PERIODIC CONSENSUS IN NETWORK SYSTEMS WITH GENERAL DISTRIBUTED PROCESSING DELAYS

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Abstract. How to understand the dynamical consensus patterns in network systems is of particular significance in both theories and applications. In this paper, we are interested in investigating the influences of distributed processing delay on the consensus patterns in a network model. As new observations, we show that the desired network model undergoes both weak consensus and periodic consensus behaviors when the parameters reach a threshold value and the connectedness of the network system may be absent. In results, some criterions of weak consensus and periodic consensus with exponential convergent rate are established by the standard functional differential equations analysis. An analytic formula is given to calculate the asymptotic periodic consensus in terms of model parameters and the initial time interval. Also, we post the threshold values for some typical distributions included uniform distribution and Gamma distribution. Finally, we give the numerical simulation and analyse the influences of different delays on the consensus.

1. **Introduction.** For a multi-node network system, consensus problems play a particular significance role in both theories and applications. Such problems are broadly investigated in fields of distributed computing [7], management science [1], flocking/swarming theory [16], distributed control [2] and sensor networks [11], and so on. Such systems also seem to have remarkable capability to regulate the flow of information from distinct and independent nodes to achieve a prescribed performance. As previous observations in both simulation and theory, the connectedness of the adjacency matrix and the processing delay play key roles to make the system achieve the emergent feature. The main motivation in current work is to analyze and explain the dynamical consensus patterns in a multi-node system, while the

connectedness of the adjacency matrix is absent and the distributed processing delays are also involved in.

In this paper, we consider a N-node network system with the distributed processing delay, reading as,

$$\dot{x}_i = \lambda \sum_{j=1}^{N} a_{ij}(\bar{x}_j(t) - \bar{x}_i(t)), \quad i = 1, 2, \cdots, N, \tag{1}$$

where $x_i \in R^n$ denotes the n dimensional state of i-th node at time t. λ is a constant measured the coupling strength. $\bar{x}_j(t) = \int_{-\tau}^{0} \varphi(s)x_j(t+s)ds$ measures the average of v_j on $[t - \tau, t]$, where τ denotes the maximum processing delay from j to i, φ is a (positive) normalized distributed function so that $\int_{-\tau}^{0} \varphi(s)ds = 1$. $\bar{x}_i(t)$ is defined similarly. In physically, a more realistic model should include a delay distribution over the time that depicts the human behavior in average. Usually, the traffic flow models are inherently time delayed because of the limited sensing and acting capabilities of drivers against velocity and position variations [9]. As we known, the delays usually follow the uniform distribution, the exponential distribution and discrete distribution. And $\varphi(s) = \frac{1}{\tau}$ for the case of the unform distribution and $\varphi(s) = \frac{\alpha}{1-e^{-\alpha\tau}}e^{\alpha s}$ for the exponential distribution, where α is a positive constant. The constant $a_{ij} \geq 0$ is the strength of the influence of node j on i and $a_{ii} = 0$. In this model, the interaction involves delayed information processing, where the difference of the average states $x_j - x_i$ influences the dynamics of the nodes after some time delay τ.

In the previously published works, consensus problems have often been studied with discrete processing delays [5, 6, 11], time-varying processing delays [8, 12] and γ-distribution delays [9, 10] and the references therein. The case of discrete delay is always viewed as a delay with Bernoulli distribution. Mathematically, there have been many contributions to the stability of network system with processing delays, see [3, 14, 15] for examples.

To understand the dynamical consensus patterns better, we assume the adjacency matrix $A = (a_{ij})_{N \times N}$ is a symmetric matrix. Let $C = \sum_{i,j=1}^{N} a_{ij}$ be the volume of the system and a_{ij} be normalized by

$$\tilde{a}_{ii} = 1 - \frac{\sum_{j=1}^{N} a_{ij}}{C}, \quad \tilde{a}_{ij} = \frac{a_{ij}}{C} \text{ for } i \neq j.$$

Let $\tilde{\lambda} = \lambda C$, then we can rewrite the system (1) in the form

$$\dot{x}_i = \tilde{\lambda} \sum_{j=1}^{N} \tilde{a}_{ij}(\bar{x}_j(t) - \bar{x}_i(t)), \quad i = 1, 2, \cdots, N. \tag{2}$$

It is easy to find that the system (1) and system (2) have the same dynamical behaviors.

Let $\tilde{A} = (\tilde{a}_{ij})_{N \times N}$, then $\tilde{A}$ is a one-row-sum matrix and stochastic matrix [13]. Also it is a diagonalizable matrix, 1 is one of its eigenvalues and all other eigenvalues are real. Throughout the paper, we assume the eigenvalue 1 of stochastic matrix $\tilde{A}$ is semi-simple with algebraic multiplicity n_0. And all different eigenvalues of $\tilde{A}$ are $\mu_i(i = 1, 2, \cdots, m_0)$ with the algebraic multiplicity p_i. All eigenvalues of $\tilde{A}$ satisfy the order

$$1 = \mu_1 > \mu_2 > \cdots > \mu_{m_0}.$$

Naturally, if $n_0 = 1$, then the matrix $\tilde{A}$ is a connected matrix. And when $n_0 > 1$, the connectedness of matrix $\tilde{A}$ will be absent. From the matrix theory, we see that there is an orthogonal matrix T_0 such that $\tilde{A} = T_0 J_0 T_0^{-1}$, where J_0 is a diagonal matrix with the first block I_{n_0}, say $J_0 = \begin{pmatrix} I_{n_0} & \mathbf{0} \\ \mathbf{0} & J^* \end{pmatrix}$, where $\mathbf{0}$ is zero matrix with matchable dimension. Define the norm of a real matrix $S \in R^{N \times m}$ by $\|S\| = \sup_{|\alpha| \neq 0} \frac{|S\alpha|}{|\alpha|}$, $\alpha \in R^m$, then $\|T_0\| = \|T_0^{-1}\| = 1$.

To find the qualitative behaviors, we finish this section by considering the equation

$$\dot{\mathbf{w}} = -\tilde{\lambda}\bar{\mathbf{w}}(t) + \tilde{\lambda} J^* \bar{\mathbf{w}}(t), \tag{3}$$

and its characteristic equation is

$$h_0(z) = Det\left(zI + \tilde{\lambda} \int_{-\tau}^0 \varphi(s)e^{zs}ds(I - J^*)\right) = 0. \tag{4}$$

Lemma 1.1. ([4], Corollary 6.1, P215) If $a_0 = \max\{Rez : h_0(z) = 0\}$, then , for any $c_0 > a_0$, there is a constant $K = K(c_0)$ such that the fundamental solution $S_w(t)$ of the equation (3) satisfies the inequality

$$\|S_w(t)\| \leq Ke^{c_0 t}.$$

2. **Main results.** To specify a solution for the network system (1), we need to specify the initial conditions

$$x_i(\theta) = f_i(\theta), \quad \text{for } \theta \in [-\tau, 0], i = 1, 2, \cdots, N, \tag{5}$$

where f_i is a given continuous vector-value function.

Definition 2.1. Suppose $\{x_i(t)\}_{i=1}^N$ is a solution to (1) and (5). The above system is said to achieve a weak periodic consensus, if there are periodic functions $\phi_{pi}(t)$ with a same period such that

$$\lim_{t \to \infty} (x_i(t) - \phi_{pi}(t)) = x_{i\infty}, i = 1, 2, \cdots, N.$$

If $x_{i\infty} = x_\infty$ for all i, then the system is said to achieve a periodic consensus, where $x_\infty \in R^n$ is a constant vector; If all $\phi_{pi}(t) = 0$, the system (1) is said to achieve a weak consensus. If both $x_{i\infty} = x_\infty$ and $\phi_{pi}(t) = 0$ hold, it is said to achieve a consensus.

Let $f_{max} = \max\{\|\mathbf{f}(\theta)\| : \theta \in [-\tau, 0]\}$ for $\mathbf{f}(\theta) = (f_1(\theta), \cdots, f_N(\theta))^T$ and

$$k^* = \frac{\tau y_{im}}{-\int_{-\tau}^0 \varphi(s)\sin(y_{im}s)ds}, \tag{6}$$

where y_{im} is a minimum positive root of equation

$$\int_{-\tau}^0 \varphi(s)\cos(ys)ds = 0. \tag{7}$$

Set

$$c_1 : = \max_{2 \leq i \leq m_0} \sup\{Re(z) : z + \tilde{\lambda}(1 - \mu_i)\int_{-\tau}^0 \varphi(s)e^{zs}ds = 0\},$$

$$c_2 : = \max_{2 \leq i \leq m_0 - 1} \sup\{Re(z) : z + \tilde{\lambda}(1 - \mu_i)\int_{-\tau}^0 \varphi(s)e^{zs}ds = 0\},$$

then we obtain the following results and the details of proof will be given in sequel.

Lemma 2.2. *Let $\tilde{\lambda} > 0$. If $0 \le \tilde{\lambda}\tau(1-\mu_{m_0}) < k^*$, then all other roots of the equation (4) have negative real parts and $c_1 < 0$. If $\tilde{\lambda}\tau(1 - \mu_{m_0}) = k^*$, then all other roots, except the pure imaginary roots, of the equations $z + \tilde{\lambda}(1 - \mu_i) \int_{-\tau}^{0} \varphi(s)e^{zs}ds = 0$ $(i = 2, \cdots, m_0 - 1)$ have negative real parts and $c_2 < 0$.*

Theorem 2.3. *Let $\mathbf{X}(t) = (x_1(t), \cdots, x_N(t))^T$ be a solution of system (1) and 1 be a n_0-multiple eigenvalue of the matrix $\tilde{A}$.*

(1) Assume $0 \le \tilde{\lambda}\tau(1 - \mu_{m_0}) < k^*$, then the system achieves a weak consensus with

$$\lim_{t \to \infty} \mathbf{X}(t) = T_0 \begin{pmatrix} I_{n_0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} T_0^{-1}\mathbf{f}(0) := \mathbf{X}_\infty,$$

and, for all $\varepsilon \in (0, -c_1)$, there is constant K_1 such that

$$\|\mathbf{X}(t) - \mathbf{X}_\infty\| \le f_{max}K_1 e^{-(|c_1|-\varepsilon)t}.$$

Especially, when $n_0 = 1$, the system achieves a consensus.

(2) Assume $\tilde{\lambda}\tau(1 - \mu_{m_0}) = k^*$, then the system achieves a weak periodic consensus with

$$\lim_{t \to \infty} (\mathbf{X}(t) - \mathbf{X}_p(t)) = \mathbf{X}_\infty.$$

and, for all $\varepsilon \in (0, -c_2)$, there is constant K_2 such that

$$\|\mathbf{X}(t) - \mathbf{X}_p(t) - \mathbf{X}_\infty\| \le f_{max}K_2 e^{-(|c_2|-\varepsilon)t},$$

where $\mathbf{X}_p(t)$ is formulated by (16). Especially, when $n_0 = 1$, the system achieves a periodic consensus.

Remark 1. For the case of uniform distribution, the distributed function is $\varphi(s) \equiv \frac{1}{\tau}$. By direct computation, we see that $k^* = \frac{\pi^2}{2}$ and $y_{im} = \frac{\pi}{\tau}$. The values of k^* and y_{im} for typical distributions are listed in following table.

TABLE 1. The values of k^* and y_{im} for some special cases

Cases	k^*	y_{im}	Descriptions
$\varphi(s) = \frac{1}{\tau}$	$\frac{\pi^2}{2}$	$\frac{\pi}{\tau}$	Uniform distribution
$\varphi(s) = \frac{2e^2}{e^2-1}e^{2s}$	116.7278	16.8680	Exponential distribution
$\varphi(s) = \frac{4e^2}{e^2-3}\|s\|e^{2s}$	3.8152	2.8801	Special γ-distribution
$\varphi(s) = \frac{4e^2}{e^2-5}s^2e^{2s}$	2.7019	2.3530	Special γ-distribution
$\varphi(s) = \begin{cases} 0, & s \in (-\tau, 0] \\ 1, & s = -\tau \end{cases}$	$\frac{\pi}{2}$	$\frac{\pi}{2\tau}$	Bernoulli distribution

3. **Proof of main results. Proof of Lemma 2.1** Assume $z = x + yi$ $(y > 0)$ is a root of $z + \tilde{\lambda}(1 - \mu_i) \int_{-\tau}^{0} \varphi(s)e^{zs}ds = 0$. Then we have

$$\begin{cases} x + \tilde{\lambda}(1 - \mu_i) \int_{-\tau}^{0} \varphi(s)e^{xs}\cos(ys)ds = 0, \\ y + \tilde{\lambda}(1 - \mu_i) \int_{-\tau}^{0} \varphi(s)e^{xs}\sin(ys)ds = 0. \end{cases} \tag{8}$$

Next, we show that $x \le 0$ for $y \in [0, y_{im}]$. Indeed, assume $x > 0$, then $\int_{-\tau}^{0} \varphi(s)e^{xs}\cos(ys)ds < 0$. Let $\tau_i(i = 1, 2, \cdots, k)$ be all the roots of equation $\cos(ys) = 0$ on $[-\tau, 0]$. Also, we assume that

$$0 = \tau_0 > \tau_1 > \cdots > \tau_k \ge \tau_{k+1} = -\tau.$$

Set

$$A_i = \int_{\tau_{i+1}}^{\tau_i} \varphi(s)|\cos(ys)|ds \text{ and } \tilde{A}_i = \int_{\tau_{i+1}}^{\tau_i} \varphi(s)e^{xs}|\cos(ys)|ds, i = 0, 1, \cdots, k,$$

then

$$\int_{-\tau}^{0} \varphi(s)\cos(ys)ds = \sum_{i=0}^{k}(-1)^i A_i \text{ and } \int_{-\tau}^{0} \varphi(s)e^{xs}\cos(ys)ds = \sum_{i=0}^{k}(-1)^i \tilde{A}_i.$$

Noting that $y \in [0, y_{im})$ and $r = y_{im}$ is a minimum positive root of equation $\int_{-\tau}^{0} \varphi(s)\cos(rs)ds = 0$, we see that $\int_{-\tau}^{0} \varphi(s)\cos(ys)ds > 0$ and $\sum_{i=0}^{l}(-1)^i A_i > 0$ for $l = 1, 2, \cdots, k$.

By direct computation, for $x > 0$, we have $\tilde{A}_0 - \tilde{A}_1 > e^{x\tau_1}(A_0 - A_1) > 0$ and

$$\tilde{A}_0 - \tilde{A}_1 + \tilde{A}_2 - \tilde{A}_3 > e^{x\tau_1}(A_0 - A_1) + e^{x\tau_3}(A_2 - A_3) > e^{x\tau_3}(A_0 - A_1 + A_2 - A_3) > 0.$$

For generally, we have

$$\int_{-\tau}^{0} \varphi(s)e^{xs}\cos(ys)ds = \sum_{i=0}^{k}(-1)^i \tilde{A}_i > e^{x\tau_k} \sum_{i=0}^{k}(-1)^i A_i$$

$$= e^{x\tau_k} \int_{-\tau}^{0} \varphi(s)\cos(ys)ds > 0.$$

It contradicts with $\int_{-\tau}^{0} \varphi(s)e^{xs}\cos(ys)ds < 0$. Thus all other roots of the equation $z + \tilde{\lambda}(1 - \mu_i) \int_{-\tau}^{0} \varphi(s)e^{zs}ds = 0$ have negative real parts when $y \in [0, y_{im})$. When $y = y_{im}$, except the pure imaginary roots, the equation $z + \tilde{\lambda}(1 - \mu_i) \int_{-\tau}^{0} \varphi(s)e^{zs}ds = 0$ have negative real parts when $i = 2, \cdots, m_0 - 1$.

On the other hand, combining

$$\tau y + \tilde{\lambda}\tau(1 - \mu_{m_0}) \int_{-\tau}^{0} \varphi(s)e^{xs}\sin(ys)ds = 0$$

and the fact $\tau y_{im} + k^* \int_{-\tau}^{0} \varphi(s)\sin(y_{im}s)ds = 0$, we see that $y \in [0, y_{im})$ if and only if $0 \leq \tilde{\lambda}\tau(1 - \mu_{m_0}) < k^*$. Also $y = y_{im}$ if and only if $\tilde{\lambda}\tau(1 - \mu_{m_0}) = k^*$. Since $\tilde{\lambda}\tau(1 - \mu_i) \leq \tilde{\lambda}\tau(1 - \mu_{m_0}) < k^*$ holds for $i = 2, \cdots, m_0$, we conclude that all other roots of the equation (4) have negative real parts when $0 \leq \tilde{\lambda}\tau(1 - \mu_{m_0}) < k^*$. Also, if $\tilde{\lambda}\tau(1 - \mu_{m_0}) = k^*$, except the pure imaginary roots, the equation $z + \tilde{\lambda}(1 - \mu_i) \int_{-\tau}^{0} \varphi(s)e^{zs}ds = 0$ have negative real parts when $i = 2, \cdots, m_0 - 1$.

Noting that the set $\{Re(z) : z = -\tilde{\lambda}(1 - \mu_i) \int_{-\tau}^{0} \varphi(s)e^{zs}ds\}$ is up-bounded when $0 \leq \tilde{\lambda}\tau(1 - \mu_{m_0}) < k^*$, and from above arguments, we see that the supremum

$$c_1 := \max_{1 \leq i \leq m_0} \sup\{Re(z) : z = -\tilde{\lambda}(1 - \mu_i) \int_{-\tau}^{0} \varphi(s)e^{zs}ds\} \leq 0.$$

Assume that $c_1 = 0$, then there is a sequence $\{z_n\}$ $(z_n = x_n + iy_n, y_n > 0)$ with

$$\lim_{n\to\infty} x_n = 0 \text{ and } z_n = -\tilde{\lambda}(1 - \mu_i) \int_{-\tau}^{0} \varphi(s)e^{z_n s}ds \text{ for some } i. \qquad (9)$$

Thus

$$x_n = -\tilde{\lambda}(1 - \mu_i) \int_{-\tau}^{0} \varphi(s)e^{x_n s}\cos(y_n s)ds$$

and

$$y_n = -\tilde{\lambda}(1-\mu_i)\int_{-\tau}^0 \varphi(s)e^{x_n s}\sin(y_n s)ds.$$

Then, for $x_n \leq 0$, the sequence $\{y_n\}$ is bounded by y_{im}. Thus there is a convergent subsequence of $\{y_n\}$. Without loss of generality, we assume $\{y_n\}$ is a convergent sequence with the limit y_∞ satisfying $y_\infty \leq y_{im}$. For $\lim_{n\to\infty} x_n = 0$, we see that

$$\int_{-\tau}^0 \varphi(s)\cos(y_\infty s)ds = 0 \ \text{ and } \ y_\infty = -\tilde{\lambda}(1-\mu_i)\int_{-\tau}^0 \varphi(s)\sin(y_\infty s)ds.$$

If $y_\infty < y_{im}$, then it contradicts that $r = y_{im}$ is a minimum positive root of equation $\int_{-\tau}^0 \varphi(s)\cos(rs)ds = 0$. If $y_\infty = y_{im}$, for $\tilde{\lambda}(1-\mu_i) < \tilde{\lambda}\tau(1-\mu_{m_0}) < k^*$, then it contradicts with $\tau y_{im} + k^* \int_{-\tau}^0 \varphi(s)\sin(y_{im}s)ds = 0$. Thus $c_1 < 0$. Similar arguments yield $c_2 < 0$. This completes the proof. $\square$

Proof of Theorem 2.1 Let $\mathbf{X} = (x_1, x_2, \cdots, x_N)^T$ and $\bar{\mathbf{X}}(t) = \int_{-\tau}^0 \varphi(s)\mathbf{X}(t+s)ds$. Thus the system (1) can be rewritten with the vector form , reading as,

$$\dot{\mathbf{X}} = -\tilde{\lambda}(I - \tilde{A})\bar{\mathbf{X}}(t), \ \ \mathbf{X}(t) = \mathbf{f}(t), t \in [-\tau, 0]. \tag{10}$$

Recalling $\tilde{A} = T_0 \begin{pmatrix} I_{n_0} & \mathbf{0} \\ \mathbf{0} & J^* \end{pmatrix} T_0^{-1}$ and let

$$\mathbf{Y}(t) = T_0^{-1}\mathbf{X}(t) = (y_1(t), y_2(t), \cdots, y_{n_0}(t), \mathbf{y}^*(t))^T,$$

then the equation of (10) yields

$$\dot{\mathbf{Y}} = -\tilde{\lambda}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I - J^* \end{pmatrix}\bar{\mathbf{Y}}(t).$$

That is, $\dot{y}_i(t) = 0$ for $i = 1, 2, \cdots, n_0$, and $\mathbf{y}^*(t)$ solves the equation (3). And then the characteristic equation $h_0(z) = 0$ becomes

$$h(z) = \prod_{i=2}^{m_0}\left(z + \tilde{\lambda}(1-\mu_i)\int_{-\tau}^0 \varphi(s)e^{zs}ds\right)^{p_i} = 0, \tag{11}$$

where p_i is the algebraic multiplicity of μ_i, m_0 is the number of the different eigenvalues of P_0.

Let $\mathbf{S}^*(t)$ be a fundamental solution operator of the equation (3). Then the solution $\mathbf{X}(t)$ of the equation (10) becomes

$$\mathbf{X}(t+\theta) = T_0\begin{pmatrix} I_{n_0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}^*(t) \end{pmatrix}T_0^{-1}\mathbf{f}(\theta), \quad \text{for} \quad t \in [0, t_1), \theta \in [-\tau, 0]. \tag{12}$$

Let

$$\mathbf{X}_a(\theta) = T_0\begin{pmatrix} I_{n_0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}T_0^{-1}\mathbf{f}(\theta) \text{ for } \ \theta \in [-\tau, 0]. \tag{13}$$

By using the equalities (12) and (13), we have

$$\|\mathbf{X}(t+\theta) - \mathbf{X}_a(\theta)\| = \|T_0\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}^*(t) \end{pmatrix}T_0^{-1}\mathbf{f}(\theta)\|. \tag{14}$$

CASE I: $\tilde{\lambda}\tau(1-\mu_{m_0}) < k^*$. Following Lemma 2.1, we see that all roots of the characteristic equation (11) have negative real parts. And from Lemma 1.1, there are a constant $K_1 > 0$ such that

$$\|\mathbf{S}^*(t)\| \leq K_1 e^{-ct} \ \text{ for all } c \in (0, -c_1).$$

Thus $\|\mathbf{S}^*(t)\| \le K_1 e^{-(|c_1|-\varepsilon)t}$ for all $\varepsilon \in (0, -c_1)$ and

$$\|\mathbf{X}(t+\theta) - \mathbf{X}_a(\theta)\| = \left\| T_0 \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}^*(t) \end{pmatrix} T_0^{-1}\mathbf{f}(\theta) \right\| \le f_{max}K_1 e^{-(|c_1|-\varepsilon)t}.$$

This implies that

$$\sup_{\theta \in [-\tau, 0]} \|\mathbf{X}(t+\theta) - \mathbf{X}_a(\theta)\| \le f_{max}K_1 e^{-(|c_1|-\varepsilon)t}, \quad \text{for } t \in [0, +\infty). \quad (15)$$

It means that $\lim_{t\to\infty} \mathbf{X}(t+\theta) = \mathbf{X}_a(\theta)$. On the other hand, noting that $\dot{y}_i(t) = 0$ for $i = 1, 2, \cdots, n_0$ and $\lim_{t\to\infty} \mathbf{y}^*(t) = \mathbf{0}$, we conclude that

$$\lim_{t\to\infty} \mathbf{X}(t) = T_0 \begin{pmatrix} I_{n_0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} T_0^{-1}\mathbf{f}(0) := \mathbf{X}_\infty.$$

Thus, we have

$$\mathbf{X}_a(\theta) = T_0 \begin{pmatrix} I_{n_0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} T_0^{-1}\mathbf{f}(0) = \mathbf{X}_\infty$$

and

$$\|\mathbf{X}(t) - \mathbf{X}_\infty\| \le \sup_{\theta \in [-\tau, 0]} \|\mathbf{X}(t+\theta) - \mathbf{X}_\infty\| \le f_{max}K_1 e^{-(|c_1|-\varepsilon)t}.$$

Thus, from Definition 2.1, the system (1) achieves a weak consensus.

Especially, when $n_0 = 1$, for T_0 is an orthogonal matrix, then $\mathbf{X}_\infty$ are formulated by

$$\mathbf{X}_\infty = \frac{1}{N} \sum_{i=1}^{N} v_i(0) \otimes \mathbf{1}_N,$$

where $\otimes$ denotes the Kronecker product. Thus the system (1) reaches a consensus.

CASE II: $\tilde{\lambda}\tau(1 - \mu_{m_0}) = k^*$. Consider the equation

$$\dot{y}(t) = -\tilde{\lambda}(1 - \mu_{m_0}) \int_{-\tau}^{0} \varphi(s)y(t+s)ds$$

and its characteristic equation is given by $z = -\tilde{\lambda}(1 - \mu_{m_0}) \int_{-\tau}^{0} \varphi(s)e^{zs}ds$. From the definition of k^*, we see that $\pm y_{im}i$ are two pure imaginary roots of above equation. Thus both $e^{ty_{im}i}$ and $e^{-ty_{im}i}$ are solutions of the given equation, and then both $\cos(y_{im}t)$ and $\sin(y_{im}t)$ are also solutions. Thus the periodic solution $y(t)$ would be formulated by $y(t) = c_1 \cos(y_{im}t) + c_2 \sin(y_{im}t)$. Substituting the initial values, we find that the basic periodic solution is

$$y(t) = \cos(y_{im}t)y(0) - \frac{\tilde{\lambda}(1 - \mu_{m_0})}{y_{im}} \sin(y_{im}t) \int_{-\tau}^{0} \varphi(s)y(s)ds, \ t \in (0, \infty).$$

Let

$$\begin{aligned} \mathbf{X}_p(t) &= \cos(y_{im}t)T_0 \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{p_{m_0}} \end{pmatrix} T_0^{-1}\mathbf{f}(0) \\ &\quad - \frac{\tilde{\lambda}(1 - \mu_{m_0})}{y_{im}} \sin(y_{im}t)T_0 \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{p_{m_0}} \end{pmatrix} T_0^{-1} \int_{-\tau}^{0} \varphi(s)\mathbf{f}(s)ds \quad (16) \end{aligned}$$

and rewrite the diagonal matrix J as

$$J = \begin{pmatrix} I_{n_0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & J_p^* & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mu_{m_0}I_{p_{m_0}} \end{pmatrix}.$$

Similarly, let $\mathbf{S}_p^*(t)$ be a fundamental solution operator of the equation

$$\dot{\mathbf{u}}^* = -\tilde{\lambda}(I - J_p^*)\bar{\mathbf{u}}^*(t). \tag{17}$$

Then the solution $\mathbf{X}(t)$ in (10) becomes

$$\mathbf{X}(t + \theta) = \mathbf{X}_\infty + \mathbf{X}_p(t) + T_0 \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_p^*(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} T_0^{-1}\mathbf{f}(\theta), \tag{18}$$

for $t \in [0, +\infty), \theta \in [-\tau, 0]$.

To find the asymptotic behaviors , we consider the characteristic equation corresponding to (17), reading as

$$Det\left(zI + \tilde{\lambda}\int_{-\tau}^0 \varphi(s)e^{zs}ds(I - J_p^*)\right) = 0.$$

By direct computation, the above equation becomes

$$h_1(z) = \prod_{i=2}^{m_0-1}\left(z + \tilde{\lambda}(1 - \mu_i)\int_{-\tau}^0 \varphi(s)e^{zs}ds\right)^{p_i} = 0. \tag{19}$$

Noting $\tilde{\lambda}\tau(1 - \mu_j) < k^*$ for $j = 2, \cdots, m_0 - 1$, and all roots of $h_1(z) = 0$ are also the roots of $h_0(z) = 0$, following Lemma 2.1, we see that all roots of the characteristic equation (19) have negative real parts when $\tilde{\lambda}\tau(1 - \mu_{m_0}) = k^*$.

Following Lemma 1.1, there is a constant $K_2 > 0$ such that

$$\|\mathbf{S}_p^*(t)\| \le K_2 e^{-ct} \text{ for all } c \in (0, -c_2).$$

Thus $\|\mathbf{S}_p^*(t)\| \le K_2 e^{-(|c_2|-\varepsilon)t}$ for all $\varepsilon \in (0, -c_2)$ and

$$\|\mathbf{X}(t + \theta) - \mathbf{X}_\infty - \mathbf{X}_p(t)\| = \|T_0 \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_p^*(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} T_0^{-1}\mathbf{f}(\theta)\|$$

$$\le f_{max}K_2 e^{-(|c_2|-\varepsilon)t}.$$

This implies that

$$\sup_{\theta \in [-\tau, 0]} \|\mathbf{X}(t + \theta) - \mathbf{X}_\infty - \mathbf{X}_p(t)\| \le f_{max}K_2 e^{-(|c_2|-\varepsilon)t}, \text{ for } t \in [0, +\infty) \tag{20}$$

Thus

$$\lim_{t\to\infty}[\mathbf{X}(t) - \mathbf{X}_p(t)] = \mathbf{X}_\infty.$$

Furthermore, when $n_0 = 1$, all the components of $\mathbf{X}_\infty$ are same. Also, all the components of $\mathbf{X}_p(t)$ are periodic functions with a same period $\frac{2\pi}{y^*}$ and

$$\lim_{t\to\infty}(x_i(t) - x_{ip}(t)) = \frac{1}{N}\sum_{i=1}^N v_i(0).$$

Thus it follows from Definition 2.1 that the system (1) achieves a periodic consensus when $n_0 = 1$. When $n_0 > 1$, the system (1) achieves a weak periodic consensus. This completes the proof. □

TABLE 2. Initial values $x_i(\theta)(i = 1, 2, ..., N)$, $\theta \in [-\tau, 0]$.

$x_1(\theta)$	$x_2(\theta)$	$x_3(\theta)$	$x_4(\theta)$	$x_5(\theta)$
7.0605	0.3183	2.7692	0.4617	0.9713
$x_6(\theta)$	$x_7(\theta)$	$x_8(\theta)$	$x_9(\theta)$	$x_{10}(\theta)$
8.2346	6.9483	3.1710	9.5022	0.3445

where the numbers are randomly selected in interval (0,10).

4. **Numerical simulation.** In this section, we verify our main conclusions by a series of numerical simulations. We consider the system (1) with 10 nodes. The initial velocities are given as follows:

Case I. Consider the adjacency $A = (a_{ij})_{N \times N}$ satisfying $a_{ij} = 1(j \neq i)$ and $a_{ii} = 0$. Then $\tilde{A} = (\tilde{a}_{ij})_{N \times N}$ satisfies $\tilde{a}_{ij} = \frac{1}{N(N-1)}(j \neq i)$ and $\tilde{a}_{ii} = \frac{N-1}{N}$. Direct calculation yields

$$\det(\mu I - \tilde{A}) = (\mu - 1)\left[\mu - \frac{(N-1)^2 - 1}{N(N-1)}\right]^{N-1}.$$

Let $N = 10$, we obtain $\mu_1 = 1(n_0 = p_1 = 1)$ and $\mu_2 = \frac{8}{9}(p_2 = 9)$. The simulation results and values of $\tilde{\lambda}$ and τ for different distribution function are listed in Table 3.

TABLE 3. The numerical simulations for Case I

Cases	$\tilde{\lambda}$	τ	Results
Uniform distribution (Fig.1)	$9\pi^2$	0.3	consensus
	$9\pi^2$	0.5	periodic consensus
Exponential distribution(Fig.2)	270	1	consensus
	1050.5502	1	periodic consensus
Special γ-distribution 1(Fig.3)	13.5	1	consensus
	34.3368	1	periodic consensus
Special γ-distribution 2(Fig.4)	9	1	consensus
	24.3171	1	periodic consensus
Bernoulli distribution(Fig.5)	9π	0.3	consensus
	9π	0.5	periodic consensus

Case II. In order to understand the dynamical behaviours when the parameter $\tilde{\lambda}$ is changing, we consider the case $A = \begin{pmatrix} A_1 & \mathbf{0} \\ \mathbf{0} & A_2 \end{pmatrix}_{2N \times 2N}$, where $A_1 = (a_{ij}^{(1)})_{N \times N}$ satisfies $a_{ij}^{(1)} = 1(j \neq i)$ and $a_{ii}^{(1)} = 0$, $A_2 = (a_{ij}^{(2)})_{N \times N}$ satisfies $a_{ij}^{(2)} = 1(j > i)$ and $a_{ii}^{(2)} = 0$. Then $\tilde{A} = (\tilde{a}_{ij})_{2N \times 2N}$ satisfies $\tilde{a}_{ij} = \frac{2}{3N(N-1)}(j \neq i)$ and

$$\tilde{a}_{ii} = 1 - \frac{2}{3N}, (i = 1, 2, ..., N) \quad \tilde{a}_{ii} = 1 - \frac{2(2N-i)}{3N(N-1)}, (i = N+1, N+2, ..., 2N).$$

Direct calculation yields

$$\det(\mu I - \tilde{A}) = (\mu - 1)^2 \left[\mu - 1 + \frac{2(N-1)+2}{3N(N-1)}\right]^{N-1} \prod_{i=N+1}^{2N} \left[\mu - 1 + \frac{2(2N-i)}{3N(N-1)}\right].$$
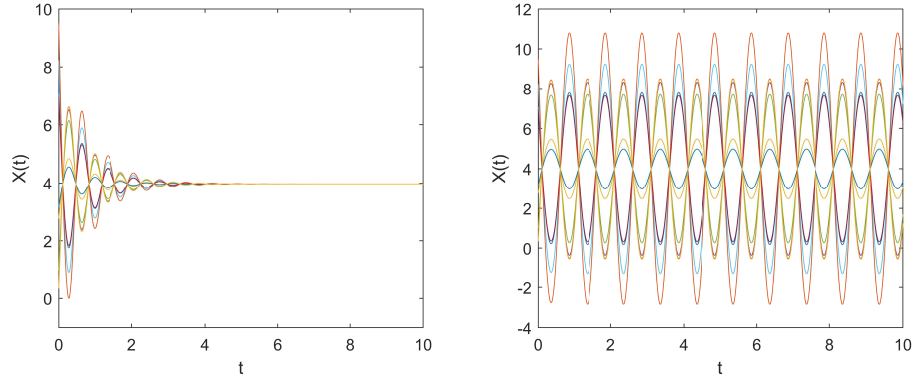
FIGURE 1. Consensus and periodic consensus with a uniform distribution delay. $\varphi(s) = \frac{1}{\tau}$, $k^* = \frac{\pi^2}{2}$(Tab.1). According to Theorem 2.1, if $\tilde{\lambda}\tau(1-\frac{8}{9}) < \frac{\pi^2}{2}$, the system achieves a consensus(left:$\tilde{\lambda} = 9\pi^2$ and $\tau = 0.3$). When $\tilde{\lambda}\tau(1-\frac{8}{9}) = \frac{\pi^2}{2}$, the system achieves a periodic consensus(right: $\tilde{\lambda} = 9\pi^2$ and $\tau = 0.5$).
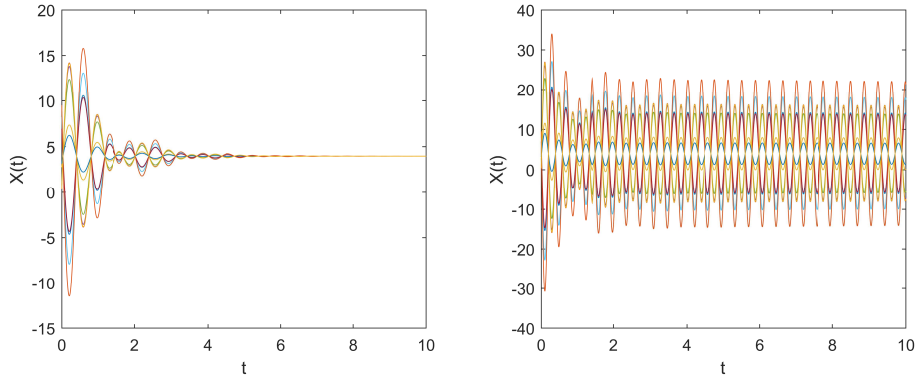

FIGURE 2. Consensus and periodic consensus with an exponential distribution delay. $\varphi(s) = \frac{\alpha e^{\alpha\tau}}{e^{\alpha\tau}-1}e^{\alpha s}(\alpha = 2, \tau = 1)$, $k^* = 116.7278$. The critical condition is that $\tilde{\lambda}\tau(1 - \frac{8}{9}) < 116.7278$. Thus, the left one is a consensus($\tilde{\lambda} = 270$ and $\tau = 1$) and the right one is a periodic consensus($\tilde{\lambda} = 1050.5502$ and $\tau = 1$).

Take $N = 5$, we obtain $\mu_1 = 1(n_0 = p_1 = 2)$, $\mu_2 = \frac{29}{30}(p_2 = 1)$, $\mu_3 = \frac{14}{15}(p_3 = 1)$, $\mu_4 = \frac{9}{10}(p_4 = 1)$, $\mu_5 = \frac{13}{15}(p_5 = 1)$ and $\mu_6 = \frac{5}{6}(p_6 = 4)$. Let $x_i(i = 6, 7, ..., 10)$ be in Group 1 (blue line) and the others $x_i(i = 1, 2, ..., 5)$ in Group 2 (red line). In this case, the numerical simulations results are listed in Table 4.
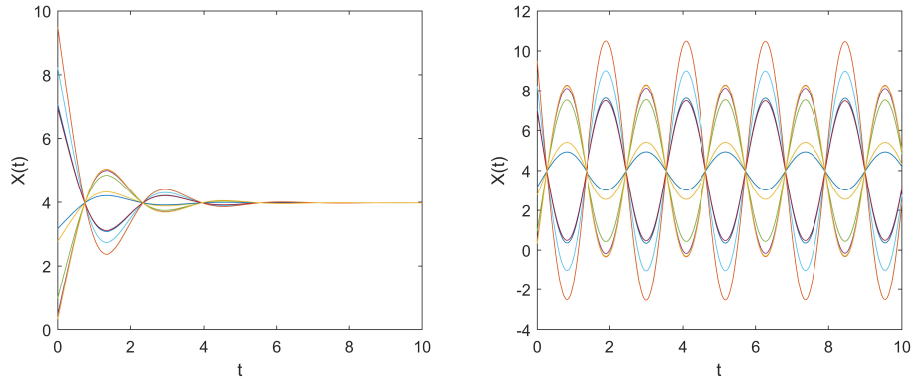
FIGURE 3. Consensus and periodic consensus with a Gamma distribution delay. $\varphi(s) = \frac{\alpha^2 e^{\alpha\tau}}{e^{\alpha\tau} - \alpha\tau - 1}|s|e^{\alpha s}(\alpha = 2, \tau = 1)$, $k^* = 3.8152$. Similarly, the left one is a consensus($\tilde{\lambda} = 13.5$ and $\tau = 1$) and the right one is a periodic consensus($\tilde{\lambda} = 34.3368$ and $\tau = 1$).
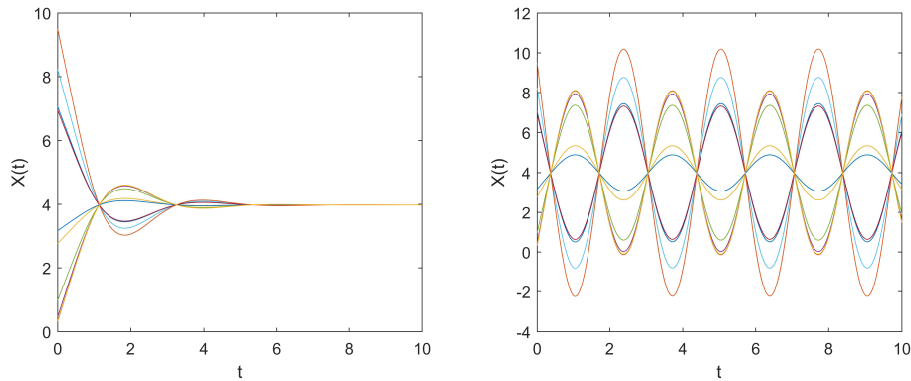

FIGURE 4. Consensus and periodic consensus with a Gamma distribution delay. $\varphi(s) = \frac{\alpha^3 e^{\alpha\tau}}{2e^{\alpha\tau} - (\alpha\tau + 1)^2 - 1}s^2 e^{\alpha s}(\alpha = 2, \tau = 1)$, $k^* = 2.7019$. The left one is the case $\tilde{\lambda} = 9$ and $\tau = 1$. The right one is the case $\tilde{\lambda} = 24.3171$ and $\tau = 1$.

REFERENCES

[1] M. H. DeGroot, Reaching a consensus, *J. Am. Stat. Assoc.*, **69** (1974), 118–121.

[2] J. A. Fax and R. M. Murray, Information flow and cooperative control of vehicle formations, *IEEE Trans. Autom. Control*, **49** (2004), 1465–1476.

[3] Q. Feng, S. K. Nguang and A. Seuret, Stability analysis of linear coupled differential–difference systems with general distributed delays, *IEEE Transactions on Automatic Control*, **65** (2020), 1356–1363.

[4] J. K. Hale and S. M. V. Lunel, *Introduction to Functional Differential Equations*, Springer-Verlag, Berlin, 1993.

[5] J. Jost, F. M. Atay and W. Lu, Consensus and synchronization in discrete-time networks of multi-agents with stochastically switching topologies and time delays, *Networks and Heterogeneous Media*, **6** (2011), 329–349.
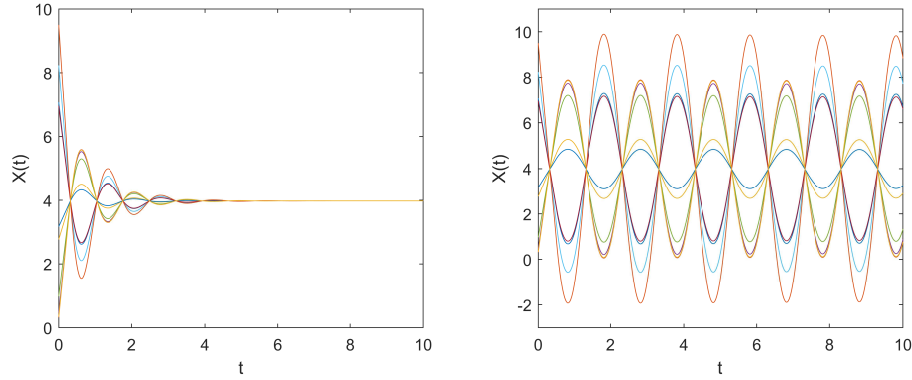
FIGURE 5. Consensus and periodic consensus with a Bernoulli distribution delay. $\varphi(s) = 0$ for $s \in (-\tau, 0]$ and $\varphi(s) = 1$ for $s = -\tau$, $k^* = \frac{\pi}{2}$. The left one is the case $\tilde{\lambda} = 9\pi$ and $\tau = 0.3$. The right one is the case $\tilde{\lambda} = 9\pi$ and $\tau = 0.5$.

TABLE 4. The numerical simulations for Case II

Distribution cases	$\tilde{\lambda}$	τ	Group 1(blue)	Group 2(red)
Uniform (Fig.6)	$6\pi^2$	0.5	consensus	periodic consensus
	$\frac{15\pi^2}{2}$	0.5	periodic consensus	divergence
Exponential (Fig.7)	700.3668	1	consensus	periodic consensus
	875.4585	1	periodic consensus	divergence
Gamma 1(Fig.8)	22.8912	1	consensus	periodic consensus
	28.614	1	periodic consensus	divergence
Gamma 2(Fig.9)	16.2114	1	consensus	periodic consensus
	20.2643	1	periodic consensus	divergence
Bernoulli(Fig.10)	6π	0.5	consensus	periodic consensus
	$\frac{15\pi}{2}$	0.5	periodic consensus	divergence

[6] F. A. Khasawneh and B. P. Mann, A spectral element approach for the stability of delay systems, *Int. J. Numer. Meth. Engng.*, **87** (2011), 566–592.

[7] N. A. Lynch, *Distributed Algorithms*, San Francisco, CA: Morgan Kaufmann, 1996.

[8] F. Mazenc, M. Malisoff and H. Özbay, Stability and robustness analysis for switched systems with time-varying delays, *SIAM J. Control Optim.*, **56** (2018), 158–182.

[9] M. Michiels, I. C. Morărescu and S.-I. Niculescu, Consensus problems with distributed delays, with application to traffic flow models, *SIAM J. Control Optim.*, **48** (2009), 77–101.

[10] I. C. Morarescu, W. Michiels and M. Jungers, Synchronization of coupled nonlinear oscillatiors with gamma-distributed delays, in *American Control Conference, ACC 2013*, (2013), Washington, United States.

[11] R. Olfati-Saber and R. M. Murray, Consensus problems in networks of agents with switching topology and time-delays, *IEEE Trans. Autom. Control*, **49** (2004), 1520–1533.

[12] A. V. Proskurnikov, Average consensus in networks with nonlinearly delayed couplings and switching topology, *Automatica*, **49** (2013) 2928–2932.

[13] D. Serre, Matrices, *Graduate Texts in Mathematics*, **216** (2010), Springer.

[14] J. W. H. So, X. Tang and X. Zou, Stability in a linear delay system without instantaneous negative feedback, *SIAM J. Math. Anal.*, **33** (2002), 1297–1304.

[15] O. Solomon and E. Fridman, New stability conditions for systems with distributed delays, *Automatica*, **49** (2013), 3467–3475.
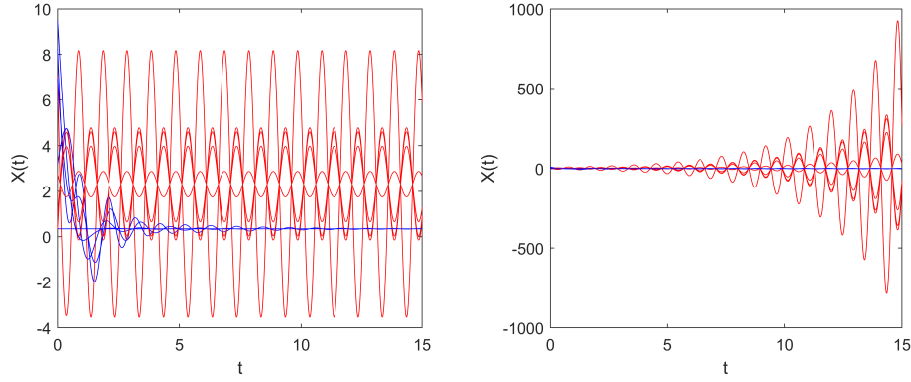
FIGURE 6. Clustering consensus with a uniform distribution delay. $\varphi(s) = \frac{1}{\tau}$, $k^* = \frac{\pi^2}{2}$. According to Theorem 2.1, if $\tilde{\lambda}\tau(1 - \frac{5}{6}) = \frac{\pi^2}{2}$, the nodes in Group 1(blue line) achieve a consensus and the ones in Group 2(red line) achieve a periodic consensus(left: $\tilde{\lambda} = 6\pi^2$ and $\tau = 0.5$). When $\tilde{\lambda}\tau(1 - \frac{13}{15}) = \frac{\pi^2}{2}$, the nodes in Group 1(blue line)achieve a periodic consensus and the others in Group 2(red line) are divergence(right: $\tilde{\lambda} = \frac{15\pi^2}{2}$ and $\tau = 0.5$).
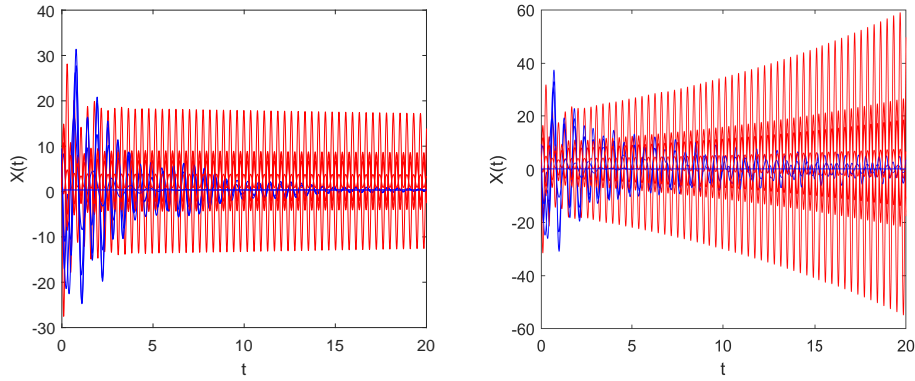
FIGURE 7. Clustering consensus with an exponential distribution delay. $\varphi(s) = \frac{\alpha e^{\alpha\tau}}{e^{\alpha\tau}-1}e^{\alpha s}(\alpha = 2, \tau = 1)$, $k^* = 116.7278$. Similarly, the left one is the case $\tilde{\lambda} = 700.3668$ and $\tau = 1$, which is that Group 1(blue line) achieves a consensus and Group 2(red line) achieves a periodic consensus. The right one is the case $\tilde{\lambda} = 875.4585$ and $\tau = 1$, which is that Group 1(blue line)achieves a periodic consensus and Group 2(red line) is divergence.

[16] T. Vicsek, A Czirók , E. Ben-Jacob, I. Cohen and O. Shochet, Novel type of phase transition in a system of self-driven particles, *Phys. Rev. Lett.*, **75** (1995), 1226–1229.
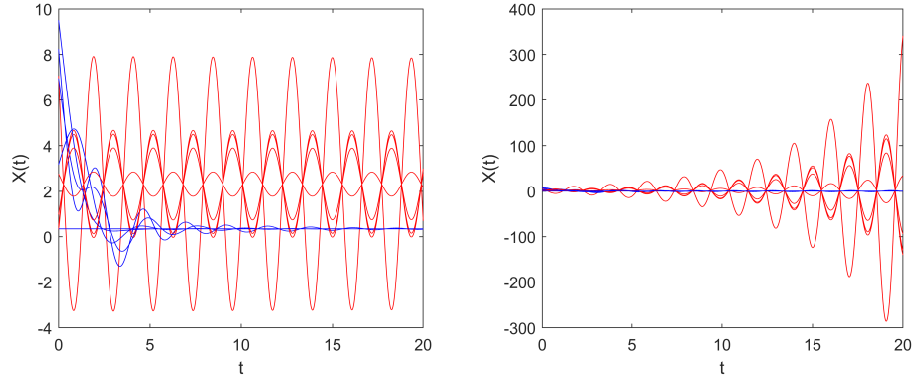
FIGURE 8. Clustering consensus with a Gamma distribution delay. $\varphi(s) = \frac{\alpha^2 e^{\alpha\tau}}{e^{\alpha\tau}-\alpha\tau-1}|s|e^{\alpha s}(\alpha = 2, \tau = 1)$, $k^* = 3.8152$. In the case of $\tilde{\lambda} = 22.8912$ and $\tau = 1$, the nodes in Group 1(blue line)achieve a consensus and the others in Group 2(red line) achieve a periodic consensus(left). In the case $\tilde{\lambda} = 28.614$ and $\tau = 1$, the nodes in Group 1(blue line)achieve a consensus and the others in Group 2(red line) are divergence(right).
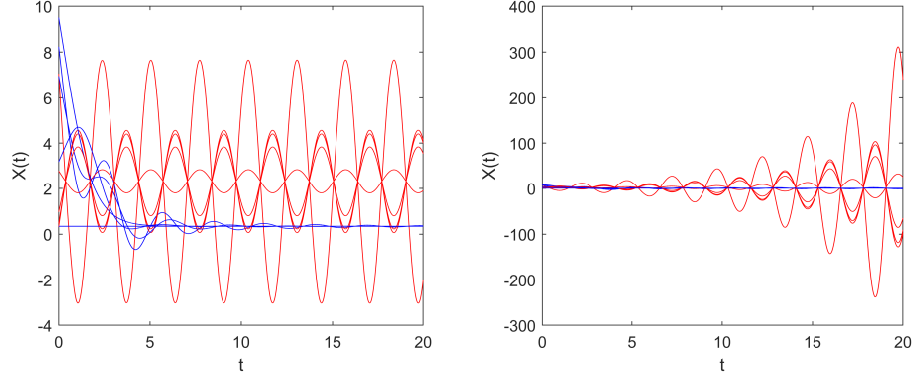

FIGURE 9. Clustering consensus with a Gamma distribution delay. $\varphi(s) = \frac{\alpha^3 e^{\alpha\tau}}{2e^{\alpha\tau}-(\alpha\tau+1)^2-1}s^2e^{\alpha s}$ $(\alpha = 2, \tau = 1)$, $k^* = 2.7019$. In the case of $\tilde{\lambda} = 16.2114$ and $\tau = 1$, the nodes in Group 1(blue line)achieve a consensus and the others in Group 2(red line) achieve a periodic consensus(left). In the case $\tilde{\lambda} = 20.2643$ and $\tau = 1$, the nodes in Group 1(blue line)achieve a consensus and the others in Group 2(red line) are divergence(right).

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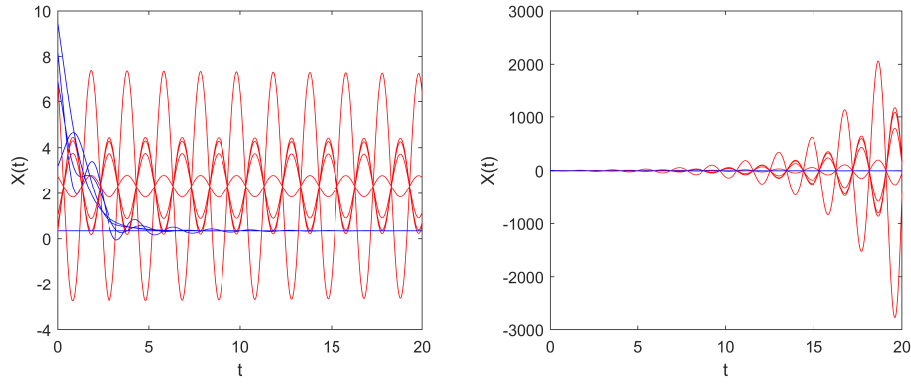*E-mail address*: junwmath@hotmail.com

FIGURE 10. Clustering consensus with a Bernoulli distribution delay. $\varphi(s) = 0$ for $s \in (-\tau, 0]$ and $\varphi(s) = 1$ for $s = -\tau$, $k^* = \frac{\pi}{2}$. In the case of $\tilde{\lambda} = 6\pi$ and $\tau = 0.5$, the nodes in Group 1(blue line)achieve a consensus and the others in Group 2(red line) achieve a periodic consensus(left). In the case $\tilde{\lambda} = \frac{15\pi}{2}$ and $\tau = 0.5$, the nodes in Group 1(blue line)achieve a consensus and the others in Group 2(red line) are divergence(right).