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Research article

Global well-posedness and asymptotic behavior of BV solutions to a system of balance laws arising in traffic flow

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Abstract: We establish global well-posedness and asymptotic behavior of BV solutions to a system of balance laws modeling traffic flow with nonconcave fundamental diagram. We prove the results by finding a convex entropy-entropy flux pair and verifying the Kawashima condition, the sub-characteristic condition, and the partial dissipative inequality in the framework of Dafermos. This problem is of specific interest since nonconcave fundamental diagrams arise naturally in traffic flow.

Keywords: hyperbolic balance laws; global BV solution; entropy-entropy flux pairs; cauchy problem; traffic flow; relaxation; nonconcave fundamental diagram

1. Introduction

This paper studies the following system of balance law arising in traffic flow

$$\begin{aligned} \rho_t + (\rho v)_x &= 0, \\ v_t + (\frac{1}{2}v^2 + g(\rho))_x + \frac{v - v_e(\rho)}{\tau} &= 0 \end{aligned} \tag{1.1}$$

with initial data

$$(\rho(x,0), v(x,0)) = (\rho_0(x), v_0(x)), \tag{1.2}$$

where $x \in \mathbb{R}, t > 0$, ρ is the density, v is the velocity, $v_e(\rho)$ is the equilibrium velocity, and ρ_0, v_0 are initial data. We assume that the relaxation time, $\tau > 0$,

$$\rho_0(x) \geq \delta_1 > 0, \tag{1.3}$$

where $\delta_1 > 0$ is a constant. In Eq (1.1), g is the anticipation factor which satisfies

$$g'(\rho) = \rho(\frac{v_e'(\rho)}{\theta})^2, \tag{1.4}$$

where

$$0 < \theta < 1. \tag{1.5}$$

We study the global existence and asymptotic behavior of bounded variation (BV) solutions to Eq (1.1) in the framework of Dafermos [6, 7]. Our goal is to verify conditions in [6, 7] that are needed to find admissible BV solutions to the Cauchy problem (1.1) and (1.2). In particular, we will find a convex entropy-entropy flux pair and verify the Kawashima condition, the sub-characteristic condition, and the partial dissipative inequality. Previously, constructing global BV solutions have been studied in [1–6, 22, 24, 25].

A traffic system can exhibit complicated behavior since it is based on interactions between roadways, vehicles, and drivers. Factors that need to be considered in analyzing such a system include nonlinear dynamics and human behavior. Microscopic [8], mesoscopic [27] and macroscopic models [4, 10, 13–21, 23, 26, 28, 31] have been utilized to deal with this phenomenon. Constructing global solutions and finding zero relaxation limits of traffic flow models have been a recent focus of study [10, 16–19, 21]. This paper is concerned with a specific macroscopic model (1.1).

The following macroscopic models have been important in the study of traffic flow: Lighthill-Whitham-Richards (LWR) model [23, 28], Payne-Whitham (PW) model [26, 30], viscous models by Kuhne, Li [14, 15, 22], and Aw-Rascle [4] and Zhang's higher continuum models [31] (ARZ).

When the state is in equilibrium, $v = v_e(\rho)$, the model (1.1) reduces to the LWR model

$$\rho_t + (\rho v_e(\rho))_x = 0, \tag{1.6}$$

where $x \in \mathbb{R}, t > 0$, and $v_e(\rho)$ is a decreasing function of ρ. The fundamental diagram is defined as

$$R(\rho) = \rho v_e(\rho). \tag{1.7}$$

In the current paper, we study Eqs (1.1) and (1.2) with nonconcave $R(\rho)$. Nonconcave flux arises naturally from traffic flow. We will solve the important problem of studying global BV solutions to nonconcave fundamental diagrams as suggested from traffic experiment data [12, 13]. In systems with nonconcave flux functions, the characteristic fields are neither linearly degenerate nor genuinely nonlinear [29].

In our model (1.1), $g(\rho)$ is a pseudo-pressure function accounting for drivers' anticipation of downstream density changes with $0 < \theta < 1$ from Eq (1.5), whereas $\theta = 1$ in the ARZ model [4, 31]. While ARZ model adopted a relative wave propagating speed to the car speed at equilibrium [18], we adopt a larger relative speed. Larger relative speed implies quicker reaction time, which leads to safer and smoother traffic conditions on highways.

The plan of the paper is as follows. We first display preliminaries in Section 2. In Section 3, we introduce symmetrizable system of balance laws and entropy-entropy flux pair. Then we find a convex entropy-entropy flux pair in Section 3 and Section 4. In Section 5, we will verify that the conditions from [6, 7] indeed hold true. We then transform our system into an equivalent form in Section 6 as required in [6, 7]. Lastly, we prove an *a priori* estimate in Section 7. We will then present our main result in Section 8 and elaborate on the implications in the conclusion, Section 9.

2. Preliminaries

In this section, we present the preliminaries.

For this study, we take the equilibrium velocity $v_e(\rho)$ in Eq (1.1) satisfying

$$v_e(0) = b, \tag{2.1}$$
$$v_e'(0) = -a, \tag{2.2}$$
$$v_e(1) = 0, \tag{2.3}$$

where $a, b > 0$ and $v_e(\rho)$ is a decreasing function i.e.

$$v_e'(\rho) < 0. \tag{2.4}$$

The equilibrium characteristic speed is the characteristic speed of Eq (1.6)

$$\lambda_*(\rho) = R'(\rho) = v_e(\rho) + \rho v_e'(\rho). \tag{2.5}$$

We consider general inhomogeneous, strictly hyperbolic system of balance laws as in [7] (Chpt. 16, Eq (16.6.1))

$$U_t + F(U)_x + P(U) = 0 \tag{2.6}$$

with initial data

$$U(x, 0) = U_0(x), \tag{2.7}$$

where $x \in \mathbb{R}, t > 0$, U is a vector in $\mathbb{R}^2$, U_0 is initial data, and $F(U), P(U)$ are vector fields in $\mathbb{R}^2$. Let O be an open subset of $\mathbb{R}^2$ containing the origin. In [7], it was assumed that

$$P(0) = 0 \tag{2.8}$$

so that $U \equiv 0$ is an equilibrium solution. In order to satisfy Eq (2.8), we make the following change of variables in Eq (1.1)

$$U = (\rho, v - b)^T = (\rho, u)^T, \tag{2.9}$$

where b is from Eq (2.1). Under Eq (2.9), system (1.1) is reduced to

$$\begin{aligned} \rho_t + (\rho(u+b))_x &= 0, \\ u_t + (\frac{1}{2}(u+b)^2 + g(\rho))_x + \frac{u + b - v_e(\rho)}{\tau} &= 0, \end{aligned} \tag{2.10}$$

where $x \in \mathbb{R}, t > 0$. Due to Eq (2.9), Eq (2.10) satisfies Eq (2.8). The initial data of Eq (1.2) is reduced to

$$(\rho(x, 0), u(x, 0)) = (\rho_0(x), u_0(x)) = (\rho_0(x), v_0(x) - b) \tag{2.11}$$

after change of variables (2.9).

We identify F and P in Eq (2.10) as

$$F(U) = (\rho(u+b), \frac{1}{2}u^2 + ub + g(\rho))^T, \tag{2.12}$$
$$P(U) = (0, \frac{u + b - v_e(\rho)}{\tau})^T. \tag{2.13}$$

From Eq (2.12), we compute the Jacobian of F as

$$DF(U) = \begin{bmatrix} u+b & \rho \\ g'(\rho) & u+b \end{bmatrix}. \tag{2.14}$$

The eigenvalues of Eq (2.14) are

$$\lambda_{1,2} = u + b \pm \rho \frac{v_e'(\rho)}{\theta}. \tag{2.15}$$

Hence, the system is strictly hyperbolic for $\rho > 0$ since $\lambda_1 \neq \lambda_2$. The corresponding right eigenvectors are

$$r_{1,2} = (\pm \frac{\theta}{v_e'(\rho)}, 1)^T. \tag{2.16}$$

The ith characteristic field is said to be genuinely nonlinear (GNL) if $\nabla\lambda_i \cdot r_i \neq 0$ for $i = 1, 2$. We calculate $\nabla\lambda_i \cdot r_i$ as follows

$$\nabla\lambda_i \cdot r_i = \frac{2v_e'(\rho) + \rho v_e''(\rho)}{v_e'(\rho)} = \frac{R''(\rho)}{v_e'(\rho)}, \quad i = 1, 2, \tag{2.17}$$

where nonconcave R is defined in Eq (1.7). Since $R''(\rho)$ changes signs, the characteristic fields are not genuinely nonlinear.

3. Entropy-entropy flux pairs

First we introduce symmetrizable system of balance laws and entropy entropy flux pairs from Dafermos [7] (Chpts. 1 and 3). We will then apply the theory to Eq (2.10) to find an entropy-entropy flux pair.

3.1. *Definition of entropy-entropy flux*

Now we present the definition of entropy-entropy flux pair from Dafermos [7] (Chpts. 1 and 3).

Let $\mathcal{K}$ be an open subset of $\mathbb{R}^k$, $k > 0$. A system of balance laws is given by

$$\text{div } G(U(X), X) = \Pi(U(X), X), \tag{3.1}$$

where G and Π are given smooth functions defined on $O \times \mathcal{K}$ taking values in $\mathbb{M}^{n\times k}$ and $\mathbb{R}^n$ respectively.

Finding entropy-entropy flux pairs is important in the study of balance laws. Our goal is find a smooth entropy-entropy flux pair $(\eta, q)(U)$ with η convex, normalized by $\eta(0) = 0, D\eta(0) = 0$. Admissible solutions U satisfy the entropy inequality [7]

$$\partial_t \eta(U(x,t)) + \partial_x q(U(x,t)) + D\eta(U(x,t))P(U(x,t)) \leq 0. \tag{3.2}$$

One way to derive entropy-entropy flux pairs is by considering the companion of G. A smooth function Q, defined on $O \times \mathcal{K}$ taking values in $\mathbb{M}^{1\times k}$ is called a *companion* of G if there is a smooth function B, defined on $O \times \mathcal{K}$ and taking values in $\mathbb{R}^n$ such that for all $U \in O$ and $X \in \mathcal{K}$,

$$DQ_\alpha(U, X) = B(U, X)^T DG_\alpha(U, X), \quad \alpha = 1, ..., k, \tag{3.3}$$

where $D = [\partial/\partial U^1, ..., \partial/\partial U^n]$ and $G_\alpha(U, X)$ denotes the α-the column vector of the matrix $G(U, X)$. The significance of companion balance laws is that any classical solution U of Eq (3.1) is automatically a classical solution of the *companion balance law*

$$\text{div } Q(U(X), X) = H(U(X), X), \tag{3.4}$$

where

$$H(U, X) = B(U, X)^T \Pi(U, X) + \nabla \cdot Q(U, X) - B(U, X)^T \nabla \cdot G(U, X). \tag{3.5}$$

Eq (3.1) is called *symmetric* when the $n \times n$ matrices $DG_\alpha(U, X), \alpha = 1, ..., k$, are symmetric, for $U \in \mathcal{O}$ and $X \in \mathcal{K}$. In [7], it was proved that a system of balance laws is endowed with nontrivial companion balance laws if and only if it is symmetrizable. When a system of balance laws (3.1) is endowed with a companion balance law (3.4), we can find an entropy-entropy flux pair.

3.2. Deriving the equation for entropy

Now we apply the theory from Subsection 3.1 to Eq (2.10) to find an entropy-entropy flux pair. For Eq (2.10), where $n = k = 2$, we have

$$\eta(U) = Q_1(U), q(U) = Q_2(U), \tag{3.6}$$

where η is called the entropy for the system and q is the entropy flux associated with η. Now we solve for an entropy-entropy flux pair (η, q) for Eq (2.10). Evaluating Eq (3.3) at Eq (2.10), we get

$$\begin{aligned} DQ_1(U) &= B(U)^T DG_1(U), \\ DQ_2(U) &= B(U)^T DG_2(U), \end{aligned} \tag{3.7}$$

where $G_1 = U, G_2 = F(U)$ from Eqs (2.9) and (2.12), and $D = [\partial/\partial U^1, \partial/\partial U^2]$. Then we have

$$\begin{aligned} G_1(\rho, u) &= \begin{bmatrix} \rho \\ u \end{bmatrix}, \\ G_2(\rho, u) &= \begin{bmatrix} \rho(u + b) \\ \frac{1}{2}u^2 + ub + g(\rho) \end{bmatrix}. \end{aligned} \tag{3.8}$$

Then, DG_1 and DG_2 are as follows

$$\begin{aligned} DG_1 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ DG_2 &= \begin{bmatrix} u + b & \rho \\ g'(\rho) & u + b \end{bmatrix}. \end{aligned} \tag{3.9}$$

From Eqs (3.7) and (3.9), we have

$$\begin{aligned} DQ_1(U) &= B(U)^T DG_1(U) \\ [Q_{1,\rho}, Q_{1,u}] &= [B_1, B_2] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Therefore

$$Q_{1,\rho} = B_1, Q_{1,u} = B_2. \tag{3.10}$$

Next we plug Eq (3.10) into the second equation of Eq (3.7) to get

$$\begin{aligned}
[Q_{2,\rho}, Q_{2,u}] &= [B_1, B_2]\begin{bmatrix} u+b & \rho \\ g'(\rho) & u+b \end{bmatrix} \\
&= [Q_{1,\rho}, Q_{1,u}]\begin{bmatrix} u+b & \rho \\ g'(\rho) & u+b \end{bmatrix} \\
&= \Big[Q_{1,\rho}(u+b) + Q_{1,u}g'(\rho), \quad Q_{1,\rho}\rho + Q_{1,u}(u+b)\Big].
\end{aligned}$$

Hence, we have the two equations for Q_2

$$Q_{2,\rho} = Q_{1,\rho}(u+b) + Q_{1,u}g'(\rho), \tag{3.11}$$

$$Q_{2,u} = Q_{1,\rho}\rho + Q_{1,u}(u+b). \tag{3.12}$$

Taking partial derivatives of Eq (3.11) with respect to u and partial derivatives of Eq (3.12) with respect to ρ and subtracting the latter from the former, we get

$$0 = g'(\rho)Q_{1,uu} - \rho Q_{1,\rho\rho}. \tag{3.13}$$

By Eq (1.4), we can rewrite Eq (3.13) as

$$Q_{1,uu} - s^2(\rho)Q_{1,\rho\rho} = 0, \tag{3.14}$$

where we define

$$s(\rho) = -\frac{\theta}{v_e'(\rho)} > 0 \tag{3.15}$$

due to Eq (1.5) and Eq (2.4).

3.3. *Solving Eq* (3.14)

Since Eq (3.14) is a second order hyperbolic partial differential equations in two variables, we follow Evans' method in *Partial differential equations* (Chpt. 7.2.5) [9]. Consider

$$\sum_{i,j=1}^{2} a^{ij}u_{x_ix_j} + \sum_{i=1}^{2} b^i u_{x_i} + cu = 0, \tag{3.16}$$

$$a^{11}a^{22} - (a^{12})^2 < 0, \tag{3.17}$$

where the coefficients a^{ij}, b^i, c $(i, j = 1, 2)$ with $a^{ij} = a^{ji}$ and the unknown u are functions for x_1 and x_2 in some region in $\mathcal{D} \subset \mathbb{R}^2$. In order to solve Eq (3.14) in Evans' framework, define

$$\tilde{Q}_1(u, \rho) = Q_1(\rho, u). \tag{3.18}$$

Letting $u = \tilde{Q}_1$, Eq (3.16) becomes

$$a^{11}\tilde{Q}_{1,uu} + a^{12}\tilde{Q}_{1,u\rho} + a^{21}\tilde{Q}_{1,\rho u} + a^{22}\tilde{Q}_{1,\rho\rho} + b^1\tilde{Q}_{1,u} + b^2\tilde{Q}_{1,\rho} + c\tilde{Q}_1 = 0. \tag{3.19}$$

In Eq (3.19), if we choose

$$a^{11} = 1, a^{12} = a^{21} = 0, a^{22} = -s^2(\rho), b^1 = b^2 = c = 0, \tag{3.20}$$

then we arrive at Eq (3.14). In particular, Eq (3.17) is satisfied since

$$a^{11}a^{22} - (a^{12})^2 = -s^2(\rho) < 0 \quad \text{in } \mathcal{D}, \tag{3.21}$$

where $s(\rho)$ is defined in Eq (3.15).

Next, we will do the following transformation. Set

$$\begin{cases} y_1 & = \Phi^1(u,\rho) \\ y_2 & = \Phi^2(u,\rho). \end{cases} \tag{3.22}$$

Let

$$\tilde{Q}_1(u,\rho) = h(y_1, y_2) \tag{3.23}$$

for some smooth function $h(y_1, y_2)$. With the change of variables (3.22), we have

$$\tilde{Q}_{1,uu} = h_{y_1y_1}(\Phi^1_u)^2 + h_{y_2y_2}(\Phi^2_u)^2 + 2h_{y_1y_2}\Phi^1_u\Phi^2_u + h_{y_1}\Phi^1_{uu} + h_{y_2}\Phi^2_{uu}, \tag{3.24}$$

$$\tilde{Q}_{1,\rho\rho} = h_{y_1y_1}(\Phi^1_\rho)^2 + h_{y_2y_2}(\Phi^2_\rho)^2 + 2h_{y_1y_2}\Phi^1_\rho\Phi^2_\rho + h_{y_1}\Phi^1_{\rho\rho} + h_{y_2}\Phi^2_{\rho\rho}. \tag{3.25}$$

Now we substitute Eqs (3.24) and (3.25) in Eq (3.14) to get

$$\begin{aligned} 0 = {} & h_{y_1y_1}((\Phi^1_u)^2 - s^2(\rho)(\Phi^1_\rho)^2) + h_{y_2y_2}((\Phi^2_u)^2 - s^2(\rho)(\Phi^2_\rho)^2) \\ & + h_{y_1}(\Phi^1_{uu} - s^2(\rho)\Phi^1_{\rho\rho}) + h_{y_2}(\Phi^2_{uu} - s^2(\rho)\Phi^2_{\rho\rho}) \\ & + 2h_{y_1y_2}(\Phi^1_u\Phi^2_u - s^2(\rho)\Phi^1_\rho\Phi^2_\rho). \end{aligned} \tag{3.26}$$

In order to simplify the first two terms in Eq (3.26), we choose (Φ^1, Φ^2) satisfying

$$(\Phi^1_u)^2 - s^2(\rho)(\Phi^1_\rho)^2 = 0, \tag{3.27}$$

$$(\Phi^2_u)^2 - s^2(\rho)(\Phi^2_\rho)^2 = 0. \tag{3.28}$$

This will only be possible if Φ^1 and Φ^2 solve the following

$$(w_u)^2 - s^2(\rho)(w_\rho)^2 = 0 \quad \text{in } \mathcal{D}, \tag{3.29}$$

where w is a solution of Eq (3.29). Note that Eq (3.29) is a product of two linear first-order PDE, namely

$$w_u + s(\rho)w_\rho = 0 \quad \text{in } \mathcal{D}, \tag{3.30}$$

$$w_u - s(\rho)w_\rho = 0 \quad \text{in } \mathcal{D}. \tag{3.31}$$

Now we calculate the third and fourth terms in Eq (3.26)

$$\partial^2_u w - s(\rho)\partial^2_\rho w - s'(\rho)s(\rho)\partial_\rho w = (\partial_u + s(\rho)\partial_\rho)(\partial_u - s(\rho)\partial_\rho)w = 0 \tag{3.32}$$

by Eqs (3.30) and (3.31). From Eqs (3.27), (3.28), and (3.32), we calculate

$$\begin{cases} (\Phi^1)_{uu} - s^2(\rho)(\Phi^1)_{\rho\rho} & = s(\rho)s'(\rho)\Phi^1_\rho, \\ (\Phi^2)_{uu} - s^2(\rho)(\Phi^2)_{\rho\rho} & = s(\rho)s'(\rho)\Phi^2_\rho, \end{cases} \tag{3.33}$$

where $s(\rho)$ is defined in Eq (3.15). Then by Eqs (3.27), (3.28), and (3.33), Eq (3.26) becomes

$$2h_{y_1y_2}(\Phi^1_u\Phi^2_u - s^2(\rho)\Phi^1_\rho\Phi^2_\rho) + h_{y_1}(s(\rho)s'(\rho)\Phi^1_\rho) + h_{y_2}(s(\rho)s'(\rho)\Phi^2_\rho) = 0. \tag{3.34}$$

3.4. Solving for Φ^1, Φ^2

We now solve for Φ^1 and Φ^2 from Eqs (3.30) and (3.31) respectively. We want a smooth solution Φ^1 of Eq (3.30) satisfying $\nabla\Phi^1 \neq 0$. Φ^1 is constant along trajectories $\mathbf{x} = (x^1, x^2)$ of Eq (3.30)

$$\begin{cases} \dot{x^1} &= 1 \\ \dot{x^2} &= s(\rho). \end{cases} \tag{3.35}$$

Then we have

$$\nabla\Phi^1 \perp \begin{pmatrix} 1 \\ s(\rho) \end{pmatrix}.$$

Thus,

$$\nabla\Phi^1 \| \begin{pmatrix} -s(\rho) \\ 1 \end{pmatrix}.$$

Hence, for some $\alpha_1(u,\rho)$, we have

$$\nabla\Phi^1 = \alpha_1(u,\rho) \begin{pmatrix} -s(\rho) \\ 1 \end{pmatrix}. \tag{3.36}$$

Indeed, we find an exact solution for Eq (3.36),

$$\Phi^1 = n_1(u + b + \frac{v_e(\rho)}{\theta}), \tag{3.37}$$

where $n_1 > 0$ and $b > 0$ from Eq (2.1). From Eqs (3.36) and (3.37), we see that $\alpha_1(u,\rho) = \dfrac{-n_1}{s(\rho)}$.

Similarly, we want a smooth solution Φ^2 of Eq (3.31) satisfying $\nabla\Phi^2 \neq 0$. Φ^2 is constant along trajectories $\mathbf{x} = (x^1, x^2)$ of Eq (3.31)

$$\begin{cases} \dot{x^1} &= 1 \\ \dot{x^2} &= -s(\rho). \end{cases} \tag{3.38}$$

Then we have

$$\nabla\Phi^2 \perp \begin{pmatrix} 1 \\ -s(\rho) \end{pmatrix},$$

$$\nabla\Phi^2 \| \begin{pmatrix} s(\rho) \\ 1 \end{pmatrix}.$$

Hence,

$$\nabla\Phi^2 = \alpha_2(u,\rho) \begin{pmatrix} -s(\rho) \\ -1 \end{pmatrix} \tag{3.39}$$

for some $\alpha_2(u,\rho)$. From Eq (3.39), we find an exact solution

$$\Phi^2 = n_2(u + b - \frac{v_e(\rho)}{\theta}), \tag{3.40}$$

where $n_2 > 0$ and $b > 0$ from Eq (2.1). From Eqs (3.39) and (3.40), we see that $\alpha_2(u,\rho) = \dfrac{-n_2}{s(\rho)}$.

4. Solving for entropy-entropy flux pairs

In this section, we solve for Q_1 by integrating factor method. Then we solve for a special case, estimate the solution of Q_1 by perturbation analysis, prove convexity, and finalize the entropy-entropy flux pair for Eq (2.10).

4.1. Solving for Q_1

In this subsection, we solve for $Q_1(\rho, u)$ by integrating factor method.

Plugging Eqs (3.37) and (3.40) into Eq (3.34), then Eq (3.34) reduces to

$$4n_1n_2h_{y_1y_2} - n_1s'(\rho)h_{y_1} + n_2s'(\rho)h_{y_2} = 0, \tag{4.1}$$

where y_1, y_2 are defined in Eq (3.22) and $h(y_1, y_2)$ is defined in Eq (3.23). Since $4n_1n_2 > 0$, (4.1) becomes

$$h_{y_1y_2} - \frac{s'(\rho)}{4n_2}h_{y_1} + \frac{s'(\rho)}{4n_1}h_{y_2} = 0. \tag{4.2}$$

Solving Eq (4.2) by integrating factor method and from Eq (3.18), we get

$$Q_1(\rho, u) = \tilde{Q}_1(u, \rho) = h(y_1, y_2) = \mu_2^{-1}(\int \mu_2(\mu_1^{-1}f(y_1))\,dy_1 + G(y_2)), \tag{4.3}$$

where f, G are C^2 functions and

$$\mu_1 = e^{\int \frac{-s'(\rho)}{4n_2}\,dy_2}, \tag{4.4}$$

$$\mu_2 = e^{\int \frac{s'(\rho)}{4n_1}\,dy_1}, \tag{4.5}$$

under the assumption

$$s''(\rho)s(\rho) + (\frac{s'(\rho)}{2})^2 = 0. \tag{4.6}$$

4.2. A special case

In this subsection, we consider the special case

$$s'(\rho) = 0, \tag{4.7}$$

where $s(\rho)$ is defined in Eq (3.15). Eq (4.7) is equivalent to $v_e''(\rho) = 0$. Indeed, we take

$$\hat{v}_e(\rho) = -a\rho + b, \tag{4.8}$$

where $a, b > 0$ from Eqs (2.1) and (2.2). In particular, $\hat{v}_e(\rho)$ satisfies Eqs (2.1) and (2.2). Then, the fundamental diagram is

$$\hat{R}(\rho) = \rho\hat{v}_e(\rho). \tag{4.9}$$

Note that $\hat{R}''(\rho) < 0$. $\hat{R}(\rho)$ in Eq (4.9) is an actual fundamental diagram observed in traffic flow, see [11].

Denote $\tilde{\hat{Q}}_1$ as a solution to Eq (3.14)

$$\tilde{\hat{Q}}_1(u,\rho) = \hat{h}(\hat{\Phi}^1(u,\rho), \hat{\Phi}^2(u,\rho)), \tag{4.10}$$

where $\hat{h}$ is a smooth function. Due to Eqs (4.7) and (4.8), Eq (4.1) reduces to

$$4n_1n_2\hat{h}_{\hat{y}_1\hat{y}_2} = 0. \tag{4.11}$$

From Eqs (3.37) and (3.40), $4n_1n_2 > 0$. Then, Eq (4.11) reduces to

$$\hat{h}_{\hat{y}_1\hat{y}_2} = 0. \tag{4.12}$$

Solving Eq (4.12), we get

$$\hat{h}(\hat{y}_1, \hat{y}_2) = \hat{\beta}(\hat{y}_1) + \hat{G}(\hat{y}_2), \tag{4.13}$$

where $\hat{\beta}$ and $\hat{G}$ are arbitrary C^2 functions and $\hat{y}_1, \hat{y}_2$ are defined in Eq (3.22).

Therefore, from Eqs (3.18), (4.7), (4.8) and (4.10), the solution to Eq (3.14) is

$$\hat{Q}_1(\rho, u) = \tilde{\hat{Q}}_1(u,\rho) = \hat{\beta}(\hat{\Phi}^1(u,\rho)) + \hat{G}(\hat{\Phi}^2(u,\rho)), \tag{4.14}$$

where

$$\hat{\Phi}^1 = n_1(u + b + \frac{\hat{v}_e(\rho)}{\theta}), \tag{4.15}$$

$$\hat{\Phi}^2 = n_2(u + b - \frac{\hat{v}_e(\rho)}{\theta}), \tag{4.16}$$

and $b > 0$ as defined in Eq (2.1).

4.3. The perturbation analysis

In this subsection, we estimate $Q_1(\rho, u)$, the solution of Eq (3.14), by $\hat{Q}_1(\rho, u)$. From Eq (3.15), we have

$$s'(\rho) = \frac{\theta v_e''(\rho)}{(v_e'(\rho))^2}. \tag{4.17}$$

Assume that

$$|s'(\rho)| \le \gamma << 1, \tag{4.18}$$

where $\gamma > 0$ is a small constant.

From Eqs (4.17) and (4.18), we derive that

$$|v_e''(\rho)| \le C\gamma, \tag{4.19}$$

where $C > 0$ is a universal constant. Eq (4.19) means that the equilibrium velocity $v_e(\rho)$ is close to $\hat{v}_e(\rho)$ defined in Eq (4.8). Hence, the fundamental diagram $R(\rho)$ in Eq (1.7) is close to $\hat{R}(\rho)$ defined in Eq (4.9).

By [29] (Chpt. 19, Sec. B), $\|(\rho,u)\|_{L_\infty} \le |(\rho,u)(\infty,t)| + |(\rho,u)(-\infty,t)| + \text{T.V.}(\rho,u)$. Since we are looking for solutions to Eq (2.10) in bounded variation space, we assume *a priori* that $\|(\rho,u)\|_{L_\infty}$ is bounded.

We first estimate

$$\Phi^1 - \hat{\Phi}^1 = \frac{n_1}{\theta}(v_e(\rho) - \hat{v}_e(\rho)), \tag{4.20}$$

$$\Phi^2 - \hat{\Phi}^2 = \frac{-n_2}{\theta}(v_e(\rho) - \hat{v}_e(\rho)), \tag{4.21}$$

where $\hat{v}_e(\rho)$ is as in Eq (4.8) and $\Phi^1, \Phi^2, \hat{\Phi}^1, \hat{\Phi}^2$ are from Eqs (3.37), (3.40), (4.15), and (4.16) respectively.

We now estimate the right hand side of Eqs (4.20) and (4.21). By Taylor expansion and Eq (4.19), there is a $\xi \in (0,\rho)$ such that

$$\begin{aligned}|v_e(\rho) - \hat{v}_e(\rho)| &= |v_e(0) - \hat{v}_e(0) + (v_e'(0) - \hat{v}_e'(0))\rho + \frac{1}{2}(v_e''(\xi) - \hat{v}_e''(\xi))\rho^2| \\ &= \frac{1}{2}|v_e''(\xi)|\rho^2 \le C\gamma,\end{aligned} \tag{4.22}$$

where ρ is bounded, $v_e(0) = b = \hat{v}_e(0)$, $v_e'(0) = -a = \hat{v}_e'(0)$ are from Eqs (2.1) and (2.2), and $\hat{v}_e''(\rho) = 0$ due to Eq (4.8).

Plugging Eq (4.22) into Eqs (4.20) and (4.21), we conclude that

$$|\Phi^1 - \hat{\Phi}^1| \le C\gamma, \tag{4.23}$$

$$|\Phi^2 - \hat{\Phi}^2| \le C\gamma. \tag{4.24}$$

Next, we estimate Q_1. From Eq (4.3) and under condition Eq (4.18), we derive

$$Q_1(\rho,u) = \beta(\Phi^1(u,\rho)) + G(\Phi^2(u,\rho)) + O(\gamma), \tag{4.25}$$

where $\gamma > 0$ is small and $\beta(y_1) = \int f(y_1)\,dy_1$, G are C^2 functions. In Eq (4.25), we choose $\beta = \hat{\beta}, G = \hat{G}$ as defined in Eq (4.14).

Now we estimate $Q_1 - \hat{Q}_1$. From Eqs (4.14), (4.25), and under the condition (4.18), by the mean value theorem, there is a $\xi_1 \in (0,\rho)$ such that

$$Q_1(\rho,u) - \hat{Q}_1(\rho,u) = \hat{\beta}'(\xi_1)(\Phi^1 - \hat{\Phi}^1) + \hat{G}'(\xi_1)(\Phi^2 - \hat{\Phi}^2) + O(\gamma), \tag{4.26}$$

where $\gamma > 0$ is small. Then from Eqs (4.23) and (4.24) under the condition Eq (4.18), we can estimate Eq (4.26) as

$$|Q_1(\rho,u) - \hat{Q}_1(\rho,u)| \le C\gamma, \tag{4.27}$$

where $\gamma > 0$ is small and C is a universal constant.

4.4. Convexity

In this subsection, we will prove convexity for $\hat{Q}_1(\rho,u)$ and $Q_1(\rho,u)$. In order for Q_1 to be convex, we need

$$Q_{1,\rho\rho} > 0, \tag{4.28}$$

$$\mathscr{D} = \det\begin{bmatrix} Q_{1,\rho\rho} & Q_{1,\rho u} \\ Q_{1,u\rho} & Q_{1,uu} \end{bmatrix} > 0. \tag{4.29}$$

From Eq (4.14), we calculate $\hat{Q}_{1,\rho\rho}$ as

$$\begin{aligned}\hat{Q}_{1,\rho\rho} &= (\frac{\hat{v}_e'(\rho)}{\theta})^2(\hat{\beta}''n_1^2 + \hat{G}''n_2^2) + \frac{\hat{v}_e''(\rho)}{\theta}(\hat{\beta}'n_1 - \hat{G}'n_2) \\ &= (\frac{\hat{v}_e'(\rho)}{\theta})^2(\hat{\beta}''n_1^2 + \hat{G}''n_2^2),\end{aligned} \tag{4.30}$$

where we used $\hat{v}_e''(\rho) = 0$ due to Eq (4.8). Next, we calculate the determinant $\hat{\mathscr{D}}$ defined in Eq (4.29) for $\hat{Q}_1$

$$\begin{aligned}\hat{\mathscr{D}} &= 4(\frac{\hat{v}_e'(\rho)}{\theta})^2\hat{\beta}''\hat{G}''n_1^2n_2^2 + \frac{\hat{v}_e''(\rho)}{\theta}(n_1^3\hat{\beta}'\hat{\beta}'' - n_2^3\hat{G}'\hat{G}'' + n_1n_2^2\hat{\beta}'\hat{G}'' - n_1^2n_2\hat{\beta}''\hat{G}') \\ &= 4(\frac{\hat{v}_e'(\rho)}{\theta})^2\hat{\beta}''\hat{G}''n_1^2n_2^2,\end{aligned} \tag{4.31}$$

where we used $\hat{v}_e''(\rho) = 0$ due to Eq (4.8). For $\hat{Q}_1$ to satisfy Eqs (4.28) and (4.29), we require

$$\hat{\beta}'', \hat{G}'' \geq m > 0, \tag{4.32}$$

where $m > 0$ is a constant. Under condition Eq (4.32), we derive from Eq (4.30) that there is $m_1 > 0$ constant such that

$$\hat{Q}_{1,\rho\rho} \geq m_1 > 0. \tag{4.33}$$

Similarly, we derive from Eq (4.31) that

$$\hat{\mathscr{D}} \geq m_1 > 0. \tag{4.34}$$

This proves the convexity conditions Eqs (4.28) and (4.29) for $\hat{Q}_1$.

Now we prove Eq (4.28) for $Q_1(\rho, u)$. Under conditions Eq (4.18), from Eqs (4.27) and (4.33) we have

$$Q_{1,\rho\rho} \geq \hat{Q}_{1,\rho\rho} - C\gamma \geq m_1 - C\gamma > 0 \tag{4.35}$$

by choosing $\gamma > 0$ small enough.

Similarly, we prove Eq (4.29) for $Q_1(\rho, u)$. Under conditions Eq (4.18), from Eqs (4.27) and (4.34) we have

$$\mathscr{D} \geq \hat{\mathscr{D}} - C\gamma \geq m_1 - C\gamma > 0 \tag{4.36}$$

by choosing $\gamma > 0$ small enough. Hence, we proved the convexity for $\hat{Q}_1(\rho, u)$ *and* $Q_1(\rho, u)$ under conditions Eqs (4.18) and (4.32).

4.5. Finalizing the entropy-entropy flux pair

In this subsection, we finalize the entropy-entropy flux pair for Eq (2.10).

We choose $\hat{\beta}(\hat{\Phi}^1) = (\hat{\Phi}^1)^2, \hat{G}(\hat{\Phi}^2) = (\hat{\Phi}^2)^2$ in $\hat{Q}_1$ defined in Eq (4.14) which satisfy Eq (4.32) with $m = 2$. Then by Eq (3.6)

$$\hat{\eta}(\rho, u) = \hat{Q}_1(\rho, u) = (\hat{\Phi}^1(\rho, u))^2 + (\hat{\Phi}^2(\rho, u))^2. \tag{4.37}$$

From Eqs (4.15) and (4.16), we have

$$\hat{\eta}(\rho, u) = (n_1(u + \frac{\hat{v}_e(\rho)}{\theta} + b))^2 + (n_2(u - \frac{\hat{v}_e(\rho)}{\theta} + b))^2. \tag{4.38}$$

Under Eq (4.8), if we integrate Eq (3.11) with respect to ρ, we get

$$\hat{q}(\rho, u) = \hat{Q}_2(\rho, u) = \hat{Q}_1(\rho, u)(u + b) + \int \hat{Q}_{1,u}\, \rho(\frac{\hat{v}_e'(\rho)}{\theta})^2\, d\rho + j(u), \tag{4.39}$$

where $\hat{Q}_1$ from Eq (4.38) and $j(u)$ is a smooth function of u. We can solve for $j(u)$ by using Eq (3.12).

Now we derive $Q_1(\rho, u)$ and $Q_2(\rho, u)$. By Eqs (3.6), (4.18), and (4.27), we have the following entropy-entropy flux pair

$$Q_1(\rho, u) = (n_1(u + \frac{v_e(\rho)}{\theta} + b))^2 + (n_2(u - \frac{v_e(\rho)}{\theta} + b))^2 + O(\gamma), \tag{4.40}$$

$$Q_2(\rho, u) = Q_1(\rho, u)(u + b) + \int Q_{1,u}\, \rho(\frac{v_e'(\rho)}{\theta})^2\, d\rho + j(u) + O(\gamma), \tag{4.41}$$

where $\gamma > 0$ is small.

5. Verifying assumptions for main results

In this section, we verify the partial dissipative inequality, the Kawashima condition, and the sub-characteristic condition.

5.1. The entropy inequality and partial dissipative inequality

We want to show that the partial dissipative inequality is satisfied for Q_1.

According to [6, 7], assume that P in Eq (2.13) is dissipative semidefinite relative to η, i.e.,

$$D\eta(U)P(U) \geq \alpha|P(U)|^2, \quad U \in O \tag{5.1}$$

with $\alpha > 0$.

For Eq (5.1) to be satisfied by $\hat{Q}_1$, we need to show that

$$\begin{bmatrix} \frac{\partial\hat{\eta}}{\partial\rho} & \frac{\partial\hat{\eta}}{\partial u} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{u + b - \hat{v}_e(\rho)}{\tau} \end{bmatrix} \geq \alpha(\frac{u + b - \hat{v}_e(\rho)}{\tau})^2, \tag{5.2}$$

where $\hat{\eta}$ is defined in Eq (4.37). From Eq (4.38), we calculate

$$\frac{\partial\hat{\eta}}{\partial u} = 2(n_1^2 + n_2^2)(u + b) + 2(n_1^2 - n_2^2)\frac{\hat{v}_e(\rho)}{\theta}. \tag{5.3}$$

Since $0 < \theta < 1$ from Eq (1.5), we can choose $n_2^2 > n_1^2$ such that

$$n_1^2 + n_2^2 = \frac{1}{\theta}(n_2^2 - n_1^2) > 0. \tag{5.4}$$

Define

$$\Gamma = n_1^2 + n_2^2. \tag{5.5}$$

Then, Eq (5.3) becomes

$$\frac{\partial \hat{\eta}}{\partial u} = 2\Gamma(u + b - \hat{v}_e(\rho)). \tag{5.6}$$

Plugging Eq (5.6) into Eq (5.2), we get

$$2\Gamma\frac{(u + b - \hat{v}_e(\rho))^2}{\tau} \geq \alpha(\frac{u + b - \hat{v}_e(\rho)}{\tau})^2.$$

Hence in order to satisfy Eq (5.2), we require

$$0 < \alpha \leq 2\Gamma\tau, \tag{5.7}$$

where Γ is as in Eq (5.5). Therefore partial dissipative condition Eq (5.1) is satisfied by $\hat{Q}_1$.

Now we prove Eq (5.1) for $\eta = Q_1(\rho, u)$ defined in Eq (4.40). We need to show

$$\begin{bmatrix} \frac{\partial \eta}{\partial \rho} & \frac{\partial \eta}{\partial u} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{u + b - v_e(\rho)}{\tau} \end{bmatrix} \geq \alpha(\frac{u + b - v_e(\rho)}{\tau})^2. \tag{5.8}$$

For $u + b - v_e(\rho) = 0$, Eq (5.8) is satisfied. Now assume $u + b - v_e(\rho) \neq 0$.

Under condition Eq (4.18), we derive from Eqs (4.40) and (5.5)

$$\frac{\partial \eta}{\partial u} - 2\Gamma(u + b - v_e(\rho)) \geq -C\gamma, \tag{5.9}$$

where $\gamma > 0$ is small and C is a universal constant. To satisfy Eq (5.8), by utilizing Eq (5.9), we require

$$2\Gamma\frac{(u + b - v_e(\rho))^2}{\tau} - C\gamma(\frac{u + b - v_e(\rho)}{\tau}) \geq \alpha(\frac{u + b - v_e(\rho)}{\tau})^2. \tag{5.10}$$

By choosing $\gamma > 0$ small enough in Eq (4.18), we have

$$\Gamma\frac{(u + b - v_e(\rho))^2}{\tau} - C\gamma\frac{u + b - v_e(\rho)}{\tau} > 0. \tag{5.11}$$

By using Eq (5.11), we require

$$\frac{\Gamma(u + b - v_e(\rho))^2}{\tau} \geq \alpha(\frac{u + b - v_e(\rho)}{\tau})^2 \tag{5.12}$$

for Eq (5.10) to be satisfied. We require

$$0 < \alpha \leq \Gamma\tau, \tag{5.13}$$

so Eq (5.12) is satisfied. Hence, partial dissipative condition (5.1) for $Q_1(\rho, u)$ is satisfied under conditions (4.18) and (5.13).

Therefore, if $\gamma > 0$ small enough in Eq (4.18), then the estimates for perturbation analysis, convexity, and partial dissipative inequality in Subsections 4.3, 4.4, and 5.1 are established respectively.

5.2. *The Kawashima condition*

Now we verify that the Kawashima condition is satisfied for Eq (2.10).

The Kawashima condition guarantees that the system resulting from linearizing Eq (2.10) does not admit solutions representing undamped traveling waves [6]. From [6, 7], the Kawashima condition is given by

$$DP(0)r_i(0) \neq 0, \quad i = 1, ..., n. \tag{5.14}$$

For Eq (5.14) to be satisfied, from Eqs (2.2), (2.13) and (2.16), we need

$$DP(0)r_i(0) = \begin{bmatrix} 0 & 0 \\ \frac{a}{\tau} & \frac{1}{\tau} \end{bmatrix} \begin{bmatrix} \pm\frac{\theta}{v_e'(0)} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \mp\frac{\theta}{\tau} + \frac{1}{\tau} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5.15}$$

since $0 < \theta < 1$ from Eq (1.5). Hence, the Kawashima condition (5.14) is satisfied.

5.3. *The subcharacteristic condition*

To get linear stability, we check if sub-characteristic inequality [7] is satisfied for Eq (2.10) i.e.,

$$\lambda_1 \leq \lambda_* \leq \lambda_2, \tag{5.16}$$

where λ_* is given in Eq (2.5). Recall that $u = v_e(\rho) - b$. Then for $v = v_e(\rho)$, from Eqs (2.5) and (2.15) the subcharacteristic condition is satisfied since

$$v_e(\rho) + \frac{\rho v_e'(\rho)}{\theta} < v_e(\rho) + \rho v_e'(\rho) < v_e(\rho) - \frac{\rho v_e'(\rho)}{\theta}. \tag{5.17}$$

Eq (5.17) is true since $0 < \theta < 1$ and $v_e'(\rho) < 0$ due to Eqs (1.5) and (2.4) respectively.

6. The equivalent form

In this section, we convert Eq (2.10) to the general form according to [7] (Chpt. 16, Eq. (16.6.10)).

$$\begin{aligned} \partial_t V + \partial_x G(V, W) &= 0, \\ \partial_t W + \partial_x H(V, W) + C(V, W)W &= 0, \end{aligned} \tag{6.1}$$

where $x \in \mathbb{R}, t > 0$ and

$$\eta_{WW}(0, 0)C(0, 0) > 0. \tag{6.2}$$

From Eq (2.13), we calculate

$$DP(0) = \frac{1}{\tau}\begin{bmatrix} 0 & 0 \\ a & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -a & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{\tau} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} = S^{-1}\Gamma S.$$

We are going to do the following change of variables.

$$\hat{U} = SU = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} \rho \\ u \end{bmatrix} = \begin{bmatrix} \rho \\ a\rho + u \end{bmatrix}.$$

Let V and W be

$$V = \rho, \tag{6.3}$$

$$W = a\rho + u. \tag{6.4}$$

Under Eqs (6.3) and (6.4), system (2.10) is transformed to

$$V_t + [V(W - aV + b)]_x = 0, \tag{6.5}$$

$$W_t + [\frac{1}{2}(W^2 - a^2V^2) + bW + g(V)]_x + C(V, W)W = 0. \tag{6.6}$$

The systems (6.5) and (6.6) is in the form of Eq (6.1) if we let

$$G(V, W) = V(W - aV + b) = VW - aV^2 + bV, \tag{6.7}$$

$$H(V, W) = \frac{1}{2}(W^2 - a^2V^2) + bW + g(V), \tag{6.8}$$

$$C(V, W) = \frac{1}{W}(\frac{W - aV + b - v_e(V)}{\tau}). \tag{6.9}$$

Since $v_e(0) = b$ and $\tau > 0$, $C(0, W) = \frac{1}{W}(\frac{W + b - v_e(0)}{\tau}) = \frac{1}{\tau} > 0$. Consequently, $C(0, 0) = \frac{1}{\tau} > 0$. Since we have a convex entropy η (4.40), we have that $\eta_{WW}C(0, 0) > 0$.

Let $Z = (V, W) = (\rho, a\rho + u)$. From Eq (2.11), the corresponding initial conditions for (6.5), (6.6) are

$$Z_0 = (V_0, W_0) = (\rho_0, a\rho_0 + u_0). \tag{6.10}$$

7. An *a priori* estimate

In this section, we show that $\rho \geq \frac{1}{2}\delta_1 > 0$ for all t under condition (1.3).

We first derive estimates for V and W. From Eq (6.6)

$$\frac{dW}{dt} + \frac{1}{\tau}W = 0, \quad t > 0, \tag{7.1}$$

where $\frac{d}{dt}$ is the derivative along trajectory of Eq (6.6). Therefore,

$$|W| = |W_0|e^{-\frac{1}{\tau}t}, \quad t > 0, \tag{7.2}$$

along trajectory of Eq (6.6).

If $W = 0$, (6.5) is a scalar conservation law. Thus, by Theorem 16.1 in [29], we have

$$V_{0,\inf} \leq V \leq V_{0,\sup}, \quad t > 0, \tag{7.3}$$

where $V_{0,\inf}$ and $V_{0,\sup}$ are infimum and supremum of V_0 respectively. Since Z_0 is of bounded variation, $V_{0,\inf}$, $V_{0,\sup}$, and $\|V\|_{L_\infty}$ are bounded.

In the general case, W is present in Eq (6.5) and $|W|$ decays exponentially with respect to t, see Eq (7.2). By modifying the proof of Lemma 16.2 in [29] and using Eq (7.2), we obtain

$$V_{0,\inf} - K\|W_0\|_{L_\infty} \le V \le V_{0,\sup} + K\|W_0\|_{L_\infty}, \quad t > 0 \tag{7.4}$$

for some $K > 0$.

By Eqs (1.3) and (6.3), we have $V_0 = \rho_0 \ge \delta_1 > 0$. Hence,

$$V_{0,\inf} \ge \delta_1 > 0. \tag{7.5}$$

Choose $\|W_0\|_{L_\infty}$ small such as

$$\|W_0\|_{L_\infty} \le \frac{1}{2K}\delta_1. \tag{7.6}$$

Then we have from Eqs (6.3), (7.4), and (7.6) that

$$\rho = V \ge V_{0,\inf} - K\|W_0\|_{L_\infty} \ge \delta_1 - K\|W_0\|_{L_\infty} \ge \frac{1}{2}\delta_1 > 0, \quad t > 0. \tag{7.7}$$

8. Main result

We verified all assumptions in Dafermos theory [6, 7] for Eq (2.10). Now we present the main result of this paper. We show the global existence and asymptotic behavior of BV solutions to the Cauchy problem (6.5), (6.6) and (6.10) for a nonconcave fundamental diagram in traffic flow.

Theorem 8.1 (Admissible BV Solution to the Cauchy Problem)**.** *Under the conditions* (1.3)*,* (1.5)*,* (4.6)*,* (4.18)*,* (5.13) *and* (7.6)*, the system of balance laws* (6.5) *and* (6.6) *is endowed with a convex entropy-entropy flux pair* (η, q) (4.40) *and* (4.41)*, satisfies the partial dissipative inequality* (5.1)*, and satisfies the Kawashima condition* (5.15)*. Consider the Cauchy problem* (6.5)*,* (6.6) *and* (6.10)*. For* $\delta_0, \sigma_0 > 0$*, suppose that* Z_0 *decays, as* $|x| \to \infty$*, sufficiently fast to render the integral*

$$\int_{-\infty}^{\infty} (x^2 + 1)|Z_0(x)|^2\, dx = \sigma^2 < \sigma_0^2, \tag{8.1}$$

with bounded variation

$$TV_{(-\infty,\infty)} Z_0(\cdot) = \delta < \delta_0, \tag{8.2}$$

and

$$\int_{-\infty}^{\infty} V_0(x)\, dx = 0, \tag{8.3}$$

then there exist positive constants $c_0, c_1, c_2, \nu > 0$ *so that the Cauchy problem* (6.5)*,* (6.6) *and* (6.10) *possesses a unique admissible BV solution Z defined on* $(-\infty, \infty) \times [0, \infty)$ *and*

$$\int_{-\infty}^{\infty} |Z(x,t)|\, dx \le c_0\sigma, \quad 0 \le t < \infty, \tag{8.4}$$

$$TV_{(-\infty,\infty)} Z(\cdot, t) \le c_1\sigma + c_2\delta e^{-\nu t}, \quad 0 \le t < \infty, \tag{8.5}$$

$$\int_{-\infty}^{\infty} |Z(x,t)|\, dx \to 0, \quad \text{as } t \to \infty, \tag{8.6}$$

$$TV_{(-\infty,\infty)} Z(\cdot, t) \to 0, \quad \text{as } t \to \infty. \tag{8.7}$$

The above results are in $Z = (V, W)$ defined in Eqs (6.5) and (6.6). Using Eqs (2.9), (6.3) and (6.4), we can transform our theorems in the original unknowns (ρ, v) satisfying system (1.1) and (1.2).

9. Conclusion

In the framework of Dafermos, we proved the existence and asymptotic behavior of global BV solutions to the Cauchy problem for a traffic flow model with nonconcave fundamental diagram. The nonconcave fundmental diagram in Eq (1.7) is close to a concave fundamental diagram in Eq (4.9). We derived the partial dissipative inequality, the sub-characteristic condition, the Kawashima condition, and a convex entropy-entropy flux pair to prove our Theorem 8.1.

We adopted the model (1.1) and (1.5) with larger anticipation factors than the ARZ model. Anticipation factor describes the effect of drivers reacting to conditions downstream. Due to higher pressure from the traffic, the driver's anticipation increases, which causes traffic flow to be more regular. Larger anticipation factors lead to safer and smoother traffic conditions on highways.

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Conflict of interest

The authors declare there is no conflict of interest.

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