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Research article

A second-order ADI method for pricing options under fractional regime-switching models

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Abstract: Fractional regime-switching option models have recently attracted much attention because they can capture the sudden state movement of the market, and deal with the non-stationary behavior. A second-order numerical scheme is proposed to solve the regime-switching option pricing models with fractional derivatives in space. The sufficient conditions of the stability and convergence of the proposed scheme are studied in details. An alternating direction implicit (ADI) method is implemented to accelerate the computation in every time layer. Numerical experiments are presented to verify the convergence and efficiency of the proposed method, compared with classical Krylov subspace solvers.

Keywords: fractional option pricing model; second-order schemes; regime-switching European options pricing

1. Introduction

Assuming that the underlying asset process follows a geometric Brownian motion, the Black-Scholes model was firstly proposed in 1973, where the option value satisfies a partial difference equation and depends only on the risk-free interest rate and the volatility of asset price [1]. In order to fit the empirical facts of practical financial markets, more extended models were introduced and studied, including jump-diffusion models [2, 3], stochastic volatility models [4], fractional differential models [5–7] and regime-switching models [8].

The idea of switching regimes is prevalently applied in order to allow Lévy processes to switch in a finite state space by a Markov chain. In option pricing models, the parameters, such as interest rate, drift and volatility, are allowed to take diverse values in a finite number of regimes [9]. For instance, the option models based on exponential Lévy processes under switching regimes were proposed and widely discussed to capture the sudden state movement from the bull market to bear market, and deal with the non-stationary behavior [10–12]. Since the partial integro-differential equation (PIDE) derived from the regime-switching exponential Lévy processes is difficult to be solved in closed form, it is essential to develop effective numerical methods.

Recently, numerical solution of fractional option models under switching regimes attract much attention from the community of financial engineering. Cartea and del-Castillo-Negrete [13] proposed a first-order shifted Grünwald difference formula for the option pricing models, including the finite moment log stable (FMLS) model [6], CGMY model [7] and KoBoL model [5]. A first-order penalty method for fractional regime-switching American option pricing models, was constructed in [14]. Further, an implicit-explicit preconditioned direct method was developed in [12] for fractional regime-switching models which was of first order in spatial approximation.

In this paper, we consider a second-order numerical scheme for fractional regime-switching option pricing models, based on the weighted and shifted Grünwald difference (WSGD) formula and Crank-Nicolson scheme. Theoretical analysis on the stability and second-order convergence of the numerical scheme is studied in detail. A second-order ADI method is proposed to accelerate the computation with a preconditioned direct solver for the discrete linear system. Numerical experiments on both fractional PDE and multi-regime FMLS and CGMY models are presented to show the convergence and efficiency of the proposed approach.

The structure of this paper is organized as follows: A second-order numerical scheme for fractional regime-switching option pricing models is presented in Section 2. Numerical analysis of stability and the second-order convergence are shown in Section 3. In Section 4, we introduce the ADI method with preconditioned direct solver for the discrete linear system. Numerical experiments in Section 5 demonstrated the convergence and efficiency of the proposed method. Finally, conclusions are drawn in Section 6.

2. A second-order finite difference method

Under the risk-neutral measure, assume that the stock price S_t follows a geometric Lévy process

$$\mathrm{d}\,(\ln S_t) = (r - \nu)\mathrm{d}t + \mathrm{d}L_t,$$

where r is the risk-free rate, ν is a convexity adjustment and $\mathrm{d}L_t$ is the increment of a Lévy process under the equivalent martingale measure [15]. Below, we discuss the general fractional regime-switching option model derivated by three particular Lévy processes: LS, CGMY and KoBoL, see [13] for more details.

Let $V_s(x,t)$ be the value of an European option in state s, the fractional regime-switching option model is defined by

$$\begin{aligned}\frac{\partial V_s(x,t)}{\partial t} + c_{s,1}\frac{\partial V_s(x,t)}{\partial x} + c_{s,2}D_{+}^{\xi_s,\alpha_s}V_s(x,t) + c_{s,3}D_{-}^{\lambda_s,\alpha_s}V_s(x,t)&\\ - d_s V_s(x,t) + \sum_{j=1}^{\bar{S}} q_{s,j}V_j(x,t) = 0,&\end{aligned} \tag{2.1}$$

where $x = \ln S_t$, $1 < \alpha_s < 2$, $s = 1, 2, \ldots, \bar{S}$. The other parameters in Eq (2.1) depend on a certain state of a Markov process in the finite set $\{1, 2, \ldots, \bar{S}\}$. The constants $q_{s,j}$ represent the elements of the state transition matrix of the Markov process, which satisfy the conditions $\sum_{j=1}^{\bar{S}} q_{s,j} = 0$ and $q_{s,j} \geq 0, \forall j \neq s$.

The left and right Riemann-Liouville tempered fractional derivatives $D_+^{\xi_s,\alpha_s}$ and $D_-^{\lambda_s,\alpha_s}$ are defined by

$$\begin{aligned} D_+^{\xi_s,\alpha_s} V_s(x,t) &= \frac{e^{\xi_s x}}{\Gamma(2-\alpha_s)} \frac{\partial^2}{\partial x^2} \int_x^{\infty} \frac{e^{-\xi_s \zeta} V_s(\zeta,t)}{(\zeta - x)^{\alpha_s - 1}} d\zeta, \\ D_-^{\lambda_s,\alpha_s} V_s(x,t) &= \frac{e^{-\lambda_s x}}{\Gamma(2-\alpha_s)} \frac{\partial^2}{\partial x^2} \int_{-\infty}^{x} \frac{e^{\lambda_s \zeta} V_s(\zeta,t)}{(x-\zeta)^{\alpha_s - 1}} d\zeta, \end{aligned} \tag{2.2}$$

where $\Gamma(\cdot)$ is the Gamma function.

In the CGMY model, the parameters in the model (2.1) are given by

$$\begin{aligned} & c_{s,1} = r - C\Gamma(\alpha_s)\left[(\xi_s - 1)^{\alpha_s} - \xi_s^{\alpha_s} + (\lambda_s + 1)^{\alpha_s} - \lambda_s^{\alpha_s}\right], \\ & c_{s,2} = c_{s,3} = C\Gamma(-\alpha_s), \quad d_s = r + C\Gamma(-\alpha_s)\left(\xi_s^{\alpha_s} + \lambda_s^{\alpha_s}\right), \\ & C > 0, \quad \lambda_s \geq 0, \quad \xi_s \geq 0. \end{aligned} \tag{2.3}$$

The model (2.1) also covers FMLS and KoBoL models by different choices, and we refer the readers to [16] for more details.

The terminal and boundary conditions for call options are given by

$$\begin{aligned} & V_s(x,T) = \max\{e^x - K, 0\}, && x_l \leq x \leq x_r, \\ & V_s(x_l,t) = 0, && 0 \leq t < T, \\ & V_s(x_r,t) = e^{x_r} - Ke^{-r(T-t)}, && 0 \leq t < T, \end{aligned} \tag{2.4}$$

where K is the strike price.

Let N and M be the number of the uniform discrete points in the space and time direction, respectively, and let $h = (x_r - x_l)/(N+1)$ and $\tau = T/M$ be the corresponding step length. Define $t_m = m\tau (m = 0, 1, 2, \cdots, M), x_n = x_l + nh (n = 0, 1, 2, \cdots, N+1)$.

Assume that the function $V_s(x,t)$ is continuously differentiable and $\partial^2 V_s(x,t)/\partial x^2$ is integrable in the interval $[0,T) \times [x_l, x_r]$, then for every α $(0 < \alpha < 2)$, the Riemann-Liouville derivative of $V_s(x,t)$ exists and coincides with the Grünwald-Letnikov type [17]. Hence, we can use the Grünwald difference approaches to approximate the tempered fractional derivatives in Eq (2.2) to avoid the strong singularity when $\zeta = x$.

The first-order shifted Grünwald difference scheme [18] was first proposed to approximate the fractional derivatives and used to solve the fractional option pricing models [12, 14]. Now we consider the second-order weighted and shifted Grünwald difference scheme [19] in the discrete process of Eq (2.1).

Based on the weighted and shifted Grünwald difference scheme, the fractional derivative can be approximated by

$$\begin{aligned} D_+^{\xi_s,\alpha_s} V_s(x_n,t_m) &= \frac{1}{h^{\alpha_s}} \sum_{k=0}^{N-n+2} \omega_k^{\alpha_s} e^{-\xi_s(k-1)h} V_s(x_{n+k-1},t_m) + O\left(h^2\right), \\ D_-^{\lambda_s,\alpha_s} V_s(x_n,t_m) &= \frac{1}{h^{\alpha_s}} \sum_{k=0}^{n+1} \omega_k^{\alpha_s} e^{-\lambda_s(k-1)h} V_s(x_{n-k+1},t_m) + O\left(h^2\right), \end{aligned} \tag{2.5}$$

where

$$\begin{cases} \omega_0^{\alpha_s} = \dfrac{\alpha_s}{2} g_0^{\alpha_s}, \quad \omega_k^{\alpha_s} = \dfrac{\alpha_s}{2} g_k^{\alpha_s} + \dfrac{2-\alpha_s}{2} g_{k-1}^{\alpha_s}, \\ g_0^{\alpha_s} = 1, \quad g_k^{\alpha_s} = \left(1 - \dfrac{\alpha_s + 1}{k}\right) g_{k-1}^{\alpha_s}, \quad k = 1, 2, \cdots, \\ \sum\limits_{k=0}^{\infty} g_k^{\alpha_s} = 0, \quad g_1^{\alpha_s} = -\alpha_s < 0 \quad g_2^{\alpha_s} > g_3^{\alpha_s} > \ldots > 0, \\ \omega_0^{\alpha_s} = \dfrac{\alpha_s}{2}, \quad \omega_1^{\alpha_s} = \dfrac{2 - \alpha_s - \alpha_s^2}{2} < 0, \quad \omega_2^{\alpha_s} = \dfrac{\alpha_s\left(\alpha_s^2 + \alpha_s - 4\right)}{4}. \end{cases} \tag{2.6}$$

Denote $V_{s,n}^m$ be the numerical solution at the discrete point (x_n, t_m) of regime s. Discretising the convection term and the time term in Eq (2.1) by central differences, and the Crank-Nicolson scheme respectively, and introducing the time-reverse transformation $t^* = T - t$, dropping $*$ for simplicity, the following fully discrete scheme is derived:

$$\begin{aligned} \frac{V_{s,n}^{m+1} - V_{s,n}^m}{\tau} = \frac{1}{2}\Bigg[& c_{s,1} \frac{V_{s,n+1}^{m+1} - V_{s,n-1}^{m+1}}{2h} + \frac{c_{s,2}}{h^{\alpha_s}} \sum_{k=0}^{N-n+2} \omega_k^{\alpha_s} e^{-\xi_s (k-1) h} V_{s,n+k-1}^{m+1} \\ & + \frac{c_{s,3}}{h^{\alpha_s}} \sum_{k=0}^{n+1} \omega_k^{\alpha_s} e^{-\lambda_s (k-1) h} V_{s,n-k+1}^{m+1} - d_s V_{s,n}^{m+1} + \sum_{j=1}^{\bar{S}} q_{s,j} V_{j,n}^{m+1} \\ & + c_{s,1} \frac{V_{s,n+1}^{m} - V_{s,n-1}^{m}}{2h} + \frac{c_{s,2}}{h^{\alpha_s}} \sum_{k=0}^{N-n+2} \omega_k^{\alpha_s} e^{-\xi_s (k-1) h} V_{s,n+k-1}^{m} \\ & + \frac{c_{s,3}}{h^{\alpha_s}} \sum_{k=0}^{n+1} \omega_k^{\alpha_s} e^{-\lambda_s (k-1) h} V_{s,n-k+1}^{m} - d_s V_{s,n}^{m} + \sum_{j=1}^{\bar{S}} q_{s,j} V_{j,n}^{m} \Bigg], \\ & n = 1, 2, \ldots, N, \quad m = 0, 1, 2, \ldots, M-1. \end{aligned} \tag{2.7}$$

Denote $V^m = \left(V_{1,1}^m, V_{1,2}^m, \ldots, V_{1,N}^m, V_{2,1}^m, \ldots, V_{2,N}^m, \ldots, V_{\bar{S},N}^m\right)^T$, $Q = \left[q_{s,j}\right]_{s,j=1}^{\bar{S}}$ and $p^{m+\frac{1}{2}} = \frac{1}{2}\tau(p^{m+1} + p^m)$, where

$$\begin{aligned} p^m &= (p_1^m, p_2^m, \ldots, p_{\bar{S}}^m)^T, \\ p_s^m &= \frac{c_{s,1}}{2h} p_{s,1}^m + \frac{c_{s,2}}{h^{\alpha_s}} p_{s,2}^m + \frac{c_{s,3}}{h^{\alpha_s}} p_{s,3}^m, \\ p_{s,1}^m &= (-V_{s,0}^m, 0, \ldots, 0, V_{s,N+1}^m)^T, \\ p_{s,2}^m &= (\omega_0^{\alpha_s} e^{\xi_s h} V_{s,0}^m + \omega_{N+1}^{\alpha_s} e^{-\xi_s N h} V_{s,N+1}^m, \ldots, \omega_2^{\alpha_s} e^{-\xi_s h} V_{s,N+1}^m)^T, \\ p_{s,2}^m &= (\omega_2^{\alpha_s} e^{-\lambda_s h} V_{s,0}^m, \ldots, \omega_{N+1}^{\alpha_s} e^{-\lambda_s N h} V_{s,0}^m + \omega_0^{\alpha_s} e^{\lambda_s h} V_{s,N+1}^m)^T. \end{aligned}$$

The matrix form of the numerical scheme (2.7) can be written as:

$$\left(I_{\bar{S}N} - \frac{1}{2}\tau(M_B + Q \otimes I_N)\right) V^{m+1} = \left(I_{\bar{S}N} + \frac{1}{2}\tau(M_B + Q \otimes I_N)\right) V^m + p^{m+\frac{1}{2}}, \tag{2.8}$$

where

$$M_B = \mathrm{diag}\left(T_1, T_2, \ldots, T_{\bar{S}}\right), \quad T_s = \tfrac{c_{s,1}}{2h} J + \tfrac{c_{s,2}}{h^{\alpha_s}} W_s + \tfrac{c_{s,3}}{h^{\alpha_s}} G_s - d_s I_N, \tag{2.9}$$

with

$$J = \text{tridiag}(-1, 0, 1),$$

$$W_s = \begin{pmatrix} \omega_1^{\alpha_s} & \omega_2^{\alpha_s} e^{-\xi_s h} & \cdots & \omega_{N-1}^{\alpha_s} e^{-\xi_s (N-2)h} & \omega_N^{\alpha_s} e^{-\xi_s (N-1)h} \\ \omega_0^{\alpha_s} e^{\xi_s h} & \omega_1^{\alpha_s} & \omega_2^{\alpha_s} e^{-\xi_s h} & \ddots & \omega_{N-1}^{\alpha_s} e^{-\xi_s (N-2)h} \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \omega_0^{\alpha_s} e^{\xi_s h} & \omega_1^{\alpha_s} & \omega_2^{\alpha_s} e^{-\xi_s h} \\ 0 & \cdots & 0 & \omega_0^{\alpha_s} e^{\xi_s h} & \omega_1^{\alpha_s} \end{pmatrix},$$

$$G_s = \begin{pmatrix} \omega_1^{\alpha_s} & \omega_0^{\alpha_s} e^{\lambda_s h} & 0 & \cdots & 0 \\ \omega_2^{\alpha_s} e^{-\lambda_s h} & \omega_1^{\alpha_s} & \omega_0^{\alpha_s} e^{\lambda_s h} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \omega_{N-1}^{\alpha_s} e^{-\lambda_s (N-2)h} & \ddots & \omega_2^{\alpha_s} e^{-\lambda_s h} & \omega_1^{\alpha_s} & \omega_0^{\alpha_s} e^{\lambda_s h} \\ \omega_N^{\alpha_s} e^{-\lambda_s (N-1)h} & \omega_{N-1}^{\alpha_s} e^{-\lambda_s (N-2)h} & \cdots & \omega_2^{\alpha_s} e^{-\lambda_s h} & \omega_1^{\alpha_s} \end{pmatrix}.$$

3. Stability and convergence of the numerical scheme

In this section, the stability and convergence of the numerical scheme (2.8) are established.

A matrix is called positive definite if, and only, if its symmetric part is positive definite, that is, all the eigenvalues are positive.

Lemma 3.1. *(Gerschgorin Disk Theorem) Suppose* $A = \left[a_{ij}\right] \in \mathbb{C}^{n\times n}$*, let*

$$G_i(A) = \left\{ z \in \mathbb{C} : |z - a_{ii}| \leqslant \sum_{j\neq i} \left|a_{ij}\right| \right\}, \quad i = 1, \dots, n,$$

then

$$\lambda(A) \subset G_1(A) \cup G_2(A) \cup \cdots \cup G_n(A).$$

Theorem 3.1. *(Stability) Assume that* $1 < \alpha_s < 2$ *and set*

$$\eta_s(x) := (\alpha_s e^{hx} + 2 - \alpha_s)(1 - e^{-hx})^{\alpha_s}. \tag{3.1}$$

If

$$\frac{c_{s,2}}{2h^{\alpha_s}} \eta_s(\xi_s) + \frac{c_{s,3}}{2h^{\alpha_s}} \eta_s(\lambda_s) \le d_s - \frac{1}{2}\left(q_{s,s} + \sum_{k=1,k\neq s}^{\bar{S}} q_{k,s} \right), \tag{3.2}$$

for all $s = 1, 2, \dots, \bar{S}$ *, then the discretisation scheme* (2.8) *is stable.*

Proof. Denote $B = -M_B - Q \otimes I_N$ and consider the matrix

$$Z = \left(I + \frac{1}{2}\tau B\right)^{-1} \left(I - \frac{1}{2}\tau B\right).$$

In order to show the stability of the scheme (2.8), it suffices to prove that the eigenvalues λ_Z of the matrix Z satisfy the estimate $|\lambda_Z| < 1$. Or equivalently, that the eigenvalues λ_B of matrix B satisfy the estimate

$$\left|\frac{1 - 1/2\tau\lambda_B}{1 + 1/2\tau\lambda_B}\right| < 1. \tag{3.3}$$

The inequality (3.3) means that any λ_B has a positive real part $\Re(\lambda_B)$. Therefore, the numerical scheme (2.8) is stable if the matrix B is positive definite, i.e., its symmetric part $\mathcal{B}$ is positive definite. Consider the symmetric Toeplitz matrix

$$\mathcal{T}_s = -\frac{c_{s,2}}{2h^{\alpha_s}}(W_s + W_s^T) - \frac{c_{s,3}}{2h^{\alpha_s}}(G_s + G_s^T) + d_s I_N,$$

and block diagonal Toeplitz matrix

$$\mathcal{T} = \operatorname{diag}(\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_{\bar{S}}),$$

thus,

$$\mathcal{B} = \mathcal{T} - \frac{Q + Q^T}{2} \otimes I_N.$$

Note that $c_{s,2}, c_{s,3}$ and d_s are nonnegative, we have

$$[\mathcal{B}]_{l,l} = -\frac{(c_{s,2} + c_{s,3})\omega_1^{\alpha_s}}{h^{\alpha_s}} + d_s - q_{s,s} > 0,$$

where $1 + (s-1)N \le l \le sN$, $s = 1, 2, \ldots, \bar{S}$.

Therefore, if the matrix $\mathcal{B}$ is strictly row diagonally dominant, then it is positive definite by Lemma 3.1, which means $\mathcal{B}$ satisfies the condition

$$[\mathcal{B}]_{l,l} > \sum_{k=1,k\neq l}^{\bar{S}N} \left|[\mathcal{B}]_{l,k}\right| \tag{3.4}$$

for $1 + (s-1)N \le l \le sN$ where $s = 1, 2, \ldots, \bar{S}$.

It is clear that the lth and the $(1 + (2s-1)N - l)$th rows are the same. Without loss of generality, we choose $1 + (s-1)N \le l \le (s-1)N + \lceil \frac{N}{2} \rceil$, then

$$\begin{aligned}\sum_{k=1,k\neq l}^{\bar{S}N} \left|[\mathcal{B}]_{l,k}\right| = &\frac{c_{s,2}}{2h^{\alpha_s}}\Big(2(\omega_0^{\alpha_s} e^{\xi_s h} + \omega_2^{\alpha_s} e^{-\xi_s h} + \cdots + \omega_\ell^{\alpha_s} e^{-\xi_s(\ell-1)h})\\ &+\omega_{\ell+1}^{\alpha_s} e^{-\xi_s \ell h} + \cdots + \omega_N^{\alpha_s} e^{-\xi_s(N-1)h}\Big)\\ &+ \frac{c_{s,3}}{2h^{\alpha_s}}\Big(2(\omega_0^{\alpha_s} e^{\lambda_s h} + \omega_2^{\alpha_s} e^{-\lambda_s h} + \cdots + \omega_\ell^{\alpha_s} e^{-\lambda_s(\ell-1)h})\\ &+\omega_{\ell+1}^{\alpha_s} e^{-\lambda_s \ell h} + \cdots + \omega_N^{\alpha_s} e^{-\lambda_s(N-1)h}\Big) + \sum_{k=1,k\neq s}^{\bar{S}} \frac{q_{s,k} + q_{k,s}}{2},\end{aligned}$$

where $\ell = l - (s-1)N$. By rearranging the sequence $\{g_k^{\alpha_s}\}$ from $\{\omega_k^{\alpha_s}\}$ and according to the properties in Eq (2.6), we have

$$\begin{aligned}&2(\omega_0^{\alpha_s} e^{\xi_s h} + \omega_2^{\alpha_s} e^{-\xi_s h} + \cdots + \omega_\ell^{\alpha_s} e^{-\xi_s(\ell-1)h}) + \omega_{\ell+1}^{\alpha_s} e^{-\xi_s \ell h} + \cdots + \omega_N^{\alpha_s} e^{-\xi_s(N-1)h}\\ =&2\sum_{k=0}^{\ell} \omega_k^{\alpha_s} e^{\xi_s(1-k)h} + \sum_{k=\ell+1}^{N} \omega_k^{\alpha_s} e^{\xi_s(1-k)h} - 2\omega_1^{\alpha_s}\\ <&(\alpha_s e^{\xi_s h} + 2 - \alpha_s)(1 - e^{-\xi_s h})^{\alpha_s} - 2\omega_1^{\alpha_s},\end{aligned}$$

and

$$\begin{aligned}&2(\omega_0^{\alpha_s}e^{\lambda_s h}+\omega_2^{\alpha_s}e^{-\lambda_s h}+\cdots+\omega_\ell^{\alpha_s}e^{-\lambda_s(\ell-1)h})+\omega_{\ell+1}^{\alpha_s}e^{-\lambda_s\ell h}+\cdots+\omega_N^{\alpha_s}e^{-\lambda_s(N-1)h}\\&<(\alpha_s e^{\lambda_s h}+2-\alpha_s)(1-e^{-\lambda_s h})^{\alpha_s}-2\omega_1^{\alpha_s}.\end{aligned}$$

Thus, the condition (3.4) becomes

$$\begin{aligned}-\frac{(c_{s,2}+c_{s,3})\omega_1^{\alpha_s}}{h^{\alpha_s}}+d_s-q_{s,s}\geq&\frac{c_{s,2}}{2h^{\alpha_s}}(\alpha_s e^{\xi_s h}+2-\alpha_s)(1-e^{-\xi_s h})^{\alpha_s}-\frac{c_{s,2}}{h^{\alpha_s}}\omega_1^{\alpha_s}\\&+\frac{c_{s,3}}{2h^{\alpha_s}}(\alpha_s e^{\lambda_s h}+2-\alpha_s)(1-e^{-\lambda_s h})^{\alpha_s}-\frac{c_{s,3}}{h^{\alpha_s}}\omega_1^{\alpha_s}\\&-\frac{q_{s,s}}{2}+\frac{1}{2}\sum_{k=1,k\neq s}^{\bar{S}}q_{k,s},\end{aligned}$$

which can be written as

$$\frac{c_{s,2}}{2h^{\alpha_s}}\eta_s(\xi_s)+\frac{c_{s,3}}{2h^{\alpha_s}}\eta_s(\lambda_s)\leq d_s-\frac{1}{2}\left(q_{s,s}+\sum_{k=1,k\neq s}^{\bar{S}}q_{k,s}\right),$$

where $\eta_s(x)$ is defined in Eq (3.1). □

It is similar that the stability condition in Theorem 3.1 from [16] is given by

$$\frac{c_{s,2}\eta_s(\xi_s)}{h^{\alpha_s}(1+e^{\alpha_s(\lambda_s-\xi_s)h})}+\frac{c_{s,3}\eta_s(\lambda_s)}{h^{\alpha_s}(1+e^{\alpha_s(\xi_s-\lambda_s)h})}\leq d_s-\frac{1}{2}\left(q_{s,s}+\sum_{k=1,k\neq s}^{\bar{S}}q_{k,s}\right).\tag{3.5}$$

Consider the specific parameters of the CGMY model from Eq (2.3), the stability condition (3.2) can be rewritten as

$$C\Gamma(-\alpha_s)\left(\frac{\eta_s(\xi_s)+\eta_s(\lambda_s)}{2h^{\alpha_s}}-\xi_s^{\alpha_s}-\lambda_s^{\alpha_s}\right)\leq r-\frac{1}{2}\left(q_{s,s}+\sum_{k=1,k\neq s}^{\bar{S}}q_{k,s}\right),$$

while condition (3.5) turns to

$$C\Gamma(-\alpha_s)\left(\frac{\eta_s(\xi_s)}{h^{\alpha_s}(1+e^{\alpha_s(\lambda_s-\xi_s)h})}+\frac{\eta_s(\lambda_s)}{h^{\alpha_s}(1+e^{\alpha_s(\xi_s-\lambda_s)h})}-\xi_s^{\alpha_s}-\lambda_s^{\alpha_s}\right)\leq r-\frac{1}{2}\left(q_{s,s}+\sum_{k=1,k\neq s}^{\bar{S}}q_{k,s}\right).$$

It is can be seen that the condition (3.2) allows a wider range of the parameters in Eq (2.1), which can describe more state movement as the financial markets change.

Consider now the convergence of the scheme (2.8). Due to the non-smoothness of the initial and boundary conditions in Eq (2.4), Eq (2.1) does not have a solution in the classical form. As a result, we consider the generalized solution, which satisfies the fractional PDE almost everywhere in $(0,T)\times(x_l,x_r)$. We define the viscosity solution of Eq (2.1) similar as Definition 2.4 in [20].

Theorem 3.2. *(Convergence) Let V_*^m be the viscosity solution of Eq* (2.1)*. The scheme* (2.8) *is of second order convergence, i.e.,*

$$\|V^m-V_*^m\|\leq C_0(h^2+\tau^2)\tag{3.6}$$

if the matrix $B = -M_B - Q \otimes I_N$ *is positive definite, i.e.,*

$$\frac{c_{s,2}}{2h^{\alpha_s}}\eta_s(\xi_s) + \frac{c_{s,3}}{2h^{\alpha_s}}\eta_s(\lambda_s) \le d_s - \frac{1}{2}\left(q_{s,s} + \sum_{k=1,k\neq s}^{\bar{S}} q_{k,s}\right),$$

where the norm $\|v\| = \sqrt{h\sum_{i=0}^{\bar{S}N} v_i^2}$ *and* C_0 *is a positive constant.*

Proof. The proof is similar as Theorem 3.2 in [16], and we omit the specific process here. □

4. The ADI method

In this section, the ADI method will be used to solve scheme (2.7). The ADI method proposed by Peaceman and Rachford [21] in 1955 was widely used to solve two dimensional problems due to its computational effectiveness. Recently, the ADI method was also applied to solve two-asset option pricing problems under fractional models [22, 23].

Denote

$$\Delta_s V_{s,n}^m = \sum_{j=1}^{\bar{S}} q_{s,j} V_{j,n}^m$$

and

$$\Delta_x^s V_{s,n}^m = c_{s,1}\frac{V_{s,n+1}^m - V_{s,n-1}^m}{2h} + \frac{c_{s,2}}{h^{\alpha_s}}\sum_{k=0}^{N-n+2}\omega_k^{\alpha_s} e^{-\xi_s(k-1)h} V_{s,n+k-1}^m + \frac{c_{s,3}}{h^{\alpha_s}}\sum_{k=0}^{n+1}\omega_k^{\alpha_s} e^{-\lambda_s(k-1)h} V_{s,n-k+1}^m - d_s V_{s,n}^m,$$

From Eq (2.7), it is easy to show

$$\left(1 - \frac{\tau}{2}\Delta_x^s\right)\left(1 - \frac{\tau}{2}\Delta_s\right) V_{s,n}^{m+1} = \left(1 + \frac{\tau}{2}\Delta_x^s\right)\left(1 + \frac{\tau}{2}\Delta_s\right) V_{s,n}^m + R, \tag{4.1}$$

where

$$R = \frac{\tau^2}{4}\Delta_x^s\Delta_s\left(V_{s,n}^{m+1} - V_{s,n}^m\right). \tag{4.2}$$

When the time step $\tau > 0$ is sufficiently small, we omit the term R and define the following ADI scheme similar to that of the Peaceman–Rachford type [21]:

$$\left(1 - \frac{\tau}{2}\Delta_x^s\right)\widehat{V}_{s,n} = \left(1 + \frac{\tau}{2}\Delta_s\right) V_{s,n}^m, \tag{4.3}$$

$$\left(1 - \frac{\tau}{2}\Delta_s\right) V_{s,n}^{m+1} = \left(1 + \frac{\tau}{2}\Delta_x^s\right)\widehat{V}_{s,n}. \tag{4.4}$$

The following theorem illustrates the convergence order of the ADI scheme (4.3) and (4.4).

Theorem 4.1. *Assume that the exact solution of the fractional PDE in Eq* (2.1) *is unique, and its partial derivatives are in* $\mathcal{L}^1(\mathbb{R})$ *and vanish outside* $[0, T) \times [x_l, x_r]$. *The ADI discretization for Eq* (2.1) *defined in Eqs* (4.3) *and* (4.4) *is also of second order convergence* $O(h^2 + \tau^2)$.

Proof. From Theorem 3.2 ,we have proved that the Crank–Nicolson method has convergence of order $O(h^2+\tau^2)$. In the ADI scheme, compared to the Crank–Nicolson scheme (2.7), the scheme (4.3) and (4.4) incurs an additional perturbation error R defined in Eq (4.2).

Since

$$R=\frac{\tau^2}{4}\Delta_x^s\Delta_s\left(V_{s,n}^{m+1}-V_{s,n}^m\right)=\frac{\tau^3}{4}\Delta_x^s\Delta_s\left(\frac{\partial V_s}{\partial t}\bigg|_{(x_n,t_m)}+O(\tau)\right)$$
$$=\frac{\tau^3}{4}\Delta_s\left(\mathcal{L}_s\,\frac{\partial V_s}{\partial t}\bigg|_{(x_n,t_m)}+O(h^2+\tau)\right),$$

where

$$\mathcal{L}_sV_s=c_{s,1}\frac{\partial V_s}{\partial x}+c_{s,2}D_+^{\xi_s,\alpha_s}V_s+c_{s,3}D_-^{\lambda_s,\alpha_s}V_s-d_sV_s.$$

When τ is sufficiently small, the perturbation error R is a higher-order term than the other terms in Eq (4.1). Therefore, the convergence order of the ADI scheme (4.3) and (4.4) is $O(h^2+\tau^2)$. □

In order to solve the ADI scheme (4.3)-(4.4), we need to define boundary conditions for $\widehat{V}_{s,n}$, which is accomplished by subtracting Eq (4.4) from Eq (4.3)

$$2\widehat{V}_{s,n}=\left(1+\frac{\tau}{2}\Delta_s\right)V_{s,n}^m+\left(1-\frac{\tau}{2}\Delta_s\right)V_{s,n}^{m+1}. \tag{4.5}$$

The corresponding algorithm is implemented as follows:

Algorithm 1 ADI method for scheme (4.3), (4.4)

1: Initialize $V_{s,n}^0$ for $s=1,2,\dots,\bar{S}$ and $n=1,2,\dots,N$ using the payoff function.
2: **for** $m=0,1,2,\dots,M-1$, **do**
3: For $s=1,2,\dots,\bar{S}$, solve the following system for $\widehat{V}_{s,*}=\left(\widehat{V}_{s,1},\widehat{V}_{s,2},\dots,\widehat{V}_{s,N}\right)^T$.

$$\left(I_N-\frac{\tau}{2}T_s\right)\widehat{V}_{s,*}=V_{s,*}^m+\frac{\tau}{2}\sum_{j=1}^{\bar{S}}q_{s,j}V_{j,*}^m+\frac{\tau}{2}\hat{p}_s, \tag{4.6}$$

where T_s is defined in Eq (2.9) , $V_{s,*}^n=\left(V_{s,1}^m,V_{s,2}^m,\dots,V_{s,N}^m\right)^T$,$\hat{p}_s=\frac{c_{s,1}}{2h}\hat{p}_{s,1}+\frac{c_{s,2}}{h^{\alpha_s}}\hat{p}_{s,2}+\frac{c_{s,3}}{h^{\alpha_s}}\hat{p}_{s,3}$, with

$$\hat{p}_{s,1}=(-\widehat{V}_{s,0},0,\dots,0,\widehat{V}_{s,N+1})^T,$$
$$\hat{p}_{s,2}=(\omega_0^{\alpha_s}e^{\xi_sh}\widehat{V}_{s,0}^m+\omega_{N+1}^{\alpha_s}e^{-\xi_sNh}\widehat{V}_{s,N+1}^m,\dots,\omega_2^{\alpha_s}e^{-\xi_sh}\widehat{V}_{s,N+1})^T,$$
$$\hat{p}_{s,2}=(\omega_2^{\alpha_s}e^{-\lambda_sh}\widehat{V}_{s,0},\dots,\omega_{N+1}^{\alpha_s}e^{-\lambda_sNh}\widehat{V}_{s,0}+\omega_0^{\alpha_s}e^{\lambda_sh}\widehat{V}_{s,N+1})^T.$$

4: For $n=1,2,\dots,N$, solve the following system for $V_{*,n}^{m+1}=\left(V_{1,n}^{m+1},V_{2,n}^{m+1},\dots,V_{\bar{S},n}^{m+1}\right)^T$.

$$\left(I_{\bar{S}}-\frac{\tau}{2}Q\right)V_{*,n}^{m+1}=\left(1+\frac{\tau}{2}\Delta_x^s\right)\widehat{V}_{*,n}, \tag{4.7}$$

where $V_{*,n}^{m+1}=\left(V_{*,n}^{m+1},V_{*,n}^{m+1},\dots,V_{*,n}^{m+1}\right)^T$.
5: **end for**

In Algorithm 1, denote $\widetilde{T}_s = I_N - \frac{\tau}{2}T_s$ to be the coefficient matrix of the linear system (4.6), which has Toeplitz structure. Using the preconditioned direct method proposed in [12, 24], the computation process of solving the linear equations with coefficient matrix $\widetilde{T}_s$ can be accelerated by fast Fourier transformations (FFT). The total computation cost to solve Eq (4.6) for each $s = 1, 2, \ldots, \bar{S}$ is $O(N \log N)$.

Since the order of the matrix Q represents the number of the regime-switching states, which is far less than N, the linear system (4.7) can be quickly solved.

For more modern ADI approaches, we refer the readers to [25, 26], which will be our future work to study them under fractional option pricing problems.

5. Numerical experiments

In this section, numerical experiments on the fractional PDE, with known exact solution and European call options under multi-regime FMLS and CGMY models, are presented to show the convergence and efficiency of the proposed ADI approach.

Compared with the ADI method, GMRES and BiCGSTAB are used to solve the linear equation on every temporal layer, where the vector V^{m-1} is taken as a initial guess and the iteration is terminated when the residual $r^{(k)}$ satisfies $\|r^{(k)}\|_2/\|r^{(0)}\|_2 \leq 10^{-7}$. All numerical experiments are carried out by Matlab R2020a.

Example 5.1. *Consider the following FPDE problem with source term:*

$$\begin{cases} \frac{\partial V_s(x,t)}{\partial t} - \frac{\partial V_s(x,t)}{\partial x} - D_{-}^{\lambda_s,\alpha_s} V_s(x,t) - \sum\limits_{j=1}^{\bar{S}} q_{s,j} V_j(x,t) = f_s(x,t), \\ V_s(0,t) = 0, \quad 0 < t \leq 1, \\ V_s(1,t) = e^{-t-\lambda_s}, \quad 0 < t \leq 1, \\ V_s(x,0) = e^{-\lambda_s x} x^{2+\alpha_s}, \quad 0 \leq x \leq 1, \end{cases} \tag{5.1}$$

with

$$f_s(x,t) = -e^{-t-\lambda_s x}\left(\frac{\Gamma(3+\alpha_s)}{\Gamma(3)}x^2 + (1-\lambda_s)x^{2+\alpha_s} + (2+\alpha_s)x^{1+\alpha_s}\right) - \sum_{j=1}^{\bar{S}} q_{s,j} V_j(x,t),$$

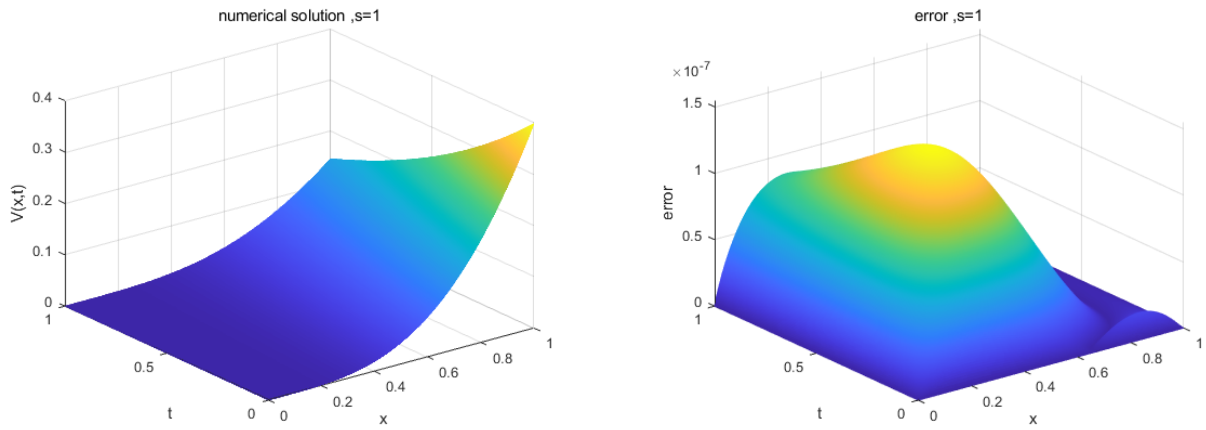
where the exact solution is $V_s(x,t) = e^{-t-\lambda_s x} x^{2+\alpha_s}$.

The following two cases are considered as the settings in [12]:

(a) $\bar{S} = 2, \quad \alpha = (1.9, 1.6), \quad \lambda = (0.92, 1.20), \quad Q = \begin{pmatrix} -6 & 6 \\ 8 & -8 \end{pmatrix}.$

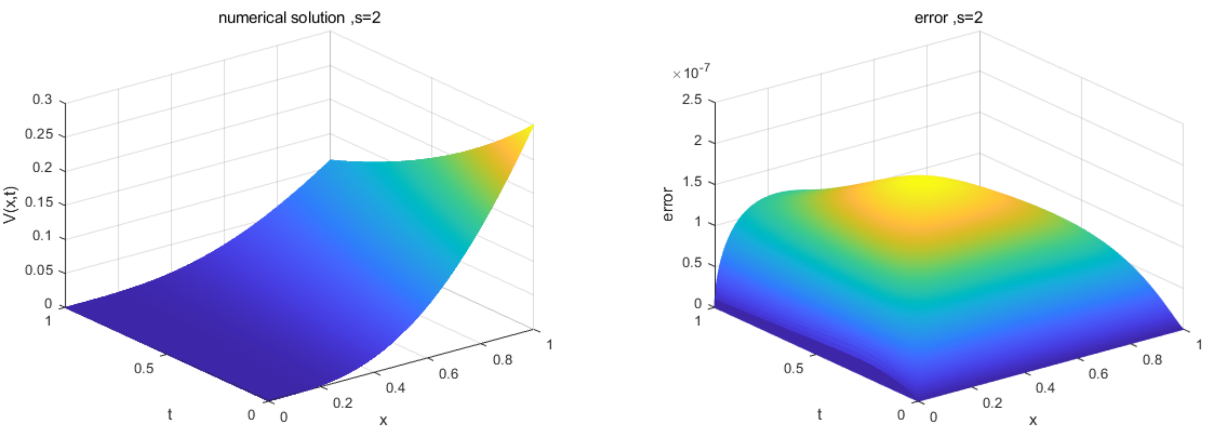
(b) $\bar{S} = 8, \quad \alpha = (1.6, 1.1, 1.9, 1.8, 1.8, 1.3, 1.6, 1.1), \quad \lambda = (2.04, 4.1, 3.6, 4.85, 2.66, 1.63, 0.53, 3.06),$

$$Q = \begin{pmatrix} -25 & 1 & 10 & 5 & 2 & 2 & 2 & 3 \\ 4 & -38 & 10 & 10 & 2 & 4 & 5 & 3 \\ 6 & 2 & -39 & 4 & 10 & 5 & 5 & 7 \\ 5 & 2 & 8 & -32 & 2 & 10 & 2 & 3 \\ 7 & 4 & 3 & 7 & -38 & 2 & 6 & 9 \\ 7 & 2 & 5 & 6 & 6 & -39 & 3 & 10 \\ 3 & 5 & 6 & 7 & 9 & 7 & -45 & 8 \\ 5 & 4 & 10 & 7 & 7 & 4 & 6 & -43 \end{pmatrix}.$$



Numerical solution　　　　Error

Figure 1. The numerical solution and error of case (a) in Example 5.1 when $s = 1$.



Numerical solution　　　　Error

Figure 2. The numerical solution and error of case (a) in Example 5.1 when $s = 2$.

In Figures 1,2, the surfaces of the numerical solution and error $|V^M - V_*^M|$ of case (a) are presented respectively, for $s = 1, 2$, when $M = N = 1024$.

Define the convergence order of the numerical scheme as

$$\text{Order}_m = \log_2 \frac{||V^{m-1} - V_*^{m-1}||}{||V^m - V_*{}^m||},$$

where V_*^m is the exact solution on t_m and $|| \cdot ||$-norm is defined in Theorem 3.2.

In Tables 1,2, the error and convergence order of Crank-Nicolson scheme and ADI scheme are listed for case (a) and case (b), respectively. We use "D-ADI" to represent the ADI Algorithm 1 with preconditioned direct method.

From Tables 1,2, it is seen that both the Crank-Nicolson and ADI schemes have second-order convergence. It is also observed that the convergence of the ADI scheme is more stable. Since the order from Tables 1,2 represents the convergence on each regime, it can further verify the theoretical analysis in Theorem 3.2.

Table 1. Error and convergence order of three numerical schemes for case (a) in Example 5.1.

		GMRES		BiCGSTAB		D-ADI	
Regime	$N = M$	$\|\|V^m - V_*{}^m\|\|$	Order	$\|\|V^m - V_*{}^m\|\|$	Order	$\|\|V^m - V_*{}^m\|\|$	Order
	2^4	2.3967E-04	—	2.3967E-04	—	1.9205E-04	—
	2^5	6.3382E-05	1.9189	6.3380E-05	1.9189	5.1484E-05	1.8993
	2^6	1.6324E-05	1.9571	1.6322E-05	1.9572	1.3344E-05	1.9479
$s = 1$	2^7	4.1466E-06	1.9770	4.1434E-06	1.9779	3.3981E-06	1.9735
	2^8	1.0494E-06	1.9824	1.0450E-06	1.9873	8.5744E-07	1.9866
	2^9	2.6886E-07	1.9646	2.6360E-07	1.9871	2.1536E-07	1.9933
	2^{10}	7.4745E-08	1.8468	6.7470E-08	1.9660	5.3965E-08	1.9967
	2^4	2.6564E-04	—	2.6563E-04	—	3.2512E-04	—
	2^5	6.9890E-05	1.9263	6.9886E-05	1.9264	8.4830E-05	1.9383
	2^6	1.7969E-05	1.9596	1.7965E-05	1.9598	2.1702E-05	1.9667
$s = 2$	2^7	4.5625E-06	1.9776	4.5583E-06	1.9786	5.4916E-06	1.9826
	2^8	1.1551E-06	1.9817	1.1500E-06	1.9869	1.3814E-06	1.9910
	2^9	2.9661E-07	1.9614	2.9041E-07	1.9854	3.4645E-07	1.9955
	2^{10}	8.3315E-08	1.8319	7.4617E-08	1.9605	8.6747E-08	1.9977

In Tables 3,4, the average of the iteration number (denoted by "IT") and the total CPU time (in seconds, denoted by "CPU") of GMRES, BiCGSTAB and ADI methods are compared when the number of discrete points N increases from 2^4 to 2^9 respectively.

From Tables 3,4, it is observed that both GMRES and BiCGSTAB require more iteration step than ADI method, and so does the CPU time. By comparing the CPU time of the three methods, it is obvious that the preconditioned direct ADI method is fast, and can significantly reduce the computation time.

Then, the proposed preconditioned direct ADI method is applied to deal with the multi-regime European option pricing model in Example 5.3. The parameters in this example change in different regimes, by which the sudden state movement and the non-stationary behavior of the market is described.

In order to verify the convergence order for non-smooth payoff function as initial conditions in Eq (2.4), we demonstrate the results of an option pricing problem in Example 5.2.

Example 5.2. *Consider the multi-regime FMLS model for pricing European call option, where* $x_l = \ln(0.1), x_r = \ln(100), K = 50, r = 0.05, T = 1$. *The regime-switching parameters are set as*

$$\bar{S} = 2, \quad \alpha = (1.9, 1.6), \quad \sigma = (0.25, 0.5), \quad Q = \begin{pmatrix} -6 & 6 \\ 8 & -8 \end{pmatrix}.$$

In Table 5, we list the error and convergence order of ADI scheme. The viscosity solution is approximated by the numerical solution using a dense mesh with $N = M = 2^{13}$.

Table 2. Error and convergence order of three numerical schemes for case (b) in Example 5.1.

		GMRES		BiCGSTAB		D-ADI	
Regime	$N = M$	$\|\|V^m - V_*^m\|\|$	Order	$\|\|V^m - V_*^m\|\|$	Order	$\|\|V^m - V_*^m\|\|$	Order
$s = 1$	2^7	2.3244E-06	—	2.3236E-06	—	2.7531E-05	—
	2^8	5.8678E-07	1.9860	5.8543E-07	1.9888	6.8850E-06	1.9995
	2^9	1.4837E-07	1.9836	1.4697E-07	1.9939	1.7221E-06	1.9993
$s = 2$	2^7	2.6927E-06	—	2.6917E-06	—	3.7611E-05	—
	2^8	6.7564E-07	1.9947	6.7401E-07	1.9977	9.3779E-06	2.0038
	2^9	1.7016E-07	1.9893	1.6848E-07	2.0002	2.3407E-06	2.0023
$s = 3$	2^7	2.0421E-06	—	2.0412E-06	—	3.2462E-05	—
	2^8	5.1579E-07	1.9852	5.1445E-07	1.9883	8.1187E-06	1.9994
	2^9	1.3055E-07	1.9822	1.2917E-07	1.9937	2.0306E-06	1.9993
$s = 4$	2^7	2.2554E-06	—	2.2544E-06	—	4.1254E-05	—
	2^8	5.6974E-07	1.9850	5.6817E-07	1.9883	1.0316E-05	1.9996
	2^9	1.4428E-07	1.9815	1.4266E-07	1.9937	2.5801E-06	1.9994
$s = 5$	2^7	2.1676E-06	—	2.1668E-06	—	2.8966E-05	—
	2^8	5.4738E-07	1.9855	5.4605E-07	1.9885	7.2445E-06	1.9994
	2^9	1.3848E-07	1.9829	1.3710E-07	1.9938	1.8120E-06	1.9993
$s = 6$	2^7	2.6192E-06	—	2.6183E-06	—	3.0412E-05	—
	2^8	6.5989E-07	1.9888	6.5841E-07	1.9916	7.5967E-06	2.0012
	2^9	1.6671E-07	1.9849	1.6517E-07	1.9951	1.8992E-06	2.0000
$s = 7$	2^7	2.7223E-06	—	2.7214E-06	—	2.5136E-05	—
	2^8	6.8708E-07	1.9862	6.8567E-07	1.9888	6.2867E-06	1.9994
	2^9	1.7361E-07	1.9846	1.7213E-07	1.9940	1.5725E-06	1.9992
$s = 8$	2^7	2.6352E-06	—	2.6343E-06	—	3.4432E-05	—
	2^8	6.6171E-07	1.9937	6.6016E-07	1.9965	8.5878E-06	2.0034
	2^9	1.6671E-07	1.9889	1.6510E-07	1.9995	2.1439E-06	2.0021

Table 3. Iteration numbers and CPU time of different solvers for case (a) in Example 5.1.

	GMRES		BiCGSTAB		D-ADI
$N = M$	IT	CPU	IT	CPU	CPU
2^4	47	0.0345	22.94	0.0059	0.0242
2^5	77	0.0539	40.47	0.0063	0.0368
2^6	119	0.1674	62.45	0.0321	0.1222
2^7	182	0.7743	84.90	0.2811	0.4580
2^8	307	5.2065	112.75	2.4813	2.0023
2^9	520	56.9593	156.25	28.1146	10.3205
2^{10}	886	922.9244	218.91	390.7879	57.3228

Table 4. Iteration numbers and CPU time of different solvers for case (b) in Example 5.1.

	GMRES		BiCGSTAB		D-ADI
$N = M$	IT	CPU	IT	CPU	CPU
2^4	57	0.0410	27.63	0.0079	0.0329
2^5	91	0.0820	45.16	0.0173	0.0792
2^6	145	0.3451	72.38	0.1356	0.3364
2^7	218	2.1381	107.41	1.3419	1.2518
2^8	340	22.8717	151.21	15.3208	5.8594
2^9	591	345.5954	232.94	237.7708	32.4836

From Table 5, it is observed that the ADI scheme can keep the second-order convergence under the non-smooth initial conditions. The convergence orders are not as steady as those in Example 5.1 perhaps because of the discontinuity at the strike price K, which can be improved by the Padé schemes proposed in [27].

Table 5. Error and convergence order of the ADI scheme in Example 5.2.

Regime	$N = M$	$\|\|V^m - V_*{}^m\|\|$	Order	$\|\|V^m - V_*{}^m\|\|_\infty$	Order
	2^7	1.1242E-02	—	1.3525E-02	—
	2^8	2.1656E-03	2.3760	2.4681E-03	2.4542
$s = 1$	2^9	5.2820E-04	2.0356	6.6201E-04	1.8985
	2^{10}	1.4432E-04	1.8719	1.5758E-04	2.0708
	2^{11}	3.0417E-05	2.2463	4.2304E-05	1.8972
	2^7	1.1462E-02	—	1.3801E-02	—
	2^8	2.2038E-03	2.3788	2.4871E-03	2.4722
$s = 2$	2^9	5.3432E-04	2.0442	6.5044E-04	1.9350
	2^{10}	1.4726E-04	1.8593	1.7085E-04	1.9287
	2^{11}	3.0424E-05	2.2751	4.1473E-05	2.0425

Example 5.3. *Consider the multi-regime CGMY model for pricing European call option where* $x_l = ln(0.1), x_r = \ln(200), K = 60, r = 0.05, T = 1, C = 0.1.$

Consider the following two cases:

(a) $\bar{S} = 4, \quad \alpha = (1.5, 1.7, 1.3, 1.8), \quad \sigma = (0.25, 0.5, 0.75, 0.5), \quad \xi = (2, 1, 5, 1), \quad \lambda = (1, 3, 2, 4),$

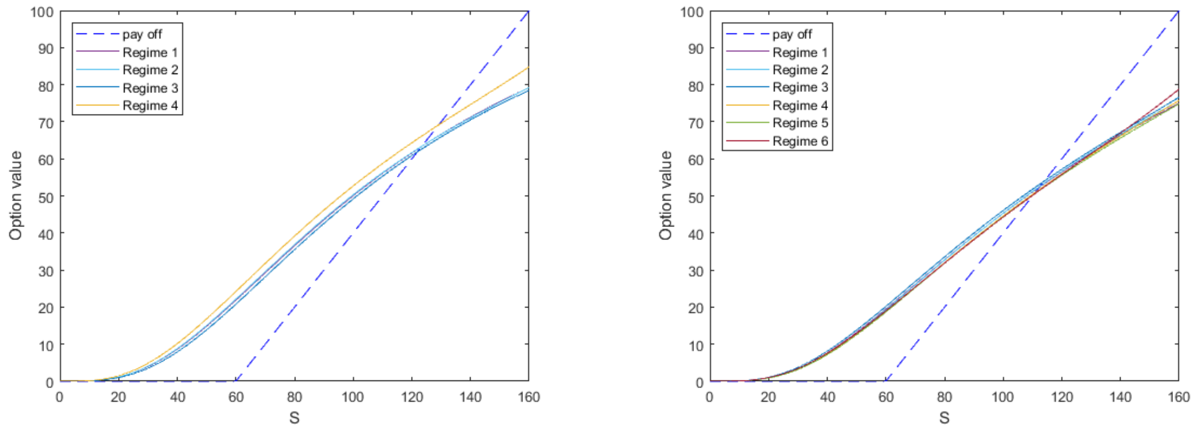
$$Q = \begin{pmatrix} -12 & 2 & 4 & 6 \\ 8 & -20 & 10 & 2 \\ 5 & 4 & -10 & 1 \\ 2 & 4 & 8 & -14 \end{pmatrix}.$$

(b) $\bar{S} = 6, \ \alpha = (1.5, 1.2, 1.7, 1.3, 1.6, 1.8), \ \sigma = (0.25, 0.5, 0.3, 0.75, 0.5, 0.2), \ \xi = (1, 2, 1, 2, 3, 3),$

$\lambda = (3, 4, 3, 1, 2, 2),$

$$Q = \begin{pmatrix} -13 & 2 & 4 & 1 & 2 & 4 \\ 3 & -20 & 8 & 3 & 4 & 2 \\ 2 & 1 & -10 & 2 & 4 & 1 \\ 2 & 4 & 2 & -16 & 1 & 7 \\ 1 & 1 & 3 & 1 & -7 & 1 \\ 4 & 4 & 1 & 1 & 2 & -12 \end{pmatrix}.$$

The option values of four regimes and six regimes are depicted, respectively, in Figure 3 when $M = N = 512$, where the blue dashed line represents the payoff of the European call option and the other colored lines represent the option prices with different regimes at the value date.



(a) Four regimes European call option (b) Six regimes European call option

Figure 3. The value of European call option under the multi-regime CGMY model and payoff function in Example 5.3.

6. Conclusions

In this paper, a second-order finite difference method is proposed to discretise a class of fractional regime-switching option pricing models. In addition, the sufficient conditions of stability and convergence of the numerical scheme are studied in detail. The ADI scheme with preconditioned direct method is considered to deal with the multi-regime structure to accelerate the computation. Numerical experiments verify the theoretical convergence order and show the efficacy of the proposed method.

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Conflict of interest

The authors declare there is no conflicts of interest.

References

1. F. Black, M. Scholes, The pricing of options and corporate liabilities, *J Polit Econ*, **81** (1973), 637–654. https://doi.org/10.1086/260062
2. R. C. Merton, Option pricing when underlying stock returns are discontinuous, *J Financ Econ*, **3** (1976), 125–144. https://doi.org/10.1016/0304-405X(76)90022-2
3. Y. H. Kwon, Y. Lee, A second-order finite difference method for option pricing under jump-diffusion models, *SIAM J Numer Anal*, **49** (2011), 2598–2617. https://doi.org/10.1137/090777529
4. A. D. White, J. C. Hull, The pricing of options on assets with stochastic volatilities, *J Finance*, **42** (1987), 281–300.
5. I. T. Koponen, Analytic approach to the problem of convergence of truncated Lévy flights towards the gaussian stochastic process, *Phys. Rev. E*, **52** (1995), 1197. https://doi.org/10.1103/PhysRevE.52.1197
6. P. Carr, L. Wu, The finite moment log stable process and option pricing, *J Finance*, **58** (2003), 753–777. https://doi.org/10.1111/1540-6261.00544
7. P. Carr, H. Geman, D. B. Madan, M. Yor, Stochastic volatility for Lévy processes, *Math Financ*, **13** (2003), 345–382. https://doi.org/10.1111/1467-9965.00020 .
8. J. Buffington, R. J. Elliott, American options with regime switching, *Int. J. Theor. Appl. Finance*, **5** (2002), 497–514. https://doi.org/10.1142/S0219024902001523
9. S. Heidari, H. Azari, A front-fixing finite element method for pricing American options under regime-switching jump-diffusion models, *Comput. Appl. Math.*, **37** (2018), 3691–3707. https://doi.org/10.1007/s40314-017-0540-z
10. R. J. Elliott, C. J. U. Osakwe, Option pricing for pure jump processes with Markov switching compensators, *Finance Stoch.*, **10** (2006), 250–275. https://doi.org/10.1007/s00780-006-0004-6
11. X. T. Gan, J. F. Yin, Pricing American options under regime-switching model with a Crank-Nicolson fitted finite volume method, *East Asian J Applied Math*, **10** (2020), 499–519. https://10.4208/eajam.170919.221219
12. X. Chen, D. Ding, S. L. Lei, W. F. Wang, An implicit-explicit preconditioned direct method for pricing options under regime-switching tempered fractional partial differential models, *Numer Algorithms*, **87** (2021), 939–965. https://doi.org/10.1007/s11075-020-00994-7
13. A. Cartea, D. del Castillo-Negrete, Fractional diffusion models of option prices in markets with jumps, *Physica A*, **374** (2007), 749–763. https://doi.org/10.1016/j.physa.2006.08.071
14. S. L. Lei, W. F. Wang, X. Chen, D. Ding, A fast preconditioned penalty method for American options pricing under regime-switching tempered fractional diffusion models, *J Sci Comput*, **75** (2018), 1633–1655. https://doi.org/10.1007/s10915-017-0602-9
15. P. Tankov, *Financial modelling with jump processes*, William Hall: CRC Press, 2003.

16. L. X. Zhang, R. F. Peng, J. F. Yin, A second order numerical scheme for fractional option pricing models, *East Asian J Applied Math*, **11** (2021), 326–348. https://10.4208/eajam.020820.121120
17. I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*, London: Academic Press,1999.
18. M. M. Meerschaert, C. Tadjeran, Finite difference approximations for fractional advection–dispersion flow equations, *J. Comput. Appl. Math.*, **172** (2004), 65–77. https://doi.org/10.1016/j.cam.2004.01.033
19. W. Y. Tian, H. Zhou, W. H. Deng, A class of second order difference approximation for solving space fractional diffusion equations, *Math. Comput.*, **84** (2012), 1703–1727. https://doi.org/10.1090/S0025-5718-2015-02917-2
20. W. Chen, Numerical methods for fractional Black-Scholes equations and variational inequalities governing option pricing, Doctoral Thesis of University of Western Australia Perth, Australia, 2014.
21. D. W. Peaceman, H. H. Rachford, The numerical solution of parabolic and elliptic differential equations, *J Soc Ind Appl Math*, **3** (1955), 28–41. https://doi.org/10.1137/0103003
22. H. M. Zhang, F. W. Liu, S. Z. Chen, M. Shen, A fast and high accuracy numerical simulation for a fractional Black-Scholes model on two assets, *Ann Appl Math*, **36** (2020), 91–110.
23. W. Chen, S. Wang, A 2nd-order ADI finite difference method for a 2D fractional Black–Scholes equation governing European two asset option pricing, *Math Comput Simul*, **171** (2020), 279–293. https://doi.org/10.1016/j.matcom.2019.10.016
24. S. L. Lei, Y. C. Huang, Fast algorithms for high-order numerical methods for space-fractional diffusion equations, *Int J Comput Math*, **94** (2017), 1062–1078. https://doi.org/10.1080/00207160.2016.1149579
25. K. J. in 't Hout, J. Toivanen, Adi schemes for valuing european options under the bates model, *Appl Numer Math*, **130** (2018), 143–156. https://doi.org/10.1016/j.apnum.2018.04.003
26. L. Boen, K. J. in 't Hout, Operator splitting schemes for American options under the two-asset merton jump-diffusion model, *Appl Numer Math*, **153** (2020), 114–131. https://doi.org/10.1016/j.apnum.2020.02.004
27. M. K. Wang, C. Wang, J. F. Yin, A class of fourth-order Padé schemes for fractional exotic options pricing model, *Electronic Research Archive*, **30** (2022), 874–897, https://10.3934/era.2022046

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