 6.75 + \sqrt{(-5t-6.75)^2 + 40}. \tag{3.24}$$
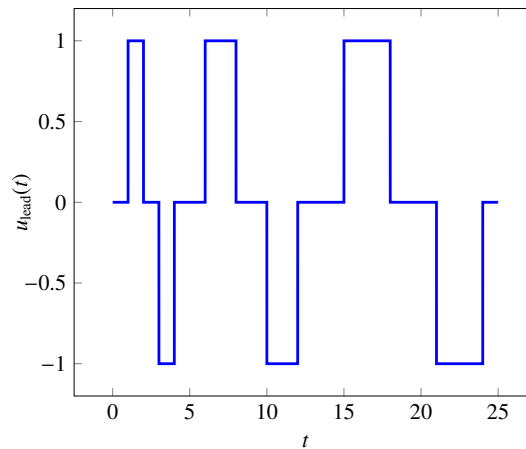

Figure 2. The acceleration profile u_{lead} of the leading vehicle.

By Remark 5 and numerical integration, we obtain another lower bound $G^{-1}(t)$ which is illustrated in Figure 3 together with $d_{\min}(t)$. As can be seen, the lower bound $G^{-1}(t)$ on the headway is rather sharp particularly for smaller time horizon, however, the lower bound of the velocity is very conservative as it is zero. We have not even computed the upper bound as it is very large.
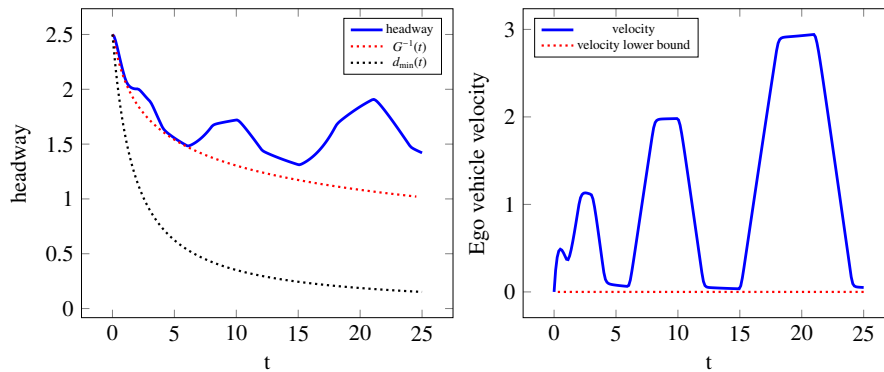
Figure 3. Left the headway and right the ego vehicle's velocity in blue while the lower bounds are presented in dotted.

As we will later on use this specific acceleration file again, we comment on its properties:

Remark 6 (The leading vehicle's acceleration)**.** The leading vehicle's acceleration u_{lead} as defined in Eq (3.22) is changing among $0, 1$, and -1 in a non-periodical way. That is, the first leading vehicle repeats the pattern "constant velocity, accelerate, constant velocity, decelerate" non-periodically, see Figure 2.

4. Well-posedness: No real-time information available–the delayed case

In this section, we introduce–based on the original Bando-FtL–a delay in the reaction to the leading vehicle's position and velocity which can be interpreted as the *information delay* the ego vehicle is

exposed to either due to their own reaction time or more realistically due to the time required to evaluate the obtained information on the leader in a suitable way. We want to underline that from a mathematical point of view, this is not a typical delay ordinary differential equation as these types of equations usually involve a delay in the solution of the modelling equation itself [13] and not some externally given dependencies (as the leading vehicle's trajectory). The model reads mathematically as follows:

Definition 4.1 (The Bando-FtL – information delay with respect to the leading vehicle)**.** For $T, v_{\ell\max}, l \in \mathbb{R}_{>0}$ and Acc as in Definition 2.1 we consider for a delay $\delta \in \mathbb{R}_{>0}$ the following initial value problem in $(x_\ell, x) \in W^{2,\infty}((-\delta, T)) \times W^{2,\infty}((0, T))$

$$\dot{x}_\ell(t) = v_\ell(t) \qquad\qquad t \in [-\delta, T] \qquad (4.1)$$
$$\dot{v}_\ell(t) = u_{\text{lead}}(t) \qquad\qquad t \in [-\delta, T] \qquad (4.2)$$
$$\dot{x}(t) = v(t) \qquad\qquad t \in [0, T] \qquad (4.3)$$
$$\dot{v}(t) = \text{Acc}(x(t), x_\ell(t - \delta), v(t), v_\ell(t - \delta)) \qquad t \in [0, T] \qquad (4.4)$$
$$(x(0), v(0)) = (x_0, v_0) \qquad\qquad (4.5)$$
$$(x_\ell(-\delta), v_\ell(-\delta)) = (x_{\ell, -\delta}, v_{\ell, -\delta}) \qquad\qquad (4.6)$$

with the leading vehicle acceleration satisfying

$$u_{\text{lead}} \in \mathcal{U}_{\text{lead}\delta} := \left\{ u \in L^\infty((-\delta, T)) : v_{\ell\max} \geq v_{\ell, -\delta} + \int_{-\delta}^t u(s)\, ds \geq 0 \ \forall t \in (-\delta, T] \right\} \qquad (4.7)$$

and the initial values $(x_0, x_{\ell, -\delta}) \in \mathbb{R}^2$ so that

$$x_{\ell, -\delta} - x_0 - l > 0$$

as well as $(v_0, v_{\ell, -\delta}) \in \mathbb{R}^2_{\geq 0}$.

Remark 7 (Information delay – a more conservative approach)**.** Interestingly, taking a delayed version of the leading vehicle trajectory into account for the ego vehicle's acceleration will lead to a more conservative approach as one will assume that the car is actually closer (as $x_\ell(t - \delta) \leq x_\ell(t) \ \forall t \in [0, T]$). However, this will mean that the ego vehicle will break sooner than in the non-delayed case and potentially increase the variation in velocity. We will dwell on this question in particular in Section 6.

Remark 8 (The definition of $\mathcal{U}_{\text{lead}\delta}$)**.** As can be seen from the previous Definition 4.1, analogously to the undelayed case in Section 3 the leading vehicle's trajectory x_ℓ is uniquely determined by u_{lead} and the leading vehicle's position and velocity at time $-\delta$, $(x_{\ell, -\delta}, v_{\ell, -\delta})$. The condition in Eq (4.7) assures that the leading vehicle's velocity is never negative and is bounded from above by $v_{\ell\max}$, a reasonable assumption for microscopic vehicle dynamics.

We again consider the extended classical solutions for the delayed case as follows:

Definition 4.2 (Extended classical solution – delayed case)**.** Assume $T, v_{\ell\max}, l \in \mathbb{R}_{>0}$, $\delta \in \mathbb{R}_{\geq 0}$, Acc as in Definition 2.1. Let the initial values $(x_0, x_{\ell-\delta}) \in \mathbb{R}^2$ be given so that

$$x_{\ell-\delta} - x_0 - l \geq 0 \text{ as well as } (v_0, v_{\ell-\delta}) \in \mathbb{R}^2_{\geq 0},$$

we call $(x_\ell, x, v) \in W^{2,\infty}((-\delta, T)) \times W^{2,\infty}((0, T)) \times W^{1,\infty}((0, T))$ the classical solution of Eqs (4.1)–(4.6) in the extended cases if the following integral equations are satisfied

$$x_\ell(t) = x_{\ell,-\delta} + (t + \delta)v_{\ell,-\delta} + \int_{-\delta}^{t} (t - s)u_{\text{lead}}(s)\,\mathrm{d}s \qquad\qquad \forall t \in [-\delta, T]$$

$$x(t) = x_0 + \int_0^t v(s)\,\mathrm{d}s \qquad\qquad \forall t \in [0, T]$$

$$v(t) = v_0 + \int_0^t \text{Acc}(x(s), x_\ell(s - \delta), v(s), v_\ell(s - \delta))\,\mathrm{d}s \qquad\qquad \forall t \in [0, T].$$

The following theorem is almost identical to the non-delayed case in Theorem 3.3, the only difference being in that we use delayed information of leading vehicle position and velocity. However, as both are delayed by the same amount of time $\delta \in \mathbb{R}_{>0}$, the model behaves identical when replacing the leading vehicle trajectory in Theorem 3.3, u_{lead} by a new $\tilde{u}_{\text{lead}}$ which is delayed, i.e., setting

$$\tilde{u}_{\text{lead}}(t) = u_{\text{lead}}(t - \delta) \; \forall t \in [0, T].$$

Theorem 4.3 (Well-posedness for the information delayed Bando-FtL model). *Given the initial value problem in Definition 4.1 with the therein stated assumptions, in particular that $(x_{\ell-\delta}, v_{\ell-\delta}) \in \mathbb{R} \times \mathbb{R}_{\geq 0}$ and $u_{lead} \in \mathcal{U}_{lead\delta}$ are chosen so that*

$$x_\ell(-\delta) - x_0 - l > 0,$$

there exists for each $T \in \mathbb{R}_{>0}$ a unique solution of the initial value problem in Definition 4.1 on the time horizon $[0, T]$ in the sense of Definition 4.2 and the solution satisfies the lower bound

$$x_\ell(t) - x(t) - l \geq x_\ell(t - \delta) - x(t) - l \geq \frac{A + \sqrt{A^2 + 4\alpha\beta}}{2\alpha} =: d_{\min} > 0 \; \forall t \in [0, T] \qquad (4.8)$$

with

$$A := -v_0 - \alpha T\|V\|_{L^\infty(\mathbb{R}_{>0})} + \alpha(x_\ell(-\delta) - x_0 - l) - \beta\frac{1}{x_\ell(-\delta) - x_0 - l}$$

and $(\alpha, \beta, l) \in \mathbb{R}_{>0}^3$ as in Definition 2.1.

Additionally, for all $t \in [0, T]$ we have for the velocity $\dot{x}$ and its derivative, the acceleration, the following lower and upper bounds $\forall t \in [0, T]$

$$v_0 \exp\left(-t\left(\alpha + \frac{\beta}{d_{\min}^2}\right)\right) \leq \dot{x}(t) \leq v_0 + t\alpha\|V\|_{L^\infty(\mathbb{R}_{>0})} + t\beta\frac{\|v_\ell\|_{L^\infty((-\delta,T))}}{d_{\min}^2}, \qquad (4.9)$$

$$-\left(\alpha + \frac{\beta}{d_{\min}^2}\right)\left(v_0 + T\alpha\|V\|_{L^\infty(\mathbb{R}_{>0})} + T\beta\frac{\|v_\ell\|_{L^\infty((-\delta,T))}}{d_{\min}^2}\right) \leq \dot{v}(t) \qquad (4.10)$$

$$\dot{v}(t) \leq \alpha\|V\|_{L^\infty(\mathbb{R}_{>0})} + \beta\frac{\|v_\ell\|_{L^\infty((-\delta,T))}}{d_{\min}^2}. \qquad (4.11)$$

Proof. Since both the velocity and position of the leading vehicle delay by the same amount of time δ, the proof of this theorem is similar to the proof of Theorem 3.3. Therefore, we omit the proof here. $\square$

Remark 9 (Generalizations of Theorem 4.3). We want to stress that the same steps as in Section 3 to improve the lower bound on the minimal distance Remark 5 can also be taken in the delayed case.

5. Convergence of solutions when delay tends to zero

The considered class of car-following models should be continuous with regard to delay as otherwise, they would prove to be not very realistic as the driver behavior is clearly subject to information delay. This is why in our following analysis we will prove that the delayed solution of Definition 4.1 converges to the un-delayed solution of Definition 3.1 in the proper topology. To this end, we require the following theorem which states a well-known result about Banach spaces and ((sub)-sub)-sequences.

Lemma 5.1 (Convergence of (sub)-subsequences to the same limit implies convergence of the sequence). *Let B be a real Banach space and $\{x_n\}_{n \in \mathbb{N}_{\geq 1}} \subset B$ be a sequence. Assume that there exists $x^* \in B$ so that for each subsequence $\{x_{n_k}\}_{k \in \mathbb{N}_{\geq 1}}$ there exists a (sub)-subsequence $\{x_{n_{k_l}}\}_{l \in \mathbb{N}_{\geq 1}}$ converging to x^*, i.e.,*

$$\lim_{l \to \infty} \|x_{n_{k_l}} - x^*\|_B = 0.$$

Then, the original sequence $\{x_n\}_{n \in \mathbb{N}_{\geq 1}}$ converges to x^*, i.e.,

$$lim_{n \to \infty} \|x_n - x^*\|_B = 0.$$

Proof. We prove by contradiction and assume that the original sequence $\{x_n\}_{n \geq 1}$ does not converge to $x^* \in B$. Then there exists $\varepsilon_0 > 0$, such that for every $M \in \mathbb{N}_{\geq 1}$, there exists $m_M \in \mathbb{N}_{\geq 1}$ with $m_M > M$, such that

$$\|x_{m_M} - x^*\|_B \geq \varepsilon_0 \quad \forall m_M > M.$$

For the thus constructed subsequence $\{x_{m_M}\}_{M \in \mathbb{N}_{\geq 1}}$, there is no (sub)-subsequence converging to $x^* \in B$ which is a contradiction to the fact that $\{x_n\}_{n \in \mathbb{N}_{\geq 1}}$ does not converge. As it converges, and as the limit of each converging subsequence is x^*, the original sequence must also converge to x^*.

Equipped with this convergence result, we state the main theorem of this section, the convergence of delayed solution to the non-delayed solution if delay converges to zero, in the $W^{1,\infty}$ topology.

Theorem 5.2 (Convergence delayed solution to non-delayed for delay to zero). *Given $\delta \in \mathbb{R}_{>0}$ and denote by $x[\delta] \in W^{2,\infty}((0,T))$ the unique solution to Definition 4.1. Let x denote the solution of the non-delayed Bando-FtL model in Definition 3.1. Then, the delayed solution converges to the non-delayed solution in the following sense:*

$$lim_{\delta \to 0} \|x - x[\delta]\|_{W^{1,\infty}((0,T))} = 0.$$

Proof. Take $t \in [0, T]$ and recall the uniform bounds (with regard to (δ)) on the acceleration as stated in Eqs (4.10) and (4.11) for a fixed finite time $T \in \mathbb{R}_{>0}$.

Then, we have that

$$\exists C \in \mathbb{R}_{\geq 0} \sup_{\delta \in \mathbb{R}_{>0}} \|\ddot{x}[\delta]\|_{L^\infty((0,T))} \leq C.$$

Using Ascoli-Arzela's theorem [9, Theorem 4.25] we have

$$\exists x^* \in W^{2,\infty}((0,T)), \; \|\ddot{x}^*\|_{L^\infty((0,T))} \leq C \; \wedge \; \{\delta_k\}_{k \in \mathbb{N}_{\geq 1}} \text{ with } \lim_{k \to \infty} \delta_k = 0 :$$

$$\lim_{k \to \infty} \|x[\delta_k] - x^*\|_{W^{1,\infty}((0,T))} = 0.$$

Recalling the definition of extended classical solutions for $x[\delta_k]$ it states that

$$x[\delta_k](t) = x_0 + \int_0^t \dot{x}[\delta_k](s) \, ds \qquad\qquad \forall t \in [0, T]$$

$$\dot{x}[\delta_k](t) = v_0 + \int_0^t \mathrm{Acc}(x[\delta_k](s), x_\ell(s - \delta_k), \dot{x}[\delta_k](s), v_\ell(s - \delta_k)) \, ds \qquad \forall t \in [0, T].$$

Thanks to the compactness result above in particular that $x[\delta_k]$ and $\dot{x}[\delta_k]$ converge in the uniform topology for $k \to \infty$, by Acc being Lipschitz (guaranteed by the lower bound on the distance as stated in Eq (3.11) in Theorem 4.3) and as $x_\ell, v_\ell \in C([-\delta, T])$, we can pass to the limit and indeed recover the non-delayed definition of the Caratheodory solution to Definition 3.1, i.e., for every $t \in [0, T]$,

$$\begin{aligned} x^*(t) &= x_0 + \int_0^t \dot{x}^*(s) \, ds \\ \dot{x}^*(t) &= v_0 + \int_0^t \mathrm{Acc}(x^*(s), x_\ell(s), \dot{x}^*(s), v_\ell(s)) \, ds. \end{aligned} \qquad (5.1)$$

As this can be done for every sequence $(\delta_{k_n})_{n \in \mathbb{N}_{\geq 1}}$ and as we know that there exists a unique solution to the non-delayed initial value problem in Eq (5.1) as guaranteed by Theorem 3.3 we can finally invoke Lemma 5.1 and obtain the claimed convergence of the entire sequence to the non-delayed solution. This concludes the proof.

Remark 10 (Convergence of the related accelerations). The convergence in Theorem 5.2 can be extended directly to the acceleration $\ddot{x}[\delta_k]$ as well, as the acceleration is according to Eq (4.4) fully determined by

$$\ddot{x}[\delta_k](t) = \mathrm{Acc}(x[\delta_k](t), x_\ell(t - \delta), \dot{x}[\delta_k](t), v_\ell(t - \delta)), \qquad t \in [0, T]$$

and as the right hand side converges uniformly for $\delta_k \to 0$, the left hand side does as well.

6. Extensions to the multi vehicle case

In this section, we will generalize our results to cover many cars. However, as there is an order of dependency, i.e., the dynamics of the current vehicle only depend on its leading vehicle, the extension is quite straight forward. Let us first define what we mean with the many-vehicle case for the original Bando-FtL model:

Definition 6.1 (The car-following model for many vehicles). Let $T > 0$ and $N \in \mathbb{N}_{\geq 1}$ be given. We assume that the parameters of the Acc satisfy what we have assumed in Definition 2.1 as well as the leading vehicle acceleration u_{lead}. Additionally, we assume for the initial datum (position and velocity)

$$\boldsymbol{x}_0 \in \mathbb{R}^N : \boldsymbol{x}_{0,i} - \boldsymbol{x}_{0,i+1} > l \ \forall i \in \{1, \dots, N-1\} \ \wedge \ \boldsymbol{v}_0 \in \mathbb{R}^N_{\geq 0}.$$

Then, the dynamics for the Bando-FtL model with many vehicles $N \in \mathbb{N}_{\geq 1}$ read in $(\boldsymbol{x}, \boldsymbol{v}) : [0, T] \to \mathbb{R}^N \times \mathbb{R}^N \ \forall t \in [0, T]$ as

$$
\begin{aligned}
\dot{\boldsymbol{x}}_1(t) &= \boldsymbol{v}_1(t), \\
\dot{\boldsymbol{v}}_1(t) &= u_{\text{lead}}(t), \\
\dot{\boldsymbol{x}}_i(t) &= \boldsymbol{v}_i(t), & i \in \{2, \dots, N\} \\
\dot{\boldsymbol{v}}_i(t) &= \text{Acc}(\boldsymbol{x}_i(t), \boldsymbol{v}_i(t), \boldsymbol{x}_{i-1}(t), \boldsymbol{v}_{i-1}(t)), & i \in \{2, \dots, N\} \\
\boldsymbol{x}(0) &= \boldsymbol{x}_0, \\
\boldsymbol{v}(0) &= \boldsymbol{v}_0.
\end{aligned}
\tag{6.1}
$$

The system for many cars is illustrated in Figure 4.
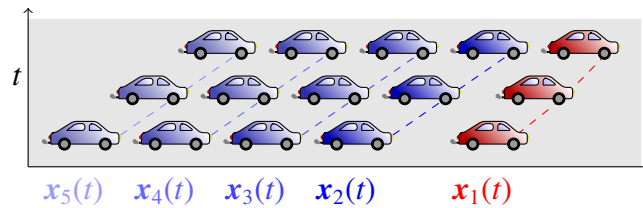
Figure 4. The leader $\boldsymbol{x}_1(t)$ with its dynamics determined by the acceleration $u_{\text{lead}}(t)$ and the following cars $\boldsymbol{x}_2(t), \boldsymbol{x}_3(t), \boldsymbol{x}_4(t), \boldsymbol{x}_5(t)$ with its dynamics governed by the Bando-FtL model in Definition 3.1. The overall dynamics is stated in Definition 6.1.

Remark 11 (The model parameters for the many car case)**.** One may choose different parameters $(\alpha_i, \beta_i) \in \mathbb{R}_{>0}^2$ in the Bando-FtL model defined in Definition 2.1 for different following vehicles $i \in \{2, \dots, N\}$. For convenience, in Definition 6.1, we have set $(\alpha_i, \beta_i) = (\alpha, \beta) \in \mathbb{R}_{>0}^2$ for each $i \in \{2, \dots, N\}$.

The following theorem gives the well-posedness of the above delayed Bando-FtL model for many vehicles, i.e., the minimum space headway of each following vehicle $i \in \{2, \dots, N\}$ and the uniform bounds on the following vehicles' accelerations and velocities.

Theorem 6.2 (Minimum space headway and bounds on acceleration and velocity)**.** *Given the initial value problem in Definition 6.1 with the therein stated assumptions, there exists for each $T \in \mathbb{R}_{>0}$ a unique solution in the sense of Definition 3.2 of the initial value problem on the time horizon $[0, T]$ and the solution satisfies the lower bound*

$$
\boldsymbol{x}_i(t) - \boldsymbol{x}_{i+1}(t) - l \geq d_{i,\min} > 0 \quad \forall (t, i) \in [0, T] \times \{1, \dots, N-1\}
\tag{6.2}
$$

for $i \in \{2, \dots, N\}$ with

$$
\begin{aligned}
d_{\min,i} &= \frac{A_i + \sqrt{A_i^2 + 4\alpha\beta}}{2\alpha} \\
A_i &= -\boldsymbol{v}_{0,i} - \alpha T \|V\|_{L^\infty(\mathbb{R}_{>0})} + \alpha(\boldsymbol{x}_{0,i-1} - \boldsymbol{x}_{0,i} - l) - \beta \frac{1}{\boldsymbol{x}_{0,i-1} - \boldsymbol{x}_{0,i} - l} \\
B_i &:= \frac{\alpha d_{\min,i}^2 + \beta}{d_{\min,i}^2}
\end{aligned}
$$

and $(\alpha, \beta, l) \in \mathbb{R}_{>0}^3$ as in Definition 2.1. Additionally, for all $t \in [0, T]$ we have for the velocity $\dot{\boldsymbol{x}}$ and its derivative, the acceleration, the following lower and upper bounds $\forall (t, i) \in [0, T] \times \{2, \dots, N\}$

$$
\boldsymbol{v}_{0,i} \exp(-B_i t) \leq \dot{\boldsymbol{x}}_i(t) \leq \max\left\{\boldsymbol{v}_{0,i}, \|V\|_{L^\infty(\mathbb{R})} + \frac{\beta}{\alpha} \frac{\dot{\boldsymbol{x}}_{i-1}(t)}{d_{\min,i}^2}\right\}
$$

and upper and lower bounded acceleration $\forall t \in [0, T]$

$$-B_i \max \left\{ v_{0,i}, \|V\|_{L^\infty(\mathbb{R})} + \frac{\beta}{\alpha} \frac{\dot{x}_{i-1}(t)}{d_{\min}^2} \right\} \leq \ddot{x}_i(t) \leq \alpha \|V\|_{L^\infty(\mathbb{R}_{\geq 0})} - \alpha v_0 e^{-B_i t} + \beta \frac{\dot{x}_{i-1}(t)}{d_{\min,i}^2}.$$

Proof. The proof is a straightforward adaption of Theorem 4.3 as the number of vehicles is finite and the system is one-sided coupled (the ego vehicle acceleration is a function of the leading vehicle position and velocity only. So we omit the proof here.

Remark 12 (Dependency of the bounds in Theorem 6.2)**.** The bounds on $\dot{x}_i$ as well as on $\ddot{x}_i$ for $i \in \{2, \ldots, N\}$ depend on the velocity $\dot{x}_i$ of the corresponding leading vehicle. However, as the system is coupled in the way that the dynamics of the ego vehicle only depend on the leading vehicle velocity, these bounds are explicit (although not very sharp). Choosing $i = 1$ we can indeed rely on the leading vehicle velocity

$$v_1(t) = v_{0,1} + \int_0^t u_{\text{lead}}(s) \, ds.$$

Then, continuing forward in $i \in \{2, \ldots, N\}$ the velocities of the previous vehicle are know or can at least be bounded by the previously stated estimates a priori.

Remark 13 (Extension to the delay case and convergence for delay to zero)**.** Following the same reasoning as in Section 4 the many car case dynamics in Definition 6.1 can be extended to the many car case with delay where each car can have a different delay for incorporating the information on the leading vehicle trajectory position and velocity into the acceleration function. It can also be shown that the solution of the delayed system of (one-sided) coupled car-following models converges to the undelayed solution when the delay of all car-following models converges simoultaneously to zero. Again, the idea is to apply inductively Theorem 5.2. We do not go into details.

7. Numerical simulation

In this section, we assume that the first leading vehicle's acceleration a at time $t \in [0, \infty)$ satisfies again the profile which has been defined in Eq (3.22) and illustrated in Figure 2. It somewhat mimics a traffic situation where the leading vehicle brakes, speeds up and remains at a constant velocity at different times and thus deals with the "extreme" cases in traffic. We consider both the non-delayed and delayed Bando-FtL model with five vehicles. For all solutions we use as parameters

$$\alpha = 0.5, \ \beta = 20, v_{\max} = 10, \ d = 2.5, \ l = 4.5 \tag{7.1}$$

with the optimal velocity function as in Eq (2.3).

The non-delayed solution which is obtained by a Runge-Kutta method of order 4 (ode45 with matlab) is illustrated in Figure 5. The left illustration demonstrates the vehicles' position, the middle the vehicles' velocity and the right one the headway. As we have chosen for the leading vehicle trajectory the acceleration in Figure 2 this type of stop and go behavior passes over to the following cars in a slightly smoothed fashion with the most "smooth" velocity that of the last vehicle, which is depicted as follower 4. In Figure 6 we introduce the mentioned delay with the delay between leading vehicle and follower 1 of $\delta_2 = 5$, the delay between follower 1 and follower 2 of $\delta_3 = 4$, the delay between follower 2 and follower 3 $\delta_4 = 3$ and finally, the delay between follower 3 and follower 4

$\delta_5 = 2$. The delay affects the velocities which are in their reaction delayed (and are further delayed with regard to the corresponding leading vehicle the larger the delay) which results in an increased headway variation and additional variation in speeds which one can interpret as an increased potential for phantom shock. In Figures 7–9 the delays are decreased more and more and indeed the headway variation reduces so that when comparing the results with the smallest delay, i.e., Figure 9, the difference to the undelayed result in Figure 5 is relatively small underlining the proved convergence in Section 5 and Theorem 5.2.
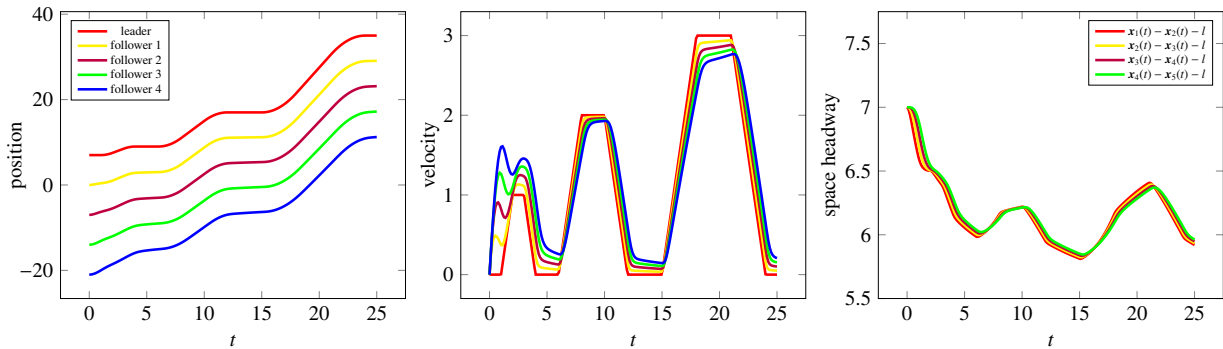
Figure 5. The un-delayed case with five vehicles: The left illustration is picturing the positions, the middle one the velocities and the right one the headway of the vehicles. As previously mentioned, there is no delay.
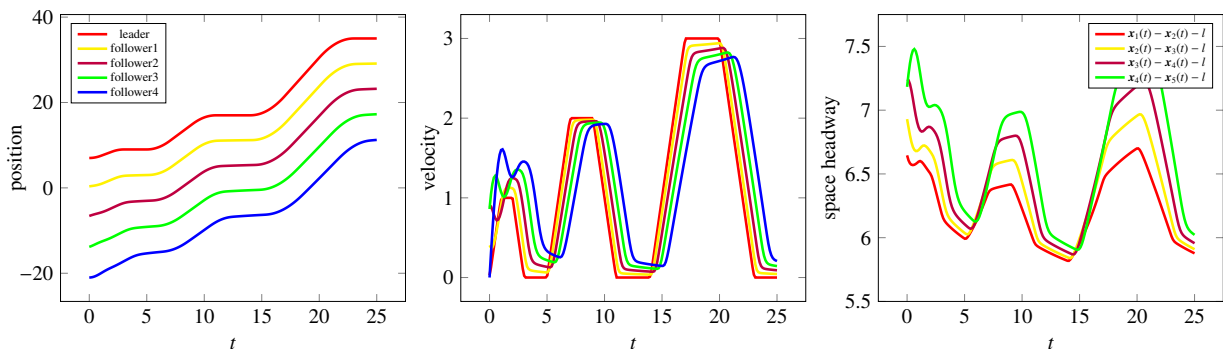
Figure 6. The delayed case with $\delta_2 = 5, \delta_3 = 4, \delta_4 = 3, \delta_5 = 2$.
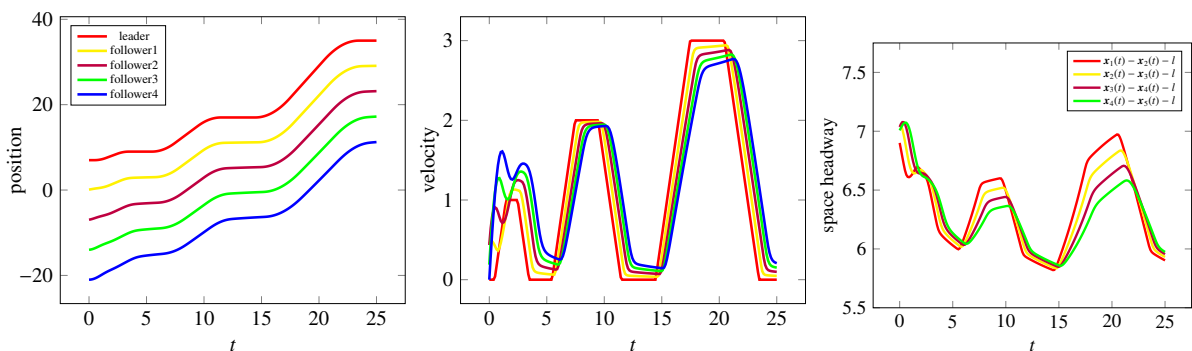

Figure 7. The delayed case with $\delta_2 = \frac{5}{25}, \delta_3 = \frac{4}{25}, \delta_4 = \frac{3}{25}, \delta_5 = \frac{2}{25}$.
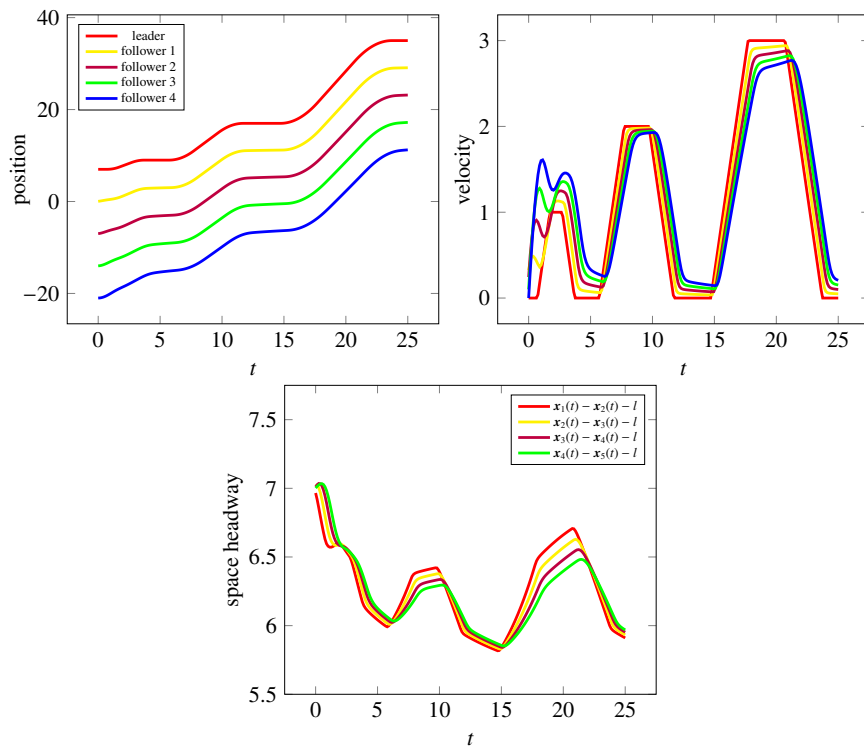
Figure 8. The delayed case with $\delta_2 = \frac{5}{50}, \delta_3 = \frac{4}{50}, \delta_4 = \frac{3}{50}, \delta_5 = \frac{2}{50}$.
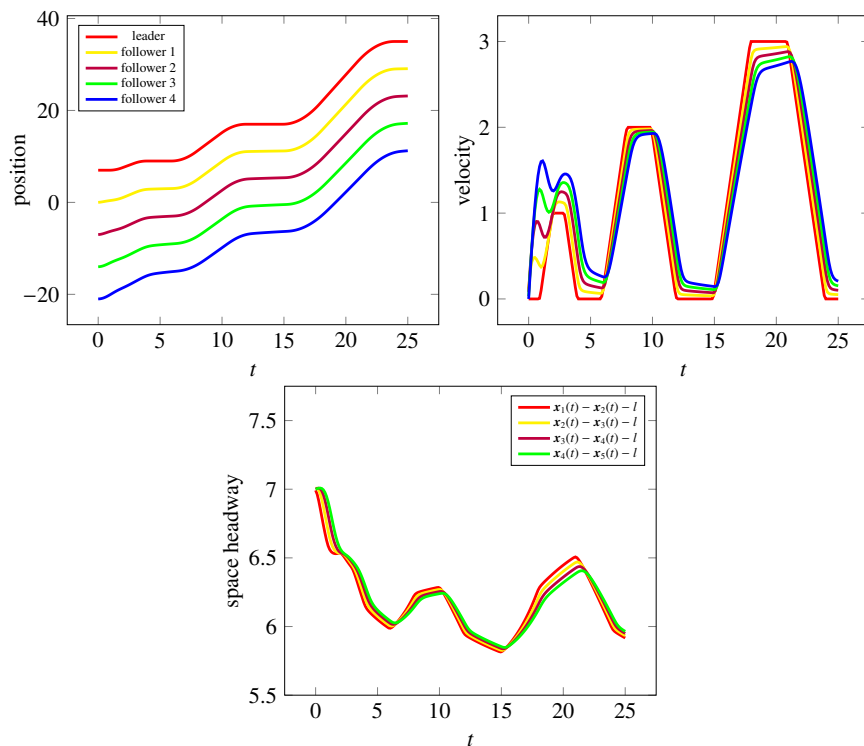

Figure 9. The delayed case with $\delta_2 = \frac{5}{150}, \delta_3 = \frac{4}{150}, \delta_4 = \frac{3}{150}, \delta_5 = \frac{2}{150}$.

This convergence is also illustrated in Figure 10 where in the left picture the L^∞ distance of the

delayed position and the non-delayed position is illustrated and in the right picture the corresponding velocity difference again in L^∞. We choose the vector of delays $\boldsymbol{\delta}_N = \frac{1}{N}(\delta_2, \delta_3, \delta_4, \delta_5)$, $N \in \mathbb{N}_{\geq 1}$. Note that the delay δ_i, $i \in \{2, 3, 4, 5\}$, approaches zero as $N \to \infty$. The convergence of the position is faster (which is easily explained as the position is the anti-derivative of the velocity).
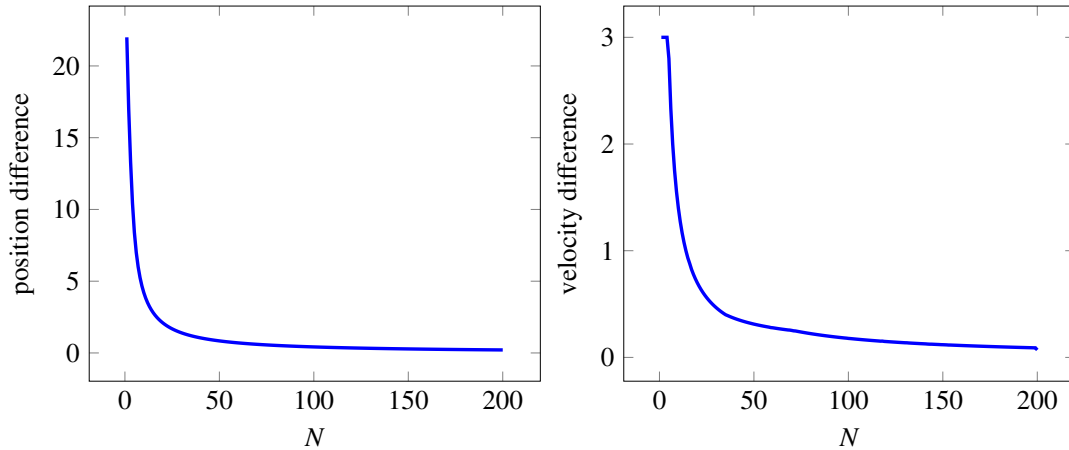

Figure 10. We choose $T = 25$. On the left, the y-axis is $\|\boldsymbol{x} - \boldsymbol{x}[\boldsymbol{\delta}_N]\|_{L^\infty((0,T);\mathbb{R}^4)}$ with $\boldsymbol{\delta}_N$ the vector of delays, the maximal difference of all positions comparing delayed with undelayed and the x-axis is N. On the right, the y-axis is $\|\dot{\boldsymbol{x}} - \dot{\boldsymbol{x}}[\boldsymbol{\delta}_N]\|_{L^\infty((0,T);\mathbb{R}^4)}$ the maximal difference in the velocity of all vehicles delayed with undelayed and the x-axis is N. Note that $\|\boldsymbol{\delta}_N\|_\infty \to 0$ as $N \to \infty$.

8. Conclusions and future research

We have studied the well-posedness of a specific car following model and have provided lower and upper bounds on velocity and acceleration which come in handy when considering particularly optimal control problems. We have introduced a delay which instantiates that the ego vehicle might react on the leading vehicle position and velocity only in a delayed sense.

Future research should address:

- different types of delays, particularly in the leading vehicle trajectory.
- the stability of traffic equilibria when introducing delays.
- the long-time behavior assuming that the leading vehicle drives with constant velocity.
- model validation, i.e., testing whether integrating a delay (and which value) into the model is more precise. Clearly, many traffic phenomena could be seen as artefacts caused by delay.
- introducing delay in other car-following models as well and understanding whether there is a substantial difference (for instance, in the intelligent driver model (IDM) [35] we cannot expect similar results as the "follow the leader part" cannot be directly integrated and the model generally has additional flaws [1]).

Conflict of interest

We declare no conflicts of interest.

References

1. S. Albeaik, A. Bayen, M. T. Chiri, X. Gong, A. Hayat, N. Kardous, et al., Limitations and improvements of the intelligent driver model (IDM), *SIAM J. Appl. Dyn. Syst.*, **21** (2022), 1862–1892. https://doi.org/10.1137/21M1406477

2. I. Bae, J. Moon and J. Seo, Toward a comfortable driving experience for a self-driving shuttle bus, *Electronics*, **8** (2019), 943. https://doi.org/10.3390/electronics8090943

3. M. Bando, K. Hasebe, K. Nakanishi, A. Nakayama, A. Shibata, Y. Sugiyama, Phenomenological study of dynamical model of traffic flow, *J. Phys. I*, **5** (1995), 1389–1399. https://doi.org/10.1051/jp1:1995206

4. M. Bando, K. Hasebe, A. Nakayama, A. Shibata, Y. Sugiyama, Dynamical model of traffic congestion and numerical simulation, *Phys. Rev. E*, **51** (1995), 1035–1042. https://doi.org/10.1103/PhysRevE.51.1035

5. M. Bando, K. Hasebe, K. Nakanishi, A. Nakayama, Analysis of optimal velocity model with explicit delay, *Phys. Rev. E*, **58** (1998), 5429–5435. https://doi.org/10.1103/PhysRevE.58.5429

6. S. Bexelius, An extended model for car-following, *Transp. Res.*, **2** (1968), 13–21. https://doi.org/10.1016/0041-1647(68)90004-X

7. P. Bosetti, M. Da Lio, A. Saroldi, On the human control of vehicles: an experimental study of acceleration, *Eur. Transport Res. Rev.*, **6** (2014), 157–170. https://doi.org/10.1007/s12544-013-0120-2

8. M. Brackstone, M. McDonald, Car-following: A historical review, *Transp. Res. Part F Psychol. Behav.*, **2** (1999), 181–196. https://doi.org/10.1016/S1369-8478(00)00005-X

9. H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, New York: Springer, 2011. https://doi.org/10.1007/978-0-387-70914-7

10. M. Burger, S. Göttlich, T. Jung, Derivation of a first order traffic flow model of lighthill-whitham-richards type, *IFAC-PapersOnLine*, **51** (2018), 49–54. https://doi.org/10.1016/j.ifacol.2018.07.009

11. R. E. Chandler, R. Herman, E. W. Montroll, Traffic dynamics: Studies in car following, *Oper. Res.*, **6** (1958), 165–184. https://doi.org/10.1287/opre.6.2.165

12. L. Davis, Modifications of the optimal velocity traffic model to include delay due to driver reaction time, *Physica A*, **319** (2003), 557–567. https://doi.org/10.1016/S0378-4371(02)01457-7

13. R. D. Driver, *Ordinary and delay differential equations*, Springer Science Business Media, 2012.

14. J. Eggert, F. Damerow, S. Klingelschmitt, The foresighted driver model, *2015 IEEE Intelligent Vehicles Symposium (IV)*, IEEE, Seoul, Korea, (2015), 322–329. https://doi.org/10.1109/IVS.2015.7225706

15. D. C. Gazis, R. Herman, R. W. Rothery, Nonlinear follow-the-leader models of traffic flow, *Oper. Res.*, **9** (1961), 545–567. https://doi.org/10.1287/opre.9.4.545

16. D. Gazis, R. Herman, The moving and "phantom" bottlenecks, *Transp. Sci.*, **26** (1992), 223–229. https://doi.org/10.1287/trsc.26.3.223

17. X. Gong, B. Piccoli, G. Visconti, Mean-field limit of a hybrid system for multi-lane multi-class traffic, 2020. https://doi.org/10.48550/arXiv.2007.14655

18. P. Hartman, *Ordinary Differential Equations: Second Edition*, Classics in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM, 3600 Market Street, Floor 6, Philadelphia, PA 19104), 1982.

19. S. Hoogendoorn, P. Bovy, State-of-the-art of vehicular traffic flow modelling, *Proc. Inst. Mech. Eng., Part I: J. Syst. Control Eng.*, **215** (2001), 283–303. https://doi.org/10.1177/095965180121500402

20. R. Jiang, M. Hu, B. Jia, R. Wang, Q. Wu, The effects of reaction delay in the nagel-schreckenberg traffic flow model, *Eur. Phys. J. B*, **54** (2006), 267–273. https://doi.org/10.1140/epjb/e2006-00449-5

21. N. Kardous, A. Hayat, S. McQuade, X. Gong, S. Truong, P. Arnold, et al., A rigorous multi-population multi-lane hybrid traffic model and its mean-field limit for dissipation of waves via autonomous vehicles, 2022. https://doi.org/10.48550/arXiv.2205.06913

22. R. Keane, H. O. Gao, Calibration for traffic microsimulation using trajectory data, part 1: Fast calibration using the adjoint method, *CoRR*, 2019. https://doi.org/10.48550/arXiv.1901.06452

23. A. Keimer, L. Pflug, Nonlocal conservation laws with time delay, *Nonlinear Differ. Equations Appl.*, **26** (2019), 1–34. https://doi.org/10.1007/s00030-019-0597-z

24. A. Keimer, L. Pflug, M. Spinola, Existence, uniqueness and regularity of multi-dimensional nonlocal balance laws with damping, *J. Math. Anal. Appl.*, **466** (2018), 18–55. https://doi.org/10.1016/j.jmaa.2018.05.013

25. A. Kesting, M. Treiber, D. Helbing, Enhanced intelligent driver model to access the impact of driving strategies on traffic capacity, *Phil. Trans. R. Soc. A*, **368** (2010), 4585–4605. https://doi.org/10.1098/rsta.2010.0084

26. Z. Li, W. Li, S. Xu, Y. Qian, Stability analysis of an extended intelligent driver model and its simulations under open boundary condition, *Physica A*, **419** (2015), 526–536. https://doi.org/10.1016/j.physa.2014.10.063

27. M. Liebner, M. Baumann, F. Klanner, C. Stiller, Driver intent inference at urban intersections using the intelligent driver model, in *2012 IEEE Intelligent Vehicles Symposium*, IEEE, Madrid, Spain, (2012), 1162–1167. https://doi.org/10.1109/IVS.2012.6232131

28. B. N. Matcha, S. Namasivayam, M. H. Fouladi, K. C. Ng, S. Sivanesan, S. Y. E. Noum, Simulation strategies for mixed traffic conditions: A review of car-following models and simulation frameworks, *J. Eng.*, **2020** (2020), 1–22. https://doi.org/10.1155/2020/8231930

29. K. Nagel, M. Schreckenberg, A cellular automaton model for freeway traffic, *J. Phys. I*, **2** (1992), 2221–2229. https://doi.org/10.1051/jp1:1992277

30. G. Orosz, B. Krauskopf, R. E. Wilson, Bifurcations and multiple traffic jams in a car-following model with reaction-time delay, *Physica D*, **211** (2005), 277–293. https://doi.org/10.1016/j.physd.2005.09.004

31. G. Orosz, G. Stépán, Subcritical hopf bifurcations in a car-following model with reaction-time delay, *Proc. Math. Phys. Eng. Sci.*, **462** (2006), 2643–2670. https://doi.org/10.1098/rspa.2006.1660

32. Oussama, T. Peter, H. Zebiri, B. Mourllion, M. Basset, Modified intelligent driver model for driver safety and traffic stability improvement, *IFAC Proceedings Volumes*, **46** (2013), 744–749. https://doi.org/10.3182/20130904-4-JP-2042.00132

33. B. Seibold, M. R. Flynn, A. R. Kasimov, R. R. Rosales, Constructing set-valued fundamental diagrams from jamiton solutions in second order traffic models, *Networks Heterogen. Media*, **8** (2013), 745–772. https://doi.org/10.3934/nhm.2013.8.745

34. R. E. Stern, S. Cui, M. L. Delle Monache, R. Bhadani, M. Bunting, M. Churchill, et al. Dissipation of stop-and-go waves via control of autonomous vehicles: Field experiments, *Transp. Res. Part C Emerging Technol.*, **89** (2018), 205–221. https://doi.org/10.1016/j.trc.2018.02.005

35. M. Treiber, A. Hennecke, D. Helbing, Congested traffic states in empirical observations and microscopic simulations, *Phys. Rev. E*, **62** (2000), 1805. https://doi.org/10.1103/PhysRevE.62.1805

36. M. Treiber, A. Kesting, *Traffic flow dynamics: Data, models and simulation*, Berlin Heidelberg: Springer-Verlag, 2013.

37. L. Yu, Z. Ke Shi, T. Li, A new car-following model with two delays, *Phys. Lett. A*, **378** (2014), 348–357. https://doi.org/10.1016/j.physleta.2013.11.030

38. C. Zhai, W. Liu, L. Huang, F. Tan, Feedback control strategy in a car-following model with two delays, in *2016 12th World Congress on Intelligent Control and Automation (WCICA)*, IEEE, Guilin, (2016), 2951–2956. https://doi.org/10.1109/WCICA.2016.7578259

39. H. Zhu, S. Dai, Analysis of car-following model considering driver's physical delay in sensing headway, *Physica A*, **387** (2008), 3290–3298. https://doi.org/10.1016/j.physa.2008.01.103