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Research article

The critical delay of the consensus for a class of multi-agent system involving task strategies

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Abstract: The time delay may induce oscillatory behaviour in multi-agent systems, which may destroy the consensus of the system. Therefore, the critical delay that is the maximum value of the delay to guarantee the consensus of the system, is an important performance index of multi-agent systems. This paper studies the influence of the processing delay on the consensus for a class of multi-agent system involving task strategies. The first-order system with a single delay and the second-order system with two different delays are investigated respectively. A critical delay independent of strategies and a critical region of the 2-D plane that depends on strategies is obtained for the first-order and the second-order system respectively. Specifically, a geometric method was used for the case of two different delays. Several numerical simulations are presented to explain the results.

Keywords: multi-agent system; consensus; processing delay; task strategies

1. Introduction

With the development of artificial intelligence, multi-agent systems have attracted extensive attention of researchers in computer science, physics, biology, social science and control engineering. The design of multi-agent systems is greatly influenced by the collective behaviour of animals in nature, such as ant colony gathering, birds flocking and fish swarming. Generally, a multi-agent system consists of multiple independent autonomous or semi-autonomous agents interconnected through a communication network, with research focuses including consensus [1], flocking [2], swarming [3], collision avoiding [4, 5], formation control [6, 7] and event-triggered control [8, 9] etc. Consensus describes the process of agents coordination which has important applications in opinion dynamics and engineering control, and has been thoroughly analysed yielding a number of conditions guaranteeing that the agents reach consensus in the past decade, see [10–14].

The early literature on consensus has mainly focused on the analysis of autonomous systems whose final state after reaching consensus depends only on the initial configuration of the system. However, the application of autonomous system is limited because it is inconvenient and inflexible to use the target state to design the initial configuration of the system when the system is required to reach a specified state. One type of intervention, adding external forces to the system to reach the desired target state, is applicable to a variety of real-world contexts, such as financial markets, public opinion and a team of UAVs, and has been prevalent in multi-agent systems. Further, such interventions usually only affect a subset of individuals in the system, resulting in the leader-follower structure of multi-agent systems [15–17].

In a multi-agent system with leader-follower structure, the leaders refer to the part of agents who master information about the target state (i.e., those affected by the intervention), and the rest of the agents in the system is called followers. Leaders should not only carry out the information communication among the agents in the system, but also track the target state, while followers only are required to join in the information communication. In most previous researches, the information communication and the target tracking were separated, and the strength of target tracking was assumed to be finite based on the actual situation that the external force is limited by energy, equipment and technology.

Different from the above studies, this paper focuses more on the agent performance than on the external force. Considering the limited ability of agents, we assume that each agent has a task strategy to properly allocate its limited energy for the information communication and the target tracking. The detail of the first-order multi-agent system involving task strategies $\alpha_i \in [0, m](i = 1, 2, \cdots, N)$, is as follows:

$$\dot{x}_i(t) = f(t) + \alpha_i(x_0(t) - x_i(t)) + (m - \alpha_i)\frac{1}{N}\sum_{j=1}^{N}(x_j(t) - x_i(t)), \quad i = 1, 2, \cdots, N, \tag{1.1}$$

where $x_i(t)$ is the position of the ith agent, $x_0(t)$ is the target state satisfying $\dot{x}_0(t) = f(t)$, $t \geq 0$, $f(t) \in C([0, \infty), \mathbb{R})$, $m \in \mathbb{R}_+$ represents the maximal strength of the information communication. For the ith agent, the strength of the information communication is $m - \alpha_i$ and the strength of the target tracking is α_i. If $\alpha_i > 0$, the ith agent is a leader, otherwise a follower. $\alpha = \sum_i \alpha_i$ is called the total strength of the target tracking of the system. The concept of the above strategy was proposed by Piccoli et al. [18] in 2016. They considered strategies $\{\alpha_i\}_{i=1}^N$ as controls and focused on finding optimal control strategies $\{\alpha_i\}_{i=1}^N$ to minimize the cost $\frac{1}{N}\sum_{i=1}^N \|x_i(T) - x_0(T)\|$, where $T > 0$ was the final time. The Eq (1.1) with the non-linear information communication was investigated in [19], and the sufficient conditions were proposed to guarantee that the system achieves consensus. In addition to the first-order system (1.1), this paper also focuses on the second-order multi-agent system involving task strategies, written as

$$\begin{aligned} \ddot{x}_i(t) = g(t) &+ \alpha_i \left[\gamma(v_0(t) - \dot{x}_i(t)) + (1 - \gamma)(x_0(t) - x_i(t))\right] \\ &+ (m - \alpha_i)\left[\frac{\gamma}{N}\sum_{j=1}^{N}(\dot{x}_j(t) - \dot{x}_i(t)) + \frac{1 - \gamma}{N}\sum_{j=1}^{N}(x_j(t) - x_i(t))\right], \end{aligned} \tag{1.2}$$

where $\gamma \in [0, 1]$ is the weight coefficient of velocities information, which measures the proportion of the velocities information in the control. Then the weight coefficient of positions information is $1 - \gamma$. $x_0(t)$ is the target state satisfying $\dot{x}_0(t) = v_0(t)$, $\dot{v}_0(t) = g(t)$, $t \geq 0$, $g(t) \in C([0, \infty), \mathbb{R})$. Then, we present the mathematical definition of the consensus of the system for Eqs (1.1) and (1.2).

Definition 1.1. Suppose $\{x_i(t)\}_{i=1}^N$ is a solution of the Eq (1.1), $x_0(t)$ is the target state satisfying $\dot{x}_0(t) = f(t)$, $t \geq 0$, $f(t) \in C([0,\infty),\mathbb{R})$. The Eq (1.1) is said to achieve consensus and reach the target state if and only if

$$\lim_{t\to\infty} |x_i(t) - x_0(t)| = 0, \quad i = 1, 2, \cdots, N.$$

Suppose $\{x_i(t)\}_{i=1}^N$ is a solution of the Eq (1.2), $x_0(t)$ is the target state satisfying $\dot{x}_0(t) = v_0(t)$, $\dot{v}_0(t) = g(t)$, $t \geq 0$, $g(t) \in C([0,\infty),\mathbb{R})$. The Eq (1.2) is said to achieve consensus and reach the target state if and only if

$$\lim_{t\to\infty} |x_i(t) - x_0(t)| = 0 \quad \textit{and} \quad \lim_{t\to\infty} |\dot{x}_i(t) - v_0(t)| = 0, \quad i = 1, 2, \cdots, N.$$

The time delay is an important topic in the research of multi-agent systems and has been widely studied. The causes of the time delay can be divided into two types: information transmission delay and information processing delay. The transmission delay means that it takes time for agents to receive information from others limited by the speed of communication, see [20–22]. The processing delay, also known as the reaction delay, refers to the time required for devices to process information, see [23–25]. The effect of time delay on consensus formation of multi-agent systems is an issue that cannot be ignored. Olfsti-Saber and Murray [26] gave a sufficient condition of consensus for a first-order system with time delay on balanced graphs and showed that there exists a trade-off between the control gain and the critical delay. Yu et al. [27] obtained some necessary and sufficient conditions for second-order consensus on directed graphs. Ma et al. studied a second-order consensus system with unstable elements over undirected graphs and maximized the critical delay by optimizing parameters [28]. A first-order consensus system with unstable elements over directed graphs was investigated in [29] and maximal critical delay was achieved through solving a nonsmooth max-min problem. For other relevant literature, see [30–32].

In this paper, we study the effect of the processing delay on the consensus of the first-order system in Eq (1.1) and the second-order system in Eq (1.2), and analyse the relationship between the processing delay and the strategies. According to the stability of linear systems in the theory of functional differential equations, the system would achieve consensus by ensuring that roots of the characteristic equation of the system have negative real parts. The specific content of the conclusion of stability is written as:

Lemma 1.2. *[33] For a linear functional equation $\dot{u}(t) = Au(t-r) + Bu(t-s)$, where $u(t) \in \mathbb{R}^N$, $A, B \in \mathbb{R}^{N\times N}$ and $r, s \in \mathbb{R}_+$. Its characteristic equation is $h(\lambda) = \mathrm{Det}\left(\lambda I - Ae^{-\lambda r} - Be^{-\lambda s}\right) = 0$. Define $a = \sup\{\mathrm{Re}\lambda : h(\lambda) = 0\}$, if $a < 0$, the zero solution of the equation is globally asymptotically stable.*

The rest of this paper is organized as follows. In Section 2, the Eq (1.1) with a single delay is investigated. Using the continuous dependence of the equation on the processing delay, we obtain the critical delay τ^* that ensures that the Eq (1.1) achieves consensus, show that the critical delay τ^* of the Eq (1.1) is independent of the strategies α_i. In Section 3, the Eq (1.2) with two different delays is investigated. Inspired by [34–36] and using the properties of plane geometry, we identify the critical region D in $\mathbb{R}^2$ that guarantees the system to achieve consensus, find that the shape of the critical region D is affected by the strategies α_i. In Section 4, several numerical simulations are presented to explain our results. Finally, we give the conclusion and discussion in Section 5.

2. The critical delay of the first-order multi-agent system

Adding the processing delay in the Eq (1.1) yields

$$\dot{x}_i(t) = f(t) + \alpha_i(x_0(t-\tau) - x_i(t-\tau)) + (m-\alpha_i)\frac{1}{N}\sum_{j=1}^{N}(x_j(t-\tau) - x_i(t-\tau)). \tag{2.1}$$

where $\tau \in \mathbb{R}_+$ represents the time required for the system to process the information of positions. Next we will transform the consensus problem of the Eq (2.1) into the stability problem of a linear autonomous system with a single delay, which further becomes the problem of judging whether the roots of the characteristic equation have negative real parts. Set $y_i(t) = x_i(t) - x_0(t)$, then the above model reduces to

$$\dot{y}_i(t) = -\alpha_i y_i(t-\tau) + (m-\alpha_i)\frac{1}{N}\sum_{j=1}^{N}(y_j(t-\tau) - y_i(t-\tau)).$$

Define

$$Y(t) = (y_1(t), y_2(t), \cdots, y_N(t))^T \in \mathbb{R}^N$$

$$and \quad \Gamma = \begin{bmatrix} \frac{m-\alpha_1}{N} & \frac{m-\alpha_1}{N} & \cdots & \frac{m-\alpha_1}{N} \\ \frac{m-\alpha_2}{N} & \frac{m-\alpha_2}{N} & \cdots & \frac{m-\alpha_2}{N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{m-\alpha_N}{N} & \frac{m-\alpha_N}{N} & \cdots & \frac{m-\alpha_N}{N} \end{bmatrix}, \tag{2.2}$$

then rewrite the above equations in matrix form

$$\dot{Y}(t) = -mY(t-\tau) + \Gamma Y(t-\tau).$$

Compute the eigenvalues of the matrix Γ

$$\begin{aligned} &|\mu I - \Gamma| \\ &= \begin{vmatrix} \mu - \frac{m-\alpha_1}{N} & -\frac{m-\alpha_1}{N} & \cdots & -\frac{m-\alpha_1}{N} \\ -\frac{m-\alpha_2}{N} & \mu - \frac{m-\alpha_2}{N} & \cdots & -\frac{m-\alpha_2}{N} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{m-\alpha_N}{N} & -\frac{m-\alpha_N}{N} & \cdots & \mu - \frac{m-\alpha_N}{N} \end{vmatrix} \\ &= \begin{vmatrix} \mu - (m-\alpha_1) & -\frac{m-\alpha_1}{N} & \cdots & -\frac{m-\alpha_1}{N} \\ \mu - (m-\alpha_2) & \mu - \frac{m-\alpha_2}{N} & \cdots & -\frac{m-\alpha_2}{N} \\ \vdots & \vdots & \ddots & \vdots \\ \mu - (m-\alpha_N) & -\frac{m-\alpha_N}{N} & \cdots & \mu - \frac{m-\alpha_N}{N} \end{vmatrix} \\ &= \begin{vmatrix} \mu - (m-\alpha_1) & -\frac{\mu}{N} & \cdots & -\frac{\mu}{N} \\ \mu - (m-\alpha_2) & \mu - \frac{\mu}{N} & \cdots & -\frac{\mu}{N} \\ \vdots & \vdots & \ddots & \vdots \\ \mu - (m-\alpha_N) & -\frac{\mu}{N} & \cdots & \mu - \frac{\mu}{N} \end{vmatrix} = \begin{vmatrix} \mu - (m-\alpha_1) & -\frac{\mu}{N} & \cdots & -\frac{\mu}{N} \\ \alpha_2 - \alpha_1 & \mu & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_N - \alpha_1 & 0 & \cdots & \mu \end{vmatrix} \\ &= \begin{vmatrix} \mu - (m - \sum_{i=1}^{N}\frac{\alpha_i}{N}) & 0 & \cdots & 0 \\ \alpha_2 - \alpha_1 & \mu & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_N - \alpha_1 & 0 & \cdots & \mu \end{vmatrix} = \mu^{N-1}\left[\mu - \left(m - \frac{\alpha}{N}\right)\right] = 0, \end{aligned}$$

where $\alpha = \sum_{i=1}^{N} \alpha_i$. Note that rank($\Gamma$) = 1, hence the matrix Γ is similarly diagonalized, i.e., there exists a non-singular matrix P such that $P\Gamma P^{-1} = J$, where $J = \text{diag}\left(m - \frac{\alpha}{N}, 0, \cdots, 0\right)$. Let $Z(t) = PY(t)$, then the equation becomes

$$\dot{Z}(t) = -mZ(t-\tau) + JZ(t-\tau).$$

The characteristic equation of the above system is

$$h(\lambda) = \left(\lambda + me^{-\lambda\tau}\right)^{N-1}\left(\lambda + \frac{\alpha}{N}e^{-\lambda\tau}\right) = 0. \tag{2.3}$$

Hence, applying Lemma 1.2 we know that the Eq (2.1) achieves consensus if all roots of the Eq (2.3) have negative real parts. Using the continuous dependence of the Eq (2.3) on the processing delay τ, we obtain the following result.

Theorem 2.1. *Assume $\alpha > 0$. Let $\Lambda = \sup\{\text{Re}\lambda : h(\lambda) = 0\}$, then there exists a critical delay*

$$\tau^* = \frac{\pi}{2m}$$

such that $\Lambda < 0$, $\forall 0 \le \tau < \tau^*$.

Proof. Owing to $m, \frac{\alpha}{N}, \tau \in \mathbb{R}$, $\overline{h(-i\omega)} = h(i\omega)$, $\forall \omega \in \mathbb{R}$, which means that if $\lambda = i\omega$ is a root of Eq (2.3), then also $\lambda = -i\omega$ is a root. Without loss of generality let $h(i\omega) = 0$, $\omega \in \mathbb{R}_+$, then we obtain

$$i\omega + me^{-i\omega\tau} = 0 \quad or \quad i\omega + \frac{\alpha}{N}e^{-i\omega\tau} = 0.$$

From $i\omega + me^{-i\omega\tau} = 0$ we have $m\cos(\omega\tau) = 0$ and $\omega = m\sin(\omega\tau)$. Adding the squares of the two equations yields $\omega = m$, then we have $\tau_m^k = \frac{\pi}{2m} + \frac{k\pi}{m}$, $k \in \mathbb{Z}$. Similarly, from $i\omega + \frac{\alpha}{N}e^{-i\omega\tau} = 0$ we could obtain $\tau_\alpha^k = \frac{\pi N}{2\alpha} + \frac{k\pi N}{\alpha}$, $k \in \mathbb{Z}$. Define

$$\tau^* = \min\left\{\tau_m^k, \tau_\alpha^k, k = 0, 1, \cdots\right\} = \frac{\pi}{2m}.$$

When $\tau = 0$,

$$h(\lambda) = (\lambda + m)^{N-1}\left(\lambda + \frac{\alpha}{N}\right) = 0.$$

By $\alpha > 0$, $\Lambda < 0$ which indicates that all roots of the Eq (2.3) lie on the left half complex plane when $\tau = 0$. Since $h(\lambda)$ is continuously dependent on τ, Λ is continuously dependent on τ. Therefore, as the increase of τ from 0 to ∞, some roots of the Eq (2.3) touch the imaginary axis of the complex plane for the first time when $\tau = \tau^*$. Then, we conclude that $\Lambda < 0$ if $0 \le \tau < \tau^*$ and $\Lambda = 0$ if $\tau = \tau^*$. □

Remark 1. Theorem 2.1 shows that the critical delay of the Eq (2.1) has nothing to do with strategies α_i, but only with the maximal strength m.

Remark 2. If $\Lambda < 0$ when $\tau = 0$, then the existence of the critical delay τ^* is equivalent to the existence of some roots of the Eq (2.3) crossing the imaginary axis of the complex plane as the increase of τ from 0 to ∞. If the critical delay τ^* exists, then it is the value of the delay when these roots first touch the imaginary axis.

3. The critical region of the second-order multi-agent system

Different from the Eq (1.1), the Eq (1.2) has to process not only the information of positions but also the information of velocities, so there are two different processing delays in the system. Adding the processing delay in the Eq (1.2) yields

$$\begin{aligned}\ddot{x}_i(t) = g(t)&+\alpha_i\left[\gamma(v_0(t-\tau_1)-\dot{x}_i(t-\tau_1))+(1-\gamma)(x_0(t-\tau_2)-x_i(t-\tau_2))\right]\\&+(m-\alpha_i)\left[\frac{\gamma}{N}\sum_{j=1}^{N}(\dot{x}_j(t-\tau_1)-\dot{x}_i(t-\tau_1))\right.\\&\left.+\frac{1-\gamma}{N}\sum_{j=1}^{N}(x_j(t-\tau_2)-x_i(t-\tau_2))\right],\end{aligned}\tag{3.1}$$

where τ_1 and τ_2 represent times required for the system to process the information of positions and velocities, respectively. Similarly, we will transform the consensus problem of the Eq (3.1) into the stability problem of a linear autonomous system with two different delay. Set $y_i(t) = x_i(t) - x_0(t)$, the above model is simplified as

$$\begin{aligned}\ddot{y}_i(t) = &-\alpha_i\left[\gamma\dot{y}_i(t-\tau_1)+(1-\gamma)y_i(t-\tau_2)\right]\\&+(m-\alpha_i)\left[\frac{\gamma}{N}\sum_{j=1}^{N}(\dot{y}_j(t-\tau_1)-\dot{y}_i(t-\tau_1))\right.\\&\left.+\frac{1-\gamma}{N}\sum_{j=1}^{N}(y_j(t-\tau_2)-y_i(t-\tau_2))\right],\end{aligned}$$

Define $Y(t)$ and Γ same as Eq (2.2), then rewrite the equations in matrix form

$$\ddot{Y}(t) = -m\gamma\dot{Y}(t-\tau_1)+\gamma\Gamma\dot{Y}(t-\tau_1)-m(1-\gamma)Y(t-\tau_2)+(1-\gamma)\Gamma Y(t-\tau_2).$$

Let $Z(t) = PY(t)$, then the above equation becomes

$$\ddot{Z}(t) = -m\gamma\dot{Z}(t-\tau_1)+\gamma J\dot{Z}(t-\tau_1)-m(1-\gamma)Z(t-\tau_2)+(1-\gamma)JZ(t-\tau_2).$$

The characteristic equation of the above system is

$$\begin{aligned}h(\lambda) = &\left[\lambda^2+\gamma m\lambda e^{-\lambda\tau_1}+(1-\gamma)me^{-\lambda\tau_2}\right]^{N-1}\\&\left[\lambda^2+\gamma\frac{\alpha}{N}\lambda e^{-\lambda\tau_1}+(1-\gamma)\frac{\alpha}{N}e^{-\lambda\tau_2}\right]=0.\end{aligned}\tag{3.2}$$

Let $\Lambda = \sup\{\mathrm{Re}\lambda : h(\lambda) = 0\}$. When $\tau_1 = \tau_2 = 0$, we have

$$h(\lambda) = \left[\lambda^2+\gamma m\lambda+(1-\gamma)m\right]^{N-1}\left[\lambda^2+\gamma\frac{\alpha}{N}\lambda+(1-\gamma)\frac{\alpha}{N}\right]=0.$$

If $\alpha > 0$, it's easy to verify that $\Lambda < 0$. Next, we firstly consider three simple cases: (1) $\tau_1 > 0$ and $\tau_2 = 0$; (2) $\tau_1 = 0$ and $\tau_2 > 0$; (3) $\tau_1 = \tau_2 > 0$.

Theorem 3.1. *Assume $\alpha > 0$, $0 < \gamma < 1$. If one of the following three cases holds:*

(1)

$$0 = \tau_2 < \tau_1 < \tau_1^* = \frac{\pi}{2\left(\frac{\gamma m}{2} + \sqrt{(1-\gamma)m + \frac{\gamma^2 m^2}{4}}\right)},$$

(2)

$$0 = \tau_1 < \tau_2 < \tau_2^* = \min_{s \in \{m, \frac{\alpha}{N}\}} \frac{\arctan\left(\frac{\gamma s}{\sqrt{-\frac{\gamma^2 s^2}{2} + \sqrt{(1-\gamma)^2 s^2 + \frac{\gamma^4 s^4}{4}}}}\right)}{\sqrt{-\frac{\gamma^2 s^2}{2} + \sqrt{(1-\gamma)^2 s^2 + \frac{\gamma^4 s^4}{4}}}},$$

(3)

$$0 < \tau_1 = \tau_2 < \tau^* = \frac{\arctan\left(\frac{\gamma}{1-\gamma}\sqrt{\frac{\gamma^2 m^2}{2} + \sqrt{(1-\gamma)^2 m^2 + \frac{\gamma^4 m^4}{4}}}\right)}{\sqrt{\frac{\gamma^2 m^2}{2} + \sqrt{(1-\gamma)^2 m^2 + \frac{\gamma^4 m^4}{4}}}}.$$

Then $\Lambda < 0$.

Proof. Because the proofs of (1) and (2) are very similar to the proof of (3), we will just prove the case $\tau_1 = \tau_2 = \tau > 0$ here. Without loss of generality let $h(i\omega) = 0$, $\omega \in \mathbb{R}_+$, then we have

$$(i\omega)^2 + i\gamma m\omega e^{-i\omega\tau} + (1-\gamma)me^{-i\omega\tau} = 0 \tag{3.3}$$

or

$$(i\omega)^2 + i\gamma\frac{\alpha}{N}\omega e^{-i\omega\tau} + (1-\gamma)\frac{\alpha}{N}e^{-i\omega\tau} = 0. \tag{3.4}$$

According to the Eq (3.3) we obtain

$$\begin{cases} \omega^2 = \gamma m\omega\sin(\omega\tau) + (1-\gamma)m\cos(\omega\tau), \\ \ \ 0 = \gamma m\omega\cos(\omega\tau) - (1-\gamma)m\sin(\omega\tau). \end{cases} \tag{3.5}$$

Adding the squares of the above two equations yields

$$\omega^4 = \gamma^2 m^2 \omega^2 + (1-\gamma)^2 m^2,$$

then we have

$$\omega = \sqrt{\frac{\gamma^2 m^2}{2} + \sqrt{(1-\gamma)^2 m^2 + \frac{\gamma^4 m^4}{4}}}.$$

On the other hand, the second equation of Eq (3.5) gives

$$\tau = \frac{\arctan\left(\frac{\gamma\omega}{1-\gamma}\right)}{\omega} = \frac{\arctan\left(\frac{\gamma}{1-\gamma}\sqrt{\frac{\gamma^2 m^2}{2} + \sqrt{(1-\gamma)^2 m^2 + \frac{\gamma^4 m^4}{4}}}\right)}{\sqrt{\frac{\gamma^2 m^2}{2} + \sqrt{(1-\gamma)^2 m^2 + \frac{\gamma^4 m^4}{4}}}} \triangleq \tau_m.$$

In the same way, according to the Eq (3.4) we could obtain

$$\tau = \frac{\arctan\left(\frac{\gamma}{1-\gamma}\sqrt{\frac{\gamma^2\alpha^2}{2N^2}+\sqrt{\frac{(1-\gamma)^2\alpha^2}{N^2}+\frac{\gamma^4\alpha^4}{4N^4}}}\right)}{\sqrt{\frac{\gamma^2\alpha^2}{2N^2}+\sqrt{\frac{(1-\gamma)^2\alpha^2}{N^2}+\frac{\gamma^4\alpha^4}{4N^4}}}} \triangleq \tau_\alpha.$$

The critical delay $\tau^* = \min\{\tau_m, \tau_\alpha\}$, so we need to determine the monotonicity of the function

$$J(s) = \frac{\arctan\left(\frac{\gamma}{1-\gamma}\sqrt{\frac{\gamma^2 s^2}{2}+\sqrt{(1-\gamma)^2 s^2+\frac{\gamma^4 s^4}{4}}}\right)}{\sqrt{\frac{\gamma^2 s^2}{2}+\sqrt{(1-\gamma)^2 s^2+\frac{\gamma^4 s^4}{4}}}}.$$

Because $\sqrt{\frac{\gamma^2 s^2}{2}+\sqrt{(1-\gamma)^2 s^2+\frac{\gamma^4 s^4}{4}}}$ is monotonically increasing with respect to s and $\frac{\arctan\xi}{\xi}$ is monotonically decreasing with respect to ξ, $J(s)$ is monotonically decreasing with respect to s. Hence, we conclude $\tau^* = \tau_m$. □

Remark 3. For the case (2), we cannot determine the monotonicity of the function

$$J(s) = \frac{\arctan\left(\frac{\gamma s}{\sqrt{-\frac{\gamma^2 s^2}{2}+\sqrt{(1-\gamma)^2 s^2+\frac{\gamma^4 s^4}{4}}}}\right)}{\sqrt{-\frac{\gamma^2 s^2}{2}+\sqrt{(1-\gamma)^2 s^2+\frac{\gamma^4 s^4}{4}}}}$$

with respect to s. Actually, the numerical curve of $J(s)$ shows that $J(s)$ is monotonically increasing with respect to s. In addition, we assume $0 < \gamma < 1$ in Theorem 3.1. By simple calculation we could obtain that if $\gamma = 1$, the critical delay of τ_1 is $\frac{\pi}{2m}$; If $\gamma = 0$, the critical delay of τ_2 is $\frac{\pi}{\sqrt{m}}$.

In the following we consider the case $\tau_1 \neq \tau_2$. In this case, the method applied in Theorems 2.1 and 3.1 is no longer feasible because we cannot obtain an explicit expression of τ^* by solving for ω. Now we use a geometric method to analyse the Eq (3.2). From Eq (3.2) we have

$$\lambda^2 + \gamma m\lambda e^{-\lambda\tau_1} + (1-\gamma)me^{-\lambda\tau_2} = 0$$

or

$$\lambda^2 + \gamma\frac{\alpha}{N}\lambda e^{-\lambda\tau_1} + (1-\gamma)\frac{\alpha}{N}e^{-\lambda\tau_2} = 0.$$

Since $\lambda = 0$ is not a root of the above equations, the deformations of the equations are

$$1 + \frac{\gamma m}{\lambda}e^{-\lambda\tau_1} + \frac{(1-\gamma)m}{\lambda^2}e^{-\lambda\tau_2} = 0 \tag{3.6}$$

and

$$1 + \frac{\gamma\alpha}{N\lambda}e^{-\lambda\tau_1} + \frac{(1-\gamma)\alpha}{N\lambda^2}e^{-\lambda\tau_2} = 0. \tag{3.7}$$

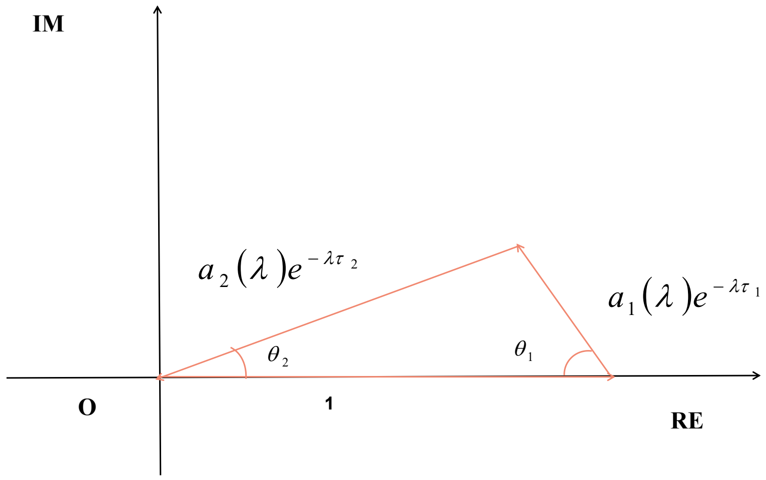

Figure 1. 1, $a_1(\lambda)e^{-\lambda\tau_1}$ and $a_2(\lambda)e^{-\lambda\tau_2}$ form a triangle in the complex plane.

For the Eq (3.6), let $a_1(\lambda) = \frac{\gamma m}{\lambda}$ and $a_2(\lambda) = \frac{(1-\gamma)m}{\lambda^2}$. Think of 1, $a_1(\lambda)e^{-\lambda\tau_1}$ and $a_2(\lambda)e^{-\lambda\tau_2}$ as three vectors in the complex plane $\mathbb{C}$, then λ is a root of the Eq (3.6) if and only if the three vectors are connected head to tail to form a triangle in the complex plane (See Figure 1).

Therefore, we could use the triangle property to get the relationship between λ, τ_1 and τ_2, and try to obtain the critical delay.

Theorem 3.2. *Assume $\alpha > 0$, $0 < \gamma < 1$. Then there exists a connected region $D \subseteq \mathbb{R}_+ \times \mathbb{R}_+$ such that $\Lambda < 0$, $\forall(\tau_1, \tau_2) \in D$. In addition, the connected region D and its boundary ∂D satisfy*

$$D_1 \triangleq D \bigcap \{(\tau_1,\tau_2)|\tau_1 \in \mathbb{R}_+, \tau_2 = 0\} = [0, \tau_1^*) \times \{0\},$$

$$D_2 \triangleq D \bigcap \{(\tau_1,\tau_2)|\tau_1 = 0, \tau_2 \in \mathbb{R}_+\} = \{0\} \times [0, \tau_2^*),$$

$$D_1 \cup D_2 \cup \{(\tau^*, \tau^*)\} \subseteq \partial D \quad and \quad \partial D \setminus (D_1 \cup D_2) \subseteq \Phi \cup \Psi,$$

where

$$\Phi = \left\{ \begin{array}{c} (\tau_1,\tau_2) \\ \in \mathbb{R}_+ \times \mathbb{R}_+ \end{array} \middle| \begin{array}{l} \tau_1 = \dfrac{\frac{3\pi}{2} + (2u-1)\pi \pm \theta_1}{\omega}, \quad \tau_2 = \dfrac{\pi + (2v-1)\pi \mp \theta_2}{\omega}, \\ u = u_0^\pm, u_0^\pm + 1, \cdots, \quad v = v_0^\mp, v_0^\mp + 1, \cdots, \quad \omega \in \Omega \end{array} \right\},$$

$$\Psi = \left\{ \begin{array}{c} (\tau_1,\tau_2) \\ \in \mathbb{R}_+ \times \mathbb{R}_+ \end{array} \middle| \begin{array}{l} \tau_1 = \dfrac{\frac{3\pi}{2} + (2p-1)\pi \pm \vartheta_1}{\omega}, \quad \tau_2 = \dfrac{\pi + (2q-1)\pi \mp \vartheta_2}{\omega}, \\ p = p_0^\pm, p_0^\pm + 1, \cdots, \quad q = q_0^\mp, q_0^\mp + 1, \cdots, \quad \omega \in \Upsilon \end{array} \right\},$$

τ_1^*, τ_2^* and τ^* are defined in Theorem 3.1, θ_1, θ_2, $u_0^\pm$, $v_0^\mp$, Ω, ϑ_1, ϑ_2, $p_0^\pm$, $q_0^\mp$ and Υ are defined in the proof.

Proof. The proof is divided into three steps.

The first step: use the triangle inequality to obtain the range of ω such that $h(i\omega) = 0$. Without loss of generality let $\lambda = i\omega$, $\omega \in \mathbb{R}_+$, then from Eq (3.6) we have

$$1 + a_1(i\omega)e^{-i\omega\tau_1} + a_2(i\omega)e^{-i\omega\tau_2} = 0,$$

where $a_1(i\omega) = -\frac{\gamma m}{\omega} i$ and $a_2(i\omega) = -\frac{(1-\gamma)m}{\omega^2}$. According to the triangle inequality that the length of any one side does not exceed the sum of the other two sides, we obtain

$$|a_1(i\omega)| + |a_2(i\omega)| \geq 1, \quad -1 \leq |a_1(i\omega)| - |a_2(i\omega)| \leq 1,$$

i.e.

$$\omega^2 - \gamma m\omega - (1-\gamma)m \leq 0, \quad \omega^2 - \gamma m\omega + (1-\gamma)m \geq 0, \quad \omega^2 + \gamma m\omega - (1-\gamma)m \geq 0.$$

Denoting the range of ω by Ω and solving the inequalities yields

$$\Omega = \begin{cases} \left[-\frac{\gamma m}{2} + \sqrt{\frac{\gamma^2 m^2}{4} + (1-\gamma)m}, \frac{\gamma m}{2} + \sqrt{\frac{\gamma^2 m^2}{4} + (1-\gamma)m}\right], & \frac{4(1-\gamma)}{\gamma^2} \geq m, \\ \left[-\frac{\gamma m}{2} + \sqrt{\frac{\gamma^2 m^2}{4} + (1-\gamma)m}, \frac{\gamma m}{2} - \sqrt{\frac{\gamma^2 m^2}{4} - (1-\gamma)m}\right] \cup & \\ \left[\frac{\gamma m}{2} + \sqrt{\frac{\gamma^2 m^2}{4} - (1-\gamma)m}, \frac{\gamma m}{2} + \sqrt{\frac{\gamma^2 m^2}{4} + (1-\gamma)m}\right], & \frac{4(1-\gamma)}{\gamma^2} < m. \end{cases}$$

Ω contains all the values of ω that make the roots of the Eq (3.6) lie on the imaginary axis of the complex plane at $i\omega$.

The second step: use Ω to calculate all the values of $(\tau_1, \tau_2) \in \mathbb{R}_+ \times \mathbb{R}_+$ that make $h(i\omega) = 0$. Define $\measuredangle a_1(i\omega) \in [0, 2\pi)$ is the angle between $a_1(i\omega)$ with the positive direction of the real axis, $\theta_1 \in [0, \pi]$ is the inner angle of the triangle formed by $a_1(i\omega)e^{-i\omega\tau_1}$ and 1. Similar definitions apply to $\measuredangle a_2(i\omega) \in [0, 2\pi)$ and $\theta_2 \in [0, \pi]$. Because $a_1(i\omega) = -\frac{\gamma m}{\omega} i$ and $a_2(i\omega) = -\frac{(1-\gamma)m}{\omega^2}$, $\measuredangle a_1(i\omega) = \frac{3\pi}{2}$ and $\measuredangle a_2(i\omega) = \pi$. By the law of cosine we have

$$\theta_1 = \arccos\left(\frac{1 + |a_1(i\omega)|^2 - |a_2(i\omega)|^2}{2|a_1(i\omega)|}\right) = \arccos\left(\frac{\omega}{2\gamma m} + \frac{\gamma m}{2\omega} - \frac{(1-\gamma)^2 m}{2\gamma\omega^3}\right),$$

$$\begin{aligned} \theta_2 &= \arccos\left(\frac{1 + |a_2(i\omega)|^2 - |a_1(i\omega)|^2}{2|a_2(i\omega)|}\right) \\ &= \arccos\left(\frac{\omega^2}{2(1-\gamma)m} + \frac{(1-\gamma)m}{2\omega^2} - \frac{\gamma^2 m}{2(1-\gamma)}\right). \end{aligned}$$

Using properties of plane geometry we know that the angle between $a_1(i\omega)e^{-i\omega\tau_1}$ with the positive direction of the real axis plus or minus θ_1 is equal to π, where plus or minus depends on whether the triangle is above or below the real axis. Similar results apply to $a_2(i\omega)e^{-i\omega\tau_2}$ and θ_2. Then, we could establish the relations between τ_1 with ω and τ_2 with ω respectively:

$$-\omega\tau_1 + 2u\pi + \measuredangle a_1(i\omega) \pm \theta_1 = \pi, \quad u \in \mathbb{Z},$$

$$-\omega\tau_2 + 2v\pi + \measuredangle a_2(i\omega) \mp \theta_2 = \pi, \quad v \in \mathbb{Z},$$

where $-\omega\tau_1 + 2u\pi \in [0, 2\pi)$ is the angle between $e^{-i\omega\tau_1}$ with the positive direction of the real axis, $-\omega\tau_2 + 2v\pi \in [0, 2\pi)$ is the angle between $e^{-i\omega\tau_2}$ with the positive direction of the real axis. Define $u_0^{\pm}$ and $v_0^{\mp}$ are the smallest positive integers to guarantee $\tau_1 > 0$ and $\tau_2 > 0$ respectively, then we obtain

$$\tau_1 = \frac{\measuredangle a_1(i\omega) + (2u-1)\pi \pm \theta_1}{\omega} = \frac{\frac{3\pi}{2} + (2u-1)\pi \pm \theta_1}{\omega}, \quad u = u_0^{\pm}, u_0^{\pm} + 1, \cdots,$$

$$\tau_2 = \frac{\angle a_2(i\omega) + (2v-1)\pi \mp \theta_2}{\omega} = \frac{\pi + (2v-1)\pi \mp \theta_2}{\omega}, \quad v = v_0^{\mp}, v_0^{\mp}+1, \cdots.$$

Define

$$\Phi = \left\{ \begin{matrix} (\tau_1,\tau_2) \\ \in \mathbb{R}_+ \times \mathbb{R}_+ \end{matrix} \middle| \begin{matrix} \tau_1 = \dfrac{\frac{3\pi}{2} + (2u-1)\pi \pm \theta_1}{\omega}, \quad \tau_2 = \dfrac{\pi + (2v-1)\pi \mp \theta_2}{\omega}, \\ u = u_0^{\pm}, u_0^{\pm}+1, \cdots, \quad v = v_0^{\mp}, v_0^{\mp}+1, \cdots, \quad \omega \in \Omega \end{matrix} \right\},$$

then $\Phi \subseteq \mathbb{R}_+ \times \mathbb{R}_+$ contains all the values of $(\tau_1, \tau_2) \subseteq \mathbb{R}_+ \times \mathbb{R}_+$ that make the roots of the Eq (3.6) lie on the imaginary axis.

Repeating the above steps for the Eq (3.7) yields that the roots of the Eq (3.7) lie on the imaginary axis if and only if $(\tau_1, \tau_2) \in \Psi$, where

$$\Psi = \left\{ \begin{matrix} (\tau_1,\tau_2) \\ \in \mathbb{R}_+ \times \mathbb{R}_+ \end{matrix} \middle| \begin{matrix} \tau_1 = \dfrac{\frac{3\pi}{2} + (2p-1)\pi \pm \vartheta_1}{\omega}, \quad \tau_2 = \dfrac{\pi + (2q-1)\pi \mp \vartheta_2}{\omega}, \\ p = p_0^{\pm}, p_0^{\pm}+1, \cdots, \quad q = q_0^{\mp}, q_0^{\mp}+1, \cdots, \quad \omega \in \Upsilon \end{matrix} \right\},$$

$$\vartheta_1 = \arccos\left(\frac{\omega N}{2\gamma\alpha} + \frac{\gamma\alpha}{2\omega N} - \frac{(1-\gamma)^2\alpha}{2\gamma\omega^3 N}\right),$$

$$\vartheta_2 = \arccos\left(\frac{\omega^2 N}{2(1-\gamma)\alpha} + \frac{(1-\gamma)\alpha}{2\omega^2 N} - \frac{\gamma^2\alpha}{2(1-\gamma)N}\right),$$

$$\Upsilon = \begin{cases} \left[-\dfrac{\gamma\alpha}{2N} + \sqrt{\dfrac{\gamma^2\alpha^2}{4N^2} + \dfrac{(1-\gamma)\alpha}{N}}, \dfrac{\gamma\alpha}{2N} + \sqrt{\dfrac{\gamma^2\alpha^2}{4N^2} + \dfrac{(1-\gamma)\alpha}{N}}\right], & \dfrac{4(1-\gamma)}{\gamma^2} \geq \dfrac{\alpha}{N}, \\ \left[-\dfrac{\gamma\alpha}{2N} + \sqrt{\dfrac{\gamma^2\alpha^2}{4N^2} + \dfrac{(1-\gamma)\alpha}{N}}, \dfrac{\gamma\alpha}{2N} - \sqrt{\dfrac{\gamma^2\alpha^2}{4N^2} - \dfrac{(1-\gamma)\alpha}{N}}\right] & \\ \cup \left[\dfrac{\gamma\alpha}{2N} + \sqrt{\dfrac{\gamma^2\alpha^2}{4N^2} - \dfrac{(1-\gamma)\alpha}{N}}, \dfrac{\gamma\alpha}{2N} + \sqrt{\dfrac{\gamma^2\alpha^2}{4N^2} + \dfrac{(1-\gamma)\alpha}{N}}\right], & \dfrac{4(1-\gamma)}{\gamma^2} < \dfrac{\alpha}{N}. \end{cases}$$

Hence, we proved that some roots of the Eq (3.2) lie on the imaginary axis if and only if $(\tau_1, \tau_2) \in \Phi \cup \Psi$.

The third step: combine Theorem 3.1 and $\Phi \cup \Psi$ to determine the connected region D. By the results in [35, 37], $\Phi \cup \Psi$ characterizes a series of continuous curves on $\mathbb{R}_+ \times \mathbb{R}_+$ and divides $\mathbb{R}_+ \times \mathbb{R}_+$ into a series of connected regions. Let D represent the connected region containing the origin $(0, 0)$, then Theorem 3.1 indicates that the connected region D and its boundary ∂D satisfy

$$D_1 \triangleq D \bigcap \{(\tau_1, \tau_2) | \tau_1 \in \mathbb{R}_+, \tau_2 = 0\} = [0, \tau_1^*) \times \{0\},$$

$$D_2 \triangleq D \bigcap \{(\tau_1, \tau_2) | \tau_1 = 0, \tau_2 \in \mathbb{R}_+\} = \{0\} \times [0, \tau_2^*),$$

$$D_1 \cup D_2 \cup \{(\tau^*, \tau^*)\} \subseteq \partial D \quad and \quad \partial D \setminus (D_1 \cup D_2) \subseteq \Phi \cup \Psi.$$

Because $\Lambda < 0$ when $(\tau_1, \tau_2) = (0, 0) \in D$, $\Lambda = 0$ when $(\tau_1, \tau_2) \in \Phi \cup \Psi$ and Λ is continuously dependent on (τ_1, τ_2), we conclude that $\Lambda < 0$, $\forall (\tau_1, \tau_2) \in D$. □

The the connected region D in Theorem 3.2 satisfies $D_1 \cup D_2 \cup \{(\tau^*, \tau^*)\} \subseteq \partial D$ and $\partial D \setminus (D_1 \cup D_2) \subseteq \Phi \cup \Psi$. Define the closure of D as $\overline{D}$, then we have $\Lambda < 0$, $\forall (\tau_1, \tau_2) \in D$ and $\Lambda = 0$, $\forall (\tau_1, \tau_2) \in \overline{D} \setminus D$. Hence, we call the connected region D the critical region. However, $\Phi \cup \Psi$ contains so many curves that we cannot imagine the approximate range of D. By further analysing $\Phi \cup \Psi$, we will show that the critical region D is contained in a bounded region.

Theorem 3.3. *Assume D is the critical region in Theorem 3.2. There exists a bounded region $D^* \subseteq \mathbb{R}_+ \times \mathbb{R}_+$ satisfying $\partial D^* \subseteq D_1 \cup D_2 \cup C_1 \cup C_2$ and $\overline{D^*} \setminus D^* \subseteq C_1 \cup C_2$ such that $D \subseteq D^*$, where*

$$C_1 = \left\{(\tau_1, \tau_2) \in \mathbb{R}_+ \times \mathbb{R}_+ \middle| \tau_1 = \frac{\frac{\pi}{2} - \theta_1}{\omega}, \quad \tau_2 = \frac{\theta_2}{\omega}, \quad \omega \in \Omega \right\} \subseteq \Phi,$$

$$C_2 = \left\{(\tau_1, \tau_2) \in \mathbb{R}_+ \times \mathbb{R}_+ \middle| \tau_1 = \frac{\frac{\pi}{2} - \vartheta_1}{\omega}, \quad \tau_2 = \frac{\vartheta_2}{\omega}, \quad \omega \in \Upsilon \right\} \subseteq \Psi.$$

D_1, D_2, θ_1, θ_2, Ω, ϑ_1, ϑ_2, and Υ are defined in Theorem 3.2.

Proof. Set

$$b_1 \triangleq a_1(i\omega)e^{-i\omega\tau_1} = -i\frac{\gamma m}{\omega}e^{-i\omega\tau_1} \quad \text{and} \quad b_2 \triangleq a_2(i\omega)e^{-i\omega\tau_2} = -\frac{(1-\gamma)m}{\omega^2}e^{-i\omega\tau_2}.$$

If $(\tau_1, \tau_2) = (0, 0)$, then $\measuredangle b_1 = \frac{3\pi}{2}$, $\measuredangle b_2 = \pi$ and the positions of 1, b_1 and b_2 on the complex plane are shown in Figure 2 Case 1. In this case, the Eq (3.6) do not have imaginary roots so that 1, b_1 and b_2 do not form a triangle in the complex plane. From Theorem 3.2 we know that 1, b_1 and b_2 could form a triangle for any $(\tau_1, \tau_2) \in \Phi \cup \Psi$. Select curves C_1 and C_2 from $\Phi \cup \Psi$, where

$$C_1 = \left\{(\tau_1, \tau_2) \in \mathbb{R}_+ \times \mathbb{R}_+ \middle| \tau_1 = \frac{\frac{\pi}{2} - \theta_1}{\omega}, \quad \tau_2 = \frac{\theta_2}{\omega}, \quad \omega \in \Omega \right\} \subseteq \Phi,$$

$$C_2 = \left\{(\tau_1, \tau_2) \in \mathbb{R}_+ \times \mathbb{R}_+ \middle| \tau_1 = \frac{\frac{\pi}{2} - \vartheta_1}{\omega}, \quad \tau_2 = \frac{\vartheta_2}{\omega}, \quad \omega \in \Upsilon \right\} \subseteq \Psi.$$

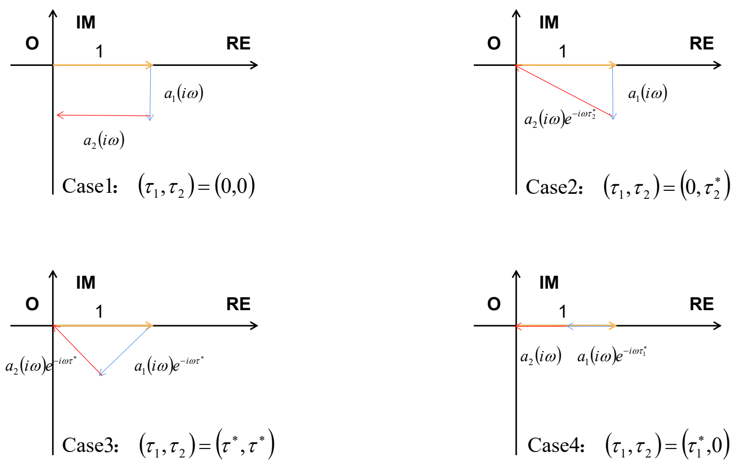

Figure 2. The positions of 1, $a_1(i\omega)e^{-i\omega\tau_1}$ and $a_2(i\omega)e^{-i\omega\tau_2}$ on the complex plane. When (τ_1, τ_2) moves clockwise from $(0, 0)$ along the curve ∂D^*, i.e., $(0, 0) \to (0, \tau_2^*) \to (\tau^*, \tau^*) \to (\tau_1^*, 0) \to (0, 0)$, the positions of 1, $a_1(i\omega)e^{-i\omega\tau_1}$ and $a_2(i\omega)e^{-i\omega\tau_2}$ change by Case 1 → Case 2 → Case 3 → Case 4 → Case 1.

For the curve C_1, let $\tau_1 = 0$ yields $\theta_1 = \frac{\pi}{2}$, then applying the properties of right triangle we obtain $1 + |a_1(i\omega)|^2 = |a_2(i\omega)|^2$ and $\tan(\theta_2) = |a_1(i\omega)|$, (See Figure 2 Case 2) i.e.,

$$1 + \frac{\gamma^2 m^2}{\omega^2} = \frac{(1-\gamma)^2 m^2}{\omega^4} \quad and \quad \tan(\theta_2) = \frac{\gamma m}{\omega}.$$

Solve the above equations yields

$$\omega = \sqrt{-\frac{\gamma^2 m^2}{2} + \sqrt{(1-\gamma)^2 m^2 + \frac{\gamma^4 m^4}{4}}} \in \Omega, \quad \tau_2 = \frac{\arctan\left(\frac{\gamma m}{\omega}\right)}{\omega} \geq \tau_2^*.$$

Let $\tau_2 = 0$ yields $\theta_2 = 0$, which means that the triangle degenerates into a line segment (See Figure 2 Case 4), so we have $1 = |a_1(i\omega)| + |a_2(i\omega)|$ and $\theta_1 = 0$, and then

$$\omega = \frac{\gamma m}{2} + \sqrt{\frac{\gamma^2 m^2}{4} + (1-\gamma)m} \in \Omega, \quad \tau_1 = \frac{\pi}{2\omega} = \tau_1^*.$$

Let $\tau_1 = \tau_2$ yields $\theta_1 + \theta_2 = \frac{\pi}{2}$, then we have $1 = |a_1(i\omega)|^2 + |a_2(i\omega)|^2$ and $\tan(\theta_2) = \frac{|a_1(i\omega)|}{|a_2(i\omega)|}$ (See Figure 2 Case 3). By solving the above equation we obtain

$$\omega = \sqrt{\frac{\gamma^2 m^2}{2} + \sqrt{(1-\gamma)^2 m^2 + \frac{\gamma^4 m^4}{4}}} \in \Omega, \quad \tau_1 = \tau_2 = \frac{\arctan\left(\frac{\gamma\omega}{1-\gamma}\right)}{\omega} = \tau^*.$$

Define a bounded region E_1 formed by the curve C_1 with the positive half of the horizontal and vertical axes, then the above statement indicates that $D \subseteq E_1$. Similarly, For the curve C_2, define a bounded region E_2 formed by the curve C_2 with the positive half of the horizontal and vertical axes, then we could verify that $D \subseteq E_2$.

Let $D^* = (E_1 \cap E_2) \setminus (C_1 \cup C_2)$, then $D \subseteq D^*$ and D is a bounded region. In addition, D^* satisfies $\partial D^* \subseteq D_1 \cup D_2 \cup C_1 \cup C_2$ and $\overline{D^*} \setminus D^* \subseteq C_1 \cup C_2$. □

Remark 4. Theorem 3.3 indicates that $(\tau_1^*, 0), (\tau^*, \tau^*) \in C_1$ and $(0, \tau_2^*) \in C_1 \cup C_2$. If curves in $\Phi \setminus C_1$ do not intersect the curve C_1 and curves in $\Psi \setminus C_2$ do not intersect the curve C_2 (except at its endpoints), then $D = D^*$. In addition, unlike the Eq (2.1), Theorem 3.3 shows that the critical region D of the Eq (3.1) is related to both the total strength α and the maximal strength m.

4. Numerical Simulation

In this section, a series of simulation examples are presented to illustrate Theorems 2.1, 3.1 and 3.3.

4.1. The first-order multi-agent system

Set the target $x_0(t) = \sin(t) + 5$ and $f(t) = \cos(t)$. Let $N = 10$ and $m = 2$, the initial positions $x_i(0)$ and strategies α_i of the Eq (2.1) are listed in Table 1.

Table 1. The initial positions $x_i(0)$ and strategies α_i of the system (2.1).

$x_1(0) = 8.1472$	$x_2(0) = 9.0579$	$x_3(0) = 1.2699$	$x_4(0) = 9.1338$	$x_5(0) = 6.3236$
$\alpha_1 = 1$	$\alpha_2 = 1$	$\alpha_3 = 0$	$\alpha_4 = 0$	$\alpha_5 = 0$
$x_6(0) = 0.9754$	$x_7(0) = 2.7850$	$x_8(0) = 5.4688$	$x_9(0) = 9.5751$	$x_{10}(0) = 9.6489$
$\alpha_6 = 0$	$\alpha_7 = 0$	$\alpha_8 = 0$	$\alpha_9 = 0$	$\alpha_{10} = 0$

According to Theorem 2.1, the critical delay $\tau^* = \frac{\pi}{2m} = \frac{\pi}{4}$. Take $\tau = 0$, $\tau = \frac{\pi}{5}$ and $\tau = \frac{\pi}{4}$ respectively to obtain simulations of the Eq (2.1) as shown in Figure 3.

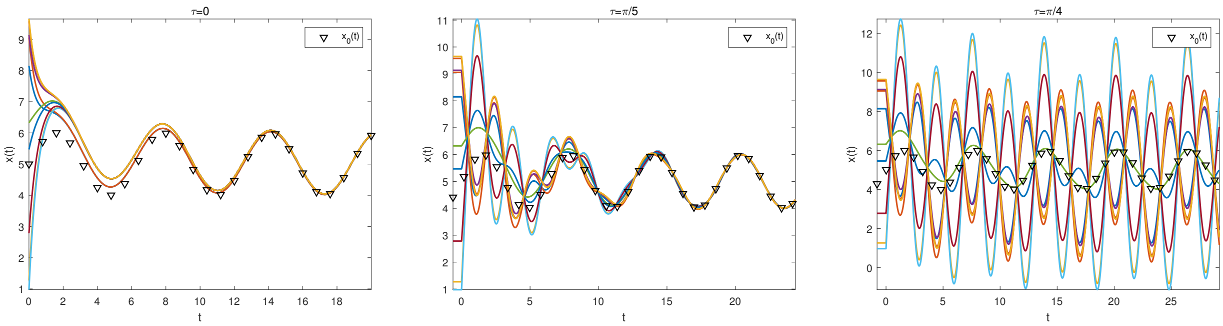

Figure 3. The figures from left to right are simulations of the system (2.1) in $\tau = 0$, $\tau = \frac{\pi}{5}$ and $\tau = \frac{\pi}{4}$ respectively. In the figures, curves represent trajectories of agents $x_i(t)(1 \le i \le N)$ in the system (2.1), and the inverted triangle describes the trajectory of the target $x_0(t)$. Oscillatory behaviour occurs in the system (2.1) due to the existence of delay τ and intensifies with the increase of τ. When $\tau < \tau^* = \frac{\pi}{4}$, the system (2.1) achieves consensus and matches the dynamic behaviour of the target $x_0(t) = \sin(t) + 5$. When $\tau = \tau^*$, periodic oscillation behaviour occurs in the system (2.1), and the system neither matches the dynamic behaviour of the target $x_0(t)$ nor achieves consensus.

4.2. The second-order multi-agent system

Set the target $x_0(t) = 3\sin(t) + 4\cos(t) + \frac{t}{2}$, $v_0(t) = 3\cos(t) - 4\sin(t) + \frac{1}{2}$ and $g(t) = -3\sin(t) - 4\cos(t)$. Let $N = 10$, $m = 2$ and $\gamma = 0.5$, the initial positions $x_i(0)$, velocities $v_i(t)$ and strategies α_i of the Eq (3.1) are listed in Table 2.

Table 2. The initial positions $x_i(0)$, velocities $v_i(t)$ and strategies α_i of the system (3.1).

$x_1(0) = 7.5127$	$x_2(0) = 2.5510$	$x_3(0) = 5.0596$	$x_4(0) = 6.9908$	$x_5(0) = 8.9090$
$v_1(0) = 2.7603$	$v_2(0) = 6.7970$	$v_3(0) = 6.5510$	$v_4(0) = 1.6261$	$v_5(0) = 1.1900$
$\alpha_1 = 1$	$\alpha_2 = 1$	$\alpha_3 = 0$	$\alpha_4 = 0$	$\alpha_5 = 0$
$x_6(0) = 9.5929$	$x_7(0) = 5.4722$	$x_8(0) = 1.3862$	$x_9(0) = 1.4929$	$x_{10}(0) = 2.5751$
$v_6(0) = 4.9836$	$v_7(0) = 9.5976$	$v_8(0) = 3.4039$	$v_9(0) = 5.8527$	$v_{10}(0) = 2.2381$
$\alpha_6 = 0$	$\alpha_7 = 0$	$\alpha_8 = 0$	$\alpha_9 = 0$	$\alpha_{10} = 0$

According to Theorem 3.1 we have $\tau_1^* = \frac{\pi}{1+\sqrt{5}}$, $\tau_2^* = \min\{1.1506, 1.0166\} = 1.0166$, $\tau^* = 0.7111$. Take $\tau_1 = \tau_2 = 0$, $\tau_1 = \tau_2 = 0.5$ and $\tau_1 = \tau_2 = 0.7111$ respectively to obtain simulations of the Eq (3.1) as shown in Figure 4.

Fix $m = 2$ and $\alpha/N = 0.2$, take $\gamma = 0.2$, $\gamma = 0.7321$ (i.e., $\frac{4(1-\gamma)}{\gamma^2} = m$) and $\gamma = 0.8$ respectively to obtain simulations of the $\Phi \cup \Psi$ by Theorem 3.2, see Figure 5.

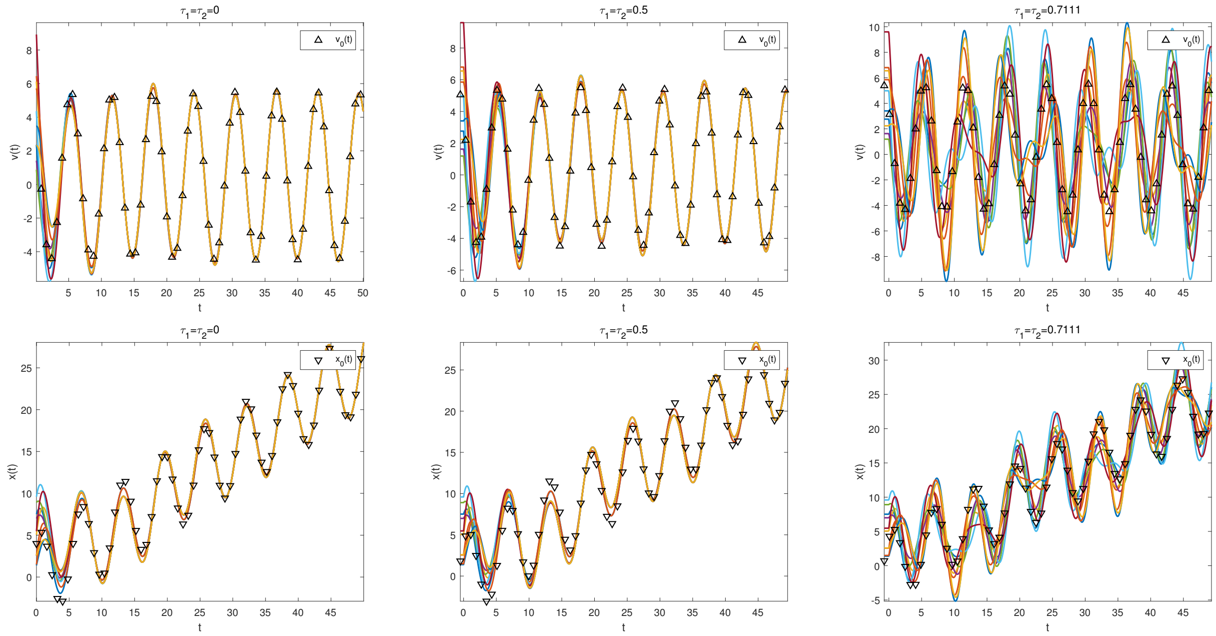

Figure 4. The figures from left to right are simulations of $x_i(t)(0 \le i \le N)$ (bottom) and $v_i(t)(0 \le i \le N)$ (top) for the system (3.1) in $\tau_1 = \tau_2 = 0$, $\tau_1 = \tau_2 = 0.5$ and $\tau_1 = \tau_2 = 0.7111$ respectively. When $\tau_1 = \tau_2 < \tau^* = 0.7111$, the velocities $v_i(t)(1 \le i \le N)$ of the system (3.1) achieve consensus and match the target velocity $v_0(t) = 3\cos(t) - 4\sin(t) + \frac{1}{2}$. And at the same time, the positions $x_i(t)(1 \le i \le N)$ of the system (3.1) achieve consensus and match the target position $x_0(t) = 3\sin(t) + 4\cos(t) + \frac{t}{2}$. When $\tau_1 = \tau_2 = \tau^* = 0.7111$, agents in the system (3.1) move around the target, but the system (3.1) neither matches the dynamic behaviour of the target and nor achieves consensus.

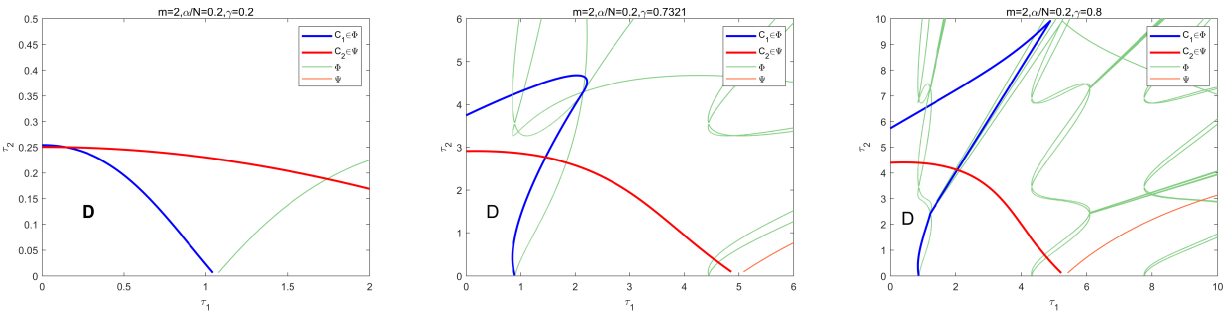

Figure 5. Fix $m = 2$ and $\alpha/N = 0.2$. The figures from left to right are simulations of $\Phi \cup \Psi$ for the system (3.1) in $\gamma = 0.2$, $\gamma = 0.7321$ and $\gamma = 0.8$ respectively. D is the critical region of the system (3.1), Denote the bounded region, containing the origin, formed by the curves C_1, C_2 and the positive half of the horizontal and vertical axes by D^*. For the one on the left, curves in $\Phi \setminus C_1$ do not intersect the curve C_1 and curves in $\Psi \setminus C_2$ do not intersect the curve C_2 (except at its endpoints), then $D = D^*$. For the middle one, even though curves in $\Phi \setminus C_1$ intersect the curve C_1, $D = D^*$. For the one on the right, $D \subsetneqq D^*$. In addition, when $\frac{4(1-\gamma)}{\gamma^2} \le m$, the curve C_1 is smooth because of the connectedness of the interval Ω. When $\frac{4(1-\gamma)}{\gamma^2} > m$, the interval Ω consists of two disconnected closed intervals, so the curve C_1 is made up of two smooth curves, such as the figure on the right.

Fix $m = 2$ and $\gamma = 0.7321$, take $\alpha/N = 0.2$, $\alpha/N = 1$ and $\alpha/N = 1.8$ respectively to obtain simulations of the $\Phi \cup \Psi$ by Theorem 3.2, see Figure 6.

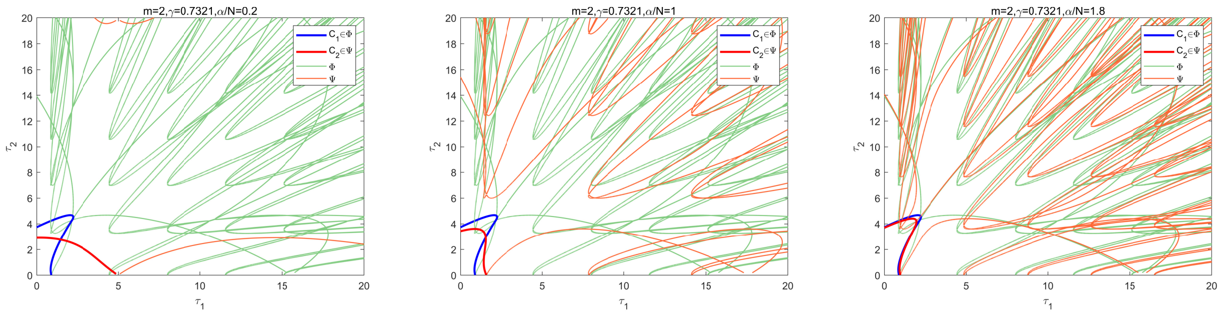

Figure 6. Fix $m = 2$ and $\gamma = 0.7321$. The figures from left to right are simulations of $\Phi \cup \Psi$ for the system (3.1) in $\alpha/N = 0.2$, $\alpha/N = 1$ and $\alpha/N = 1.8$ respectively. The critical region D of the system (3.1) changes as α changes. In particular, as α/N approaches m, curves Φ gradually converges to curves Ψ.

5. Conclusion and Future Direction

In this paper, we analysed the influence of the processing delay on the consensus for the first-order system in Eq (1.1) and the second-order system in Eq (1.2). For the first-order system in Eq (2.1), by the continuous dependence of the equation on τ, we obtained the critical delay τ^* that ensures the Eq (2.1) to achieve consensus and showed that the critical delay τ^* is independent of the strategies α_i. For the second-order system in Eq (3.1), from the properties of plane geometry, we identified the critical region D in $\mathbb{R}^2$ that guarantees the Eq (3.1) to achieve consensus and found that the shape of the critical region D is affected by the strategies α_i.

The concept of strategies α_i was firstly proposed by Piccoli et al. [18] to study the problem of optimal strategy. Using Pontryagin's minimum principle in optimal control theory, they found optimal strategies $\{\alpha_i\}_{i=1}^N$ to minimize the cost $\frac{1}{N}\sum_{i=1}^N \|x_i(T) - x_0(T)\|$, where $T > 0$ was the final time. More importantly, they showed that optimal strategies are sparse. From the present work, we want to find optimal strategies α_i for the Eq (2.1) and analyse the influence of delays on the selection of optimal strategies α_i.

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Conflict of interest

The authors declare there is no conflict of interest.

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