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Research article

Error estimate of BDF2 scheme on a Bakhvalov-type mesh for a singularly perturbed Volterra integro-differential equation

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Abstract: A singularly perturbed Volterra integro-differential problem is considered. The variable two-step backward differentiation formula is used to approximate the first-order derivative term and the trapezoidal formula is used to discretize the integral term. Then, the stability and convergence analysis of the proposed numerical method are proved. It is shown that the proposed scheme is second-order uniformly convergent with respect to perturbation parameter ε in the discrete maximum norm. Finally, a numerical experiment verifies the theoretical results.

Keywords: Volterra integro-differential equation; singularly perturbed; two-step backward differentiation formula; Bakhvalov mesh

1. Introduction

Volterra integro-differential equations (VIDEs) with a first-order derivative term arise from population growth models, biology, medicine, physics, and so on [1–3]. When the first-order derivative of such problems is multiplied by a perturbation parameter ε, these problems are called singularly perturbed Volterra integro-differential equations (SPVIDEs), which are given by

$$\begin{cases} \mathcal{L}u := \varepsilon u'(x) + a(x)u(x) + \int_0^x K(x,t)u(t)dt = f(x), \ x \in \Omega := (0,1], \\ u(0) = w, \end{cases} \tag{1.1}$$

where $0 < \varepsilon \ll 1$ is a perturbation parameter and functions $a(x)$, $f(x)$, $K(x,t)$ are sufficiently smooth. In additional, we assume that there exists a positive constant β such that $a(x) \geq \beta > 0$. Under these conditions, Eq (1.1) has a unique solution [4]. Furthermore, the derivatives of $u(x)$ have following bounds (see [4, Lemma 1.1])

$$|u^{(k)}(x)| \leq C(1 + \varepsilon^{-k} e^{-\beta x/\varepsilon}), \quad x \in \Omega, \quad k = 0, 1, 2, 3, \tag{1.2}$$

where C is a positive constant, which is independent of ε. Obviously, it can be seen from Eq (1.2) that the exact solution $u(x)$ of Eq (1.1) exists a exponential boundary layer at $x = 0$ as $\varepsilon \to 0$. It is well known that SPVIDEs are widely used to describe a nonlocal reaction-diffusion model in spatially inhomogeneous media that takes into account feedback and nonlocal interactions [5].

Due to the presence of perturbation parameter ε, some traditional finite difference and finite element methods on a uniform mesh could not obtain reliable numerical results. Therefore, the layer-adaptive grid (Bakhvalov and Shishkin meshes) and adaptive grid methods are widely used to construct some parameter-uniform numerical methods for solving SPVIDEs, see [6–12] and references therein. Among these existing methods, the accuracy of most of these methods is only first-order. For this reason, many researchers studied some second-order accurate numerical methods for Eq (1.1). For example, the authors in [13] used the Richardson extrapolation technique to improve the ε-uniform convergence of the adaptive grid method from first-order to second-order. Yapman [14] designed a homogeneous (nonhybrid) type difference scheme on Shishkin-type mesh for Eq (1.1), and confirmed that the numerical method was almost second-order convergent.

It is well known that the variable two-step backward differential formula (BDF2) technique has widely used to approximate the time derivative with second-order accurate for linear parabolic equations and differential-algebraic problems [15–17]. In particular, in [17], the authors introduced discrete orthogonal convolution kernels for the first time and utilized them to prove the stability and convergent of the adaptive BDF2 time-stepping scheme in L^2 norm. Recently, the authors [18] applied the BDF2 technique on a Shishkin mesh to solve the problem in Eq (1.1) and proved that the proposed method was almost second-order uniformly convergent. In addition, the result in [19] suggests that the BDF2 all-at-once system utilizing the preconditioned Krylov subspace solvers can be a competitive solution method for Riesz fractional diffusion equations.

Inspired by the BDF2 that we recently used in [18], where we designed a novel finite difference scheme to prove ε-uniform convergence for a Shishkin mesh. In this paper, we prove ε-uniform convergence of this finite difference scheme on a Bakhvalov-type (B-type) mesh. The advantage of this paper is that the convergence rate of our numerical method do not contain $\ln N$-factors, where N is the number of mesh steps.

The paper is organized as follows: The construction and characteristics of Bakhvalov-type mesh is introduced in Section 2. This is followed by the stability analysis of the discretization scheme in Section 3. Then the bound of truncation error and the parameter-uniform convergence of the numerical method are proved in Section 4. In Section 5, the statement of the numerical experiment illustrates that our theoretical findings are effective. Finally, some concluding discussions are presented in Section 6.

To simplify the notation we set $g_i = g(x_i)$ for any function g and set $K_{i,k} = K(x_i, x_k)$. The maximum norm denoted by $\|K\| := \max_{(x,t)\in\Omega^2} |K(x,t)|$ and $\|g\| := \max_{x\in\Omega} |g(x)|$. In addition, for any sequences $\{\gamma_k\}$ if the index $j > i$, we assume the summation $\sum_{k=j}^{i} \gamma_k = 0$ and $\prod_{k=j}^{i} \gamma_k = 1$.

2. The Bakhvalov-type mesh

Let $\bar{\Omega}^N := \{0 = x_0 < x_1 < \ldots < x_N = 1\}$ be a B-type mesh with the local mesh step size $h_i = x_i - x_{i-1}$, $i = 1, 2, \ldots, N$. Then, for $t_i = i/N$, the B-type mesh points x_i can be obtained by the following generating function [20, 21]

$$x_i = \begin{cases} \mu\varepsilon\varphi(t_i), & \text{for } i = 0, 1, \ldots, J, \\ 1 - 2(1 - x_J)(1 - i/N) & \text{for } i = J + 1, \ldots, N, \end{cases} \tag{2.1}$$

where $\varphi(t) = -\ln[1 - 2(1 - \varepsilon)t]$, $J = N/2$ and μ is a positive mesh-parameter that satisfy $\mu\beta \geq 2$. For the sake of simplicity, $r_i = h_i/h_{i-1}(i = 2, \ldots, N)$ represents the i-th step-size ratios.

Next, the next two lemmas list some characteristics of the above B-type mesh.

Lemma 2.1. *The step sizes of the B-type mesh defined by Eq (2.1) satisfy*

$$h_i \leq h_{i+1} \leq CN^{-1},\ i = 1, \ldots, N - 1, \tag{2.2}$$

$$h_1 \leq C\varepsilon N^{-1}. \tag{2.3}$$

$$h_i \leq \mu\varepsilon,\ i = 1, \ldots, J - 1, \tag{2.4}$$

$$1 \leq r_i \leq 3,\ i = 2, \ldots, J - 1, \tag{2.5}$$

Proof. The proof of Eqs (2.2)–(2.4) can be found in [22, Lemma 1]. Here, we just need to prove Eq (2.5). In fact, by using Eqs (2.1) and (2.2), one has

$$\begin{aligned} 1 \leq r_i &= \frac{\varphi(t_i) - \varphi(t_{i-1})}{\varphi(t_{i-1}) - \varphi(t_{i-2})} = \frac{\varphi'(\xi_i)}{\varphi'(\xi_{i-1})} \leq \frac{\varphi'(t_i)}{\varphi'(t_{i-2})} \\ &= \frac{1 - 2(1 - \varepsilon)(i - 2)/N}{1 - 2(1 - \varepsilon)i/N} = 1 + \frac{4(1 - \varepsilon)/N}{1 - 2(1 - \varepsilon)i/N} \\ &\leq 1 + \frac{4(1 - \varepsilon)/N}{1 - 2(1 - \varepsilon)(J - 1)/N} \\ &= 1 + \frac{4(1 - \varepsilon)/N}{\varepsilon + 2(1 - \varepsilon)/N} \leq 3, \end{aligned}$$

where $\xi_i \in [t_{i-1}, t_i]$, $i = 2, \cdots, J - 1$, which completes the proof. □

Lemma 2.2. *Let $\{x_i\}_{i=0}^N$ be the B-type mesh generated by Eq (2.1). Then we have*

$$\int_{x_{i-1}}^{x_i} \left(1 + \varepsilon^{-1} e^{-\frac{\beta x}{2\varepsilon}}\right) dx \leq CN^{-1},\ i = 1, \cdots, N.$$

Proof. The proof can be found in [23, 24]. □

3. Discretization scheme and stability analysis

Before constructing a second-order discretization scheme for Eq (1.1), we first give the definition of two-step backward differential formula as follows: For a given function g, let $P_{i,k}g$ be the corresponding Lagrange interpolating polynomial over points $x_i, x_{i-1}, \ldots, x_{i-k}$. Then the first-order derivative of this polynomial at point $x = x_i$ can be obtained by

$$D_2 g_i := (P_{i,2}g)'(x_i) = \sum_{k=1}^{i} b_{i-k}^i (g_k - g_{k-1}) \quad \text{for } i \geq 1, \tag{3.1}$$

where $b_0^1 := 1/h_1$, $b_0^i := \frac{1+2r_i}{h_i(1+r_i)}$, $b_1^i := -\frac{r_i^2}{h_i(1+r_i)}$ and $b_j^i := 0$ for $2 \leq j \leq i-1, \quad i \geq 2$. Therefore, by using Eq (3.1) and the trapezoidal formula to approximate the first-order derivative and the integral term, respectively, we obtain the following discretization scheme

$$\begin{cases} \mathcal{L}^N u_i^N := \varepsilon D_2 u_i^N + a_i u_i^N + \sum_{k=1}^{i} \frac{h_k}{2} \left[K_{i,k-1} u_{k-1}^N + K_{i,k} u_k^N \right] = f_i, \\ u_0^N = \mu, \end{cases} \tag{3.2}$$

where u_i^N is the approximation solution of $u(x)$ at point $x = x_i$.

To facilitate the analysis of the stability and convergence of the discretization scheme in Eq (3.2), we first define a discrete orthogonal convolution (DOC) kernels $\left\{\theta_{i-j}^i\right\}_{j=1}^i$, which is given by (see [17, Lemma 2.3])

$$\theta_{i-j}^i = \frac{h_i}{b_0^j h_j} \prod_{k=j+1}^{i} \frac{r_k}{1+2r_k}. \tag{3.3}$$

Then by using the following discrete orthogonal identity

$$\sum_{j=k}^{i} \theta_{i-j}^i b_{j-k}^j = \delta_{ik} \text{ for } \forall \ 1 \leq k \leq i, \tag{3.4}$$

where δ_{ik} is the Kronecker delta symbol, we obtain

$$\sum_{j=1}^{i} \theta_{i-j}^i D_2 u_j = \sum_{k=1}^{i} (u_k - u_{k-1}) \sum_{j=k}^{i} \theta_{i-j}^i b_{j-k}^j = u_i - u_{i-1}, \ \ 1 \leq i \leq N. \tag{3.5}$$

Furthermore, the next lemma list a characteristics of DOC kernels:

Lemma 3.1. *[17, Corollary 2.1] The DOC kernels θ_{i-j}^i has following property*

$$\theta_{i-j}^i > 0, \ \ 1 \leq j \leq i \ \textit{and} \ \sum_{j=1}^{i} \theta_{i-j}^i = h_i, \ \ i \geq 1. \tag{3.6}$$

Now, based on the above definition of DOC kernels in Eq (3.3) and the proof of Lemma 2 in [18], we list the stability of the above discretization scheme in Eq (3.2).

Lemma 3.2. *Under the condition that there exists a constant α^* such that*

$$\beta + \frac{h_i K_{i,i}}{2} \geq \alpha^* > 0, \tag{3.7}$$

the solution u_i^N of Eq (3.2) satisfies

$$\max_{0 \leq i \leq N} \left| u_i^N \right| \leq C \left(\max_{0 \leq i \leq N} |f_i| + \left| u_0^N \right| \right).$$

Proof. For $i = 1, \cdots, N$, multiplying both sides of Eq (3.2) by the DOC kernels θ^i_{i-j}, and summing j from 1 to i, we get

$$\varepsilon \sum_{j=1}^{i} \theta^i_{i-j} D_2 u^N_j + \sum_{j=1}^{i} \theta^i_{i-j} a_j u^N_j + \sum_{j=1}^{i} \theta^i_{i-j} \sum_{k=1}^{j} \frac{h_k}{2} \left[K_{i,k-1} u^N_{k-1} + K_{i,k} u^N_k\right] = \sum_{j=1}^{i} \theta^i_{i-j} f_j.$$

Then, combining with Eq (3.5), yields,

$$\begin{aligned} \left|\left(\varepsilon + \theta^i_0 a_i + \theta^i_0 \frac{K_{i,i} h_i}{2}\right) u^N_i\right| = \Bigg| &\sum_{j=1}^{i} \theta^i_{i-j} f_j + \varepsilon u^N_{i-1} - \sum_{j=1}^{i-1} \theta^i_{i-j} a_j u^N_j - \theta^i_0 \sum_{k=1}^{i-1} \frac{h_k}{2} K_{i,k} u^N_k \\ &- \sum_{j=1}^{i} \theta^i_{i-j} \sum_{k=1}^{j} \frac{h_k}{2} K_{i,k-1} u^N_{k-1} - \sum_{j=1}^{i-1} \theta^i_{i-j} \sum_{k=1}^{j} \frac{h_k}{2} K_{j,k} u^N_k \Bigg|. \end{aligned} \tag{3.8}$$

Based on Eq (3.7), we have

$$C\left|(\varepsilon + h_i) u^N_i\right| \le \left|\left(\varepsilon + \theta^i_0 \alpha^*\right) u^N_i\right| \le \left|\left(\varepsilon + \theta^i_0 a_i + \theta^i_0 \frac{K_{i,i} h_i}{2}\right) u^N_i\right|. \tag{3.9}$$

where we have applied the fact that

$$\varepsilon + \theta^i_0 \alpha^* = \varepsilon + \frac{h_i(1 + r_i)}{1 + 2r_i} \alpha^* \ge \varepsilon + h_i \alpha^*/2 \ge \min\{1, \alpha^*/2\}(\varepsilon + h_i) = C(\varepsilon + h_i).$$

Then, from Eqs (3.8),(3.9), it is easy to obtain

$$\begin{aligned} \left|(\varepsilon + h_i) u^N_i\right| \le C \Bigg(&\sum_{j=1}^{i} \theta^i_{i-j} \left|f_j\right| + \varepsilon \left|u^N_{i-1}\right| + \|a\| \sum_{j=1}^{i-1} \theta^i_{i-j} \left|u^N_j\right| + \|K\| \theta^i_0 \sum_{k=1}^{i-1} h_k \left|u^N_k\right| \\ &+ \|K\| \sum_{j=1}^{i} \theta^i_{i-j} \sum_{k=1}^{j} h_k \left|u^N_{k-1}\right| + \|K\| \sum_{j=1}^{i-1} \theta^i_{i-j} \sum_{k=1}^{j} h_k \left|u^N_k\right| \Bigg). \end{aligned} \tag{3.10}$$

Furthermore, we have

$$\left|u^N_i\right| \le C\left(\frac{1}{\varepsilon + h_i} \sum_{j=1}^{i} \theta^i_{i-j} \left|f_j\right| + \left|u^N_{i-1}\right| + \frac{1}{\varepsilon + h_i} \sum_{j=1}^{i-1} \theta^i_{i-j} \left|u^N_j\right| + \sum_{j=1}^{3} I_j\right), \tag{3.11}$$

where

$$\begin{aligned} I_1 &= \frac{1}{(\varepsilon + h_i)} \sum_{j=1}^{i} \theta^i_{i-j} \sum_{k=1}^{j} h_k \left|u^N_{k-1}\right|, \\ I_2 &= \frac{1}{(\varepsilon + h_i)} \sum_{j=1}^{i-1} \theta^i_{i-j} \sum_{k=1}^{j} h_k \left|u^N_k\right|, \\ I_3 &= \frac{1}{(\varepsilon + h_i)} \theta^i_0 \sum_{k=1}^{i-1} h_k \left|u^N_k\right|. \end{aligned}$$

Obviously, to derive the bound for $\left|u_i^N\right|$, we only need to estimate the bounds of $I_j,\ j = 1,2,3,$ respectively. For I_1, it follows from Lemma 3.1 that

$$\begin{aligned} I_1 &\le \frac{1}{\varepsilon + h_i}\sum_{k=1}^{i} h_k \left|u_{k-1}^N\right| \sum_{j=k}^{i} \theta_{i-j}^i \\ &\le \frac{h_i}{\varepsilon + h_i}\sum_{k=1}^{i} h_k \left|u_{k-1}^N\right| \le \sum_{k=1}^{i} h_k \left|u_{k-1}^N\right|, \end{aligned} \tag{3.12}$$

where we have interchanged the order of summation. Similar to Eq (3.12), we get

$$I_2 \le \sum_{k=1}^{i-1} h_k \left|u_k^N\right| \text{ and } I_3 \le \sum_{k=1}^{i-1} h_k \left|u_k^N\right|. \tag{3.13}$$

From Eqs (3.11)–(3.13), we obtain

$$\left|u_i^N\right| \le C\left(\frac{1}{\varepsilon + h_i}\sum_{j=1}^{i} \theta_{i-j}^i \left|f_j\right| + \sum_{k=0}^{i-1} \lambda_k \left|u_k^N\right|\right), \tag{3.14}$$

where the sequences $\{\lambda_k\}$ are defined by

$$\lambda_k = \begin{cases} h_1, & k = 0, \\ h_k + h_{k+1} + \dfrac{\theta_{i-k}^i}{\varepsilon + h_i}, & 1 \le k \le i-2, \\ 1 + h_k + h_{k+1} + \dfrac{\theta_{i-k}^i}{\varepsilon + h_i}, & k = i-1. \end{cases}$$

Finally, by using the Grönwall inequality, see [25, Lemma 3.2], one has

$$\begin{aligned} \left|u_i^N\right| &\le \frac{C}{\varepsilon + h_i}\sum_{j=1}^{i} \theta_{i-j}^i \left|f_j\right| \exp\left(\sum_{k=0}^{i-1} \lambda_k\right) \\ &\le \frac{C \max\limits_{1\le j\le i}\left|f_j\right|}{\varepsilon + h_i}\sum_{j=1}^{i} \theta_{i-j}^i \exp\left(\sum_{k=0}^{i-1} h_{k+1} + \sum_{k=1}^{i-1} h_k + 1 + \frac{\sum\limits_{j=1}^{i-1} \theta_{i-j}^i}{\varepsilon + h_i}\right) \\ &\le C \max_{1\le j\le i}\left|f_j\right| \frac{h_i}{\varepsilon + h_i} \exp\left(x_i + x_{i-1} + 1 + \frac{h_i}{\varepsilon + h_i}\right) \\ &\le C \max_{1\le j\le i}\left|f_j\right|,\ i \ge 1, \end{aligned} \tag{3.15}$$

which completes the proof. □

In order to derive convergence analysis below, we also give the bound of $\left|u_i^N\right|$ for $i = 1, \cdots, J-1$ as follows:

Lemma 3.3. *For $i = 1, \cdots, J-1$, we have*

$$\left|u_i^N\right| \le C\left(\left|u_0^N\right| + |f_i|\right).$$

Proof. Firstly, for $i = 1$, it follows from Eq (3.2) that

$$\left|\left(\frac{\varepsilon}{h_1} + a_1 + \frac{h_1 K_{1,1}}{2}\right) u_1^N\right| = \left|f_1 + \frac{\varepsilon}{h_1} u_0^N - \frac{h_1 K_{1,0}}{2} u_0^N\right|.$$

Then, by using Eq (2.3), yields,

$$\begin{aligned}
\left|u_1^N\right| &\le \left(|f_1| + \left|\frac{\varepsilon}{h_1}\right|\left|u_0^N\right| + \left|\frac{h_1 K_{1,0}}{2}\right|\left|u_0^N\right|\right) / \left|\frac{\varepsilon}{h_1} + a_1 + \frac{h_1 K_{1,1}}{2}\right| \\
&\le C\left[|f_1| + N\left|u_0^N\right|\right] / \left|CN + \alpha^*\right| \\
&\le C\left(|f_1| N^{-1} + \left|u_0^N\right|\right).
\end{aligned} \tag{3.16}$$

In addition, set $\eta_i = \frac{\varepsilon}{h_i}\frac{1+2r_i}{1+r_i} + a_i + \frac{h_i K_{i,i}}{2}$, by using the Eq (2.5), we have $\eta_i \ge C\left(\varepsilon/h_i + \alpha^*\right)$. Furthermore, similarly to Eq (3.11), one has

$$\begin{aligned}
\left|u_i^N\right| &= \left|f_i + \frac{\varepsilon}{h_i}(1 + r_i) u_{i-1}^N - \frac{\varepsilon}{h_i}\frac{r_i^2}{1 + r_i} u_{i-2}^N - \sum_{k=1}^{i}\frac{h_k K_{i,k-1}}{2} u_{k-1}^N - \sum_{k=1}^{i-1}\frac{h_k K_{i,k}}{2} u_k^N\right| / |\eta_i| \\
&\le C\left(|f_i| + \frac{\varepsilon}{h_i}\left|u_{i-1}^N\right| + \frac{\varepsilon}{h_i}\left|u_{i-2}^N\right| + \sum_{k=1}^{i} h_k\left|u_{k-1}^N\right| + \sum_{k=1}^{i-1} h_k\left|u_k^N\right|\right) / |\eta_i|, \\
&\le C\left(|f_i| + \left|u_{i-1}^N\right| + \left|u_{i-2}^N\right| + \sum_{k=1}^{i} h_k\left|u_{k-1}^N\right| + \sum_{k=1}^{i-1} h_k\left|u_k^N\right|\right) \\
&= C\left(|f_i| + \sum_{k=0}^{i-1}\lambda_k\left|u_k^N\right|\right),
\end{aligned}$$

where the sequences $\{\lambda_k\}$ are defined by

$$\lambda_k = \begin{cases} h_{k+1}, & k = 0, \\ h_k + h_{k+1}, & 1 \le k \le i-3, \\ 1 + h_k + h_{k+1}, & k = i-2, i-1. \end{cases}$$

Using the Grönwall inequality, we get

$$\left|u_i^N\right| \le C|f_i| \exp\left(\sum_{k=0}^{i-1}\lambda_k\right) \le C|f_i|, \ i \ge 2. \tag{3.17}$$

The result of this lemma can be implied by Eqs (3.16),(3.17). □

4. Convergence analysis

Let $e_i^N := u_i - u_i^N$, $i = 1, 2, 3, \ldots, N$, denote the error at point $x = x_i$ in the numerical solution. Then we have

$$\mathcal{L}^N e_i^N = R_{1,i} + R_{2,i}, \tag{4.1}$$

where

$$R_{1,i} = \varepsilon D_2 u_i - \varepsilon u_i', \tag{4.2}$$

$$R_{2,i} = \sum_{k=1}^{i} \frac{h_k}{2}\left[K_{i,k-1}u_{k-1}^N + K_{i,k}u_k^N\right] - \sum_{k=1}^{i}\int_{x_{k-1}}^{x_k} K(x_i,t)u(s)ds. \tag{4.3}$$

Lemma 4.1. *Under the B-type mesh defined in Eq (2.1), we obtain the following estimations*

$$\left|R_{1,i}\right| \le CN^{-1},\ i = 1, \tag{4.4}$$

$$\left|R_{1,i}\right| \le CN^{-2},\ i \ge 2, \tag{4.5}$$

and

$$\left|R_{2,i}\right| \le CN^{-2},\ i \ge 1. \tag{4.6}$$

Proof. We first consider the truncation error $\left|R_{1,i}\right|$ when $i = 1$, by using Taylor's expansion formula, Eq (1.2), Lemma 2.1 and Lemma 2.2, we have

$$\left|R_{1,1}\right| \le \frac{\varepsilon}{h_1}\int_0^{x_1} t\,|u''(t)|\,dt \le C\int_0^{x_1}\left(1 + \varepsilon^{-1}e^{-\beta t/(2\varepsilon)}\right)dt \le CN^{-1}. \tag{4.7}$$

For $2 \le i \le J-1$ and $i \ge J+2$, the step-size ratios $1 \le r_i \le 3$, then for the reason of Eq (4.7), we can obtain

$$\begin{aligned}
\left|R_{1,i}\right| &= \varepsilon\left|\frac{1+r_i}{2h_i}\int_{x_{i-1}}^{x_i}(t-x_{i-1})^2\,u'''(t)dt - \frac{r_i^2}{2h_i(1+r_i)}\int_{x_{i-2}}^{x_i}(t-x_{i-2})^2\,u'''(t)dt\right| \\
&\le C\left[\int_{x_{i-1}}^{x_i}(t-x_{i-1})\,\varepsilon\,|u'''(t)|\,dt + \int_{x_{i-2}}^{x_i}(t-x_{i-2})\,\varepsilon\,|u'''(t)|\,dt\right] \\
&\le C\left[\int_{x_{i-1}}^{x_i}(t-x_{i-1})\left(1+\varepsilon^{-2}e^{-\beta t/\varepsilon}\right)dt + \int_{x_{i-2}}^{x_i}(t-x_{i-2})\left(1+\varepsilon^{-2}e^{-\beta t/\varepsilon}\right)dt\right] \\
&\le C\left[\left(\int_{x_{i-1}}^{x_i}\left(1+\varepsilon^{-1}e^{-\beta t/(2\varepsilon)}\right)dt\right)^2 + \left(\int_{x_{i-2}}^{x_i}\left(1+\varepsilon^{-1}e^{-\beta t/(2\varepsilon)}\right)dt\right)^2\right] \\
&\le CN^{-2},
\end{aligned} \tag{4.8}$$

where we have used fact that

$$\int_c^d \phi(s)(s-c)ds \le \frac{1}{2}\left[\int_c^d \sqrt{\phi(s)}ds\right]^2$$

for any positive monotonically decreasing function ϕ on $[c, d]$, see [26].

Furthermore, for $i = J,\ J+1$, the truncation error $R_{1,i}$ can also be estimated in the following form:

$$\begin{aligned}\left|R_{1,i}\right| &= \varepsilon\left|\frac{1+2r_i}{2(1+r_i)h_i}\int_{x_{i-1}}^{x_i}(t-x_{i-1})^2\,u'''(t)dt - \frac{r_i^2}{2h_i(1+r_i)}\int_{x_{i-2}}^{x_{i-1}}(t-x_{i-2})^2\,u'''(t)dt\right. \\ &\quad\left. - \frac{r_i}{2(1+r_i)}\int_{x_{i-1}}^{x_i}(2(t-x_{i-1})+h_{i-1})\,u'''(t)dt\right| \\ &\leq C\left[\int_{x_{i-1}}^{x_i}(t-x_{i-1})\left(1+\varepsilon^{-2}e^{-\beta t/\varepsilon}\right)dt + \int_{x_{i-2}}^{x_{i-1}}(t-x_{i-2})\left(1+\varepsilon^{-2}e^{-\beta t/\varepsilon}\right)dt\right. \\ &\quad\left. + \int_{x_{i-1}}^{x_i}h_{i-1}\left(1+\varepsilon^{-2}e^{-\beta t/\varepsilon}\right)dt\right] \\ &\leq CN^{-2} + Ch_{i-1}\int_{x_{i-1}}^{x_i}\varepsilon^{-2}e^{-\beta t/\varepsilon}dt.\end{aligned} \tag{4.9}$$

Now, we need to estimate the second term on the right-hand side of Eq (4.9) to obtain a estimate of $\left|R_{1,i}\right|$ for $i = J,\ J+1$. Firstly, for $i = J+1$, one has

$$h_J\int_{x_J}^{x_{J+1}}\varepsilon^{-2}e^{-\beta t/\varepsilon}dt \leq h_J h_{J+1}\varepsilon^{-2}e^{-\beta x_J/\varepsilon} = CN^{-2}\varepsilon^{\mu\beta-2} \leq CN^{-2}. \tag{4.10}$$

Secondly, for $i = J$, if $\varepsilon \leq N^{-1}$, by using Eq (2.4) it is easy to get that

$$h_{J-1}\int_{x_{J-1}}^{x_J}\varepsilon^{-2}e^{-\beta t/\varepsilon}dt \leq \frac{\mu}{\beta}e^{-\beta x_{J-1}/\varepsilon} = \frac{\mu}{\beta}\left(\varepsilon+2(1-\varepsilon)N^{-1}\right)^{\mu\beta} \leq CN^{-2}. \tag{4.11}$$

On the other hand, if $\varepsilon > N^{-1}$, we can get that

$$\begin{aligned}h_{J-1}\int_{x_{J-1}}^{x_J}\varepsilon^{-2}e^{-\beta t/\varepsilon}dt &\leq h_{J-1}h_J\varepsilon^{-2}e^{-\beta x_{J-1}/\varepsilon} \leq CN^{-2}\varepsilon^{-2}\left(\varepsilon+2(1-\varepsilon)N^{-1}\right)^{\mu\beta} \\ &\leq CN^{-2}\varepsilon^{-2}\varepsilon^{\mu\beta} \leq CN^{-2}.\end{aligned} \tag{4.12}$$

From Eqs (4.8)–(4.12), we get $|R_{1,i}| \leq CN^{-2}$ for $2 \leq i \leq N$.

Different from BDF2, the discretization of the integral part in Eq (3.2) does not require two starting points, we can estimate $R_{2,i}$ for $1 \leq i \leq N$.

$$\begin{aligned}
\left|R_{2,i}\right| &= \left|\sum_{k=1}^{i} \int_{x_{k-1}}^{x_k} \left[\frac{x_k - s}{h_k} \int_{x_{k-1}}^{x_k} (t - x_{k-1}) \left[K(x_i, t)u(t)\right]'' \, dt \right.\right. \\
&\quad \left.\left. + \int_{s}^{x_k} (t - s) \left[K(x_i, t)u(t)\right]'' \, dt\right] ds\right| \\
&\leq C \sum_{k=1}^{i} \int_{x_{k-1}}^{x_k} \left[\int_{x_{k-1}}^{x_k} (t - x_{k-1}) \left(1 + |u(t)| + |u'(t)| + |u''(t)|\right) dt \right. \\
&\quad \left. + \int_{s}^{x_k} (t - s) \left(1 + |u(t)| + |u'(t)| + |u''(t)|\right) dt\right] ds \\
&\leq C \sum_{k=1}^{i} \int_{x_{k-1}}^{x_k} \int_{x_{k-1}}^{x_k} (t - x_{k-1}) \left(1 + \varepsilon^{-1} e^{-\beta t/\varepsilon} + \varepsilon^{-2} e^{-\beta t/\varepsilon}\right) dt ds \\
&\leq C \max_{1 \leq k \leq i} \int_{x_{k-1}}^{x_k} (t - x_{k-1}) \left(1 + \varepsilon^{-2} e^{-\beta t/\varepsilon}\right) dt \sum_{k=1}^{i} \int_{x_{k-1}}^{x_k} ds \\
&\leq CN^{-2},
\end{aligned} \tag{4.13}$$

which completes the proof. □

Finally, we can get the main result of this paper as follows:

Theorem 4.1. *Let $u(x)$ be the solution of Eq (1.1) and u_i^N be the solution of Eq (3.2) on the mesh $\bar{\Omega}^N$. Then ,under the condition $\beta + \frac{h_i K_{i,i}}{2} \geq \alpha^* > 0$, the error satisfy*

$$|e_i^N| \leq CN^{-2}, \ i \geq 1. \tag{4.14}$$

Proof. From Eq (3.16), we have $\left|e_1^N\right| \leq CN^{-1} \left|R_{1,1} + R_{2,1}\right| \leq CN^{-2}$. In particular, Lemma 3.3 imply that

$$\left|e_i^N\right| \leq C \left|R_{1,i} + R_{2,i}\right| \leq C \left|R_{1,i}\right| + C \left|R_{2,i}\right| \leq CN^{-2} \text{ for } 2 \leq i \leq J - 1. \tag{4.15}$$

Next, for $i \geq J$, it follows from Eq (3.3) and Eq (3.15) that

$$\begin{aligned}
\left|e_i^N\right| &\leq \frac{C}{\varepsilon + h_i} \sum_{j=1}^{i} \theta_{i-j}^i \left|R_{1,j} + R_{2,j}\right| \\
&\leq \frac{C}{\varepsilon + h_i} \left(\max_{1 \leq j \leq i} \left|R_{2,j}\right| \sum_{j=1}^{i} \theta_{i-j}^i + \max_{2 \leq j \leq i} \left|R_{1,j}\right| \sum_{j=2}^{i} \theta_{i-j}^i + \theta_{i-1}^i \left|R_{1,1}\right|\right) \\
&\leq CN^{-2} + \frac{CN^{-1}}{\varepsilon + h_i} \frac{h_i}{b_0^1 h_1} \prod_{k=2}^{i} \frac{r_k}{1 + 2r_k} \\
&\leq CN^{-2} + CN^{-1} \prod_{k=2}^{i} \frac{r_k}{2r_k} \leq CN^{-2} + CN^{-1} \left(\frac{1}{2}\right)^{J-1} \\
&\leq CN^{-2} + CN^{-1} (\sqrt{2})^{-N} \leq CN^{-2},
\end{aligned} \tag{4.16}$$

where Lemma 3.1 are used. This completes the proof. □

5. Numerical results and discussion

In this section, to confirm our theoretical results, we present numerical results for two test examples. As the exact solution of this test problem is available, the computed errors and rates of convergence are defined by

$$E^N := \max_{0\le i\le N}\left|u_i^N - u(x_i)\right|, \ \ \rho = \log_2\left(\frac{E^N}{E^{2N}}\right), \tag{5.1}$$

respectively, where u_i^N is the solution of the discretization scheme in Eq (3.2). Here, in all numerical experiments below, we choose $\varepsilon = 10^{-k},\ k = 1,\dots,7$ and $N = 2^m,\ m = 5,\dots,9$. Meanwhile, to obtain the Bakhvalov mesh $\bar{\Omega}^N$, we choose $\mu = 2$ for Example 1 and $\mu = \frac{2}{3}$ for Example 2. All experiments were performed on a Windows 10(64 bit)PC-Intel(R) Core(TM) i7-7500U CPU 2.70GHz 8 GB of RAM using MATLAB R2017a.

Example 1. The first example is taken from [14]:

$$\begin{cases}\varepsilon u'(x) + u(x) + \displaystyle\int_0^x xu(t)dt = f(x), & 0 < x < 1,\\ u(0) = 2, & \end{cases}$$

where

$$f(x) = -\frac{\varepsilon}{(1+x)^2} + \frac{1}{1+x} + x\varepsilon\left(1 - e^{-x/\varepsilon}\right) + x\ln(1+x).$$

The exact solution is $u(x) = 1/(1+x) + e^{-x/\varepsilon}$. The numerical results of Example 1 are listed in Table 1.

Table 1. The maximum errors and convergence orders for Example 1.

ε		Number of mesh-intervals, N				
		2^5	2^6	2^7	2^8	2^9
10^{-1}	E^N	6.8315e-03	1.9417e-03	5.3305e-04	1.4033e-04	3.6211e-05
	ρ	1.8148	1.8650	1.9254	1.9543	
10^{-2}	E^N	7.8978e-03	2.2569e-03	6.2633e-04	1.6537e-04	4.2822e-05
	ρ	1.8071	1.8494	1.9212	1.9493	
10^{-3}	E^N	8.0206e-03	2.2931e-03	6.3704e-04	1.6825e-04	4.3582e-05
	ρ	1.8064	1.8479	1.9208	1.9488	
10^{-4}	E^N	8.0331e-03	2.2968e-03	6.3813e-04	1.6854e-04	4.3659e-05
	ρ	1.8063	1.8477	1.9207	1.9488	
10^{-5}	E^N	8.0343e-03	2.2972e-03	6.3824e-04	1.6857e-04	4.3667e-05
	ρ	1.8063	1.8477	1.9207	1.9488	
10^{-6}	E^N	8.0345e-03	2.2972e-03	6.3825e-04	1.6858e-04	4.3667e-05
	ρ	1.8063	1.8477	1.9207	1.9488	
10^{-7}	E^N	8.0345e-03	2.2972e-03	6.3825e-04	1.6858e-04	4.3667e-05
	ρ	1.8063	1.8477	1.9207	1.9488	

Example 2. We consider the following singularly perturbed Volterra integro-differential equation:

$$\begin{cases} \varepsilon u'(x) + (3 + e^{-x})u(x) + \displaystyle\int_0^x \sin(t + x)u(t)dt = -10\left(\varepsilon x + \varepsilon^2\right)e^{-x/\varepsilon} + 5x^2, \quad 0 < x < 1, \\ u(0) = 1. \end{cases}$$

Since the exact solution of this test problem is not available, we use the double-mesh principle [14] to calculate the maximum point-wise errors as follows:

$$E^N := \max_{0 \le i \le N} \left| u_i^N - u_i^{2N} \right|,$$

where u_i^{2N} is the solution of the Eq (3.2) on the following mesh:

$$\widetilde{\Omega}^N := \bar{\Omega}^N \cup \left\{ x_{i-1/2} = \frac{x_i + x_{i-1}}{2} \right\}_{i=1}^N .$$

For different values of ε and N, the numerical results of Example 2 are displayed in Table 2.

Table 2. The maximum errors and convergence orders for Example 2.

ε		Number of mesh-intervals, N				
		2^5	2^6	2^7	2^8	2^9
10^{-1}	E^N	8.0584e-03	2.4171e-03	6.8617e-04	1.8417e-04	4.8068e-05
	ρ	1.7372	1.8167	1.8975	1.9379	
10^{-2}	E^N	9.1666e-03	2.7679e-03	7.9616e-04	2.1448e-04	5.6218e-05
	ρ	1.7276	1.7977	1.8922	1.9317	
10^{-3}	E^N	9.2972e-03	2.8091e-0	8.0908e-04	2.1804e-04	5.7175e-05
	ρ	1.7267	1.7958	1.8917	1.9311	
10^{-4}	E^N	9.3105e-03	2.8133e-03	8.1040e-04	2.1840e-04	5.7272e-05
	ρ	1.7266	1.7956	1.8916	1.9311	
10^{-5}	E^N	9.3118e-03	2.8138e-03	8.1053e-04	2.1844e-04	5.7282e-05
	ρ	1.7266	1.7956	1.8916	1.9311	
10^{-6}	E^N	9.3119e-03	2.8138e-03	8.1054e-04	2.1844e-04	5.7283e-05
	ρ	1.7266	1.7956	1.8916	1.9311	
10^{-7}	E^N	9.3120e-03	2.8138e-03	8.1054e-04	2.1844e-04	5.7283e-05
	ρ	1.7266	1.7956	1.8916	1.9311	

Tables 1–2 illustrate that the errors are robust with respect to ε, and the convergence order is close to 2. Meanwhile, Figures 1–2 display the log-log plot of these errors computed by our presented numerical method. In summary, our numerical results confirm our theoretical results.

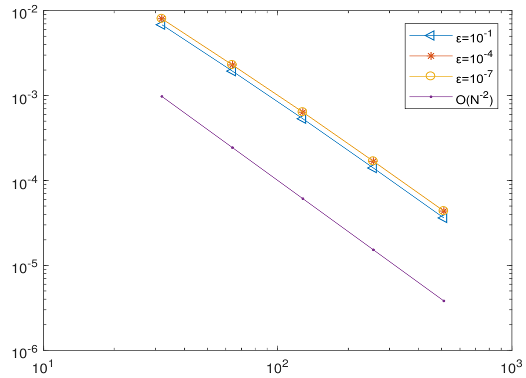

Figure 1. Loglog plot for order of convergence for Example 1.

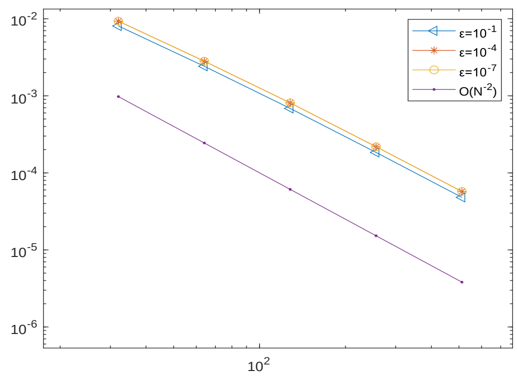

Figure 2. Loglog plot for order of convergence for Example 2.

6. Conclusion

We have discussed the ε-uniform convergence of the variable two-step backward differentiation formula of a reduced linear singularly perturbed Volterra integro-differential equation on a Bakhvalov-type mesh. We first proved the stability of our discretization scheme. Then, the proof of ε-uniform convergence can be given by the stability of the numerical method.

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Conflict of interest

The authors declare there is no conflict of interest.

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