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Research article

A fully discrete local discontinuous Galerkin method for variable-order fourth-order equation with Caputo-Fabrizio derivative based on generalized numerical fluxes

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Abstract: In this paper, an effective numerical method for the variable-order(VO) fourth-order problem with Caputo-Fabrizio derivative will be constructed and analyzed. Based on generalized alternating numerical flux, appropriate spatial and temporal discretization, we get a fully discrete local discontinuous Galerkin(LDG) scheme. The theoretic properties of the fully discrete LDG scheme are proved in detail by mathematical induction, and the method is proved to be unconditionally stable and convergent with $O(\tau + h^{k+1})$, where h is the spatial step, τ is the temporal step and k is the degree of the piecewise P^k polynomial. In order to show the efficiency of our method, some numerical examples are carried out by Matlab.

Keywords: Caputo-Fabrizio derivative; generalized alternating numerical flux; fractional fourth-order model

1. Introduction

In this paper, we will analyze the following variable-order fourth-order equations

$$\begin{aligned} &u_t + \rho(t){}_0^{CF}\partial_t^{1-\alpha(t)}u + u_{xxxx} = w(x,t), \quad (x,t) \in (a,b)\times(0,T], \\ &u(x,0) = u_0(x), \quad x \in [a,b], \end{aligned} \tag{1.1}$$

where $0 < \alpha(t) < 1$, $\rho(t)$ is a continuous function. The solution of the problem (1.1) is periodic or compactly supported.

The variable-order Caputo-Fabrizio derivative is defined as [27]

$${}_0^{CF}\partial_t^{1-\alpha(t)}u(x,t) = \left(\frac{1}{(\alpha(t))}\int_0^{\zeta}\frac{\partial u(x,\xi)}{\partial \xi}\exp\left[\frac{\alpha(t)-1}{\alpha(t)}(\zeta-\xi)\right]d\xi\right)_{\zeta=t}.$$

Fractional calculus can better reflect the reality of nature. Many physical problems are regulated by fractional order differential equations (FDEs) and finding analytical solutions to these equations has been the subject of research by many researchers in recent years [5, 16]. The main reason for which is that the theory of derivatives of fractions (non-integers) has aroused considerable interest in mathematics, physics, engineering and other scientific fields [14, 21, 31].

Designing an effective numerical method is meaningful for fractional order differential equations. Some numerical methods such as finite difference methods [4, 24], orthogonal spline collocation method [28], finite element methods [18], finite volume methods [12], spectral methods [7, 11], discontinuous Galerkin method [17, 20], orthogonal spline collocation methods [28] and so on, have been attempted to approximate the exact solution.

Fourth-order problems as a significant part of FDEs are studied by some scholars. Combining appropriate spatial and temporal discretization, some methods have been developed to solve fourth-order FDEs, including compact difference methods [8, 22, 30], orthogonal spline collocation methods [23], the homotopy perturbation methods [3], Galerkin-Legendre spectral methods [2], non-polynomial quintic spline methods [9], finite element methods [13, 15], LDG methods [6, 17, 25]. However, the report about numerical methods for variable-order fourth-order FDEs with Caputo-Fabrizio derivative is limited. We will study a LDG method for the problem (1.1) based on generalized numerical fluxes.

The rest of this paper is as follows. First in Section 2, some notations and necessary lemmas will be introduced. Then in Section 3, we propose a LDG scheme for solving the above problem (1.1), and discuss the stability and convergence of the method by mathematical induction. In Section 4, we shall give some numerical experiments which is made by using Matlab procedure to show the efficiency of our method. Finally in Section 5, the conclusion is given.

2. Preliminaries

Let $a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = b$ be a partition of the domain $\bar{\Omega} = [a, b]$, $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, for $j = 1, \cdots N$, and define $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}},\ 1 \leq j \leq N$, $h = \max\limits_{1\leq j\leq N} h_j$.

We denote $u^+_{j+\frac{1}{2}} = \lim\limits_{t\to 0^+} u(x_{j+\frac{1}{2}} + t)$ and $u^-_{j+\frac{1}{2}} = \lim\limits_{t\to 0^+} u(x_{j+\frac{1}{2}} - t)$. Futhermore, the weighted average of a function v is defined by $(v)^{(\delta)}_{j+\frac{1}{2}} = \delta v^-_{j+\frac{1}{2}} + (1-\delta)v^+_{j+\frac{1}{2}}$, where δ is the given weight.

The local discontinuous Galerkin space V^k_h is shown below

$$V^k_h = \{\varsigma : \varsigma \in P^k(I_j), j = 1, 2, \cdots N\}.$$

For a periodic function ϑ which is defined on the domain $[a, b]$, the generalized Gauss-Radau projections [1, 19], denoted by $\mathcal{P}_\delta$. Let $\vartheta^e = \mathcal{P}_\delta\vartheta - \vartheta$ be the projection error. For the case $\delta \neq \frac{1}{2}$, it has the following properties

$$\int_{I_j} (\vartheta^e)\eta(x)dx = 0, \quad \forall\ \eta \in P^{k-1}(I_j), \quad (\vartheta^e)^{\delta}_{j+\frac{1}{2}} = 0, \quad j = 1, 2, \cdots N. \tag{2.1}$$

The following conclusion can be obtained from [1].

Lemma 2.1. *If $\delta \neq \frac{1}{2}$, $\vartheta \in H^{s+1} \in [a,b]$, then there holds*

$$\|\vartheta^e\| + h^{\frac{1}{2}}\|\vartheta^e\|_{L^2(\tau_h)} \leq Ch^{\min(k+1,s+1)}\|\vartheta\|_{s+1}, \tag{2.2}$$

the constant C which is independent of h, is solely dependent on the function ϑ. τ_h is the union of element boundary points, and $\|\vartheta^e\|_{\tau_h}$ can be defined by

$$\|\vartheta^e\|_{\tau_h} = \left(\frac{1}{2}\sum_{j=1}^{N}(((\vartheta^e)^+_{j+\frac{1}{2}})^2 + ((\vartheta)^{e-}_{j+\frac{1}{2}})^2)\right)^{\frac{1}{2}}.$$

Throughout this paper, the notation C represents a positive constant that may have a different value at each time. The usual notations in Sobolev space are used in the paper. Let the scalar inner product on $L^2(E)$ be denoted by $(\cdot,\cdot)_E$, and the associated norm by $\|\cdot\|_E$. If $E = \Omega$, we drop E.

3. Fully discrete LDG method

Firstly, we rewrite Eq (1.1) as a system

$$p = u_x, \quad q = p_x, \quad s = q_x, \quad u_t + \rho(t)_0^{CF}\partial_t^{1-\alpha(t)}u + s_x = w(x,t). \tag{3.1}$$

Let $t_n = \frac{n}{M}T$, and $\tau = t_n - t_{n-1}$. The temporal derivatives u_t and $\rho(t)_0^{CF}\partial_t^{1-\alpha(t)}u$ at t_n are approximated as follows

$$\begin{aligned}
u_t(x,t_n) &= \frac{u(x,t_n) - u(x,t_{n-1})}{\tau} + \Phi_1^n(x),\\
\rho(t_n)_0^{CF}\partial_t^{1-\alpha(t)}u(x,t_n) &= \frac{\rho(t_n)}{\alpha(t_n)}\int_0^{t_n}\frac{\partial u(x,\varsigma)}{\partial\varsigma}\exp\left[\frac{\alpha(t_n)-1}{\alpha(t_n)}(t_n-\varsigma)\right]d\varsigma\\
&= \frac{\rho(t_n)}{(1-\alpha(t_n))\tau}\sum_{k=1}^{n}(u(x,t_k)-u(x,t_{k-1}))(\exp\left[\frac{(\alpha(t_n)-1)\tau}{\alpha(t_n)}(n-k)\right]\\
&- \exp\left[\frac{(\alpha(t_n)-1)\tau}{\alpha(t_n)}(n-k+1)\right])\\
&+ \frac{\rho(t_n)}{\alpha(t_n)}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}(\varsigma - t_{k-\frac{1}{2}})\frac{\partial^2 u(x,c_k)}{\partial\varsigma^2}exp\left[\frac{\alpha(t_n)-1}{\alpha(t_n)}(t_n-\varsigma)\right]d\varsigma\\
&= \frac{\rho(t_n)}{(1-\alpha(t_n))\tau}\sum_{k=1}^{n}(u(x,t_k)-u(x,t_{k-1}))b_k^n + \Phi_2^n(x)\\
&= \frac{\rho(t_n)}{(1-\alpha(t_n))\tau}\left(b_n^n u(x,t_n) + \sum_{k=1}^{n-1}(b_k^n - b_{k+1}^n)u(x,t_k) - b_1^n u(x,t_0)\right) + \Phi_2^n(x),
\end{aligned} \tag{3.2}$$

where $c_k \in (t_{k-1}, t_k)$,

$$b_k^n = \exp\left[\frac{(\alpha(t_n)-1)\tau}{\alpha(t_n)}(n-k)\right] - \exp\left[\frac{(\alpha(t_n)-1)\tau}{\alpha(t_n)}(n-k+1)\right]$$

and $\Phi^n(x) = \Phi_1^n(x) + \Phi_2^n(x)$ is the truncation error. According to [27], we can know the following conclusion

$$\|\Phi^n(x)\| \leq C\tau, \tag{3.3}$$

here the constant $C > 0$, depending on T and the function u.

Furthermore, by some calculations, we could find that b_k^n in Eq (3.2) has the following property

$$0 < b_1^n < b_2^n < b_3^n < \cdots < b_n^n = b_{n-1}^{n-1}. \tag{3.4}$$

Then we could define the fully-discrete LDG method for the problem (1.1). Let $u_h^n,\ p_h^n,\ q_h^n,\ s_h^n \in V_h^k$ be the approximations of $u(\cdot,t_n),\ p(\cdot,t_n),\ q(\cdot,t_n),\ s(\cdot,t_n)$, respectively, $w^n(x) = w(x,t_n)$. Find $u_h^n,\ p_h^n,\ q_h^n,\ s_h^n \in V_h^k$, such that for $v,\ w,\ \rho,\ \varphi \in V_h^k$,

$$\begin{aligned}
&(\frac{1}{\tau}+\frac{\rho(t_n)b_n^n}{(1-\alpha(t_n))\tau})\int_\Omega u_h^n v dx - \left(\int_\Omega s_h^n v_x dx - \sum_{j=1}^{N}((\widehat{s_h^n}v^-)_{j+\frac{1}{2}} - (\widehat{s_h^n}v^+)_{j-\frac{1}{2}})\right)\\
&\quad = \frac{\rho(t_n)}{(1-\alpha(t_n))\tau}\left(\sum_{k=1}^{n-1}(b_{k+1}^n - b_k^n)\int_\Omega u_h^k v dx + b_1^n\int_\Omega u_h^0 v dx\right)\\
&\quad + \frac{1}{\tau}\int_\Omega u_h^{n-1} v dx + \int_\Omega w^n v dx,\\
&\int_\Omega s_h^n w dx + \int_\Omega q_h^n w_x dx - \sum_{j=1}^{N}((\widehat{q_h^n}w^-)_{j+\frac{1}{2}} - (\widehat{q_h^n}w^+)_{j-\frac{1}{2}}) = 0,\\
&\int_\Omega q_h^n \rho dx + \int_\Omega p_h^n \rho_x dx - \sum_{j=1}^{N}((\widehat{p_h^n}\rho^-)_{j+\frac{1}{2}} - (\widehat{p_h^n}\rho^+)_{j-\frac{1}{2}}) = 0,\\
&\int_\Omega p_h^n \varphi dx + \int_\Omega u_h^n \varphi_x dx - \sum_{j=1}^{N}((\widehat{u_h^n}\varphi^-)_{j+\frac{1}{2}} - (\widehat{u_h^n}\varphi^+)_{j-\frac{1}{2}}) = 0.
\end{aligned} \tag{3.5}$$

The hat terms in Eq (3.5) which are from integration by parts are numerical fluxes. In order to guarantee stability, we will choose the following generalized numerical fluxes [26]

$$\widehat{u_h^n} = (u_h^n)^{(\delta)},\ \widehat{p_h^n} = (p_h^n)^{(1-\delta)},\ \widehat{q_h^n} = (q_h^n)^{(\delta)},\ \widehat{s_h^n} = (s_h^n)^{(1-\delta)}, \tag{3.6}$$

where $\delta \neq \frac{1}{2}$. If $\delta = 0$ or 1, the flux Eq (3.6) will be purely alternating numerical fluxes [29].

In order to simplify the notations in the stability and convergence, we could denote

$$\begin{aligned}
\Theta_\Omega(u_h^n, p_h^n; w, v) = &\int_\Omega u_h^n w_x dx - \sum_{j=1}^{N}\left(((u_h^n)^{(\delta)}w^-)_{j+\frac{1}{2}} - ((u_h^n)^{(\delta)}w^+)_{j-\frac{1}{2}}\right)\\
&+ \int_\Omega p_h^n v_x dx - \sum_{j=1}^{N}\left(((p_h^n)^{(1-\delta)}v^-)_{j+\frac{1}{2}} - ((p_h^n)^{(1-\delta)}v^+)_{j-\frac{1}{2}}\right).
\end{aligned} \tag{3.7}$$

3.1. Stability analysis

Without loss of generality, the case $w = 0$ is considered in the numerical analysis of the scheme (3.5).

Theorem 3.1. *Assume that the solution of the problem (1.1) is compactly supported or periodic, then the LDG method (3.5) is stable and satisfies the following inequalities*

$$\|u_h^n\| \le \|u_h^0\|, \quad n = 1, 2 \cdots, M. \tag{3.8}$$

Proof. In scheme (3.5), we first take the test functions $v = u_h^n$, $w = p_h^n$, $\rho = q_h^n$, $\varphi = -s_h^n$, we could have

$$\begin{aligned}
&(\frac{1}{\tau}+\frac{\rho(t_n)b_n^n}{(1-\alpha(t_n))\tau})\|u_h^n\|^2 + \|q_h^n\|^2 + \Theta_\Omega(q_h^n, p_h^n; p_h^n, q_h^n) - \Theta_\Omega(u_h^n, s_h^n; s_h^n, u_h^n)\\
=&\frac{\rho(t_n)}{(1-\alpha(t_n))\tau}((\sum_{i=1}^{n-1}(b_{k+1}^n - b_k^n)\int_\Omega u_h^k u_h^n dx\\
&+ b_1^n\int_\Omega u_h^0 u_h^n dx) + \frac{1}{\tau}\int_\Omega u_h^{n-1}u_h^n dx.
\end{aligned} \tag{3.9}$$

In each cell I_j, we can obtain

$$\begin{aligned}
\Theta_{I_j}(q_h^n, p_h^n; p_h^n, q_h^n) =& \int_{I_j} q_h^n (p_h^n)_x dx - \Big(((q_h^n)^{(\delta)}(p_h^n)^-)_{j+\frac{1}{2}} - ((q_h^n)^{(\delta)}(p_h^n)^+)_{j-\frac{1}{2}}\Big)\\
&+ \int_{I_j} p_h^n (q_h^n)_x dx - \Big(((p_h^n)^{(1-\delta)}(q_h^n)^-)_{j+\frac{1}{2}} - ((p_h^n)^{(1-\delta)}(q_h^n)^+)_{j-\frac{1}{2}}\Big)\\
=&((q_h^n)^-(p_h^n)^-)_{j+\frac{1}{2}} - ((q_h^n)^+(p_h^n)^+)_{j-\frac{1}{2}} - ((q_h^n)^{(\delta)}(p_h^n)^-)_{j+\frac{1}{2}}\\
&+ ((q_h^n)^{(\delta)}(p_h^n)^+)_{j-\frac{1}{2}} - ((p_h^n)^{(1-\delta)}(q_h^n)^-)_{j+\frac{1}{2}} + ((p_h^n)^{(1-\delta)}(q_h^n)^+)_{j-\frac{1}{2}}.
\end{aligned} \tag{3.10}$$

Summing (3.10) from 1 to N over j, we can get

$$\Theta_\Omega(q_h^n, p_h^n; p_h^n, q_h^n) = 0, \quad \Theta_\Omega(u_h^n, s_h^n; s_h^n, u_h^n) = 0. \tag{3.11}$$

Combining Eqs (3.4),(3.11) and Cauchy-Schwarz inequality, the equality (3.9) will become

$$\begin{aligned}
(\frac{1}{\tau} + \frac{\rho(t_n)b_n^n}{(1-\alpha(t_n))\tau})\|u_h^n\| \le& \frac{\rho(t_n)}{(1-\alpha(t_n))\tau}\left(\sum_{k=1}^{n-1}(b_{k+1}^n - b_k^n)\|u_h^k\| + b_1^n\|u_h^0\|\right)\\
&+ \frac{1}{\tau}\|u_h^{n-1}\|.
\end{aligned} \tag{3.12}$$

In what follows we will prove Theorem 3.1 by using mathematical induction. For the case $n = 1$ in Eq (3.12), we can easily get

$$(\frac{1}{\tau} + \frac{\rho(t_1)b_1^1}{(1-\alpha(t_1))\tau})\|u_h^1\| \le \frac{\rho(t_1)}{(1-\alpha(t_1))\tau}b_1^1\|u_h^0\| + \frac{1}{\tau}\|u_h^0\|, \tag{3.13}$$

that is

$$\|u_h^1\| \leq \|u_h^0\|. \tag{3.14}$$

Next assume that the following inequality holds

$$\|u_h^k\| \leq \|u_h^0\|,\ k = 1, 2, 3 \cdots, n-1, \tag{3.15}$$

we need to prove

$$\|u_h^n\| \leq \|u_h^0\|.$$

According to Eq (3.12), we could have the following inequality

$$\begin{aligned}
(\frac{1}{\tau} + \frac{\rho(t_n) b_n^n}{(1-\alpha(t_n))\tau})\|u_h^n\| \leq & \frac{\rho(t_n)}{(1-\alpha(t_n))\tau}\left(\sum_{k=1}^{n-1}(b_{k+1}^n - b_k^n)\|u_h^k\| + b_1^n\|u_h^0\|\right) \\
& + \frac{1}{\tau}\|u_h^{n-1}\| \\
\leq & \frac{\rho(t_n)}{(1-\alpha(t_n))\tau}\left(\sum_{k=1}^{n-1}(b_{k+1}^n - b_k^n) + b_1^n\right)\|u_h^0\| \\
& + \frac{1}{\tau}\|u_h^0\| \\
= & \frac{\rho(t_n)}{(1-\alpha(t_n))\tau} b_n^n \|u_h^0\| + \frac{1}{\tau}\|u_h^0\|.
\end{aligned} \tag{3.16}$$

Obviously, we can directly obtain

$$\|u_h^n\| \leq \|u_h^0\|.$$

This finishes the proof of Theorem 3.1.

3.2. Error estimate

Theorem 3.2. *Suppose that $u(x, t_n)$ is the exact solution of the problem (1.1), u_h^n is the numerical solution of the fully discrete LDG scheme (3.5), then the following result holds*

$$\|u(x, t_n) - u_h^n\| \leq C(\tau + h^{k+1}),$$

where $C > 0$ is a constant depending on u, T.

Proof.

$$\begin{aligned}
e_u^n &= u(x, t_n) - u_h^n = \xi_u^n - \eta_u^n, \quad \xi_u^n = \mathcal{P}_\delta e_u^n, \qquad \eta_u^n = \mathcal{P}_\delta u(x, t_n) - u(x, t_n), \\
e_p^n &= p(x, t_n) - p_h^n = \xi_p^n - \eta_p^n, \quad \xi_p^n = \mathcal{P}_{1-\delta} e_p^n, \quad \eta_p^n = \mathcal{P}_{1-\delta} p(x, t_n) - p(x, t_n), \\
e_q^n &= q(x, t_n) - q_h^n = \xi_q^n - \eta_q^n, \quad \xi_q^n = \mathcal{P}_\delta e_q^n, \qquad \eta_q^n = \mathcal{P}_\delta q(x, t_n) - q(x, t_n), \\
e_s^n &= s(x, t_n) - s_h^n = \xi_s^n - \eta_s^n, \quad \xi_s^n = \mathcal{P}_{1-\delta} e_s^n, \quad \eta_s^n = \mathcal{P}_{1-\delta} s(x, t_n) - s(x, t_n).
\end{aligned} \tag{3.17}$$

Here η_u^n , η_p^n, η_q^n and η_s^n have been estimated by the inequality (2.2). Next we will estimate ξ_u^n, ξ_p^n, ξ_q^n and ξ_s^n. Based on the fluxes (3.6), we have

$$
\begin{aligned}
&(\frac{1}{\tau}+\frac{\rho(t_n)b_n^n}{(1-\alpha(t_n))\tau})\int_\Omega e_u^n v dx-\int_\Omega e_s^n v_x dx+\sum_{j=1}^{N}\Big(((e_s^n)^{(1-\delta)}v^-)_{j+\frac{1}{2}}-((e_s^n)^{(1-\delta)}v^+)_{j-\frac{1}{2}}\Big)\\
&+\int_\Omega \Phi^n(x)v dx-\frac{1}{\tau}\int_\Omega e_u^{n-1}v dx-\frac{\rho(t_n)}{(1-\alpha(t_n))\tau}((\sum_{k=1}^{n-1}(b_{k+1}^n-b_k^n)\int_\Omega e_u^k v dx+b_1^n\int_\Omega e_u^0 v dx)\\
&+\int_\Omega e_s^n w dx+\int_\Omega e_q^n w_x dx-\sum_{j=1}^{N}\Big(((e_q^n)^{(\delta)}w^-)_{j+\frac{1}{2}}-((e_q^n)^{(\delta)}w^+)_{j-\frac{1}{2}}\Big)+\int_\Omega e_q^n\rho dx \qquad (3.18)\\
&+\int_\Omega e_p^n\rho_x dx-\sum_{j=1}^{N}\Big(((e_p^n)^{(1-\delta)}\rho^-)_{j+\frac{1}{2}}-((e_p^n)^{(1-\delta)}\rho^+)_{j-\frac{1}{2}}\Big)+\int_\Omega e_p^n\varphi dx\\
&+\int_\Omega e_u^n\varphi_x dx-\sum_{j=1}^{N}\Big(((e_u^n)^{(\delta)}\varphi^-)_{j+\frac{1}{2}}-((e_u^n)^{(\delta)}\varphi^+)_{j-\frac{1}{2}}\Big)=0.
\end{aligned}
$$

Based on Eq (3.17), and taking $v=\xi_u^n, w=\xi_p^n, \rho=\xi_q^n, \varphi=-\xi_s^n$, the above error Eq (3.18) could be written as

$$
\begin{aligned}
&(\frac{1}{\tau}+\frac{\rho(t_n)b_n^n}{(1-\alpha(t_n))\tau})\|\xi_u^n\|^2+\|\xi_q^n\|^2+\Theta_\Omega(\xi_q^n,\xi_p^n;\xi_p^n,\xi_q^n)-\Theta_\Omega(\xi_u^n,\xi_s^n;\xi_s^n,\xi_u^n)\\
&=\frac{\rho(t_n)}{(1-\alpha(t_n))\tau}\left(\sum_{k=1}^{n-1}(b_{k+1}^n-b_k^n)\int_\Omega \xi_u^k\xi_u^n dx+b_1^n\int_\Omega \xi_u^0\xi_u^n dx\right)-\int_\Omega\Phi^n(x)\xi_u^n dx\\
&\quad+(\frac{1}{\tau}+\frac{\rho(t_n)b_n^n}{(1-\alpha(t_n))\tau})\int_\Omega \eta_u^n\xi_u^n dx+\Theta_\Omega(\eta_q^n,\eta_p^n;\xi_p^n,\xi_q^n)-\Theta_\Omega(\eta_u^n,\eta_s^n;\xi_s^n,\xi_u^n) \qquad (3.19)\\
&\quad+\frac{1}{\tau}\int_\Omega \xi_u^{n-1}\xi_u^n dx-\frac{1}{\tau}\int_\Omega \eta_u^{n-1}\xi_u^n dx+\int_\Omega \eta_q^n\xi_q^n dx+\int_\Omega \eta_s^n\xi_p^n dx-\int_\Omega \eta_p^n\xi_s^n dx\\
&\quad-\frac{\rho(t_n)}{(1-\alpha(t_n))\tau}\left(\sum_{k=1}^{n-1}(b_{k+1}^n-b_k^n)\int_\Omega \eta_u^k\xi_u^n dx+b_1^n\int_\Omega \eta_u^0\xi_u^n dx\right).
\end{aligned}
$$

Then, taking $v=-\xi_q^n$, $w=\xi_s^n$, $\rho=\beta\xi_u^n$, $\varphi=\beta\xi_p^n$ in Eq (3.18), we can get the following equation from the error Eq (3.18)

$$
\begin{aligned}
&(\frac{1}{\tau}+\frac{\rho(t_n)b_n^n}{(1-\alpha(t_n))\tau})\|\xi_p^n\|^2+\|\xi_s^n\|^2+\beta\Theta_\Omega(\xi_u^n,\xi_p^n;\xi_p^n,\xi_u^n)+\Theta_\Omega(\xi_q^n,\xi_s^n;\xi_s^n,\xi_q^n)\\
&=-\frac{\rho(t_n)}{(1-\alpha(t_n))\tau}\left(\sum_{k=1}^{n-1}(b_{k+1}^n-b_k^n)\int_\Omega \xi_u^k\xi_q^n dx+b_1^n\int_\Omega \xi_u^0\xi_q^n dx\right)+\int_\Omega\Phi^n(x)\xi_q^n dx\\
&\quad-\beta\int_\Omega \eta_u^n\xi_q^n dx+\beta\Theta_\Omega(\eta_u^n,\eta_p^n;\xi_p^n,\xi_u^n)+\Theta_\Omega(\eta_q^n,\eta_s^n;\xi_s^n,\xi_q^n) \qquad (3.20)\\
&\quad-\frac{1}{\tau}\int_\Omega \xi_u^{n-1}\xi_q^n dx+\frac{1}{\tau}\int_\Omega \eta_u^{n-1}\xi_q^n dx+\beta\int_\Omega \eta_q^n\xi_u^n dx+\int_\Omega \eta_s^n\xi_s^n dx+\beta\int_\Omega \eta_p^n\xi_p^n dx\\
&\quad+\frac{\rho(t_n)}{(1-\alpha(t_n))\tau}\left(\sum_{k=1}^{n-1}(b_{k+1}^n-b_k^n)\int_\Omega \eta_u^k\xi_q^n dx+b_1^n\int_\Omega \eta_u^0\xi_q^n dx\right).
\end{aligned}
$$

By applying Eqs (3.19),(3.20), stability results and $\xi_u^0 = 0$, we have the following equality

$$\beta\|\xi_u^n\|^2 + \|\xi_q^n\|^2 + \beta\|\xi_p^n\|^2 + \|\xi_s^n\|^2 = \frac{\rho(t_n)}{(1-\alpha(t_n))\tau}\Big(\sum_{k=1}^{n-1}(b_{k+1}^n - b_k^n)\int_\Omega \xi_u^k(\xi_u^n - \xi_q^n)dx + b_1^n\int_\Omega \xi_u^0(\xi_u^n - \xi_q^n)dx\Big)$$
$$+ \frac{1}{\tau}\int_\Omega \xi_u^{n-1}(\xi_u^n - \xi_q^n)dx - \int_\Omega \Phi^n(x)(\xi_u^n - \xi_q^n)dx + \sum_{i=1}^{4} T_i \tag{3.21}$$

where $\beta = (\frac{1}{\tau} + \frac{\rho(t_n)b_n^n}{(1-\alpha(t_n))\tau})$,

$$T_1 = \int_\Omega \eta_q^n\xi_q^n dx + \int_\Omega \eta_s^n\xi_p^n dx - \int_\Omega \eta_p^n\xi_s^n dx + \beta\int_\Omega \eta_q^n\xi_u^n dx + \int_\Omega \eta_s^n\xi_s^n dx + \beta\int_\Omega \eta_p^n\xi_p^n dx,$$
$$T_2 = \frac{\rho(t_n)}{(1-\alpha(t_n))\tau}\Big(b_n^n\int_\Omega \eta_u^n(\xi_u^n - \xi_q^n)dx\sum_{k=1}^{n-1}(b_{k+1}^n - b_k^n)\int_\Omega \eta_u^k\xi_q^n dx + b_1^n\int_\Omega \eta_u^0(\xi_u^n - \xi_q^n)dx\Big)$$
$$- \frac{1}{\tau}\int_\Omega \xi_u^{n-1}(\xi_u^n - \xi_q^n)dx,$$
$$T_3 = \int_\Omega \frac{(\eta_u^n - \eta_u^{n-1})}{\tau}(\xi_u^n - \xi_q^n)dx$$
$$T_4 = \Theta_\Omega(\eta_q^n, \eta_p^n; \xi_p^n, \xi_q^n) - \Theta_\Omega(\eta_u^n, \eta_s^n; \xi_s^n, \xi_u^n) + \beta\Theta_\Omega(\eta_u^n, \eta_p^n; \xi_p^n, \xi_u^n) + \Theta_\Omega(\eta_q^n, \eta_s^n; \xi_s^n, \xi_q^n).$$

Next we begin to discuss the terms T_i, $i = 1, 2, 3, 4$.

Bsaed on the approximation property, Cauchy-Schwarz inequality, we have

$$T_1 \le Ch^{k+1}\Big(\Big\|\xi_u^n\Big\| + \Big\|\xi_q^n\Big\| + \Big\|\xi_s^n\Big\| + \Big\|\xi_p^n\Big\|\Big).$$

and

$$T_2 \le Ch^{k+1}\Big\|\xi_u^n - \xi_q^n\Big\|.$$

With the help of

$$\|\frac{\eta_u^i - \eta_u^{i-1}}{\tau}\| \le \|\frac{1}{\tau}\int_{t_{i-1}}^{t_i}\frac{\partial}{\partial t}(Q_\delta u(x,t) - u(x,t))dt\| \le Ch^{k+1}\|u_t\|_{L^\infty(H^2(\Omega))},$$

$$T_3 \le Ch^{k+1}\|u_t\|_{L^\infty(H^2(\Omega))}\Big\|\xi_u^n - \xi_q^n\Big\|,$$

According to the properties (2.1), we can obtain

$$T_4 = 0.$$

Noticing the fact that

$$ab \le \frac{1}{4\varepsilon}a^2 + \varepsilon b^2,$$

and

$$(a-b)^2 \le 2(a^2+b^2).$$

Based on the above results, and Cauchy-Schwarz inequalities in Eq (3.21), we could have

$$\begin{aligned} \beta\left\|\xi_u^n\right\|^2 + \left\|\xi_q^n\right\|^2 \le & \frac{1}{4\varepsilon_1}\left(\frac{\rho(t_n)}{(1-\alpha(t_n))\tau}\Big(\sum_{k=1}^{n-1}(b_{k+1}^n - b_k^n)\left\|\xi_u^k\right\| + b_1^n\left\|\xi_u^0\right\|\Big) + \frac{1}{\tau}\left\|\xi_u^{n-1}\right\| + \left\|\Phi^n(x)\right\|dx\right)^2 \\ & + \varepsilon_1\left\|\xi_u^n - \xi_q^n\right\|^2 + \varepsilon_2\left\|\xi_u^n\right\|^2 + \varepsilon_3\left\|\xi_q^n\right\|^2 + Ch^{2k+2} \end{aligned} \tag{3.22}$$

that is

$$\begin{aligned} \beta\left\|\xi_u^n\right\|^2 + \left\|\xi_q^n\right\|^2 \le & \frac{1}{4\varepsilon_1}\left(\frac{\rho(t_n)}{(1-\alpha(t_n))\tau}\Big(\sum_{k=1}^{n-1}(b_{k+1}^n - b_k^n)\left\|\xi_u^k\right\| + b_1^n\left\|\xi_u^0\right\|\Big) + \frac{1}{\tau}\left\|\xi_u^{n-1}\right\| + \left\|\Phi^n(x)\right\|dx\right)^2 \\ & + (2\varepsilon_1+\varepsilon_2)\left\|\xi_u^n\right\|^2 + (2\varepsilon_1+\varepsilon_3)\left\|\xi_q^n\right\|^2 + Ch^{2k+2}, \end{aligned} \tag{3.23}$$

so we can obtain the following inequality

$$\begin{aligned} (\beta - 2\varepsilon_1 - \varepsilon_2)\left\|\xi_u^n\right\|^2 \le & \frac{1}{4\varepsilon_1}\left(\frac{\rho(t_n)}{(1-\alpha(t_n))\tau}\Big(\sum_{k=1}^{n-1}(b_{k+1}^n - b_k^n)\left\|\xi_u^k\right\| + b_1^n\left\|\xi_u^0\right\|\Big) + \frac{1}{\tau}\left\|\xi_u^{n-1}\right\|\right. \\ & \left. + \left\|\Phi^n(x)\right\|dx\right)^2 + Ch^{2k+2} \end{aligned} \tag{3.24}$$

that is

$$\left\|\xi_u^n\right\| \le \frac{1}{2\sqrt{(\beta-2\varepsilon_1-\varepsilon_2)\varepsilon_1}\tau}\Big(\frac{\rho(t_n)}{(1-\alpha(t_n))}\sum_{k=1}^{n-1}(b_{k+1}^n - b_k^n)\left\|\xi_u^k\right\| + \left\|\xi_u^{n-1}\right\|\Big) + C(h^{k+1}+\tau). \tag{3.25}$$

Finally, by applying discrete Gronwall inequality and the triangle inequality, the proof Theorem 3.2 is completed.

4. Numerical experiment

Consider the problem (1.1)

$$\begin{aligned} & u_t + \rho(t)\,{}_0^{CF}\partial_t^{1-\alpha(t)}u + u_{xxxx} = w(x,t), \quad (x,t)\in(0,2\pi)\times(0,1], \\ & u(x,0) = \sin(x), \quad x\in(0,2\pi). \end{aligned}$$

Let $\rho(t)=\frac{1}{2}$, and the right term $w(x,t)$ is taken such that the exact solution for the problem (1.1) is $u(x,t)=e^t\sin(x)$.

In the following numerical experiments, the following basis functions for $x\in I_j$ are choosed

$$\phi_1^j = 1, \quad \phi_2^j = \frac{x - (\frac{x_{j-\frac{1}{2}}+x_{j+\frac{1}{2}}}{2})}{h_j}, \quad \phi_3^j = (\frac{x - (\frac{x_{j-\frac{1}{2}}+x_{j+\frac{1}{2}}}{2})}{h_j})^2. \tag{4.1}$$

Table 1. Errors versus N, for different δ with $M = 5000,\ T = 1$.

$\alpha(t)$	δ	P^k	N	L^∞-error	order	L^2-error	order
$\alpha(t) = \frac{3+5t}{10}$	$\delta = 0.1$	P^0	5	1.740531150383093	-	1.856872679569707	-
			10	0.864569387772493	1.0095	0.895561146790427	1.0520
			15	0.572851975309558	1.0151	0.589558568838264	1.0311
			20	0.428542766345249	1.0089	0.439974710819443	1.0173
		P^1	5	0.684519250525526	-	0.494306358929751	-
			10	0.192652231688883	1.8291	0.131115425232428	1.9146
			15	0.086980076197341	1.9612	0.059094992390308	1.9655
			20	0.049356492303964	1.9696	0.033413520766790	1.9820
		P^2	5	0.084986517667719	-	0.044925635579710	-
			10	0.010076694499805	3.0762	0.005290980027691	3.0859
			15	0.002978268712921	3.0061	0.001558719171324	3.0142
			20	0.001265853161807	2.9741	6.727656547770956E-04	2.9207
	$\delta = 0.3$	P^0	5	1.885198974510259	-	2.070042187044406	-
			10	0.890498263410747	1.0820	0.943149454109176	1.1341
			15	0.580949986392324	1.0534	0.605185028386995	1.0943
			20	0.432018364333049	1.0296	0.446811629365803	1.0546
		P^1	5	0.630265500474040	-	0.517981933170420	-
			10	0.235126887392044	1.4225	0.185662688702393	1.4802
			15	0.120574166167753	1.6471	0.093525685668037	1.6911
			20	0.072032377351253	1.7907	0.055457129576016	1.8167
		P^2	5	0.076329772694388	-	0.040857513944066	-
			10	0.008202425617554	3.2181	0.004355936326285	3.2295
			15	0.002330058182918	3.1039	0.001269089557106	3.0415
			20	9.896100371102395E-04	2.9767	5.526844252958554E-04	2.8895
	$\delta = 0.8$	P^0	5	1.826393769789430	-	1.974561867576591	-
			10	0.879694345160746	1.0539	0.920703715362021	1.1007
			15	0.577573781928998	1.0377	0.597751317918225	1.0654
			20	0.430569533399183	1.0210	0.443549658602455	1.0371
		P^1	5	0.651288720319747	-	0.512561963971504	-
			10	0.214043573323575	1.6054	0.153386050326162	1.7406
			15	0.100449966106176	1.8658	0.071549136282852	1.8807
			20	0.057413974281730	1.9444	0.040999588158489	1.9355
		P^2	5	0.080177976833727	-	0.042899183979938	-
			10	0.009242097354639	3.1169	0.004782445115423	3.1651
			15	0.002714243368699	3.0219	0.001399042040896	3.0315
			20	0.001132120309036	3.0395	6.060712650631764E-04	2.9079

Table 2. Errors versus N, for different δ with $M = 5000,\ T = 1$.

$\alpha(t)$	δ	P^k	N	L^∞-error	order	L^2-error	order
			5	1.810302729631866	-	1.950431137529400	-
			10	0.877399118498201	1.0449	0.916404251319563	1.0897
		P^0	15	0.576885850871822	1.0342	0.596395642065351	1.0594
			20	0.430278309350828	1.0192	0.442964518772469	1.0338
			5	0.651519200140843	-	0.512199596387930	-
	$\delta = 0.2$		10	0.213787037852454	1.6076	0.153374062256062	1.7397
		P^1	15	0.099726789743246	1.8807	0.071548068227604	1.8806
			20	0.057317676171240	1.9251	0.040999460569314	1.9355
			5	0.079817658439521	-	0.042896459836405	-
			10	0.009197629002603	3.1174	0.004782155144824	3.1651
		P^2	15	0.002658885254297	3.0608	0.001398217766249	3.0328
			20	0.001126035045434	2.9866	6.041931292179050E-04	2.9166
			5	1.865747594446795	-	2.037246844551530	-
			10	0.887707777972510	1.0716	0.937011068141408	1.1205
		P^0	15	0.580116988315355	1.0492	0.603231846217448	1.0861
			20	0.431666440723307	1.0274	0.445965926728114	1.0500
			5	0.633708004317878	-	0.516741062515626	-
$\alpha(t) = \frac{2+\sin(t)}{7}$	$\delta = 0.7$		10	0.235919326182138	1.4255	0.185598771207892	1.4773
		P^1	15	0.121501316888998	1.6366	0.093518217061097	1.6905
			20	0.072176715017533	1.8104	0.055455799964511	1.8165
			5	0.076672827151093	-	0.040853928772116	-
			10	0.008243823024788	3.2173	0.004355603274341	3.2295
		P^2	15	0.002379576236816	3.0645	0.001268176450632	3.0431
			20	9.947572460603090E-04	3.0317	5.506224366806317E-04	2.9000
			5	1.729087893311004	-	1.843367998709562	-
			10	0.862950112030744	1.0027	0.893326833848479	1.0451
		P^0	15	0.572364336168310	1.0126	0.588862758645951	1.0279
			20	0.428335813433843	1.0076	0.439675530183568	1.0156
			5	0.686293159773277	-	0.494248023341584	-
	$\delta = 0.9$		10	0.193058217003713	1.8298	0.131114165235490	1.9144
		P^1	15	0.087682768453988	1.9466	0.059094911534041	1.9655
			20	0.049450035210393	1.9910	0.033413504585291	1.9820
			5	0.085360832701889	-	0.044923377907586	-
			10	0.010123790019360	3.0758	0.005290732336462	3.0859
		P^2	15	0.003036086784551	2.9702	0.001557984170353	3.0152
			20	0.001270534625058	3.0281	6.710762184142701E-04	2.9278

Table 3. Errors versus M, order for different $\alpha(t)$ with $N = 1000,\ \delta = 0.1\ and\ T = 1$.

δ	$\alpha(t)$	P^k	M	L^∞-error	order	L^2-error	order
$\delta = 0.1$	$\frac{2+\sin(t)}{7}$	P^0	5	0.090087363149624	-	0.159181356644604	-
			10	0.046080582339233	0.9672	0.080719555365971	0.9797
			20	0.024318500122052	0.9221	0.041278317744786	0.9675
			40	0.014263331789890	0.7697	0.022039911004129	0.9053
		P^1	5	0.089685315404578	-	0.158930336103160	-
			10	0.045285538906416	0.9858	0.080234299480900	0.9861
			20	0.022772885774461	0.9917	0.040331924115622	0.9923
			40	0.011427368276094	0.9948	0.020222651900295	0.9960
		P^2	5	0.089666839925252	-	0.158930335672200	-
			10	0.045267355562751	0.9861	0.080234298636511	0.9861
			20	0.022754850556509	0.9923	0.040331922445057	0.9923
			40	0.011409407707825	0.9960	0.020222648577901	0.9960
	$\frac{3+e^t}{8}$	P^0	5	0.092278282392325	-	0.163075784277075	-
			10	0.047588731867645	0.9554	0.083422842300834	0.9670
			20	0.025137243620514	0.9208	0.042790838444117	0.9631
			40	0.014637973590646	0.7801	0.022797986729060	0.9084
		P^1	5	0.091877239195643	-	0.162817382350204	-
			10	0.046811118261484	0.9729	0.082940253881480	0.9731
			20	0.023637091702674	0.9858	0.041865614610574	0.9863
			40	0.011883655939878	0.9921	0.021033322221825	0.9931
		P^2	5	0.091859870837349	-	0.162817381813411	-
			10	0.046794026688965	0.9731	0.082940252808304	0.9731
			20	0.023620142459790	0.9863	0.041865612465820	0.9863
			40	0.011866778883860	0.9931	0.021033317934436	0.9931
	$\frac{3+5t}{10}$	P^0	5	0.092426147173819	-	0.163338581702283	-
			10	0.047626659435398	0.9565	0.083490784014887	0.9682
			20	0.025142841044313	0.9216	0.042801172563352	0.9640
			40	0.014635468540203	0.7807	0.022792929313315	0.9091
		P^1	5	0.092033960297031	-	0.163093173386189	-
			10	0.046857711782370	0.9739	0.083020884986288	0.9742
			20	0.023651002086529	0.9864	0.041888334452081	0.9869
			40	0.011888473556514	0.9923	0.021039934224149	0.9934
		P^2	5	0.092015469364094	-	0.163093172965469	-
			10	0.046839518094077	0.9742	0.083020884169484	0.9742
			20	0.023632961090627	0.9869	0.041888332842850	0.9869
			40	0.011870509954076	0.9934	0.021039931030017	0.9934

The convergence results are showed for both L^∞ norm and L^2 norm of the error. By varying the value of the parameter δ, the different $\alpha(t)$, and the polynomial approximations from P^0 to P^2. For uniform meshes, numerical errors and convergence rates with different δ *and* $\alpha(t)$ are shown in Tables 1–3 for $k = 0$, 1 and 2, respectively. The convergent results in Tables 1 and 2 illustrate that the spatial convergence rate can attain $O(h^{k+1})$ for piecewise P^k polynomials. Table 3 shows the temporal convergence rate is closed to $O(\Delta t)$ by using our scheme, this is consistent with the theoretical results.

5. Conclusions

In this article, an accurate numerical method is presented to solve a class of variable-order (VO) fourth-order problem with Caputo-Fabrizio derivative. Based on finite difference method in time and LDG method in space, we obtain a fully discrete scheme. By taking the generalized alternating numerical fluxes, the unconditional stability and convergence are proved in detail. Some numerical experiments are shown which illustrates the effectiveness of our scheme.

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Conflicts of interest

The authors declare that they have no conflicts of interest.

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