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Research article

A Hilliges-Weidlich-type scheme for a one-dimensional scalar conservation law with nonlocal flux

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Abstract: The simulation model proposed in [M. Hilliges and W. Weidlich. A phenomenological model for dynamic traffic flow in networks. *Transportation Research Part B: Methodological*, **29** (6): 407–431, 1995] can be understood as a simple method for approximating solutions of scalar conservation laws whose flux is of density times velocity type, where the density and velocity factors are evaluated on neighboring cells. The resulting scheme is monotone and converges to the unique entropy solution of the underlying problem. The same idea is applied to devise a numerical scheme for a class of one-dimensional scalar conservation laws with nonlocal flux and initial and boundary conditions. Uniqueness of entropy solutions to the nonlocal model follows from the Lipschitz continuous dependence of a solution on initial and boundary data. By various uniform estimates, namely a maximum principle and bounded variation estimates, along with a discrete entropy inequality, the sequence of approximate solutions is shown to converge to an entropy weak solution of the nonlocal problem. The improved accuracy of the proposed scheme in comparison to schemes based on the Lax-Friedrichs flux is illustrated by numerical examples. A second-order scheme based on MUSCL methods is presented.

Keywords: Nonlocal conservation law; well-posedness; Hilliges-Weidlich numerical type scheme; initial-boundary value problem; convolution kernel functions; MUSCL methods

1. Introduction

1.1. Scope

Nonlocal conservation laws model various phenomena, such as the dynamics of crowds [8–10], vehicular traffic [7, 18], supply chains [3], granular materials [1], and sedimentation phenomena [5]. These nonlocal equations are given by expressions of the type

$$\partial_t \boldsymbol{\rho} + \mathrm{div}_{\boldsymbol{x}} F(\boldsymbol{x}, t, \boldsymbol{\rho}, W) = 0, \quad t > 0, \quad \boldsymbol{x} \in \mathbb{R}^d, \quad d \geq 1,$$

where $\boldsymbol{\rho} = \boldsymbol{\rho}(\boldsymbol{x}, t) \in \mathbb{R}^N$, $N \geq 1$ is the vector of the conserved quantities and the variable $W = W(\boldsymbol{x}, t, \boldsymbol{\rho})$ depends on an integral evaluation of $\boldsymbol{\rho}$. The aim of this work is to propose an approach for a rigorous treatment of boundary conditions in the case of a spatially one-dimensional nonlocal problems, through development of new numerical schemes that are more accurate and less diffusive. The strategies that we employ are inspired by the results obtained in [5, 6, 17]. Particularly, we propose to study a simplification of the problem studied in [5], we adopt the treatment of the boundary conditions proposed in [17] and we present a numerical scheme based on local one studied in [6, 19]. Our proposed scheme takes advantage of the form in which the flow is written, namely density ρ times a local decreasing factor $g(\rho)$ times a nonlocal convolution term $V(x, t) = (\omega * v(\rho))$, where v is a given velocity function and ω is a convolution kernel such that the governing conservation law becomes

$$\partial_t \rho + \partial_x (\rho g(\rho) V(x, t)) = 0. \tag{1.1}$$

In the case of a standard (local) conservation law, captured by setting $V =$ const., the above-mentioned approach results in a monotone scheme [6], so it is possible to invoke standard arguments to prove its convergence to an entropy solution. This idea is extended herein to the nonlocal equation (1.1), although we emphasize that the resulting scheme is *not* monotone.

1.2. Related work.

There are many works about existence and uniqueness results for nonlocal equations, see e.g., [2, 14, 15, 17] for the scalar case in one space dimension. In these papers a first-order Lax-Friedrichs (LxF)-type numerical scheme is used to approximate the problem and to prove the existence of solutions and the nonlocal term is considered as a convolution between a kernel function and the unknown (mean downstream density approach). LxF-type schemes are the most common approach used to solve nonlocal conservation laws because they are easy to implement and due to their monotonicity, they make it possible to numerically analyze nonlocal flux problems. Their well-known main disadvantage is, however, their large amount of numerical diffusion that smears out sharp features of the exact solution. To reduce this phenomenon, Friedrich et al. [16] proposed a Godunov-type numerical scheme where the nonlocal term is considered as a convolution between a kernel function and the velocity of unknown (mean downstream velocity approach). We adopt this idea about the convolution to propose and develop our model and computations. A well-known early analysis of initial-boundary value problems (IBVP) for conservation laws is due to Bardos et al. [4], where existence, uniqueness and continuous dependence of the solution on initial data in the case of zero boundary data are proved. These results were extended to more general but smooth boundary data by Colombo and Rossi [11]. Rossi [21] studied an IBVP for a general scalar balance law in one space dimension. Under rather general assumptions on the flux and source functions, the author proves the well-posedness of the problem and stability of its solutions with respect to variations in the flux and the source terms. For both results, the initial and boundary data are required to be bounded functions with bounded total variation. In [14] a global well-posedness result for a class of weak entropy solutions of bounded variations of scalar conservation laws with nonlocal flux on bounded domains is established under suitable regularity assumptions on the flux function. The nonlocal operator is the standard convolution product. The existence of solutions is obtained by proving the convergence of an adapted LxF algorithm. Lipschitz continuous dependence from initial and boundary data is derived applying Kružkov's doubling of variables technique. In [17] Goatin and Rossi study the same problem as De Filippis and Goatin [14], but with a different approach, namely following the treatment of the boundary conditions proposed by Colombo and Rossi in [12] where a particular multi-dimensional system of conservation laws in bounded domains with zero boundary conditions was considered. More specifically, the nonlocal operator in the flux function is not a mere convolution product, but it is assumed to be 'aware' of boundaries and by introducing an adapted LxF algorithm, various estimates on the approximate solutions that allow to prove the existence of solutions to the original IBVP are introduced. Uniqueness was derived from the Lipschitz continuous dependence on initial and boundary data, which is proved exploiting results available for the local problem.

1.3. Outline of the paper.

This work is organized as follows: In Section 2 we present the class of nonlocal conservation laws considered and the assumptions needed the problem studied as well as the main result of this paper, whose proof is postponed to end of Section 3. Lipschitz continuous dependence of solutions to the problem on initial and boundary data is proved in Section 3. Afterwards, in Section 4 we introduce the numerical scheme and derive some of its important properties such as the maximum principle and BV and L^1 Lipschitz continuity in time estimates. These imply convergence of the scheme proposed, which in turn covers existence part of the well-posedness of the governing model. Throughout the paper we address the new scheme as a "HW scheme" according to the proponents of the original idea (Hilliges and Weidlich [19]), and in Section 5 we provide a second-order version of a HW-type numerical scheme. Finally, in Section 6 we present some numerical examples, analyzing the L^1-error of the approximate solutions of studied problem computed with different schemes. Appendix A collects some estimates necessary throughout the paper.

2. Initial-boundary value problem

We consider a particular initial-boundary value problem which is a version of a nonlocal model of sedimentation proposed in [5]. Our model has the following structure:

$$\begin{aligned}
&\partial_t\rho + \partial_x(f(\rho)V(x,t)) = 0, \quad (x,t) \in]a,b[\times\mathbb{R}^+,\\
&\rho(x,0) = \rho_0(x), \quad x \in]a,b[,\\
&\rho(a,t) = \rho_a(t), \quad \rho(b,t) = \rho_b(t), \quad t \in \mathbb{R}^+,
\end{aligned} \tag{2.1}$$

where

$$f(\rho) := \rho g(\rho), \tag{2.2}$$

$$V(x,t) := (\omega * v(\rho))(x,t) = \frac{1}{W(x)} \int_a^b v(\rho(y,t))\omega(y-x)\,\mathrm{d}y \tag{2.3}$$

with $W(x) := \int_a^b \omega(y-x)\,\mathrm{d}y$ for a suitable convolution kernel ω.

Remark 2.1. *The particular combination of local and nonlocal evaluations of ρ present in Eqs* (2.2) *and* (2.3) *can be motivated by following the discussion of [5, Sect. 1.2] for a model of sedimentation. Namely, if we assume that the nonlocal model describes the volume fraction of solids $\rho \in [0,1]$ within a solid-fluid two-phase flow system, then the solid and fluid conservation equations in differential form are $\partial_t\rho + \partial_x(\rho v_\mathrm{s}) = 0$ and $\partial_t(1-\rho) + \partial_x((1-\rho)v_\mathrm{f}) = 0$, where v_s and v_f are the solid and fluid phase velocities and x is the vertical spatial coordinate. One then defines the volume average velocity of the mixture $q := \rho v_\mathrm{s} + (1-\rho)v_\mathrm{f}$ and the solid-fluid relative velocity $v_\mathrm{r} = v_\mathrm{s} - v_\mathrm{f}$. Now for the particular case of batch settling in a closed column, we have $q = 0$ for all x and t, and then $\rho v_\mathrm{s} = \rho(1-\rho)v_\mathrm{r}$, so that the unique PDE to be solved is*

$$\partial_t\rho + \partial_x(\rho(1-\rho)v_\mathrm{r}) = 0, \tag{2.4}$$

where v_r is specified by some constitutive function. This scenario corresponds to Eqs (2.1)–(2.3) *if we choose $g(\rho) = 1-\rho$ and assume that v_r is given through the nonlocal convolution*

$$v_\mathrm{r} = v_\mathrm{r}(x,t) = (\omega * v(\rho))(x,t), \tag{2.5}$$

where $\rho \mapsto v(\rho)$ is a given, in general nonlinear function. In other words, the local and nonlocal evaluations of ρ in Eq (2.1) *arise from the combination of properly defined volume fractions in mixture theory with the constitutive assumption in Eq* (2.5). *The standard local evaluation $v_\mathrm{r}(x,t) = v(\rho(x,t))$ corresponds to the well-known kinematic sedimentation model, while utilizing $v(\omega * \rho)(x,t)$ instead of $(\omega * v(\rho))(x,t)$ is the model alternative explored in [5].*

Assumptions 2.1. *The initial-boundary value problem* (2.1) *is studied under the following assumptions:*

(i) The initial function satisfies $\rho_0 \in BV(I;\mathbb{R}^+)$, where $I :=]a,b[\subseteq \mathbb{R}^+$.
(ii) The function g satisfies $g \in C^2([0,1];\mathbb{R}_0^+)$, $g'(\rho) \le 0$ for $\rho \in [0,1]$, and $g(1) = 0$.
(iii) The function v satisfies $v \in C^2([0,1];\mathbb{R}^+)$, $v'(\rho) \le 0$ for $\rho \in [0,1]$, and $0 = v(1) \le v(\rho) \le v(0) = 1$.
(iv) The convolution kernel ω satisfies $\omega \in (C^2 \cap W^{2,1} \cap W^{2,\infty})(\mathbb{R};\mathbb{R})$ such that $\int_\mathbb{R} \omega(y)\,\mathrm{d}y = 1$, and there exists $K_\omega > 0$ such that for all $x \in I$, $W(x) = \int_a^b \omega(y-x)\,\mathrm{d}y \ge K_\omega$.

In what follows, we denote $\|\cdot\|_{L^\infty([0,1])} := \|\cdot\|_\infty$. The weak entropy solution of problem (2.1) is defined, as in [14, 17], in the following sense:

Definition 2.1. *A function $\rho \in (L^1 \cap L^\infty \cap BV)(]a,b[\times\mathbb{R}^+;\mathbb{R})$ is an* entropy weak solution *if for all $\varphi \in C_\mathrm{c}^1(\mathbb{R}^2;\mathbb{R}^+)$ and $k \in \mathbb{R}$,*

$$\begin{aligned}&\int_0^\infty \int_a^b \Big(|\rho - k|\varphi_t + \mathrm{sgn}(\rho-k)(f(\rho)-f(k))V\varphi_x - \mathrm{sgn}(\rho-k)f(k)V_x\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t\\&+ \int_a^b \Big|\rho_0(x) - k\Big|\varphi(x,0)\,\mathrm{d}x + \int_0^\infty \mathrm{sgn}(\rho_a - \kappa)\big(f(\rho(a+,t)) - f(\kappa)\big)V(a,t)\varphi(a,t)\,\mathrm{d}t\\&- \int_0^\infty \mathrm{sgn}(\rho_b - \kappa)\big(f(\kappa) - f(\rho(b-,t))\big)V(b,t)\varphi(b,t)\,\mathrm{d}t \ge 0.\end{aligned}$$

Definition 2.2. *A function* $\rho \in L^\infty(]a,b[\times\mathbb{R}^+;[0,1])$ *is an* entropy weak solution to problem (2.1) *if, for all* $\varphi \in C^1_c(\mathbb{R}^2,\mathbb{R}^+)$ *and* $k \in \mathbb{R}$,

$$\int_0^\infty \int_a^b \Big((\rho-k)^\pm \partial_t\varphi(x,t) + \mathrm{sgn}^\pm(\rho-k)\big(f(\rho)-f(k)\big)V(x,t)\partial_x\varphi(x,t)$$
$$- \mathrm{sgn}^\pm(\rho-k)f(k)\varphi(x,t)\partial_x V(x,t)\Big)\,\mathrm{d}x\,\mathrm{d}t + \int_a^b (\rho_0(x)-k)^\pm\varphi(x,0)\,\mathrm{d}x$$
$$+ L\Big(\int_0^\infty (\rho_a(t)-k)^\pm\varphi(a,t)\,\mathrm{d}t + \int_0^\infty (\rho_b(t)-k)^\pm\varphi(b,t)\,\mathrm{d}t\Big) \geq 0,$$

where

$$L := \|v\|_\infty(\|g\|_\infty + \|g'\|_\infty). \tag{2.6}$$

Here we have used the notation $s^+ := \max\{s,0\}$, $s^- := \max\{-s,0\}$, and

$$\mathrm{sgn}^+(s) := \begin{cases} 1 & \text{if } s>0, \\ 0 & \text{if } s\leq 0, \end{cases} \qquad \mathrm{sgn}^-(s) := \begin{cases} 0 & \text{if } s\geq 0, \\ -1 & \text{if } s<0. \end{cases}$$

Definition 2.1 will be useful in the existence proof, while Definition 2.2 will be used in the uniqueness proof. We need to remark that in the frame of functions in L^∞, Definition 2.2 implies Definition 2.1, for more details see [20]. In the rest of the paper, we denote $\mathcal{I}(r,s) := [\min\{r,s\},\max\{r,s\}]$ for any $r,s \in \mathbb{R}$.

Our main result concerning the new model is given by the following theorem, which states the well-posedness of the problem.

Theorem 2.1 (Well-posedness). *Let* $\rho_0 \in BV(I;\mathbb{R}^+)$, $\rho_a,\rho_b \in BV(\mathbb{R}^+;[0,1])$ *and Assumptions 2.1 be in effect. Then, for all* $T>0$, *the Cauchy problem* (2.1) *admits a unique entropy weak solution* $\rho \in (L^1\cap L^\infty\cap BV)(I\times[0,T];\mathbb{R}^+)$ *in the sense of the Definitions 2.1 and 2.2. Moreover, the following estimates hold: for any* $t\in[0,T]$,

$$0 \leq \rho(x,t) \leq 1 \quad \text{for all } x\in I, \tag{2.7}$$

$$\big\|\rho(\cdot,t)\big\|_{L^1(I)} \leq \mathcal{R}_1, \tag{2.8}$$

$$\mathrm{TV}(\rho(\cdot,t);I) \leq \mathrm{e}^{t\mathcal{T}_1}\Big(\mathrm{TV}(\rho_0;I) + \mathrm{TV}(\rho_a;]0,T[) + \big|\rho_0(a+)-\rho_a(0+)\big|$$
$$+ \mathrm{TV}(\rho_b;]0,T[) + \big|\rho_0(b-)-\rho_b(0+)\big|\Big) + \frac{\mathcal{T}_2(t)}{\mathcal{T}_1(t)}(\mathrm{e}^{t\mathcal{T}_1}-1), \tag{2.9}$$

and for $\tau>0$,

$$\big\|\rho(.,t)-\rho(.,t-\tau)\big\|_{L^1(I)} \leq \tau\Big(C_t(t) + L(\mathrm{TV}(\rho_a;]t-\tau,t[) + \mathrm{TV}(\rho_a;]t-\tau,t[)\Big)$$

where L is defined by Eq (2.6) *and*

$$\mathcal{R}_1(t) := c\|\rho_0\|_{L^1(I)} + L(\|\rho_a\|_{L^1([0,t])} + \|\rho_b\|_{L^1([0,t])}),$$
$$\mathcal{T}_1 := \mathcal{L}(\|g\|_\infty + \|g'\|_\infty),$$

$$\mathcal{L} := 2K_\omega^{-1}\|v\|_\infty \|\omega'\|_{L^1(\mathbb{R})}, \tag{2.10}$$
$$\mathcal{T}_2 := (\mathcal{W}\mathcal{R}_1(t) + 2\mathcal{L})\|g\|_\infty,$$
$$\mathcal{W} := 2K_\omega^{-1}\|v\|_\infty \|\omega''\|_{L^1(\mathbb{R})} + 4K_\omega^{-2}\|\omega'\|^2_{L^1(I)}\|v\|_\infty, \tag{2.11}$$
$$C_t(t) := \|v\|_\infty(\|g'\|_\infty + \|g\|_\infty)C_x(t) + \|g\|_\infty \mathcal{L}\mathcal{R}_1(t).$$

The proof consists of two parts: existence and uniqueness of entropy solutions. While uniqueness follows from the Lipschitz continuous dependence of weak entropy solutions on initial and boundary data, existence is based on a construction of a converging sequence of approximate solutions defined by a numerical scheme.

3. Uniqueness of entropy solutions

One part of the proof of Theorem 2.1 is to show uniqueness of weak entropy solutions for the model (2.1). Therefore, we prove the Lipschitz continuous dependence of weak entropy solutions with respect to initial and boundary data. Here, we follow [17]. We define $V(x,t)$ by Eq (2.3) and analogously $U(x,t)$ by replacing ρ in Eq (2.3) by another function σ. Furthermore, we let $r(x,t,u) := ug(u)V(x,t)$ and $h(x,t,u) := ug(u)U(x,t)$. Observe that by the definition of V and U,

$$V(x,t) \leq \|v\|_\infty, \quad U(x,t) \leq \|v\|_\infty,$$

furthermore, we have the following estimates derived in Appendix A:

$$\left|\partial_x V(x,t)\right| \leq 2K_\omega^{-1}\|v\|_\infty \|\omega'\|_{L^1(I)} =: \mathcal{P}_1, \tag{3.1}$$
$$\begin{aligned}\left|\partial^2_{xx} V\right| \leq\ & K_\omega^{-2}\|v\|_\infty\|\omega''\|_{L^1(I)}\|\omega\|_{L^1(I)} \\ & + \mathcal{P}_1 K_\omega^{-2}\|\omega\|_{L^1(I)} + \mathcal{P}_1 K_\omega^{-1} + K_\omega^{-1}\|v\|_\infty\|\omega''\|_{L^1(I)} =: \mathcal{P}_2,\end{aligned} \tag{3.2}$$
$$\left|V(x,t) - U(x,t)\right| \leq \mathcal{P}_3 \int_a^b \left|\rho(y,t) - \sigma(y,t)\right| \mathrm{d}y, \quad \mathcal{P}_3 := K_\omega^{-1}\|\omega\|_{L^\infty(\mathbb{R})}\|v'\|_\infty, \tag{3.3}$$
$$\begin{aligned}\left|\partial_x V(x,t) - \partial_x U(x,t)\right| &\leq \mathcal{M} \int_a^b \left|\rho(y,t) - \sigma(y,t)\right| \mathrm{d}y, \\ \mathcal{M} &:= K_\omega^{-2}\|\omega'\|_{L^1(\mathbb{R})}\|v'\|_\infty\|\omega\|_{L^\infty(\mathbb{R})} + K_\omega^{-1}\|v'\|_\infty\|\omega'\|_{L^\infty(\mathbb{R})}.\end{aligned} \tag{3.4}$$

In order to obtain the desired estimate, we first consider the local initial-boundary value problem

$$\begin{aligned} &\partial_t\phi + \partial_x r(x,t,\phi) = 0, \quad (x,t) \in I\times]0,T[, \\ &\phi(x,0) = \sigma_0(x), \quad x \in I; \quad \phi(a,t) = \sigma_a(t), \quad \phi(b,t) = \sigma_b(t), \quad t \in]0,T[. \end{aligned} \tag{3.5}$$

By Assumptions 2.1, $r \in C^2(I \times [0,T] \times \mathbb{R}; \mathbb{R})$ and $\partial_u r \in L^\infty(I \times [0,T] \times \mathbb{R}; \mathbb{R})$ and by estimation Eq (3.1), $\partial^2_{xu} r(x,t,u) < \infty$. Thus, we may apply Theorem 2.4 of [21] to deduce that problem (3.5) admits a unique solution in $(L^\infty \cap BV)(I\times]0,T[;\mathbb{R})$ which satisfies, for all $t \in [0,T[$, $0 \leq \phi(x,t) \leq 1$ for all $x \in I$ and

$$\begin{aligned}\mathrm{TV}(\phi(t)) \leq \mathrm{e}^{tC_2(t)}\big(&\mathrm{TV}(\sigma_0) + \mathrm{TV}(\sigma_a;]0,t[) + \left|\sigma_0(a+) - \sigma_a(0+)\right| \\ &+ \mathrm{TV}(\sigma_b;]0,t[) + \left|\sigma_0(b-) - \sigma_b(0+)\right| + \mathcal{K}t\big),\end{aligned} \tag{3.6}$$

where

$$C_2(t) := \mathcal{P}_1(\|g\|_\infty + \|g'\|_\infty), \quad \mathcal{K} := 2((\mathcal{P}_1 + \mathcal{P}_2)\|g\|_\infty + \mathcal{P}_1\|g'\|_\infty).$$

Assume that ρ is a solution to the IBVP

$$\begin{aligned}
&\partial_t\rho + \partial_x r(x,t,\rho) = 0, \quad (x,t) \in I\times]0,T[,\\
&\rho(x,0) = \rho_0(x), \quad x \in I; \quad \rho(a,t) = \rho_a(t), \quad \rho(b,t) = \rho_b(t), \quad t \in]0,T[
\end{aligned}$$

and that σ is a solution to the analogous IBVP

$$\begin{aligned}
&\partial_t\sigma + \partial_x h(x,t,\sigma) = 0, \quad (x,t) \in I\times]0,T[,\\
&\sigma(x,0) = \sigma_0(x), \quad x \in I; \quad \sigma(a,t) = \sigma_a(t), \quad \sigma(b,t) = \sigma_b(t), \quad t \in]0,T[.
\end{aligned}$$

Therefore, for $t > 0$ we compute

$$\left\|\rho(\cdot,t) - \sigma(\cdot,t)\right\|_{L^1(I)} \le \left\|\rho(\cdot,t) - \phi(\cdot,t)\right\|_{L^1(I)} + \left\|\phi(\cdot,t) - \sigma(\cdot,t)\right\|_{L^1(I)}, \tag{3.7}$$

where the first term on the right-hand side of Eq (3.7) evaluates the distance between solutions to IBVPs with the same flux function, but different initial and boundary data. Then, we can apply Proposition 3.7 of [21] to get

$$\left\|\rho(\cdot,t) - \phi(\cdot,t)\right\|_{L^1(I)} \le \|\rho_0 - \sigma_0\|_{L^1(I)} + L(\|\rho_a - \sigma_a\|_{L^1([0,t])} + \|\rho_b - \sigma_b\|_{L^1([0,t])}) =: A(t).$$

Now, the second term on the right-hand side of Eq (3.7) evaluates the distance between solutions to IBVPs with different flux functions, but same initial and boundary data. Therefore, we apply Theorem 2.6 of [21] to obtain

$$\begin{aligned}
\left\|\phi(\cdot,t) - \sigma(\cdot,t)\right\|_{L^1(I)} \le &\int_0^t \int_a^b \left\|\partial_x(r-h)(x,s,\cdot)\right\|_{L^\infty(\mathcal{U})} \mathrm{d}x\,\mathrm{d}s\\
&+ \int_0^t \left\|\partial_u(r-h)(\cdot,s,\cdot)\right\|_{L^\infty(I\times\mathcal{U}))} \min\{\mathrm{TV}(\sigma(\cdot,s)), \mathrm{TV}(\phi(\cdot,s))\}\,\mathrm{d}s\\
&+ 2\int_0^t \left\|(r-h)(a,s,\cdot)\right\|_{L^\infty(\mathcal{U})} \mathrm{d}s + 2\int_0^t \left\|(r-h)(b,s,\cdot)\right\|_{L^\infty(\mathcal{U})} \mathrm{d}s,
\end{aligned} \tag{3.8}$$

where

$$\mathcal{U} := [-\max\{\|\pi(s)\|_{L^\infty(I)}, \|\sigma(s)\|_{L^\infty(I)}\}, \max\{\|\pi(s)\|_{L^\infty(I)}, \|\sigma(s)\|_{L^\infty(I)}\}] = [-1,1].$$

Next, we estimate all terms appearing in Eq (3.8). First of all, by Theorem 2.1,

$$\begin{aligned}
\mathrm{TV}(\sigma(\cdot,t)) \le \mathrm{e}^{t\mathcal{T}_1(t)}\Big(&\mathrm{TV}(\sigma_0;I) + \mathrm{TV}(\sigma_a;(0,t)) + \left|\sigma_0(a+) - \sigma_a(0+)\right|\\
&+ \mathrm{TV}(\sigma_b;(0,t)) + \left|\sigma_0(b-) - \sigma_b(0+)\right|\Big) + \frac{\mathcal{T}_2(t)}{\mathcal{T}_1(t)}(\mathrm{e}^{t\mathcal{T}_1(t)} - 1)
\end{aligned} \tag{3.9}$$

with

$$\mathcal{T}_1(t) := \mathcal{L}(\|g\|_\infty + \|g'\|_\infty), \quad \mathcal{T}_2(t) := (\mathcal{W}\mathcal{S}_1(t) + 2\mathcal{L})\|g\|_\infty,$$

$$\mathcal{S}_1(t) := \|\sigma_0\|_{L^1(I)} + L\big(\|\sigma_a\|_{L^1([0,t])} + \|\sigma_b\|_{L^1([0,t])}\big).$$

Thus, by Eqs (3.6) and (3.9), we get

$$\begin{aligned} &\min\{\mathrm{TV}(\sigma(\cdot,s)), \mathrm{TV}(\phi(\cdot,s))\} \\ &\leq e^{t\mathcal{T}_3(t)}\Big(\mathrm{TV}(\sigma_0; I) + \mathrm{TV}(\sigma_a;(0,t)) + \big|\sigma_0(a+) - \sigma_a(0+)\big| + \mathrm{TV}(\sigma_b;(0,t)) \\ &\quad + \big|\sigma_0(b-) - \sigma_b(0+)\big|\Big) + \min\{\mathcal{K}te^{C_2(t)t}, (\mathcal{T}_2(t)/\mathcal{T}_1(t))(e^{t\mathcal{T}_1} - 1)\} =: \mathcal{T}_4(t). \end{aligned} \tag{3.10}$$

To handle the first term on the right-hand side of Eq (3.8), we use the estimate

$$\big|\partial_x(r-h)(x,t,u)\big| = \big|ug(u)\partial_x(V-U)\big| \leq C|u|\mathcal{M}\int_a^b \big|\rho(y,t) - \sigma(y,t)\big|\,\mathrm{d}y,$$

which implies

$$\big\|\partial_x(r-h)(x,s,\cdot)\big\|_{L^\infty(\mathcal{U})} \leq C\mathcal{M}\int_a^b \big|\rho(y,t) - \sigma(y,t)\big|\,\mathrm{d}y. \tag{3.11}$$

Next, in view of $\partial_u(r-h)(x,t,u) = \partial_u(ug(u))(V-U)$ we get

$$\begin{aligned} \big\|\partial_u(r-h)(\cdot,s,\cdot)\big\|_{L^\infty(I\times\mathcal{U})} &\leq \big\|\partial_u(ug(u))\big\|_\infty \mathcal{P}_3 \int_a^b \big|\rho(y,s) - \sigma(y,s)\big|\,\mathrm{d}y \\ &\leq (\|g\|_\infty + \|g'\|_\infty)\mathcal{P}_3 \int_a^b \big|\rho(y,s) - \sigma(y,s)\big|\,\mathrm{d}y. \end{aligned} \tag{3.12}$$

The third integral on the right-hand side of Eq (3.8) is estimated by considering that

$$\big|(r-h)(a,t,u)\big| = \big|ug(u)(V-U)\big| \leq C|u|\mathcal{P}_3\int_a^b \big|\rho(y,t) - \sigma(y,t)\big|\,\mathrm{d}y,$$

hence

$$\big\|(r-h)(a,s,\cdot)\big\|_{L^\infty([-1,1])} \leq C\mathcal{P}_3\int_a^b \big|\rho(y,t) - \sigma(y,t)\big|\,\mathrm{d}y; \tag{3.13}$$

the fourth integral is treated similarly. Finally, combining Eq (3.10) to Eq (3.13) we get

$$\big\|\phi(\cdot,t) - \sigma(\cdot,t)\big\|_{L^1(I)} \leq B(t)\int_0^t\int_a^b \big|\rho(y,t) - \sigma(y,t)\big|\,\mathrm{d}y\,\mathrm{d}s, \tag{3.14}$$

where we define

$$B(t) = C\mathcal{M} + \mathcal{P}_3\Big((\|g\|_\infty + \|g'\|_\infty)\mathcal{T}_4(t) + 4C\Big). \tag{3.15}$$

Inserting $A(t)$ and Eq (3.14) into Eq (3.7) yields

$$\big\|\rho(\cdot,t) - \sigma(\cdot,t)\|_{L^1(I)} \leq A(t) + B(t)\int_0^t \big\|\rho(\cdot,s) - \sigma(\cdot,s)\big\|_{L^1(I)}\mathrm{d}s,$$

so by an application of Gronwall's lemma we arrive at the estimate

$$\left\|\rho(\cdot,t)-\sigma(\cdot,t)\right\|_{L^1(I)} \leq A(t) + \int_0^t A(s)B(s)\exp\left(\int_s^t B(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s$$
$$\leq A(t) + B(t)\int_0^t A(s)\mathrm{e}^{B(t)(t-s)}\,\mathrm{d}s \leq A(t)(1+B(t)t\mathrm{e}^{B(t)t}).$$

Consequently, we have proved the following lemma.

Lemma 3.1 (Lipschitz continuous dependence on initial and boundary data)**.** *If Assumptions Eq* (2.1) *are in effect and ρ and σ are two entropy solutions to Eq* (2.1) *with initial data $\rho_0,\ \sigma_0 \in BV(I;\mathbb{R}^+)$ and $\rho_a, \rho_b, \sigma_a, \sigma_b \in BV(]0,T[;[0,1])$, then the estimate*

$$\left\|\rho(\cdot,T)-\sigma(\cdot,T)\right\|_{L^1(I)} \leq \left(\|\rho_0-\sigma_0\| + L(\|\rho_a-\sigma_a\|_{L^1([0,T])} + \|\rho_b-\sigma_b\|_{L^1([0,T])})\right)(1+B(T)Te^{B(T)T}) \tag{3.16}$$

holds for any $T>0$, where $B(T)$ is defined in Eq (3.15)*.*

4. Existence of solutions

The proof of existence of solutions consists of several steps that are developed in this section. We construct a sequence of approximate solutions to Eq (2.1) and derive the compactness estimates necessary to prove its convergence by Helly's theorem. We then show that the limit function is a weak entropy solution to the IBVP (2.1).

4.1. Numerical scheme

Fix $T>0$, we take a space step $\Delta x = (b-a)/M$ with $M \in \mathbb{N}$ and a time step Δt that is subject to a CFL condition specified later, and we set $\lambda = \Delta t/\Delta x$. We denote the center of the cells by $x_j := a + (j-1/2)\Delta x$ for $j = 1,\dots,M$, and $x_{j+1/2} = a + j\Delta x$, $j = 0,\dots,M$ are the cells interfaces. Moreover, we set $N_T = \lfloor T\Delta t \rfloor$ and, for $n = 0,\dots,N_T$ let $t^n = n\Delta t$ be the time mesh. The initial datum and the boundary data are approximated as

$$\rho_j^0 := \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \rho_0(x)\,\mathrm{d}x, \quad j = 1,\dots,M;$$
$$\rho_a^n := \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \rho_a(t)\,\mathrm{d}t, \quad \rho_b^n := \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \rho_b(t)\,\mathrm{d}t \quad n = 0,\dots,N_T-1,$$

furthermore, we set $\rho_0^n := \rho_a$ and $\rho_{M+1}^n := \rho_b$. For $n = 0,\dots,N_T-1$, we set $\omega^k := \omega((k-1/2)\Delta x)$ for $k \in \mathbb{Z}$ and define

$$W_{j+1/2} := \Delta x \sum_{k=1}^{M} \omega^{k-j}, \quad V_{j+1/2}^n := \frac{\Delta x}{W_{j+1/2}} \sum_{k=1}^{M} \omega^{k-j} v(\rho_k) \quad \text{for } j = 0,\dots,M.$$

We define a piecewise constant approximate solution $\rho_\Delta(x,t)$ to Eq (2.1) as

$$\rho_\Delta(x,t) = \rho_j^n \quad \text{for } t \in [t^n, t^{n+1}[,\ x \in [x_{j-1/2}, x_{j+1/2}[, \tag{4.1}$$

where $n = 0, \ldots, N_T - 1$, $j = 1, \ldots, M$, through the numerical scheme

$$\rho_j^{n+1} = \rho_j^n - \lambda(F_{j+1/2}^n(\rho_j^n, \rho_{j+1}^n) - F_{j-1/2}^n(\rho_{j-1}^n, \rho_j^n)), \quad j = 1, \ldots, M, \tag{4.2}$$

where a nonlocal version of the monotone numerical flux proposed in [19] and also used in [6] is employed, namely

$$F_{j+1/2}^n(u, w) = ug(w)V_{j+1/2}^n. \tag{4.3}$$

Next, we study the properties of the numerical scheme (4.2) and (4.3). Particularly, we are going to prove that the sequence of approximate solutions $\rho_\Delta(x, t)$ satisfies the assumptions of Helly's compactness theorem.

4.2. Uniform bounds of numerical solutions

Lemma 4.1 (Maximum principle)**.** *If Assumptions 2.1 and the CFL condition*

$$\lambda\|v\|_\infty(\|g\|_\infty + \|g'\|_\infty) \leq 1 \tag{4.4}$$

hold, then if $\rho_0(x) \in [0, 1]$ *for* $x \in I$*, the approximate solution satisfies*

$$0 \leq \rho_j^n \leq 1 \quad \text{for all } j = 1, \ldots, M \text{ and } n = 1, \ldots, N_T.$$

Proof. We assume that $0 \leq \rho_j^n \leq 1$ for $j = 1, \ldots, M$. From Eq (4.2) we have

$$\begin{aligned}\rho_j^{n+1} &= \rho_j^n - \lambda(\rho_j^n g(\rho_{j+1}^n)V_{j+1/2}^n - \rho_{j-1}^n g(\rho_j^n)V_{j-1/2}^n) \\ &\leq \rho_j^n + \lambda\rho_{j-1}^n g(\rho_j^n)V_{j-1/2}^n \leq \rho_j^n + \lambda g(\rho_j^n)V_{j-1/2}^n =: G(\rho_j^n).\end{aligned}$$

In view of $G'(\rho) = 1 + \lambda g'(\rho)V_{j-1/2}^n$, under the CFL condition (4.4), G is a non-decreasing function of ρ. Thus

$$\max_{\rho_j^n \in [0,1]} G(\rho_j^n) = G(1) = 1,$$

which implies that $\rho_j^{n+1} \leq 1$. Returning to Eq (4.2), we obtain that

$$\rho_j^{n+1} \geq \rho_j^n - \lambda\rho_j^n g(\rho_{j+1}^n)V_{j+1/2}^n = (1 - \lambda g(\rho_{j+1}^n)V_{j+1/2}^n)\rho_j^n, \quad j = 1, \ldots, M.$$

Consequently, if Eq (4.4) is in effect, then $\rho_j^{n+1} \geq 0$. □

Lemma 4.2 (L^1 bound)**.** *Let Assumptions 2.1 and the CFL condition* (4.4) *hold. If* $\rho_0 \in L^\infty(I; [0, 1])$ *and* $\rho_a, \rho_b \in L^\infty(\mathbb{R}^+; [0, 1])$*, then, for all* $t > 0$*,* ρ_Δ *satisfies*

$$\left\|\rho_\Delta(\cdot, t)\right\|_{L^1(I)} \leq \|\rho_0\|_{L^1(I)} + L(\|\rho_a\|_{L^1([0,t])} + \|\rho_b\|_{L^1([0,t])}) =: C_1(t), \tag{4.5}$$

where L is defined in Eq (2.6)*.*

Proof. Lemma 4.1 (for $n = 0, \dots, N$) and the assumption $g(1) = 0$ imply

$$\begin{aligned}\left\|\rho_\Delta(\cdot, t^{n+1})\right\|_{L^1(I)} &= \Delta x(\rho_1^n + \cdots + \rho_M^n) + \Delta t(\rho_a^n g(\rho_1^n)V_{1/2} - \rho_M^n g(\rho_b^n)V_{M+1/2}^n) \\ &= \|\rho_\Delta(\cdot, t^n)\|_{L^1(I)} + \Delta t(\rho_a^n g(\rho_1^n)V_{1/2}^n + \rho_M^n(g(1) - g(\rho_b^n))V_{M+1/2}^n) \\ &= \|\rho_\Delta(\cdot, t^n)\|_{L^1(I)} + \Delta t\rho_a^n g(\rho_1^n)V_{1/2}^n + \Delta t\rho_M^n g'(\zeta_j^n)(1 - \rho_b^n)V_{M+1/2}^n,\end{aligned}$$

where $\zeta_j^n \in \mathcal{I}(\rho_b^n, 1)$. Now, using item (ii) of Assumptions 2.1 and the nonnegativity of ρ_a^n and ρ_b^n we have

$$\left\|\rho_\Delta(\cdot, t^{n+1})\right\|_{L^1(I)} \leq \left\|\rho_\Delta(\cdot, t^n)\right\|_{L^1(I)} + \Delta t\|v\|_\infty(\|g\|_\infty + \|g'\|)(\rho_a^n + \rho_b^n).$$

An iterative argument yields the desired estimate (4.5). □

Lemma 4.3 (*BV* estimate in space)**.** *Let Assumptions 2.1 hold, $\rho_0 \in BV(I; \mathbb{R}^+)$, $\rho_a, \rho_b \in BV(\mathbb{R}^+, [0, 1])$ and let ρ_Δ be given by Eq* (4.2). *If the CFL condition* (4.4) *holds, then for all $n = 1, \dots, N_T$ the discrete space BV estimate*

$$\sum_{j=0}^{M}\left|\rho_{j+1}^n - \rho_j^n\right| \leq C_x(t^n) \tag{4.6}$$

is satisfied, where we define the time-dependent bound

$$C_x(t^n) := \mathrm{e}^{\mathcal{K}_1 t^n}\left(\sum_{j=0}^{M}\left|\rho_{j+1}^0 - \rho_j^0\right| + \sum_{m=1}^{n}\left|\rho_a^m - \rho_a^{m-1}\right| + \sum_{m=1}^{n}\left|\rho_b^m - \rho_b^{m-1}\right|\right) + \mathcal{K}_1^{-1}\mathcal{K}_2(\mathrm{e}^{\mathcal{K}_1 t^n} - 1) \tag{4.7}$$

and $\mathcal{K}_1$ and $\mathcal{K}_2$ are defined as

$$\mathcal{K}_1 := \mathcal{L}(\|g'\|_\infty + \|g\|_\infty), \quad \mathcal{K}_2 := (\mathcal{W}C_1(t) + 2\mathcal{L})\|g\|_\infty. \tag{4.8}$$

Proof. Subtracting two versions of Eq (4.2) from each other, we obtain

$$\rho_{j+1}^{n+1} - \rho_j^{n+1} = \mathcal{A}_j^n - \lambda\mathcal{B}_j^n,$$

where

$$\begin{aligned}\mathcal{A}_j^n &:= \rho_{j+1}^n - \rho_j^n - \lambda(\rho_{j+1}^n g(\rho_{j+2}^n)V_{j+3/2}^n - \rho_j^n g(\rho_{j+1}^n)V_{j+1/2}^n - \rho_j^n g(\rho_{j+1}^n)V_{j+3/2}^n + \rho_{j-1}^n g(\rho_j^n)V_{j+1/2}^n), \\ \mathcal{B}_j^n &:= \rho_j^n g(\rho_{j+1}^n)V_{j+3/2}^n - \rho_j^n g(\rho_{j+1}^n)V_{j+1/2}^n + \rho_{j-1}^n g(\rho_j^n)V_{j-1/2}^n - \rho_{j-1}^n g(\rho_j^n)V_{j+1/2}^n.\end{aligned}$$

A straightforward computation reveals that $\mathcal{A}_j^n$ can be written in the form

$$\begin{aligned}\mathcal{A}_j^n = &\left(1 - \lambda(g(\rho_{j+1}^n)V_{j+3/2}^n - \rho_j^n g'(\xi_{j+1/2}^n)V_{j+1/2}^n)\right)(\rho_{j+1}^n - \rho_j^n) \\ &- \lambda\rho_{j+1}^n g'(\xi_{j+3/2}^n)V_{j+3/2}^n(\rho_{j+2}^n - \rho_{j+1}^n) + \lambda g(\rho_j^n)V_{j+1/2}^n(\rho_j^n - \rho_{j-1}^n),\end{aligned} \tag{4.9}$$

where $\xi_{j+3/2}^n \in \mathcal{I}(\rho_{j+1}^n, \rho_{j+2}^n)$. By the CFL condition (4.4), the first term in the right-hand side of Eq (4.9) is positive, thus summing over $j \in \{1, \dots, M-1\}$ yields

$$\begin{aligned}\sum_{j=1}^{M-1}\left|\mathcal{A}_j^n\right| \leq &\sum_{j=1}^{M-1}\left|\rho_{j+1}^n - \rho_j^n\right| + \lambda g(\rho_1^n)V_{3/2}^n|\rho_1^n - \rho_a^n| - \lambda g(\rho_M^n)V_{M+1/2}^n|\rho_M^n - \rho_{M-1}^n| \\ &- \lambda\rho_M^n g'(\xi_{M+1/2}^n)V_{M+1/2}^n|\rho_b^n - \rho_M^n| + \lambda\rho_1^n g'(\xi_{3/2}^n)V_{3/2}^n|\rho_2^n - \rho_1^n|.\end{aligned} \tag{4.10}$$

On the other hand,

$$\mathcal{B}_j^n = -\rho_j^n(V_{j+1/2}^n - V_{j+3/2}^n)g'(\xi_{j+1/2}^n)(\rho_{j+1}^n - \rho_j^n) \\ + g(\rho_j^n)(V_{j+1/2}^n - V_{j-1/2}^n)(\rho_j^n - \rho_{j-1}^n) + \rho_j^n g(\rho_j^n)(V_{j+3/2}^n - 2V_{j+1/2}^n + V_{j-1/2}^n).$$

Taking absolute values and summing over $j \in \{1, \dots, M-1\}$ we have

$$\begin{aligned}\lambda\sum_{j=1}^{M-1}\left|\mathcal{B}_j^n\right| &\le -\lambda\sum_{j=1}^{M-1}\rho_j^n g'(\xi_{j+1/2}^n)\left|V_{j+3/2}^n - V_{j+1/2}^n\right|\left|\rho_{j+1}^n - \rho_j^n\right| + \lambda\sum_{j=1}^{M-1} g(\rho_j^n)\left|V_{j-1/2}^n - V_{j+1/2}^n\right|\left|\rho_j^n - \rho_{j-1}^n\right| \\ &\quad + \lambda\sum_{j=1}^{M-1}\rho_j^n g(\rho_j^n)\left|V_{j+3/2}^n - 2V_{j+1/2}^n + V_{j-1/2}^n\right| \\ &= -\lambda\sum_{j=1}^{M-1}\left(\rho_j^n g'(\xi_{j+1/2}^n) - g(\rho_{j+1}^n)\right)\left|V_{j+3/2}^n - V_{j+1/2}^n\right|\left|\rho_{j+1}^n - \rho_j^n\right| \\ &\quad + \lambda\sum_{j=1}^{M-1}\rho_j^n g(\rho_j^n)\left|V_{j+3/2}^n - 2V_{j+1/2}^n + V_{j-1/2}^n\right| + \lambda g(\rho_1^n)|V_{3/2}^n - V_{1/2}^n||\rho_1^n - \rho_a^n| \\ &\quad - \lambda g(\rho_M^n)|V_{M+1/2}^n - V_{M-1/2}^n||\rho_M^n - \rho_{M-1}^n|.\end{aligned}$$

By using the following estimations (which are proved in Appendix A)

$$\left|V_{j+3/2}^n - V_{j+1/2}^n\right| \le \mathcal{L}\Delta x, \quad \left|V_{j+3/2}^n - 2V_{j+1/2}^n + V_{j-1/2}^n\right| \le \Delta x^2\mathcal{W}, \tag{4.11}$$

we obtain

$$\begin{aligned}\lambda\sum_{j=1}^{M-1}\left|\mathcal{B}_j^n\right| &\le -\lambda\mathcal{L}\Delta x\sum_{j=1}^{M-1}\left(\rho_j^n g'(\xi_{j+1/2}^n) - g(\rho_{j+1}^n)\right)\left|\rho_{j+1}^n - \rho_j^n\right| \\ &\quad + \lambda\Delta x^2\mathcal{W}\sum_{j=1}^{M-1}\rho_j^n g(\rho_j^n) + \lambda g(\rho_1^n)|V_{3/2}^n - V_{1/2}^n||\rho_1^n - \rho_a^n| \\ &\quad - \lambda g(\rho_M^n)|V_{M+1/2}^n - V_{M-1/2}^n||\rho_M^n - \rho_{M-1}^n| \\ &\le -\mathcal{L}\Delta t\sum_{j=1}^{M-1}\left(\rho_j^n g'(\xi_{j+1/2}^n) - g(\rho_{j+1}^n)\right)\left|\rho_{j+1}^n - \rho_j^n\right| + \Delta t\mathcal{W}\|g\|_\infty\|\rho\|_{L^1(I)} + \mathcal{L}\Delta t g(\rho_1^n)|\rho_1^n - \rho_a^n|.\end{aligned} \tag{4.12}$$

We now deal with the boundary terms, first for the left boundary term. By the definition of the scheme (4.2),

$$\begin{aligned}&\rho_1^{n+1} - \rho_a^{n+1} \\ &= \rho_1^n - \rho_a^n + \rho_a^n - \rho_a^{n+1} - \lambda\Big(\left(\rho_1^n g(\rho_2^n) - \rho_1^n g(\rho_1^n)\right)V_{3/2}^n + \left(\rho_1^n g(\rho_1^n) - \rho_a^n g(\rho_1^n)\right)V_{3/2}^n + \rho_a^n g(\rho_1^n)(V_{3/2}^n - V_{1/2}^n)\Big) \\ &= \rho_1^n - \rho_a^n + \rho_a^n - \rho_a^{n+1} - \lambda\Big(\rho_1^n g'(\xi_{3/2}^n)V_{3/2}^n(\rho_2^n - \rho_1^n) + (\rho_1^n - \rho_a^n)g(\rho_1^n)V_{3/2}^n + \rho_a^n g(\rho_1^n)(V_{3/2}^n - V_{1/2}^n)\Big) \\ &= \rho_a^n - \rho_a^{n+1} + (1 - \lambda g(\rho_1^n)V_{3/2}^n)(\rho_1^n - \rho_a^n) - \lambda\rho_1^n g'(\xi_{3/2}^n)(\rho_2^n - \rho_1^n) - \lambda\rho_a^n g(\rho_1^n)(V_{3/2}^n - V_{1/2}^n).\end{aligned}$$

Taking absolute values, invoking Eq (4.4) and using Eq (4.11) we obtain

$$\left|\rho_1^{n+1}-\rho_a^{n+1}\right| \le \left|\rho_a^n-\rho_a^{n+1}\right| + (1-\lambda g(\rho_1^n)V_{3/2}^n)\left|\rho_1^n-\rho_a^n\right| - \lambda\rho_1^n g'(\xi_{3/2}^n)V_{3/2}^n\left|\rho_2^n-\rho_1^n\right| + \mathcal{L}\Delta t\rho_a^n g(\rho_1^n). \quad (4.13)$$

An analogous discussion of the other boundary term yields

$$\begin{aligned}\left|\rho_b^{n+1}-\rho_M^{n+1}\right| &\le \left|\rho_b^n-\rho_b^{n+1}\right| + (1+\lambda\rho_M^n g'(\xi_{M+1/2}^n)V_{M+1/2}^n)\left|\rho_M^n-\rho_b^n\right| \\ &\quad + \lambda g(\rho_M^n)V_{M+1/2}^n\left|\rho_M^n-\rho_{M-1}^n\right| + \Delta t\rho_{M-1}^n g(\rho_M^n)\mathcal{L}.\end{aligned} \quad (4.14)$$

Finally, collecting the estimates Eqs (4.10), (4.12), (4.13) and (4.14) we arrive at

$$\begin{aligned}\sum_{j=0}^{M}\left|\rho_{j+1}^{n+1}-\rho_j^{n+1}\right| &\le \left|\rho_a^n-\rho_a^{n+1}\right| + \sum_{j=0}^{M}\left|\rho_{j+1}^n-\rho_j^n\right| - \mathcal{L}\Delta t\sum_{j=1}^{M-1}(\rho_j^n g'(\xi_{j+1/2}^n)-g(\rho_{j+1}^n))\left|\rho_{j+1}^n-\rho_j^n\right| \\ &\quad + \Delta t\mathcal{W}\|g\|_\infty\|\rho\|_{L^1(I)} + \Delta t\mathcal{L}g(\rho_1^n)|\rho_1^n-\rho_a^n| + |\rho_b^n-\rho_b^{n+1}| + \Delta t\mathcal{L}(\rho_a^n g(\rho_1^n)+\rho_{M-1}^n g(\rho_M^n)) \\ &\le \left|\rho_a^n-\rho_a^{n+1}\right| + \left(1+\Delta t\mathcal{L}(\|g'\|_\infty+\|g\|_\infty)\right)\sum_{j=0}^{M}|\rho_{j+1}^n-\rho_j^n| \\ &\quad + \Delta t\mathcal{W}\|g\|_\infty\|\rho\|_{L^1(I)} + |\rho_b^n-\rho_b^{n+1}| + 2\Delta t\mathcal{L}\|g\|_\infty \\ &= \left|\rho_a^n-\rho_a^{n+1}\right| + (1+\Delta t\mathcal{K}_1)\sum_{j=0}^{M}|\rho_{j+1}^n-\rho_j^n| + |\rho_b^n-\rho_b^{n+1}| + \Delta t\mathcal{K}_2,\end{aligned}$$

with $\mathcal{L}$, $\mathcal{W}$, $\mathcal{K}_1$ and $\mathcal{K}_2$ defined as in Eqs (2.10), (2.11), and (4.8). The previous estimate implies Eqs (4.6) and (4.7) by standard arguments.

□

Lemma 4.4 (BV estimate in space and time)**.** *Let $\rho_0 \in BV(I;\mathbb{R}^+)$ and ρ_a, $\rho_b \in BV(\mathbb{R}^+;[0,1])$. If Assumptions 2.1 and the CFL condition* (4.4) *hold, then for all $n = 0,\dots,N_T$, the estimate*

$$\sum_{m=0}^{n-1}\sum_{j=0}^{M}\Delta t\left|\rho_{j+1}^m-\rho_j^m\right| + \sum_{m=0}^{n-1}\sum_{j=0}^{M+1}\Delta x\left|\rho_j^{m+1}-\rho_j^m\right| \le C_{xt}(t^n) \quad (4.15)$$

holds, where

$$C_{xt}(t^n) = t^n C_x(t^n) + C_t(t^n) + \Delta x(\,\mathrm{TV}(\rho_a;[0,T]) + \mathrm{TV}(\rho_b;[0,T])).$$

Proof. By Lemma 4.3 we have

$$\sum_{m=0}^{n-1}\sum_{j=0}^{M}\Delta t\left|\rho_{j+1}^m-\rho_j^m\right| \quad\le\quad n\Delta t C_x(n\Delta t). \quad (4.16)$$

By the definition of numerical scheme (4.2), for $m \in \{0,\dots,n-1\}$ and $j \in \{1,\dots,M\}$ we get

$$\begin{aligned}\left|\rho_j^{m+1}-\rho_j^m\right| &= \left|\lambda\rho_j^m g'(\xi_{j+1/2}^m)V_{j+1/2}^m(\rho_{j+1}^m-\rho_j^m) + \lambda\rho_j^m g(\rho_j^m)(V_{j+1/2}^m-V_{j-1/2}^m) + \lambda g(\rho_j^m)V_{j-1/2}^m(\rho_j^m-\rho_j^m)\right| \\ &\le \lambda\|g'\|_\infty\|v\|_\infty\left|\rho_{j+1}^m-\rho_j^m\right| + \lambda\|g\|_\infty\mathcal{L}\Delta x\rho_j^m + \lambda\|g\|_\infty\|v\|_\infty\left|\rho_j^m-\rho_{j-1}^m\right|.\end{aligned}$$

Multiplying the last inequality by Δx and summing for j from 1 to M we get

$$\sum_{j=1}^{M} \Delta x \left|\rho_j^{m+1} - \rho_j^m\right| \leq \Delta t \|v\|_\infty (\|g'\|_\infty + \|g\|_\infty) \sum_{j=0}^{M} \left|\rho_{j+1}^m - \rho_j^m\right| + \Delta t \|g\|_\infty \mathcal{L} \|\rho^m\|_{L^1(I)}.$$

Lemmas 4.2 and 4.3 now imply that

$$\sum_{j=1}^{M} \Delta x \left|\rho_j^{m+1} - \rho_j^m\right| \leq \Delta t \|v\|_\infty (\|g'\|_\infty + \|g\|_\infty) C_x(m\Delta t) + \Delta t \|g\|_\infty \mathcal{L} C_1(m\Delta t) = \Delta t C_t(m\Delta t),$$

where we define

$$C_t(\tau) := \|v\|_{L^\infty([0,1])} (\|g'\|_\infty + \|g\|_\infty) C_x(\tau) + \|g\|_\infty \mathcal{L} C_1(\tau).$$

In particular,

$$\begin{aligned} \sum_{j=0}^{M+1} \Delta x \left|\rho_j^{m+1} - \rho_j^m\right| &= \Delta x \left|\rho_a^{m+1} - \rho_a^m\right| + \Delta x \left|\rho_b^{m+1} - \rho_b^m\right| + \sum_{j=1}^{M} \Delta x \left|\rho_j^{m+1} - \rho_j^m\right| \\ &\leq \Delta x \left|\rho_a^{m+1} - \rho_a^m\right| + \Delta x \left|\rho_b^{m+1} - \rho_b^m\right| + \Delta t C_t(m\Delta t), \end{aligned} \tag{4.17}$$

which, summed over $m = 0, \ldots, n-1$, yields

$$\sum_{m=0}^{n-1} \sum_{j=0}^{M+1} \Delta x \left|\rho_j^{m+1} - \rho_j^m\right| \leq \Delta x \sum_{m=0}^{n-1} \left(\left|\rho_a^{m+1} - \rho_a^m\right| + \left|\rho_b^{m+1} - \rho_b^m\right| \right) + n\Delta t C_t(n\Delta t). \tag{4.18}$$

Summing Eqs (4.16) and (4.18) we get the desired estimate Eq (4.15). □

4.3. Convergence analysis

Lemmas 4.1 and 4.4 allow us to apply Helly's compactness theorem that ensures the existence of a subsequence of ρ_Δ, still denoted by ρ_Δ, that converges in L^1 to a function $\rho \in L^\infty(I \times [0,T])$, for all $T > 0$. Now we need to prove that this limit function is indeed an entropy weak solution to Eq (2.1) in the sense of Definition 2.2. First we will show that the approximate solutions obtained by the scheme (4.2) satisfies a discrete entropy inequality. To this end, for $j = 1, \ldots, M$, $n = 0, \ldots, N_T - 1$, and $k \in \mathbb{R}$, we define

$$\begin{aligned} H_j^n(u,w,z) &:= w - \lambda(F_{j+1/2}^n(w,z) - F_{j-1/2}^n(u,w)), \\ G_{j+1/2}^{n,k}(u,w) &:= F_{j+1/2}^n(u \vee k, w \vee k) - F_{j+1/2}^n(k,k), \\ L_{j+1/2}^{n,k}(u,w) &:= F_{j+1/2}^n(k,k) - F_{j+1/2}^n(u \wedge k, w \wedge k), \end{aligned}$$

where $w \wedge z := \min\{w,z\}$, $w \vee z := \max\{w,z\}$, and $F_{j+1/2}^n(u,w)$ is defined as in Eq (4.3). Observe that due to the definition of the scheme,

$$\rho_j^{n+1} = H_j^n(\rho_{j-1}^n, \rho_j^n, \rho_{j+1}^n),$$

and we also recall the equivalence $(s-k)^+ = s \vee k - k$ and $(s-k)^- = k - s \wedge k$.

Lemma 4.5 (Discrete entropy inequalities)**.** *If Assumptions 2.1 and the CFL condition* (4.4) *are in effect, then the approximate solution* ρ_Δ *in Eq* (4.1) *satisfies the discrete entropy inequalities*

$$\begin{aligned}
&(\rho_j^{n+1}-k)^+ - (\rho_j^n - k)^+ + \lambda(G^{n,k}_{j+1/2}(\rho_j^n,\rho_{j+1}^n) - G^{n,k}_{j-1/2}(\rho_{j-1}^n,\rho_j^n))\\
&\quad + \lambda \operatorname{sgn}^+(\rho_j^{n+1}-k) f(k)(V^n_{j+1/2} - V^n_{j-1/2}) \le 0 \quad \text{and} \qquad (4.19)\\
&(\rho_j^{n+1}-k)^- - (\rho_j^n - k)^- + \lambda(L^{n,k}_{j+1/2}(\rho_j^n,\rho_{j+1}^n) - L^{n,k}_{j-1/2}(\rho_{j-1}^n,\rho_j^n))\\
&\quad + \lambda \operatorname{sgn}^-(\rho_j^{n+1}-k) f(k)(V^n_{j+1/2} - V^n_{j-1/2}) \le 0, \qquad (4.20)
\end{aligned}$$

for $j = 1,\dots,M,\ n = 0,\dots,N_T-1$ *and* $k \in \mathbb{R}$.

Proof. By the CFL condition (4.4), the map $(u,w,z) \mapsto H_j^n(u,w,z)$ satisfies

$$\begin{aligned}
\partial_u H_j^n(u,w,z) &= \lambda g(w) V^n_{j-/2} \ge 0,\\
\partial_w H_j^n(u,w,z) &= 1 - \lambda(g(z)V^n_{j+1/2} - u g'(w) V^n_{j-1/2}) \ge 0, \qquad (4.21)\\
\partial_z H_j^n(u,w,z) &= -\lambda w g'(z) V^n_{j+1/2} \ge 0.
\end{aligned}$$

Notice that

$$H_j^n(k,k,k) = k - \lambda f(k)(V^n_{j+1/2} - V^n_{j-1/2}).$$

The monotonicity properties Eq (4.21) imply that

$$\begin{aligned}
&H_j^n(\rho_{j-1}^n \vee k, \rho_j^n \vee k, \rho_{j+1}^n \vee k) - H_j^n(k,k,k)\\
&\ge H_j^n(\rho_{j-1}^n,\rho_j^n,\rho_{j+1}^n) \vee H_j^n(k,k,k) - H_j^n(k,k,k)\\
&= (H_j^n(\rho_{j-1}^n,\rho_j^n,\rho_{j+1}^n) - H_j^n(k,k,k))^+ = (\rho_j^{n+1} - k + \lambda f(k)(V^n_{j+1/2} - V^n_{j-1/2}))^+,
\end{aligned}$$

moreover, we also have

$$\begin{aligned}
&H_j^n(\rho_{j-1}^n \vee k, \rho_j^n \vee k, \rho_{j+1}^n \vee k) - H_j^n(k,k,k)\\
&= (\rho_j^n \vee k) - \lambda(F^n_{j+1/2}(\rho_j^n \vee k, \rho_{j+1}^n \vee k) - F^n_{j-1/2}(\rho_{j-1}^n \vee k, \rho_j^n \vee k))\\
&\quad - (k - \lambda(F^n_{j+1/2}(k,k) - F^n_{j-1/2}(k,k)))\\
&= (\rho_j^n \vee k) - k - \lambda(F^n_{j+1/2}(\rho_j^n \vee k, \rho_{j+1}^n \vee k) - F^n_{j-1/2}(\rho_{j-1}^n \vee k, \rho_j^n \vee k) - F^n_{j+1/2}(k,k) + F^n_{j-1/2}(k,k))\\
&= (\rho_j^n - k)^+ - \lambda(G^{n,k}_{j+1/2}(\rho_j^n,\rho_{j+1}^n) - G^{n,k}_{j-1/2}(\rho_{j-1}^n,\rho_j^n)),
\end{aligned}$$

hence

$$\begin{aligned}
&(\rho_j^n - k)^+ - \lambda(G^{n,k}_{j+1/2}(\rho_j^n,\rho_{j+1}^n) - G^{n,k}_{j-1/2}(\rho_{j-1}^n,\rho_j^n))\\
&\ge (\rho_j^{n+1} - k + \lambda f(k)(V^n_{j+1/2} - V^n_{j-1/2}))^+\\
&= \operatorname{sgn}^+(\rho_j^{n+1} - k + \lambda f(k)(V^n_{j+1/2} - V^n_{j-1/2}))(\rho_j^{n+1} - k + \lambda f(k)(V^n_{j+1/2} - V^n_{j-1/2}))\\
&\ge (\rho_j^{n+1} - k) + \lambda \operatorname{sgn}^+(\rho_j^{n+1} - k) f(k)(V^n_{j+1/2} - V^n_{j-1/2}),
\end{aligned}$$

which proves Eq (4.19), while Eq (4.20) is proven in an entirely analogous way. □

Remark 4.1. *Lemma 4.5 and its proof mimic standard arguments known for monotone schemes of local conservation laws [13], although the scheme is* not *monotone in the proper sense since the argument Eq* (4.21) *suppresses the presence of* ρ^n_{j-1}, ρ^n_j *and* ρ^n_{j+1} *within* $V^n_{j-1/2}$ *and* $V^n_{j+1/2}$.

Lemma 4.6. *Let* $\rho_0 \in BV(]a, b[; \mathbb{R}^+)$, $\rho_a, \rho_b \in BV(\mathbb{R}^+; \mathbb{R}^+)$, *and Assumptions 2.1 and the CFL condition* (4.4) *be in effect. Then the piecewise constant approximate solutions* ρ_Δ *in Eq* (4.1) *resulting from the HW scheme* (4.2) *converge, as* $\Delta x \to 0$, *towards an entropy weak solution of initial boundary value problem* (2.1).

Proof. Adding and subtracting $\lambda G^{n,k}_{j+1/2}(\rho^n_j, \rho^n_j)$ we may rewrite Eq (4.19) as

$$\begin{aligned}0 \geq (\rho^{n+1}_j - k)^+ - (\rho^n_j - k)^+ + \lambda(G^{n,k}_{j+1/2}(\rho^n_j, \rho^n_{j+1}) - G^{n,k}_{j+1/2}(\rho^n_j, \rho^n_j))\\ + \lambda(G^{n,k}_{j+1/2}(\rho^n_j, \rho^n_j) - G^{n,k}_{j-1/2}(\rho^n_{j-1}, \rho^n_j)) + \lambda \operatorname{sgn}^+(\rho^{n+1}_j - k) f(k)(V^n_{j+1/2} - V^n_{j-1/2}).\end{aligned}$$

Let $\varphi \in C^1_c([0, T] \times I; \mathbb{R}^+)$ for some $T > 0$. Multiplying the inequality above by $\Delta x \varphi(x_j, t^n)$ and summing over $j = 1, \ldots, M$ and $n \in \mathbb{N}$ yields

$$0 \geq T_1 + T_2 + T_3 + T_4,$$

where we define the terms

$$\begin{aligned}T_1 &:= \Delta x \sum_{n=0}^{\infty} \sum_{j=1}^{M} ((\rho^{n+1}_j - k)^+ - (\rho^n_j - k)^+)\varphi(x_j, t^n),\\ T_2 &:= \Delta t \sum_{n=0}^{\infty} \sum_{j=1}^{M} (G^{n,k}_{j+1/2}(\rho^n_j, \rho^n_{j+1}) - G^{n,k}_{j+1/2}(\rho^n_j, \rho^n_j))\varphi(x_j, t^n),\\ T_3 &:= -\Delta t \sum_{n=0}^{\infty} \sum_{j=1}^{M} (G^{n,k}_{j-1/2}(\rho^n_{j-1}, \rho^n_j) - G^{n,k}_{j+1/2}(\rho^n_j, \rho^n_j))\varphi(x_j, t^n),\\ T_4 &:= \Delta t \sum_{n=0}^{\infty} \sum_{j=1}^{M} \operatorname{sgn}^+(\rho^{n+1}_j - k) f(k)(V^n_{j+1/2} - V^n_{j-1/2})\varphi(x_j, t^n).\end{aligned}$$

Summing by parts, we obtain

$$\begin{aligned}T_1 &= \Delta x \sum_{n=1}^{\infty} \sum_{j=1}^{M} (\rho^n_j - k)^+ \varphi(x_j, t^{n-1}) - \Delta x \sum_{n=0}^{\infty} \sum_{j=1}^{M} (\rho^n_j - k)^+ \varphi(x_j, t^n)\\ &= -\Delta x \sum_{j=1}^{M} (\rho^0_j - k)^+ \varphi(x_j, 0) - \Delta x \Delta t \sum_{n=1}^{\infty} \sum_{j=1}^{M} (\rho^n_j - k)^+ \frac{\varphi(x_j, t^n) - \varphi(x_j, t^{n-1})}{\Delta t},\end{aligned}$$

and by the dominated convergence theorem,

$$T_1 \xrightarrow{\Delta x \to 0^+} -\int_a^b (\rho_0(x) - k)^+ \varphi(x_j, 0)\,\mathrm{d}x - \int_0^\infty \int_a^b (\rho(x, t) - k)^+ \partial_t \varphi(x, t)\,\mathrm{d}x\,\mathrm{d}t.$$

Again by the dominated convergence theorem,

$$\begin{aligned} T_4 &= \Delta t \Delta x \sum_{n=0}^{\infty} \sum_{j=1}^{M} \operatorname{sgn}^+(\rho_j^{n+1} - k) f(k) \frac{V_{j+1/2}^n - V_{j-1/2}^n}{\Delta x} \varphi(x_j, t^n) \\ &\xrightarrow{\Delta x \to 0^+} \int_0^\infty \int_a^b \operatorname{sgn}^+(\rho(x,t) - k) f(k) (\partial_x V) \varphi(x,t) \, \mathrm{d}x \, \mathrm{d}t. \end{aligned}$$

Concerning T_2 and T_3, we get

$$\begin{aligned} T_2 + T_3 &= \Delta t \sum_{n=0}^{\infty} \sum_{j=1}^{M} (G_{j+1/2}^{n,k}(\rho_j^n, \rho_{j+1}^n) - G_{j+1/2}^{n,k}(\rho_j^n, \rho_j^n)) \varphi(x_j, t^n) \\ &\quad - \Delta t \sum_{n=0}^{\infty} \sum_{j=0}^{M-1} (G_{j+1/2}^{n,k}(\rho_j^n, \rho_{j+1}^n) - G_{j+3/2}^{n,k}(\rho_{j+1}^n, \rho_{j+1}^n)) \varphi(x_{j+1}, t^n) = T_{20} + T_{30} = T_{23}, \end{aligned}$$

where we define

$$\begin{aligned} T_{20} &:= \Delta t \sum_{n=0}^{\infty} \sum_{j=1}^{M-1} \Big((G_{j+1/2}^{n,k}(\rho_j^n, \rho_{j+1}^n) - G_{j+1/2}^{n,k}(\rho_j^n, \rho_j^n)) \varphi(x_j, t^n) \\ &\qquad - (G_{j+1/2}^{n,k}(\rho_j^n, \rho_{j+1}^n) - G_{j+3/2}^{n,k}(\rho_{j+1}^n, \rho_{j+1}^n)) \varphi(x_{j+1}, t^n) \Big), \\ T_{30} &:= \Delta t \sum_{n=0}^{\infty} \Big((G_{M+1/2}^{n,k}(\rho_M^n, \rho_b^n) - G_{M+1/2}^{n,k}(\rho_M^n, \rho_M^n)) \varphi(x_M, t^n) \\ &\qquad - (G_{1/2}^{n,k}(\rho_a^n, \rho_1^n) - G_{3/2}^{n,k}(\rho_1^n, \rho_1^n)) \varphi(x_1, t^n) \Big). \end{aligned}$$

Now we define

$$\begin{aligned} S &:= -\Delta x \Delta t \sum_{n=0}^{+\infty} \sum_{j=1}^{M} G_{j+1/2}^{n,k}(\rho_j^n, \rho_j^n) \frac{\varphi(x_{j+1}, t^n) - \varphi(x_j, t^n)}{\Delta x} \\ &\quad - L \Delta t \sum_{n=0}^{+\infty} ((\rho_a^n - k)^+ \varphi(a, t^n) + (\rho_b^n - k)^+ \varphi(b, t^n)). \end{aligned} \tag{4.22}$$

Since

$$\begin{aligned} G_{j+1/2}^{n,k}(\rho_j^n, \rho_j^n) &= F_{j+1/2}^n(\rho_j^n \vee k, \rho_j^n \vee k) - F_{j+1/2}^n(k,k) \\ &= (f(\rho_j^n \vee k) - f(k)) V_{j+1/2}^n = \operatorname{sgn}^+(\rho_j^n - k)(f(\rho_j^n) - f(k)) V_{j+1/2}^n, \end{aligned}$$

it follows that

$$\begin{aligned} S &\xrightarrow{\Delta x \to 0^+} - \int_0^{+\infty} \int_a^b \operatorname{sgn}^+(\rho(x,t) - k)(f(\rho_j^n) - f(k)) V \partial_x \varphi(x,t) \, \mathrm{d}x \, \mathrm{d}t \\ &\quad - L \Big(\int_0^{+\infty} (\rho_a(t) - k)^+ \varphi(a,t) \, \mathrm{d}t + \int_0^{+\infty} (\rho_b(t) - k)^+ \varphi(b,t) \, \mathrm{d}t \Big). \end{aligned}$$

Let us rewrite S in Eq (4.22) as follows:

$$\begin{aligned}S &= -\Delta t\sum_{n=0}^{+\infty}\sum_{j=1}^{M} G^{n,k}_{j+1/2}(\rho^n_j,\rho^n_j)(\varphi(x_{j+1},t^n)-\varphi(x_j,t^n)) - L\Delta t\sum_{n=0}^{+\infty}((\rho^n_a-k)^+\varphi(a,t^n)+(\rho^n_b-k)^+\varphi(b,t^n))\\ &= -\Delta t\sum_{n=0}^{+\infty}\left(\sum_{j=1}^{M} G^{n,k}_{j+1/2}(\rho^n_j,\rho^n_j)\varphi(x_{j+1},t^n) - \sum_{j=0}^{M-1} G^{n,k}_{j+3/2}(\rho^n_{j+1},\rho^n_{j+1})\varphi(x_{j+1},t^n)\right)\\ &\quad - L\Delta t\sum_{n=0}^{+\infty}((\rho^n_a-k)^+\varphi(a,t^n)+(\rho^n_b-k)^+\varphi(b,t^n)) = S_{20}+S_{30},\end{aligned}$$

where we define

$$\begin{aligned}S_{20} &:= -\Delta t\sum_{n=0}^{\infty}\sum_{j=1}^{M-1}\{G^{n,k}_{j+1/2}(\rho^n_j,\rho^n_j) - G^{n,k}_{j+3/2}(\rho^n_{j+1},\rho^n_{j+1})\}\varphi(x_{j+1},t^n),\\ S_{30} &:= -\Delta t\sum_{n=0}^{+\infty}(G^{n,k}_{M+1/2}(\rho^n_M,\rho^n_M)\varphi(x_{M+1},t^n) - G^{n,k}_{3/2}(\rho^n_1,\rho^n_1)\varphi(x_1,t^n))\\ &\quad - L\Delta t\sum_{n=0}^{+\infty}((\rho^n_a-k)^+\varphi(a,t^n)+(\rho^n_b-k)^+\varphi(b,t^n)).\end{aligned}$$

Adding and subtracting $G^{n,k}_{j+1/2}(\rho^n_j,\rho^n_j)$, we may rewrite S_{20} as

$$\begin{aligned}S_{20} &= -\Delta t\sum_{n=0}^{\infty}\sum_{j=1}^{M-1}(G^{n,k}_{j+1/2}(\rho^n_j,\rho^n_{j+1}) - G^{n,k}_{j+1/2}(\rho^n_j,\rho^n_j))\varphi(x_{j+1},t^n)\\ &\quad - \Delta t\sum_{n=0}^{\infty}\sum_{j=1}^{M-1}(G^{n,k}_{j+1/2}(\rho^n_j,\rho^n_{j+1}) - G^{n,k}_{j+3/2}(\rho^n_{j+1},\rho^n_{j+1}))\varphi(x_{j+1},t^n).\end{aligned}$$

We evaluate now the distance between T_{20} and S_{20}:

$$\left|T_{20}-S_{20}\right| \le \Delta t\sum_{n=0}^{\infty}\sum_{j=1}^{M-1}\left|G^{n,k}_{j+1/2}(\rho^n_j,\rho^n_{j+1}) - G^{n,k}_{j+1/2}(\rho^n_j,\rho^n_j)\right|\left|\varphi(x_{j+1},t^n)-\varphi(x_j,t^n)\right|,$$

since

$$\begin{aligned}\left|G^{n,k}_{j+1/2}(\rho^n_j,\rho^n_{j+1}) - G^{n,k}_{j+1/2}(\rho^n_j,\rho^n_j)\right| &= \left|F^n_{j+1/2}(\rho^n_j\vee k,\rho^n_{j+1}\vee k) - F^n_{j+1/2}(\rho^n_j\vee k,\rho^n_j\vee k)\right|\\ &= \left|(\rho^n_j\vee k)(g(\rho^n_{j+1}\vee k)-g(\rho^n_j\vee k))V^n_{j+1/2}\right|\\ &= \left|(\rho^n_j\vee k)g'(\eta^n_{j+1/2})((\rho^n_{j+1}\vee k)-(\rho^n_j\vee k))V^n_{j+1/2}\right|\\ &\le \|v\|_\infty\|g'\|_\infty\left|(\rho^n_j\vee k)((\rho^n_{j+1}\vee k)-(\rho^n_j\vee k))\right|\\ &\le \|v\|_\infty\|g'\|_\infty\left|\rho^n_{j+1}-\rho^n_j\right| \le L\left|\rho^n_{j+1}-\rho^n_j\right|,\end{aligned}$$

in light of the uniform BV estimate Eq (4.6) we deduce that

$$\begin{aligned}\left|T_{20}-S_{20}\right| &\le L\Delta x\Delta t\|\partial_x\varphi\|_\infty\sum_{n=0}^{\infty}\sum_{j=1}^{M-1}\left|\rho^n_{j+1}-\rho^n_j\right|\\ &\le L\Delta x T\|\partial_x\varphi\|_\infty\max_{0\le n\le T/\Delta t}\mathrm{TV}(\rho_\Delta(\cdot,t^n)) = \mathcal{O}(\Delta x).\end{aligned}\tag{4.23}$$

Furthermore, we obtain

$$\begin{aligned}S_{30}-T_{30} &= -\Delta t\sum_{n=0}^{\infty}(G^{n,k}_{M+1/2}(\rho^n_M,\rho^n_M)\varphi(x_{M+1},t^n)-G^{n,k}_{3/2}(\rho^n_1,\rho^n_1)\varphi(x_1,t^n))\\ &\quad - L\Delta t\sum_{n=0}^{\infty}((\rho^n_a-k)^+\varphi(a,t^n)+(\rho^n_b-k)^+\varphi(b,t^n))\\ &\quad - \Delta t\sum_{n=0}^{\infty}\Big((G^{n,k}_{M+1/2}(\rho^n_M,\rho^n_b)-G^{n,k}_{M+1/2}(\rho^n_M,\rho^n_M))\varphi(x_M,t^n)\\ &\qquad - (G^{n,k}_{1/2}(\rho^n_a,\rho^n_1)-G^{n,k}_{3/2}(\rho^n_1,\rho^n_1))\varphi(x_1,t^n)\Big),\end{aligned}$$

which we can write as $S_{30}-T_{30}=R_1+R_2+R_3$ with

$$\begin{aligned}R_1 &:= \Delta t\sum_{n=0}^{\infty}(G^{n,k}_{1/2}(\rho^n_a,\rho^n_1)\varphi(x_1,t^n)-L(\rho^n_a-k)^+\varphi(a,t^n)),\\ R_2 &:= -\Delta t\sum_{n=0}^{\infty}(L(\rho^n_b-k)^+\varphi(b,t^n)+G^{n,k}_{M+1/2}(\rho^n_M,\rho^n_b)\varphi(x_M,t^n)),\\ R_3 &:= -\Delta t\sum_{n=0}^{\infty}G^{n,k}_{M+1/2}(\rho^n_M,\rho^n_M)(\varphi(x_{M+1},t^n)-\varphi(x_M,t^n)).\end{aligned}$$

Observe that

$$\begin{aligned}\partial_u F^n_{j+1/2}(u,z) &= \partial_u(ug(z)V^n_{j+1/2}) = g(z)V^n_{j+1/2}\geq 0,\\ \partial_z F^n_{j+1/2}(u,z) &= \partial_z(ug(z)V^n_{j+1/2}) = ug'(z)V^n_{j+1/2}\leq 0,\end{aligned}$$

meaning that the numerical flux is increasing with respect to the first variable and the decreasing with respect to the second one. Consequently,

$$\begin{aligned}G^{n,k}_{j+1/2}(u,z) &= F^n_{j+1/2}(u\vee k,z\vee k)-F^n_{j+1/2}(k,k)\\ &\geq F^n_{j+1/2}(k,z\vee k)-F^n_{j+1/2}(k,k)=(kg(z\vee k)-kg(k))V^n_{j+1/2}\\ &= kg'(v^n_{j+1/2})((z\vee k)-k)V^n_{j+1/2}\geq -\|v\|_\infty\|g'\|_\infty(z-k)^+\geq -L(z-k)^+,\\ G^{n,k}_{j+1/2}(u,z) &= F^n_{j+1/2}(u\vee k,z\vee k)-F^n_{j+1/2}(k,k)\\ &\leq F^n_{j+1/2}(u\vee k,k)-F^n_{j+1/2}(k,k)=(u\vee k)g(k)V^n_{j+1/2}-kg(k)V^n_{j+1/2}\\ &= ((u\vee k)-k)g(k)V^n_{j+1/2}\leq \|v\|_\infty\|g\|_\infty(u-k)^+\leq L(u-k)^+,\end{aligned}$$

hence

$$\begin{aligned}R_1 &= \Delta t\sum_{n=0}^{\infty}G^{n,k}_{1/2}(\rho^n_a,\rho^n_1)(\varphi(x_1,t^n)-\varphi(a,t^n))+\Delta t\sum_{n=0}^{\infty}(G^{n,k}_{1/2}(\rho^n_a,\rho^n_1)-L(\rho^n_a-k)^+)\varphi(a,t^n)\\ &\leq LT\Delta x\|\partial_x\varphi\|_\infty\sup_{0\leq n\leq T/\Delta t}(\rho^n_a-k)^+ + L\Delta t\sum_{n=0}^{\infty}((\rho^n_a-k)^+-(\rho^n_a-k)^+)\varphi(a,t^n)\end{aligned}$$

$$
\begin{aligned}
&\leq T\Delta x\|\partial_x\varphi\|_{L^\infty}\|\rho_a\|_{L^\infty([0,t])} = O(\Delta x),\\
R_2 &= -\Delta t\sum_{n=0}^{\infty}(L(\rho_b^n - k)^+ + G^{n,k}_{M+1/2}(\rho_M^n,\rho_b^n))\varphi(b,t^n) - \Delta t\sum_{n=0}^{\infty}G^{n,k}_{M+1/2}(\rho_M^n,\rho_b^n)(\varphi(x_M,t^n) - \varphi(b,t^n))\\
&\leq -L\Delta t\sum_{n=0}^{\infty}((\rho_b^n - k)^+ - (\rho_b^n - k)^+)\varphi(b,t^n) + LT\Delta x\|\partial_x\varphi\|_{L^\infty}\sup_{0\leq n\leq T/\Delta t}(\rho_b^n - k)^+\\
&\leq LT\Delta x\|\partial_x\varphi\|_{L^\infty}\|\rho_b\|_{L^\infty([0,t])} = O(\Delta x),\\
R_3 &\leq \Delta t\left|\sum_{n=0}^{\infty}G^{n,k}_{M+1/2}(\rho_M^n,\rho_M^n)(\varphi(x_{M+1},t^n) - \varphi(x_M,t^n))\right| \leq \Delta t\Delta x\|\partial_x\varphi\|_{L^\infty}\sum_{n=0}^{+\infty}\left|G^{n,k}_{M+1/2}(\rho_M^n,\rho_M^n)\right|.
\end{aligned}
$$

Taking into account that

$$
\begin{aligned}
G^{n,k}_{M+1/2}(\rho_M^n,\rho_M^n) &= F^n_{M+1/2}(\rho_M^n\vee k,\rho_M^n\vee k) - F^n_{M+1/2}(k,k)\\
&= (\rho_M^n\vee k)g(\rho_M^n\vee k)V^n_{M+1/2} - kg(k)V^n_{M+1/2}\\
&= ((\rho_M^n\vee k)(g(\rho_M^n\vee k) - g(k)) + (\rho_M^n\vee k - k)g(k))V^n_{M+1/2}\\
&= ((\rho_M^n\vee k)g'(\eta^n_{j+1/2})(\rho_M^n\vee k - k) + (\rho_M^n\vee k - k)g(k))V^n_{M+1/2}\\
&= ((\rho_M^n\vee k)g'(\eta^n_{j+1/2}) + g(k))(\rho_M^n\vee k - k)V^n_{M+1/2},
\end{aligned}
$$

we get

$$
\begin{aligned}
R_3 &= \Delta t\Delta x\|\partial_x\varphi\|_{L^\infty}\sum_{n=0}^{+\infty}\left|((\rho_M^n\vee k)g'(\eta^n_{j+1/2}) + g(k))(\rho_M^n\vee k - k)V^n_{M+1/2}\right|\\
&\leq \Delta t\Delta x\|\partial_x\varphi\|_{L^\infty}\sum_{n=0}^{+\infty}\left|(g'(\eta^n_{j+1/2}) + g(k))(\rho_M^n\vee k - k)V^n_{M+1/2}\right| \leq L\Delta t\Delta x\|\partial_x\varphi\|_{L^\infty}\|v\|_\infty\sum_{n=0}^{\infty}\left|\rho_M^n\vee k - k\right|\\
&= L\Delta t\Delta x\|\partial_x\varphi\|_{L^\infty}\|v\|_\infty\sum_{n=0}^{\infty}(\rho_M^n - k)^+ \leq LT\Delta x\|\partial_x\varphi\|_{L^\infty}\|v\|_\infty\sup_{0\leq n\leq T/\Delta t}\|\rho^n\|_\infty\\
&\leq LT\Delta x\|\partial_x\varphi\|_{L^\infty}\|v\|_\infty = O(\Delta x),
\end{aligned}
$$

thanks to the maximum principle estimate. Hence, $S_{30} - T_{30} \leq O(\Delta x)$, so that we finally get

$$
0 \geq T_1 + T_2 + T_3 + T_4 = T_1 + T_4 + T_{23} \pm S = T_1 + T_4 + S - O(\Delta x).
$$

This concludes the proof. □

Proof of Theorem 2.1. The existence of solutions to problem (2.1) follows from the results of Section 4. The uniqueness is ensured by the Lipschitz continuous dependence of solutions to Eq (2.1) on initial and boundary data, see Section 3. The estimates on the solution to Eq (2.1) are obtained from the corresponding discrete estimates passing to the limit. In particular, the L^1 bound follows from Eq (4.2), the maximum principle from Eq (4.1), the total variation bound from Eq (4.3) and the Lipschitz continuity in time from Eq (4.17), since $\Delta x = \Delta t/\lambda$ and taking $\lambda = 1/L$. □

5. A second-order scheme

The scheme given by Eqs (4.2), (4.3) is only first-order accurate. We propose here a second-order accuracy scheme, constructed using MUSCL-type variable extrapolation and Runge-Kutta temporal differencing. To implement it, we approximate $\rho(x,t^n)$ by a piecewise linear functions in each cell, i.e. $\hat{\rho}_j(x,t^n) = \rho_j^n + \sigma_j^n(x-x_j)$, where the slopes σ_j^n are calculated via the generalized minmod limiter, i.e.

$$\sigma_j^n = \frac{1}{\Delta x}\operatorname{minmod}\Big(\vartheta(\rho_j^n-\rho_{j-1}^n), \frac{1}{2}(\rho_{j+1}^n-\rho_{j-1}^n), \vartheta(\rho_{j+1}^n-\rho_j^n)\Big),$$

where $\vartheta \in [1,2]$ and

$$\operatorname{minmod}(a,b,c) := \begin{cases} \operatorname{sgn}(a)\min\{|a|,|b|,|c|\} & \text{if } \operatorname{sgn}(c)=\operatorname{sgn}(b)=\operatorname{sgn}(a),\\ 0 & \text{otherwise.}\end{cases}$$

This extrapolation enables one to define left and right values at the cell interfaces respectively by

$$\begin{aligned}\rho_{j+1/2}^{\mathrm{L}} &:= \hat{\rho}_j\left(x_j+\frac{\Delta x}{2},t^n\right) = \rho_j^n + \sigma_j^n\frac{\Delta x}{2},\\ \rho_{j-1/2}^{\mathrm{R}} &:= \hat{\rho}_j\left(x_j-\frac{\Delta x}{2},t^n\right) = \rho_j^n - \sigma_j^n\frac{\Delta x}{2}.\end{aligned}$$

and

$$\begin{aligned}\hat{V}_{j+1/2}^n &= \frac{1}{\hat{W}_{j+1/2}}\int_a^b v(\hat{\rho}(y,t))\omega(y-x_{j+1/2})\,\mathrm{d}y\\ &= \frac{1}{\hat{W}_{j+1/2}}\sum_{k=1}^{M}\int_{x_{k-1/2}}^{x_{k+1/2}} v(\hat{\rho}_k(y,t))\omega(y-x_{j+1/2})\,\mathrm{d}y\\ &= \frac{\Delta x}{2\hat{W}_{j+1/2}}\sum_{k=1}^{M}\int_{-1}^{1} v\left(\hat{\rho}_k\Big(\frac{\Delta x}{2}y+x_k,t^n\Big)\right)\omega\left(\frac{\Delta x}{2}y+(k-j+1/2)\Delta x\right)\mathrm{d}y\\ &= \frac{\Delta x}{2\hat{W}_{j+1/2}}\sum_{k=1}^{M}\sum_{e=1}^{N_G} p_e v\left(\hat{\rho}_k\Big(\frac{\Delta x}{2}y_e+x_k,t^n\Big)\right)\omega\left(\frac{\Delta x}{2}y_e+(k-j+1/2)\Delta x\right)\\ &= \frac{\Delta x}{2\hat{W}_{j+1/2}}\sum_{k=1}^{M}\sum_{e=1}^{N_G} p_e v\left(\rho_k^n+\frac{\Delta x}{2}y_e\sigma_k^n\right)\omega\left(\frac{\Delta x}{2}y_e+(k-j+1/2)\Delta x\right)\end{aligned}$$

where $\hat{W}_{j+1/2} = \int_a^b \omega(y-x_{j+1/2})\mathrm{d}y$ is computed in exact form, and y_e are the Gauss-Lobatto-Quadrature points. The MUSCL version of the numerical flux reads

$$F_{j+1/2}^n := \rho_{j+1/2}^{\mathrm{L}} g(\rho_{j+1/2}^{\mathrm{R}})\hat{V}_{j+1/2}^n.$$

To achieve formal second-order accuracy also in time, we use second-order Runge-Kutta (RK) time stepping. More precisely, if we write our scheme with first-order Euler time differences and second-order spatial differences formally as

$$\rho_j^{n+1} = \rho_j^n - \lambda L_j(\rho^n) := \rho_j^n - \lambda(F_{j+1/2}^n - F_{j-1/2}^n), \tag{5.1}$$

then the RK version takes the two-step form

$$\rho_j^1 = \rho_j^n - \lambda L_j(\rho^n); \quad \rho_j^{n+1} = \frac{1}{2}(\rho_j^n + \rho_j^1) - \frac{\lambda}{2} L_j(\rho_j^1). \tag{5.2}$$

6. Numerical examples

In this section we solve Eq (2.1) by using the numerical scheme (4.2) on the x-interval $I = [0, 1]$ with suitable boundary conditions and $t \in [0, T]$, with T specified later. In the numerical examples we consider the equation (2.1) with $g(\rho) = 1 - \rho$ and $v(\rho) = (1 - \rho)^4$, where we recall that $f(\rho) = \rho g(\rho)$ and $V(x, t) = (\omega * v(\rho))(x, t)$, where the kernel function $\omega(x)$ is specified later in each case. For numerical experiments the interval I is subdivided into M subintervals of length $\Delta x = 1/M$, and the time step is computed taking account the CFL condition (4.4), and for each numerical experiment, we specify the inital and boundary conditions.

6.1. Example 1

In this example we compare numerical approximations obtained by scheme (4.2) with those generated by an adapted LxF-type scheme proposed by Goatin and Rossi in [17], starting from the initial and boundary conditions

$$\rho_0(x) = 0.2 \quad \text{for } x \in I; \quad \rho_a(t) = 0.1, \quad \rho_b(t) = 0.5 \quad \text{for } t > 0.$$

Here we employ the symmetric kernel function

$$\omega(y) = \omega_1(y) := \frac{3}{4\eta}\left(1 - \frac{y^2}{\eta^2}\right)\chi_{[-\eta,\eta]}(y) \tag{6.1}$$

with $\eta = 0.05$. In Figure 1 we display the numerical approximations for $M = 800$ at simulated times $T = 2$ and $T = 8$ and compare them with the reference solution which is computed with the LxF scheme with $M_{\text{ref}} = 12800$. We observe better accuracy for the proposed scheme. This property also becomes apparent in Table 1 where we show the corresponding approximate L^1 errors for discretizations $M = 100 \times 2^l$, $l = 0, 1, \ldots, 4$ and respective experimental orders of convergence (E.O.C.). We observe that the approximate L^1 errors decrease as the grid is refined and E.O.C. assumes values close to one, in agreement with the formal first order of acccuracy of the scheme.

Table 1. Example 1: approximate L^1-error $e_M(u)$ and E.O.C. for the LxF and HW numerical fluxes with $\Delta x = 1/M$ and symmetric kernel (6.1) with $\eta = 0.05$ at simulated times $T = 2$ and $T = 8$.

	$T = 2$				$T = 8$			
	LxF		HW		LxF		HW	
M	$e_M(\rho_\Delta)$	E.O.C.	$e_M(\rho_\Delta)$	E.O.C.				
100	1.71e-01	—	1.02e-01	—	6.34e-01	—	5.28e-01	—
200	1.11e-01	0.63	5.42e-02	0.92	5.96e-01	0.89	5.80e-01	-0.14
400	4.64e-02	1.25	2.74e-02	0.98	4.89e-01	0.28	3.69e-01	0.65
800	2.02e-02	1.20	1.39e-02	0.98	2.86e-01	0.78	1.19e-01	1.63
1600	9.58e-03	1.07	6.84e-03	1.03	1.11e-01	1.37	4.18e-02	1.51

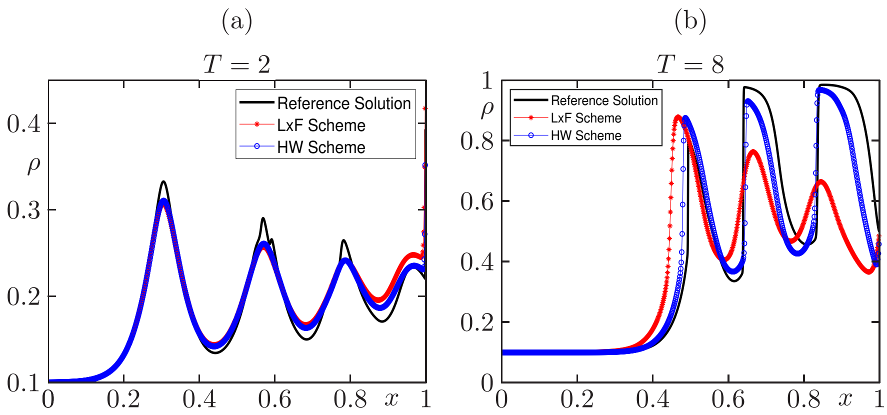

Figure 1. Example 1: numerical approximations obtained with HW and LxF numerical flux with $M = 800$ and symmetric kernel $\omega_1(y)$ with $\eta = 0.05$ at simulated times (a) $T = 2$, (b) $T = 8$.

6.2. *Example 2*

We now compare the dynamics in the solution of model (2.1) by using various kernel functions. We consider the symmetric kernel ω_1 Eq (6.1) as in Example 1 along with a non-symmetric kernel

$$\omega_2(y) := \frac{20}{\eta}\left(\frac{5y}{\eta} + \frac{1}{2}\right)\exp\left(-\frac{10y}{\eta} - 1\right)\chi_{[-\frac{\eta}{10},\eta]}(y) \tag{6.2}$$

and the anisotropic discontinuous kernels

$$\omega_3(y) := \frac{1}{\eta}\chi_{[0,\eta]}(y), \quad \omega_4(y) := \frac{3}{\eta^3}(\eta - y)^2\chi_{[0,\eta]}(y), \tag{6.3}$$

where in all cases we choose $\eta = 0.2$. In Figure 2 we display the different kernel functions. The initial and boundary conditions are given by

$$\rho_0(x) = 0.1 \quad \text{for } x \in I; \quad \rho_a(t) = 0.1, \quad \rho_b(t) = 1 \quad \text{for } t > 0.$$

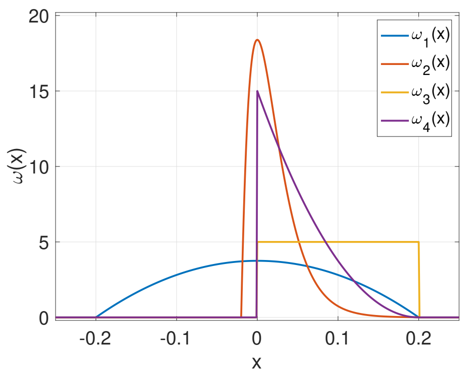

Figure 2. Example 2: kernel functions $\omega(x) = \omega_i(x)$, $i = 1, \ldots, 4$, given by Eqs (6.1), (6.2) and (6.3) with $\eta = 0.2$.

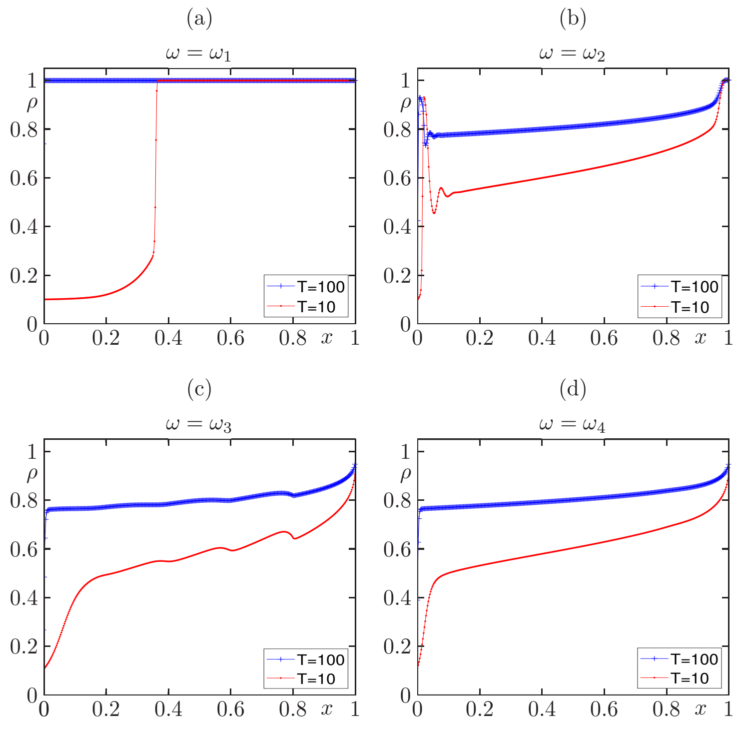

Figure 3. Example 2: dynamics of the model (2.1) for various kernel functions (ω_1, ω_2, ω_3 or ω_4). Numerical solutions with $M = 800$ at simulated times $T = 10$ and $T = 100$.

In Figure 3 we display numerical approximations with $\Delta x = 1/400$ at times $T = 10$ and $T = 100$. We can evidence that the dynamics of the solution is different for each kernel functions; by using

ω_1 we observe that numerical solution goes faster to a stationary state solution than for other kernel functions, in which this stationary state is not observed for enough large simulation time. Regarding the kernel ω_2 we can see the formation of some oscillations. Now, for the kernel ω_3 we can see the formation of some layers on the numerical solution and that the period of these layers are proportional to η. Finally, for ω_4 we can observe a numerical solution more smooth than in the previous solutions.

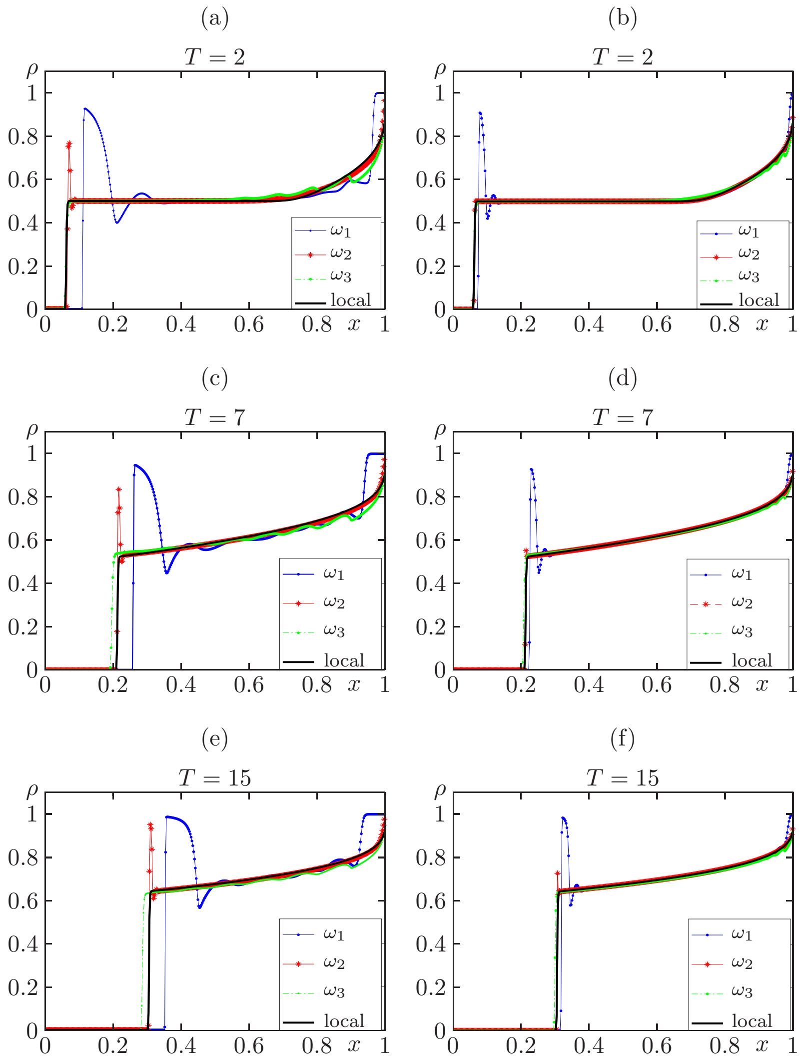

Figure 4. Example 3: numerical solutions of (2.1) for $M = 400$ at indicated simulated times with (left) $\eta = 0.1$ and (right) $\eta = 0.025$.

6.3. Example 3

The aim of the present example is to investigate the behavior of numerical solutions considering the kernel functions ω_1, ω_2, and ω_3 as well as for two different values of the parameter η, namely $\eta = 0.1$ and $\eta = 0.025$. The initial and boundary conditions are

$$\rho_0(x) = 0.5 \quad \text{for } x \in I; \quad \rho_a(t) = 0, \quad \rho_b(t) = 1 \quad \text{for } t > 0,$$

which leaves zero flow conditions at boundary, i.e., $f_{1/2} = f_{M+1/2} = 0$. Numerical approximations are computed at simulated times $T = 2$, $T = 7$ and $T = 15$ with discretization $M = 400$. In Figure 4, first we observe that numerical solutions for the nonlocal problem (2.1) get closer to the solution of the local problem as η takes a smaller value, but ω_1 and ω_2 make it slower due to the presence of the oscillations that the numerical solutions present when we use these functions.

Table 2. Example 4: approximate L^1 errors $e_M(\rho)$ and E.O.C for the first and second order version of the HW scheme with $\Delta x = 1/M$, at $T = 0.1$.

	$T = 0.1$			
	HW first-order version		HW second-order version	
M	$e_M(\rho_\Delta)$	E.O.C.	$e_M(\rho_\Delta)$	E.O.C.
100	8.71e-03	–	3.20e-04	–
200	4.60e-03	9.19e-01	8.63e-05	1.89
400	2.38e-03	9.54e-01	2.24e-05	1.95
800	1.21e-03	9.77e-01	5.53e-06	2.02

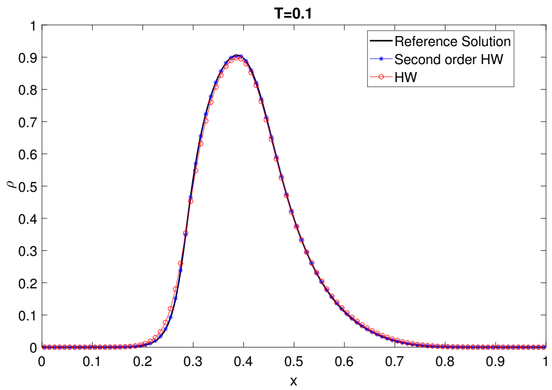

Figure 5. Example 4: comparison of the numerical solutions at $T = 0.1$ corresponding to initial condition (6.4), computed with $1/\Delta x = 100$ and kernel function ω_1.

6.4. Example 4: Error test for second order scheme

We consider the problem (2.1), with a smooth initial datum

$$\rho_0(x) = 0.9 \exp(-70(x - 0.4)^2) \quad \text{for } x \in [0, 1], \tag{6.4}$$

with boundary conditions $\rho_a = \rho_b = 0$, and with the symmetric kernel function ω_1 with $\eta = 0.2$. In Figure 5 we display the numerical approximations obtained with the second order scheme (5.1), (5.2), computed with $M = 100$ at $T = 0.1$. The reference solution is computed with $M = 6400$. As expected, the numerical solutions obtained with second order version of the HW scheme capture the reference solution better than the first order one. In Table 2 we compute the L^1−error and E.O.C. We recover the correct order of accuracy for the second order HW scheme. Instead, we obtain just first order accuracy for HW scheme (4.2). We also can observe for scheme (4.2) that the L^1−error for each level of refinement is bigger than the error of the second order version scheme.

7. Conclusions

In this paper we extend to its nonlocal version a numerical scheme presented in [6, 19] where we take advantages of the form in which the flow is written, $\rho v(\rho)V(x,t)$, where $v(\rho)$ is a positive and non-increasing function, and $V(x,t)$ is a positive function containing the nonlocal terms. We have proved maximum principle, L^1-bound, and BV estimations, also, a discrete entropy inequality in order to prove the well-posedness of an 1-D and nonlocal IBVP. In future work, we aim to investigate how to extend this numerical scheme in applications such as crowd dynamics models where the function $V(x,t)$ is a vector field containing the preference directions and nonlocal corrections terms.

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Conflict of interest

The authors declare there is no conflict of interest.

References

1. D. Amadori, W. Shen, An integro-differential conservation law arising in a model of granular flow, *J. Hyperbolic Differ. Equ.*, **9** (2012), 105–131. https://doi.org/10.1142/S0219891612500038
2. P. Amorim, R. Colombo, A. Teixeira, On the numerical integration of scalar nonlocal conservation laws, *ESAIM M2AN*, **49** (2015), 19–37. http://dx.doi.org/10.1051/m2an/2014023
3. D. Armbruster, P. Degond, C. Ringhofer, A model for the dynamics of large queuing networks and supply chains, *SIAM J. Appl. Math.*, **66** (2006), 896–920. https://doi.org/10.1137/040604625
4. C. Bardos, A. Y. le Roux, J. C. Nédélec, First order quasilinear equations with boundary conditions, *Commun. Partial. Differ. Equ.*, **4** (1979), 1017–1034. https://doi.org/10.1080/03605307908820117
5. F. Betancourt, R. Bürger, K. H. Karlsen, E. M. Tory, On nonlocal conservation laws modelling sedimentation, *Nonlinearity*, **24** (2011), 855–885. https://doi.org/10.1088/0951-7715/24/3/008

6. R. Bürger, A. García, K. Karlsen, J. Towers, A family of numerical schemes for kinematic flows with discontinuous flux, *J Eng Math*, **60** (2008), 387–425. https://doi.org/10.1007/s10665-007-9148-4
7. C. Chalons, P. Goatin, L. M. Villada, High-order numerical schemes for one-dimensional nonlocal conservation laws, *SIAM J. Sci. Comput.*, **40** (2018), A288–A305. https://doi.org/10.1137/16M110825X
8. R. M. Colombo, M. Garavello, M. Lécureux-Mercier, A class of nonlocal models for pedestrian traffic, *Math Models Methods Appl Sci*, **22** (2012), 1150023. https://doi.org/10.1142/S0218202511500230
9. R. M. Colombo, M. Herty, M. Mercier, Control of the continuity equation with a non local flow, *ESAIM Control Optim. Calc. Var.*, **17** (2011), 353–379. https://doi.org/10.1051/cocv/2010007
10. R. M. Colombo, M. Lécureux-Mercier, Nonlocal crowd dynamics models for several populations, *Acta Math. Sci. Ser. B Engl. Ed.*, **32** (2012), 177–196. https://doi.org/10.1016/S0252-9602(12)60011-3
11. R. M. Colombo, E. Rossi, Rigorous estimates on balance laws in bounded domains, *Acta Math. Sci. Ser. B Engl. Ed.*, **35** (2015), 906–944. https://doi.org/10.1016/S0252-9602(15)30028-X
12. R. M. Colombo, E. Rossi, Nonlocal conservation laws in bounded domains, *Math Models Methods Appl Sci*, **50** (2018), 4041–4065. https://doi.org/10.1137/18M1171783
13. M. G. Crandall, A. Majda, Monotone difference approximations for scalar conservation laws, *Math. Comp.*, **34** (1980), 1–21. https://doi.org/10.1090/S0025-5718-1980-0551288-3
14. C. De Filippis, P. Goatin, The initial–boundary value problem for general non-local scalar conservation laws in one space dimension, *Nonlinear Analysis*, **161** (2017), 131–156. https://doi.org/10.1016/j.na.2017.05.017
15. F. A. Chiarello, P. Goatin, Global entropy weak solutions for general non-local traffic flow models with anisotropic kernel, *ESAIM: M2AN*, **52** (2018), 163–180. https://doi.org/10.1051/m2an/2017066
16. J. Friedrich, O. Kolb, S. Göttlich, A Godunov type scheme for a class of LWR traffic flow models with non-local flux, *Netw. Heterog. Media*, **13** (2018), 531–547. https://doi.org/10.3934/nhm.2018024
17. P. Goatin, E. Rossi, Well-posedness of IBVP for 1D scalar non-local conservation laws, *Z. Angew. Math. Mech.*, **99** (2019), e201800318. https://doi.org/10.1002/zamm.201800318
18. P. Goatin, S. Scialanga, Well-posedness and finite volume approximations of the LWR traffic flow model with non-local velocity, *Netw. Heterog. Media*, **11** (2016), 107–121. https://doi.org/10.3934/nhm.2016.11.107
19. M. Hilliges, W. Weidlich, A phenomenological model for dynamic traffic flow in networks, *TRANSPORT RES B-METH*, **29** (1995), 407–431.https://doi.org/10.1016/0191-2615(95)00018-9
20. E. Rossi, Definitions of solutions to the IBVP for multi-dimensional scalar balance laws, *J. Hyperbolic Differ. Equ.*, **15** (2018), 349–374. https://doi.org/10.1142/S0219891618500133
21. E. Rossi, Well-posedness of general 1d initial boundary value problems for scalar balance laws, *Discrete Contin Dyn Syst Ser A*, **39** (2019), 3577–3608. https://doi.org/10.3934/dcds.2019147

Appendix A: Technical estimates

In this appendix we show the computations for some estimates used in the proofs of above sections. First, we prove estimates Eq (4.11) in the proof of Lemma 4.3. We denote $y_k := (k-1/2)\Delta x$, for $k \in \mathbb{Z}$ and then we compute

$$\begin{aligned}
\left|V^n_{j+3/2} - V^n_{j+1/2}\right| &= \left|\frac{\Delta x}{W_{j+3/2}} \sum_{k=1}^{M} \omega^{k-j-1} v(\rho^n_k) - \frac{\Delta x}{W_{j+1/2}} \sum_{k=1}^{M} \omega^{k-j} v(\rho^n_k)\right| \\
&= \left|\frac{\Delta x}{W_{j+3/2}} \sum_{k=1}^{M} (\omega^{k-j-1} - \omega^{k-j}) v(\rho^n_k) + \Delta x\Big(\frac{1}{W_{j+3/2}} - \frac{1}{W_{j+1/2}}\Big) \sum_{k=1}^{M} \omega^{k-j} v(\rho^n_k)\right| \\
&\le \left|\frac{\Delta x}{W_{j+3/2}} \sum_{k=1}^{M} (\omega^{k-j-1} - \omega^{k-j}) v(\rho^n_k)\right| + \left|\frac{\Delta x^2}{W_{j+3/2} W_{j+1/2}}\Big(\sum_{k=1}^{M} \omega^{k-j} v(\rho^n_k)\Big) \sum_{k=1}^{M} (\omega^{k-j} - \omega^{k-j-1})\right| \\
&\le \frac{\Delta x \|v\|_\infty}{|W_{j+3/2}|} \sum_{k=1}^{M} \left|\omega^{k-j-1} - \omega^{k-j}\right| + \frac{\Delta x \|v\|_\infty}{W_{j+3/2} W_{j+1/2}} \left|\Delta x \Big(\sum_{k=1}^{M} \omega^{k-j}\Big) \sum_{k=1}^{M} (\omega^{k-j} - \omega^{k-j-1})\right| \\
&\le \frac{\Delta x}{K_\omega} \|v\|_\infty \Big(\sum_{k=1}^{M} \left|\int_{y_{k-j-1}}^{y_{k-j}} \omega'(y) \mathrm{d}y\right| + \sum_{k=1}^{M} \left|\omega^{k-j} - \omega^{k-j-1}\right|\Big) \le \mathcal{L}\Delta x
\end{aligned}$$

with $\mathcal{L} = 2K_\omega^{-1} \|v\|_\infty \|\omega'\|_{L^1(\mathbb{R})}$. Now following closely [17], we compute

$$\begin{aligned}
&\left|V^n_{j+3/2} - 2V^n_{j+1/2} + V^n_{j-1/2}\right| \\
&= \left|\Delta x \sum_{k=1}^{M} v(\rho^n_k) \Big(\frac{\omega^{k-j-1}}{W_{j+3/2}} - \frac{\omega^{k-j}}{W_{j+1/2}} - \frac{\omega^{k-j}}{W_{j+1/2}} + \frac{\omega^{k-j+1}}{W_{j-1/2}}\Big)\right| \\
&\le \left|\frac{\Delta x^3}{W_{j+3/2} W_{j+1/2}}\bigg(\Big(\sum_{k=1}^{M} v(\rho^n_k) \int_{\xi_{k-j-1/2}}^{\xi_{k-j+1/2}} \omega''(y)\,\mathrm{d}y\Big) \sum_{l=1}^{M} \omega^{l-j} - \Big(\sum_{k=1}^{M} v(\rho^n_k) \omega^{k-j}\Big) \sum_{l=1}^{M} \int_{\xi_{l-j-1/2}}^{\xi_{l-j+1/2}} \omega''(y)\,\mathrm{d}y\bigg)\right| \\
&\quad + \left|\frac{2\Delta x^4 \|\omega'\|_{L^1}}{W_{j+3/2} W_{j+1/2} W_{j-1/2}}\bigg(\Big(\sum_{k=1}^{M} v(\rho^n_k) \omega^{k-j}\Big) \sum_{l=1}^{M} \omega'(\xi_{l-j+1/2}) - \Big(\sum_{k=1}^{M} v(\rho^n_k) \omega'(\xi_{k-j+1/2})\Big) \sum_{l=1}^{M} \omega^{l-j}\bigg)\right| \\
&\le \left|\frac{\Delta x^2 \|v\|_\infty}{W_{j+3/2} W_{j+1/2}}\Big(\sum_{k=1}^{M} \int_{\xi_{k-j-1/2}}^{\xi_{k-j+1/2}} \omega''(y)\,\mathrm{d}y\Big) \Delta x \sum_{l=1}^{M} \omega^{l-j}\right| \\
&\quad + \left|\frac{\Delta x^2 \|v\|_\infty}{W_{j+3/2} W_{j+1/2}}\Big(\Delta x \sum_{k=1}^{M} \omega^{k-j}\Big) \sum_{l=1}^{M} \int_{\xi_{l-j-1/2}}^{\xi_{l-j+1/2}} \omega''(y)\,\mathrm{d}y\right| \\
&\quad + \left|\frac{2\Delta x^2 \|\omega'\|_{L^1(I)} \|v\|_\infty}{W_{j+3/2} W_{j+1/2} W_{j-1/2}}\Big(\Delta x \sum_{k=1}^{M} \omega^{k-j}\Big) \Delta x \sum_{l=1}^{M} \omega'(\xi_{l-j+1/2})\right| \\
&\quad + \left|\frac{2\Delta x^2 \|\omega'\|_{L^1(I)} \|v\|_\infty}{W_{j+3/2} W_{j+1/2} W_{j-1/2}}\Big(\Delta x \sum_{k=1}^{M} \omega'(\xi_{k-j+1/2})\Big) \Delta x \sum_{l=1}^{M} \omega^{l-j}\right| \\
&\le \Delta x^2 \|v\|_\infty \bigg(\left|\frac{1}{W_{j+3/2}} \sum_{k=1}^{M} \int_{\xi_{k-j-1/2}}^{\xi_{k-j+1/2}} \omega''(y)\,\mathrm{d}y\right| + \left|\frac{1}{W_{j+3/2}} \sum_{l=1}^{M} \int_{\xi_{l-j-1/2}}^{\xi_{l-j+1/2}} \omega''(y)\,\mathrm{d}y\right|
\end{aligned}$$

$$+\left|\frac{2\|\omega'\|_{L^1(I)}}{W_{j+3/2}W_{j-1/2}}\Delta x\sum_{l=1}^{M}\omega'(\xi_{l-j+1/2})\right|+\left|\frac{2\|\omega'\|_{L^1(I)}}{W_{j+3/2}W_{j-1/2}}\Delta x\sum_{k=1}^{M}\omega'(\xi_{k-j+1/2})\right|\Bigg)$$

$$\le\frac{\Delta x^2\|v\|_\infty}{K_\omega}\left(\left|\sum_{k=1}^{M}\int_{\xi_{k-j-1/2}}^{\xi_{k-j+1/2}}\omega''(y)\mathrm{d}y\right|+\left|\sum_{l=1}^{M}\int_{\xi_{l-j-1/2}}^{\xi_{l-j+1/2}}\omega''(y)\mathrm{d}y\right|\right)$$

$$+2K_\omega^{-2}\Delta x^2\|\omega'\|^2_{L^1(I)}\|v\|_\infty+2K_\omega^{-2}\Delta x^2\|\omega'\|^2_{L^1(I)}\|v\|_\infty\le\Delta x^2\mathcal{W},$$

where we set $\mathcal{W}:=2K_\omega^{-1}\|v\|_\infty\|\omega''\|_{L^1(\mathbb{R})}+4K_\omega^{-2}\|\omega'\|^2_{L^1(I)}\|v\|_\infty$ and $\xi_{k-j+1/2}\in]y_{k-j-1},y_{k-j}[$. This concludes the proof of estimates Eq (4.11) in the proof of Lemma 4.3. Next, we establish estimates Eq (3.1) to Eq (3.4) in Section 3.1. To this end we calculate

$$\left|\partial_x V(x,t)\right|$$

$$=\left|-\frac{W'(x)}{(W(x))^2}\int_a^b v(\rho(y,t))\omega(y-x)\,\mathrm{d}y-\frac{1}{W(x)}\int_a^b v(\rho(y,t))\omega'(y-x)\,\mathrm{d}y\right|\le 2K_\omega^{-1}\|v\|_\infty\|\omega'\|_{L^1(I)},$$

$$\left|\partial^2_{xx}V\right|$$

$$=\left|\frac{-(W(x))^2W''(x)+2(W'(x))W(x)}{(W(x))^4}\int_a^b v(\rho(y,t))\omega(y-x)\,\mathrm{d}y+\frac{W'(x)}{(W(x))^2}\int_a^b v(\rho(y,t))\omega'(y-x)\,\mathrm{d}y\right.$$

$$\left.+\frac{W'(x)}{(W(x))^2}\int_a^b v(\rho(y,t))\omega'(y-x)\,\mathrm{d}y+\frac{1}{W(x)}\int_a^b v(\rho(y,t))\omega''(y-x)\,\mathrm{d}y\right|$$

$$\le K_\omega^{-2}\|v\|_\infty\|\omega''\|_{L^1(I)}\|\omega\|_{L^1(I)}+2K_\omega^{-3}\|v\|_\infty\|\omega'\|_{L^1(I)}\|\omega\|_{L^1(I)}+2K_\omega^{-2}\|v\|_\infty\|\omega'\|_{L^1(I)}+K_\omega^{-1}\|v\|_\infty\|\omega''\|_{L^1(I)},$$

$$\left|V(x,t)-U(x,t)\right|$$

$$=\left|\frac{1}{W(x)}\int_a^b(v(\rho(y,t))-v(\sigma(y,t)))\omega(y-x)\,\mathrm{d}y\right|=\left|\frac{1}{W(x)}\int_a^b v'(\xi)(\rho-\sigma)(y,t)\omega(y-x)\,\mathrm{d}y\right|$$

$$\le\frac{\|\omega\|_{\infty(\mathbb{R})}\|v'\|_\infty}{K_\omega}\int_a^b|\rho(y,t)-\sigma(y,t)|\,\mathrm{d}y,$$

$$\left|\partial_x V(x,t)-\partial_x U(x,t)\right|$$

$$=\left|-\frac{W'(x)}{(W(x))^2}\int_a^b(v(\rho(y,t))-v(\sigma(y,t)))\omega(y-x)\,\mathrm{d}y-\frac{1}{W(x)}\int_a^b(v(\rho(y,t))-v(\sigma(y,t)))\omega'(y-x)\,\mathrm{d}y\right|$$

$$=\left|-\frac{W'(x)}{(W(x))^2}\int_a^b v'(\xi)(\rho-\sigma)(y,t)\omega(y-x)\,\mathrm{d}y-\frac{1}{W(x)}\int_a^b v'(\xi)(\rho-\sigma)(y,t)\omega'(y-x)\,\mathrm{d}y\right|$$

$$\le\mathcal{M}\int_a^b|\rho(y,t)-\sigma(y,t)|\,\mathrm{d}y,$$

with $\mathcal{M}$ as in Eq (3.4).

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