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THE LAX-FRIEDRICHS SCHEME FOR INTERACTION BETWEEN THE INVISCID BURGERS EQUATION AND MULTIPLE PARTICLES

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Abstract. We propose a finite difference method based on the Lax-Friedrichs scheme for a model of interaction between multiple solid particles and an inviscid fluid. The single-particle version has been studied extensively during the past decade. The model studied here consists of the inviscid Burgers equation with multiple nonconservative moving source terms that are singular and account for drag force interaction between the fluid and the particles. Each particle trajectory satisfies a differential equation that ensures conservation of momentum of the entire system. To deal with the singular source terms we discretize a model that associates with each particle an advection PDE whose solution is a shifted Heaviside function. This alternative model is well known but has not previously been used in numerical methods. We propose a definition of entropy solution which directly generalizes the previously defined single-particle notion of entropy solution. We prove convergence (along a subsequence) of the Lax-Friedrichs approximations, and also prove that if the set of times where the particle paths intersect has Lebesgue measure zero, then the limit is an entropy solution. We also propose a higher resolution version of the scheme, based on MUSCL processing, and present the results of numerical experiments.

1. **Introduction.** This paper concerns a one-dimensional model of solid-fluid interaction:

$$\begin{cases} \partial_t u + \partial_x f(u) = \displaystyle\sum_{k=1}^{K} \lambda_k \left(h_k'(t) - u\right) \delta(x - h_k(t)), \quad (x,t) \in \mathbb{R} \times (0,T) := \Pi_T, \\ m_k {h_k}''(t) = \lambda_k \left(u(h_k(t),t) - h_k'(t)\right), \quad t \in (0,T), \quad k = 1,\dots,K, \\ u(x,0) = u_0(x), \quad (h_k(0), h_k'(0)) = (h_{k,0}, v_{k,0}), \quad k = 1,\dots,K. \end{cases} \tag{1.1}$$

Here $f(u) = u^2/2$, and $\delta(x)$ denotes the Dirac delta measure concentrated at $x = 0$. The function $u = u(x,t)$ models the velocity of the fluid, $h_k(t)$ models the location of the kth solid particle at time t, $\lambda_k > 0$ is a drag coefficient associated with the

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kth particle, and $m_k > 0$ is the mass of the kth particle. Study of the single-particle version of (1.1) was initiated in [11], and has been the subject of a number of additional papers.

The fluid velocity is governed by the inviscid Burgers equation $u_t + f(u)_x = 0$, and the particle-fluid coupling is due to friction, more specifically the drag terms $\lambda_k (u - h_k')$ which appear in both the PDE and the ODEs in (1.1). Since there is no viscosity, the velocity $u(x,t)$ admits entropy weak solutions, meaning that shock waves occur. This leads to complex interactions between the resulting shock wave and the particles. When multiple particles are present there are interesting features of the solutions that include particles drafting and passing by one another; see Figure 4 or Figure 5.

There are some difficulties associated with (1.1), in addition to the well-known ones associated with a nonlinear conservation law. The source terms on the right side of the first equation are nonconservative products of distributions; their meaning is not immediately clear. The differential equations appearing in the second line are coupled to the conservation law. Due to discontinuities in u the meaning of the right side of the DE's is also not readily apparent. There are related difficulties in designing practical numerical algorithms.

Notwithstanding these difficulties there has been much progress on the single-particle version of (1.1). A notion of solution has been developed, well-posedness has been proven, and numerical algorithms have been designed whose approximations are known to converge to the unique solution. In this paper we focus on the multiple-particle problem, which has not been studied as thoroughly. We propose a notion of entropy solution suitable for multiple particles, present a Lax-Friedrichs difference scheme for the multiple-particle problem, and prove that the resulting approximations converge to an entropy solution. This is accomplished under the assumption that the particle paths do not intersect except possibly at a set of times whose Lebesgue measure is zero.

Reference [4] developed a unifying framework for the jump conditions that hold across a spatial flux discontinuity for a conservation law with discontinuous flux, using the theory of L^1-dissipative (L^1D) admissibility germs. The relevant L^1D admissible germ for the problem discussed here is $\mathcal{G}(\lambda, c)$, which was identified in [7].

Definition 1.1 (the germ $\mathcal{G}(\lambda, c)$, [7])**.** The germ $\mathcal{G}(\lambda, c)$ is the subset of $\mathbb{R}^2$ defined by

$$\mathcal{G}(\lambda, c) = (c, c) + \{(a, b) \in \mathbb{R}^2 \,|b = a - \lambda\} \cup \{(a, b) \in \mathbb{R}^2 \,|a \geq 0,\, b \leq 0,\, -\lambda \leq a + b \leq \lambda\}. \tag{1.2}$$

Reference [6] gives a definition of entropy solution for the single-particle version of (1.1). The following is a direct generalization of that definition to the multiple-particle problem.

Definition 1.2 (entropy solution)**.**
(i) Given $h_k \in W^{1,\infty}([0,T], \mathbb{R})$, $k = 1, \ldots, K$, let $\Gamma = \bigcup_{k=1}^{K} \{(h_k(t), t)) : t \in [0,T)\}$. A function u is a solution of the first equation of (1.1) with initial data u_0 if $u \in L^\infty(\Pi_T) \cap C([0,T]); L^1_{\text{loc}}(\mathbb{R}))$, if u is a Kružkov entropy solution in $\Pi_T \setminus \Gamma$ of the Burgers equation with initial data u_0, and if for a.e. $t \in (0,T)$ the one-sided traces of u at each particle position satisfy

$$\big(u(h_k(t)^-, t), u(h_k(t)^+, t)\big) \in \mathcal{G}\left(\lambda_k(t), h_k'(t)\right), \quad k = 1, \ldots, K. \tag{1.3}$$

(ii) A function h_k is a solution of the second equation of (1.1) with initial data $(h_{k,0}, v_{k,0})$ if $h_k \in W^{2,\infty}([0,T])$, if $h_k(0) = h_{k,0}$, $h_k'(0) = v_{k,0}$, and if given given u a Kružkov entropy solution of the Burgers equation in $\Pi_T \setminus \Gamma$ we have for a.e. $t \in (0,T)$

$$\begin{aligned} m_k h_k''(t) = &\left(\frac{1}{2}u(h_k(t)^-,t)^2 - h_k'(t)u(h_k(t)^-,t)\right) \\ &- \left(\frac{1}{2}u(h_k(t)^+,t)^2 - h_k'(t)u(h_k(t)^+,t)\right). \end{aligned} \tag{1.4}$$

(iii) With the notation $\vec{h} = (h_1, \ldots, h_K)$, a pair $(u, \vec{h})$ satisfying (i) and (ii) above is an entropy solution of the system (1.1).

Remark 1. Definition 1.2 requires strong one-sided traces $u(h_k(t)^\pm, t)$ along each path $x = h_k(t)$. Assuming that the particle trajectories do not intersect except possibly on a subset of $(0,T)$ having Lebesgue measure zero, the results of [13] guarantee existence of the required traces. This is due to the regularity of the paths $x = h_k(t)$ and the fact that u is a Kružkov entropy solution of the Burgers equation in $\Pi_T \setminus \Gamma$.

Assumption 1.1. The initial data satisfies $u_0 \in \mathrm{BV}(\mathbb{R})$.

Above we have used the notation $\mathrm{BV}(\mathbb{R})$ to denote the set of functions of bounded variation on $\mathbb{R}$, i.e., those functions $\rho : \mathbb{R} \mapsto \mathbb{R}$ for which

$$\mathrm{TV}(\rho) := \sup\left\{\sum_{i=1}^{M} |\rho(\xi_i) - \rho(\xi_{i-1})|\right\} < \infty,$$

where the sup extends over all $M \geq 1$ and all partitions $\{\xi_0 < \xi_1 < \ldots < \xi_M\}$ of $\mathbb{R}$.

Theorem 1.3 (Main theorem)**.** *The Lax-Friedrichs scheme described in Section 2 produces approximations that converge as the mesh size approaches zero, along a subsequence, to a pair $(u, \vec{h})$ where $u \in L^\infty(\Pi_T) \cap C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}))$ and $h_k \in W^{2,\infty}([0,T])$, $k = 1, \ldots, K$. If the particle trajectories $h_k(t)$ do not intersect except possibly on a subset of $(0,T)$ having Lebesgue measure zero, then $(u, \vec{h})$ is an entropy solution in the sense of Definition 1.2.*

As mentioned above, there has been significant progress on the single-particle version of (1.1) [1, 5, 6, 7, 11]. The study of (1.1) started with reference [11]. Among other things the authors completely solved the Riemann problem for $K = 1$, and described the asymptotic behavior of solutions.

In reference [5], the authors introduce two finite volume methods for computing approximate solutions. One is a Glimm-like scheme, and the other is a well-balanced scheme that uses nonrectangular space-time cells near the interface. These methods employ random sampling for placing the particle at a mesh interface at each time step. The nonconservative source term is handled by using a certain well-balanced scheme that was analyzed in [7]. They avoid the use of a moving mesh, and also avoid the use of a Riemann solver for the full model. The case of multiple particles is addressed, and is handled via a splitting method.

Reference [14] presents a finite volume scheme that is based on the well-balanced scheme of [5, 7], but uses an adaptive stencil as an alternative to using a moving grid. The multiple-particle case is handled by splitting.

Reference [7] proves well-posedness for the problem

$$u_t + \left(u^2/2\right)_x = -\lambda u \delta(x), \quad u(x,0) = u_0(x). \tag{1.5}$$

This is a simplification of (1.1), but its analysis provides an important step in analyzing the full problem. As mentioned above the germ $\mathcal{G}(\lambda, c)$, which is required for the correct defintion of entropy solution, was identified in [7].

Reference [6] proves well-posedness of the model (1.1) for $K = 1$, assuming that the initial data is of bounded variation. Approximate solutions are generated via a wave-front tracking algorithm. Definition 1.2 is a direct generalization of the definition for $K = 1$ appearing in [6].

Reference [1] presents a class of finite volume schemes for (1.1) when $K = 1$. The schemes are similar to those in [5], but a moving grid is used, which keeps the particle located at a fixed cell boundary. The approximations are shown to converge to the unique entropy solution.

References [2] and [3] concern a generalized version of (1.1) (again, for $K = 1$), where the fluid is governed by the inviscid compressible Euler equations.

Reference [10] specifically deals with a multiple-particle problem. The authors prove well-posedness for a version of (1.1) where the particle paths $h_k(t)$ are given, i.e., the second equation of (1.1) does not appear.

Let $H(\cdot)$ denote the Heaviside function, i.e., the characteristic function of $[0, \infty)$. The system (1.1) has the following equivalent formulation [5, 11]:

$$\begin{cases} \partial_t u + \partial_x (u^2/2) = \displaystyle\sum_{k=1}^{K} \lambda_k \left(h_k'(t) - u\right) \partial_x w_k, \quad (x,t) \in \Pi_T, \\ \partial_t w_k + h_k'(t) \partial_x w_k = 0, \quad (x,t) \in \Pi_T, \quad k = 1, \ldots, K, \\ m_k {h_k}''(t) = \lambda_k \left(u(h_k(t), t) - h_k'(t)\right), \quad t \in (0,T), \quad k = 1, \ldots, K, \\ u(x,0) = u_0(x), \quad (h_k(0), h_k'(0)) = (h_{k,0}, v_{k,0}), \quad k = 1, \ldots, K, \\ w_k(x,0) = H(x - h_{k,0}), \quad k = 1, \ldots, K. \end{cases} \tag{1.6}$$

Although the splitting approach for multiple particles used in [5] and [14] gives good numerical results, extending the convergence analysis from the single-particle to the multiple-particle problem seems difficult. Various bounds required for convergence are not preserved by the splitting steps. The numerical schemes in those papers are based on the model (1.1). In this paper we instead discretize (1.6), using Lax-Friedrichs differencing for each of the PDEs. The advantage of this approach is that the case of multiple particles is accommodated without splitting. This makes it possible to obtain a number of estimates which taken together give a convergence proof for the multiple-particle model. On the other hand, while the schemes of [1], [5], and [14] give very sharply resolved shocks at the particle locations, our Lax-Friedrichs method results in a substantial amount of smearing. With this in mind, we additionally propose a higher resolution version of the scheme, based on MUSCL processing.

The rest of the paper is organized as follows. In Section 2 we describe the Lax-Friedrichs scheme mentioned above. In Section 3 we prove convergence, modulo a subsequence, of the approximations for u, as well as the approximations for h_k. In Section 4 we prove convergence of the approximations for w_k. In Section 5 we verify that the subsequential limit u is a Kružkov entropy solution in $\Pi_T \setminus \Gamma$ and satisfies the jump condition (1.3). In Section 6 we prove that the limit h_k satisfies the differential equation (1.4). Section 6 concludes with the proof of Theorem 1.3.

Section 7 describes the MUSCL processing mentioned above. Section 8 presents the results of some numerical experiments.

2. **The Lax-Friedrichs scheme applied to (1.6).** We use a uniform spatial mesh size Δx, and temporal step size Δt. Define

$$x_j = j\Delta x, \quad j \in \mathbb{Z}, \quad t^n = n\Delta t, \quad 0 \le n \le N, \tag{2.1}$$

where the integer N is such that $N\Delta t \in [T, T+\Delta t)$. Define $I_j = [x_j - \Delta x/2, x_j + \Delta x/2)$, $I^n = [t^n, t^{n+1})$. Let $\chi_j(x)$ denote the characteristic function of I_j, and $\chi^n(t)$ the characteristic function of I^n We denote by U_j^n the finite difference approximation of $u(x_j, t^n)$, $U_j^n \approx u(x_j, t^n)$. Similarly $W_{k,j}^n \approx w_k(x_j, t^n)$. Let $\{Q_j^n\}$ be a grid-defined function such as $\{U_j^n\}$ or $\{W_{k,j}^n\}$. We will use the following notational abbreviations:

$$\begin{aligned} &\Delta_+ Q_j^n = Q_{j+1}^n - Q_j^n, \quad \Delta_- Q_j^n = Q_j^n - Q_{j-1}^n, \quad \hat{Q}_j^n = \frac{1}{2}\left(Q_{j-1}^n + Q_{j+1}^n\right), \\ &Q_{\min}^n = \inf_{j\in Z} Q_j^n, \quad Q_{\max}^n = \sup_{j\in Z} Q_j^n, \quad \|Q^n\|_\infty = \sup_{j\in\mathbb{Z}} \left|Q_j^n\right|. \end{aligned} \tag{2.2}$$

Let $v_0(x)$ denote the initial data $u_0(x)$ or $H(x - h_{k,0})$. The data $v_0(x)$ is discretized via $V_j^0 = \frac{1}{\Delta x}\int_{I_j} v_0(x)\,dx$, implying that

$$\inf_{x\in I_j} v_0(x) \le V_j^0 \le \sup_{x\in I_j} v_0(x), \text{ and } \sum_{j\in\mathbb{Z}} \chi_j(x) V_j^0 \to v_0(x) \text{ in } L^1_{\text{loc}}(\mathbb{R}) \text{ as } \Delta x \to 0. \tag{2.3}$$

With the notation $v_{\min}^0 = \inf_{y\in\mathbb{R}} v_0(y)$, $v_{\max}^0 = \sup_{y\in\mathbb{R}} v_0(y)$, we have $-\infty < v_{\min}^0, v_{\max}^0 < \infty$. Due to our method of discretizing v_0, $v_{\min}^0 \le V_{\min}^0$, $V_{\max}^0 \le v_{\max}^0$, $\left\|V^0\right\|_\infty \le \|v_0\|_\infty$, and $\sum_{j\in\mathbb{Z}} \left|\Delta_+ V_j^0\right| \le \mathrm{TV}(v_0)$.

We extend $\{U_j^n\}$ and $\{W_{k,j}^n\}$ from grid-defined functions to functions defined on all of Π_T via

$$u^\Delta(x,t) = \sum_{n=0}^N \sum_{j\in\mathbb{Z}} \chi_j(x)\chi^n(t) U_j^n, \quad w_k^\Delta(x,t) = \sum_{n=0}^N \sum_{j\in\mathbb{Z}} \chi_j(x)\chi^n(t) W_{k,j}^n. \tag{2.4}$$

Similarly,

$$c_k^\Delta(t) = \sum_{n=0}^N \chi^n(t) c_k^n, \quad h_k^\Delta(t) = \sum_{n=0}^N \chi^n(t)\left(h_k^n + (t - t^n)c_k^n\right), \tag{2.5}$$

where $c_k^n \approx h_k'(t^n)$ and $h_k^n \approx h_k(t^n)$, with the initialization $(h_k^0, c_k^0) = (h_{k,0}, v_{k,0})$

Let $\mu = \Delta t/\Delta x$. The algorithm that we propose discretizes the first two equations of (1.6) via the Lax-Friedrichs scheme, the third equation using Euler's method:

$$\begin{cases} U_j^{n+1} = U_j^n - \mu\Delta_- \bar{f}_{j+1/2}^n + \displaystyle\sum_{k=1}^K \frac{\lambda_k \mu}{2}\left(c_k^n - \hat{U}_j^n\right)\left(W_{k,j+1}^n - W_{k,j-1}^n\right), \\ W_{k,j}^{n+1} = W_{k,j}^n - \mu\Delta_- \bar{g}_{k,j+1/2}^n, \\ c_k^{n+1} = c_k^n - \displaystyle\frac{1}{m_k}\sum_{j\in\mathbb{Z}} \frac{\Delta t \lambda_k}{2}\left(c_k^n - \hat{U}_j^n\right)\left(W_{k,j+1}^n - W_{k,j-1}^n\right), \\ h_k^{n+1} = h_k^n + c_k^n \Delta t. \end{cases} \tag{2.6}$$

Here

$$\begin{aligned}\bar{f}^n_{j+1/2} &= \bar{f}(U^n_{j+1}, U^n_j) = \frac{1}{2}\left(\left(U^n_{j+1}\right)^2/2 + \left(U^n_j\right)^2/2\right) - \frac{q}{2\mu}\left(U^n_{j+1} - U^n_j\right),\\ \bar{g}^n_{k,j+1/2} &= \frac{1}{2}\left(c^n_k W^n_{k,j+1} + c^n_k W^n_{k,j}\right) - \frac{q}{2\mu}\left(W^n_{k,j+1} - W^n_{k,j}\right),\end{aligned} \tag{2.7}$$

where q is a parameter. For our purposes $q \in (0, 1/2]$. The numerical fluxes in (2.7) result by applying the Lax-Friedrichs flux [12] to $f(u) = u^2/2$ and $g^n_k(w) = c^n_k w$.

Remark 2. The scheme (2.6) preserves solutions where the fluid velocity and particle velocities are equal to the same constant: $U^n_j = v$ for all $j \in \mathbb{Z}$, $c^n_k = v$ for $k = 1, \ldots, K$.

Remark 3. Some explanation of the third equation of (2.6) is in order. Based on the third equation of (1.6), the third equation of (2.6) should be (approximately) equivalent to

$$c^{n+1}_k = c^n_k - \frac{1}{m_k}\Delta t \lambda_k c^n_k + \frac{1}{m_k}\Delta t \lambda_k \tilde{u}(h_k(t^n), t^n),$$

where $\tilde{u}(h_k(t^n), t^n) \approx u(h_k(t^n), t^n)$. To see that the third equation of (2.6) is actually of this form, note that since $W^n_{k,j} \approx H(x_j - h_k(t^n))$, the grid function $\{(1/2)(W^n_{k,j+1} - W^n_{k,j-1})/\Delta x\}$ approximates $\delta(x - h_k(t^n))$, a delta function concentrated at $x = h_k(t^n)$. In particular, we expect $(1/2)\sum_{j\in\mathbb{Z}}\left(W^n_{k,j+1} - W^n_{k,j-1}\right) \approx 1$ (in fact this holds with "$\approx$" replaced by "$=$"; this follows from (3.5) of Lemma 3.1), and so we can write the third equation of (2.6) in the form

$$c^{n+1}_k = c^n_k - \frac{1}{m_k}\Delta t \lambda_k c^n_k + \frac{1}{m_k}\Delta t \lambda_k (1/2)\sum_{j\in\mathbb{Z}} \hat{U}^n_j \left(W^n_{k,j+1} - W^n_{k,j-1}\right).$$

Thus, by defining

$$\tilde{u}(h_k(t^n), t^n) := (1/2)\sum_{j\in\mathbb{Z}} \hat{U}^n_j \left(W^n_{k,j+1} - W^n_{k,j-1}\right) \approx \int_{\mathbb{R}} u(x, t^n)\delta(x - h_k(t^n))\, dx,$$

we have the desired approximation $\tilde{u}(h_k(t^n), t^n) \approx u(h_k(t^n), t^n)$. Clearly there are other, possibly simpler, methods of discretizating the third equation of (1.6). The reason for choosing this particular approximation is to ensure the discrete conservation of momentum property discussed below.

From the first two equations of (1.1) it follows that, at least formally, the total momentum of the system is conserved:

$$\frac{d}{dt}\left(\int_{\mathbb{R}} u(x,t)\, dx + \sum_{k=1}^{K} m_k h'_k(t)\right) = 0. \tag{2.8}$$

The scheme (2.6) enforces a discrete version of (2.8).

Proposition 1. *Assume that there is a $0 < J \in \mathbb{Z}$ such that $U^n_j = 0$ for $|j| > J$, and that $\|U^n\|_\infty < \infty$. Define the discrete momentum:*

$$\mathcal{M}^n = \Delta x \sum_{j\in\mathbb{Z}} U^n_j + \sum_{k=1}^{K} m_k c^n_k. \tag{2.9}$$

The discrete momentum is conserved: $\mathcal{M}^{n+1} = \mathcal{M}^n$ for $0 \le n \le N$.

Proof. Multiplying by Δx and summing the first equation of (2.6) over $j \in \mathbb{Z}$ gives

$$\Delta x \sum_{j\in\mathbb{Z}} U_j^{n+1} = \Delta x \sum_{j\in\mathbb{Z}} U_j^n + \sum_{j\in\mathbb{Z}} \sum_{k=1}^{K} \frac{\lambda_k \Delta t}{2} \left(c_k^n - \hat{U}_j^n\right) \left(W_{k,j+1}^n - W_{k,j-1}^n\right). \quad (2.10)$$

Multiplying the third equation of (2.6) by m_k and then summing over k gives

$$\sum_{k=1}^{K} m_k c_k^{n+1} = \sum_{k=1}^{K} m_k c_k^n - \sum_{k=1}^{K} \sum_{j\in\mathbb{Z}} \frac{\Delta t \lambda_k}{2} \left(c_k^n - \hat{U}_j^n\right) \left(W_{k,j+1}^n - W_{k,j-1}^n\right). \quad (2.11)$$

The proof is completed by adding (2.10) and (2.11). □

Define

$$Z_j^n = U_j^n + \sum_{k=1}^{K} \lambda_k W_{k,j}^n, \quad z^\Delta(x,t) = \sum_{n=0}^{N} \sum_{j\in\mathbb{Z}} \chi_j(x)\chi^n(t) Z_j^n. \quad (2.12)$$

Lemma 2.1. *Z_j^n satisfies the following (equivalent) evolution equations:*

$$Z_j^{n+1} = Z_j^n - \mu \Delta_- \bar{f}(Z_{j+1}^n, Z_j^n) + \frac{\mu}{2} \sum_{k=1}^{K} \lambda_k \hat{W}_{k,j}^n \left(Z_{j+1}^n - Z_{j-1}^n\right), \quad (2.13)$$

$$Z_j^{n+1} = Z_j^n + \frac{1}{2}\left(q - \mu \hat{U}_j^n\right) \Delta_+ Z_j^n - \frac{1}{2}\left(q + \mu \hat{U}_j^n\right) \Delta_- Z_j^n. \quad (2.14)$$

Remark 4. From (1.6) and the definition $z = u + \sum_{k=1}^{K} \lambda_k w_k$, one can derive (formally) the PDE

$$\partial_t z + \partial_x f(z) = \sum_{k=1}^{K} \lambda_k w_k \partial_x z. \quad (2.15)$$

Evidently (2.13) is a discretization of (2.15).

Remark 5. It is clear by inspection of either (2.13) or (2.14) that the scheme (2.6) preserves solutions of the form $Z_j^n =$ constant.

Proof. Using (2.12) and (2.6) we find that

$$\begin{aligned} Z_j^{n+1} &= U_j^n - \mu \Delta_- \bar{f}_{j+1/2}^n + \sum_{k=1}^{K} \frac{\lambda_k \mu}{2} \left(c_k^n - \hat{U}_j^n\right) \left(W_{k,j+1}^n - W_{k,j-1}^n\right) \\ &\quad + \sum_{k=1}^{K} \lambda_k \left(W_{k,j}^n - \mu \Delta_- \bar{g}_{k,j+1/2}^n\right) \\ &= Z_j^n - \mu \Delta_- \bar{f}_{j+1/2}^n + \sum_{k=1}^{K} \frac{\lambda_k \mu}{2} \left(c_k^n - \hat{U}_j^n\right) \left(W_{k,j+1}^n - W_{k,j-1}^n\right) \\ &\quad - \mu \sum_{k=1}^{K} \lambda_k \Delta_- \bar{g}_{k,j+1/2}^n. \end{aligned} \quad (2.16)$$

Next we use

$$\begin{aligned} \Delta_- \bar{f}_{j+1/2}^n &= \frac{1}{2} \hat{U}_j^n \left(U_{j+1}^n - U_{j-1}^n\right) - \frac{q}{2\mu} \Delta_+ \Delta_- U_j^n, \\ \Delta_- \bar{g}_{k,j+1/2}^n &= \frac{1}{2} c_k^n \left(W_{k,j+1}^n - W_{k,j-1}^n\right) - \frac{q}{2\mu} \Delta_+ \Delta_- W_{k,j}^n. \end{aligned} \quad (2.17)$$

Substituting (2.17) into (2.16) and canceling $(\mu/2)\sum_{k=1}^{K}\lambda_k c_k^n(W_{k,j+1}^n - W_{k,j-1}^n)$, the result is

$$\begin{aligned} Z_j^{n+1} &= Z_j^n - \frac{\mu}{2}\hat{U}_j^n\left(U_{j+1}^n - U_{j-1}^n\right) + \frac{q}{2}\Delta_+\Delta_- U_j^n \\ &\quad - \frac{\mu}{2}\hat{U}_j^n\sum_{k=1}^{K}\lambda_k\left(W_{k,j+1}^n - W_{k,j-1}^n\right) + \frac{q}{2}\sum_{k=1}^{K}\lambda_k\Delta_+\Delta_- W_{k,j}^n \\ &= Z_j^n - \frac{\mu}{2}\hat{U}_j^n\left(Z_{j+1}^n - Z_{j-1}^n\right) + \frac{q}{2}\Delta_+\Delta_- Z_j^n \\ &= Z_j^n - \frac{\mu}{2}\hat{U}_j^n\left(\Delta_+ Z_j^n + \Delta_- Z_j^n\right) + \frac{q}{2}\left(\Delta_+ Z_j^n - \Delta_- Z_j^n\right). \end{aligned} \tag{2.18}$$

The identity (2.14) is immediate from (2.18).

For the proof of (2.13), we start from the second equality of (2.18) and substitute $\hat{U}_j^n = \hat{Z}_j^n - \sum_{k=1}^{K}\lambda_k\hat{W}_{k,j}^n$, which results in

$$\begin{aligned} Z_j^{n+1} &= Z_j^n - \frac{\mu}{2}\left(\hat{Z}_j^n - \sum_{k=1}^{K}\lambda_k\hat{W}_{k,j}^n\right)\left(Z_{j+1}^n - Z_{j-1}^n\right) + \frac{q}{2}\Delta_+\Delta_- Z_j^n \\ &= Z_j^n - \frac{\mu}{2}\hat{Z}_j^n\left(Z_{j+1}^n - Z_{j-1}^n\right) + \frac{\mu}{2}\left(\sum_{k=1}^{K}\lambda_k\hat{W}_{k,j}^n\right)\left(Z_{j+1}^n - Z_{j-1}^n\right) + \frac{q}{2}\Delta_+\Delta_- Z_j^n \\ &= Z_j^n - \frac{\mu}{2}\left(f(Z_{j+1}^n) - f(Z_{j-1}^n)\right) + \frac{q}{2}\Delta_+\Delta_- Z_j^n \\ &\quad + \frac{\mu}{2}\left(\sum_{k=1}^{K}\lambda_k\hat{W}_{k,j}^n\right)\left(Z_{j+1}^n - Z_{j-1}^n\right). \end{aligned} \tag{2.19}$$

The identity (2.13) now follows directly from (2.19). □

3. **Convergence of u^Δ and h_k^Δ.** Let $\Delta = (\Delta x, \Delta t)$. For our convergence analysis we will assume that $\Delta \to 0$ with μ fixed, and satisfying the following CFL condition:

$$\mu\max\left(\max_{1\le k\le K}\left|c_k^0\right|, \|z_0\|_\infty + \sum_{k=1}^{K}\lambda_k, \|u_0\|_\infty + \sum_{k=1}^{K}\lambda_k\right) \le q \le 1/2. \tag{3.1}$$

Additionally we assume that

$$\Delta t \le m_k/\lambda_k, \quad k = 1,\dots,K, \tag{3.2}$$

which will be satisfied automatically for Δ sufficiently small.

Define $z_0(x) = u_0(x) + \sum_{k=1}^{K}\lambda_k H(x - h_k(0))$. Due to the method of discretizing u_0 and $H(x - h_k(0))$, it follows from from (2.12) that $Z_j^0 = \frac{1}{\Delta x}\int_{I_j} z_0(x)\,dx$. Using the notation $z_{\min}^0 = \inf_{y\in\mathbb{R}} z_0(y)$, $z_{\max}^0 = \sup_{y\in\mathbb{R}} z_0(y)$, we have $-\infty < z_{\min}^0, z_{\max}^0 < \infty$, and $z_{\min}^0 \le Z_{\min}^0$, $Z_{\max}^0 \le z_{\max}^0$, and $\left\|Z^0\right\|_\infty \le \|z_0\|_\infty$.

Lemma 3.1. *The following properties hold:*

$$z_{\min}^0 \le Z_j^n \le z_{\max}^0, \quad \|Z^n\|_\infty \le \|z_0\|_\infty, \tag{3.3}$$

$$u_{\min}^0 - \sum_{k=1}^{K}\lambda_k \le U_j^n \le u_{\max}^0 + \sum_{k=1}^{K}\lambda_k, \quad \|U^n\|_\infty \le \|u_0\|_\infty + \sum_{k=1}^{K}\lambda_k, \tag{3.4}$$

$$W_{k,j}^n \in [0,1], \quad \Delta_+ W_{k,j}^n \ge 0, \quad \sum_{j\in\mathbb{Z}}\Delta_+ W_{k,j}^n = 1, \tag{3.5}$$

$$|c_k^n| \le \max\left(\left|c_k^0\right|, \|u_0\|_\infty + \sum_{k=1}^K \lambda_k\right). \tag{3.6}$$

Proof. The proof is by induction on n. Clearly all of (3.3), (3.4), (3.5), and (3.6) hold at $n = 0$. Assume that those assertions hold at time step n. From (3.1) and the induction hypothesis it follows that

$$\mu\left(\|Z^n\|_\infty + \sum_{k=1}^K \lambda_k\right) \le q, \qquad \mu\,|c_k^n| \le q, \quad k = 1, \dots, K. \tag{3.7}$$

To prove that (3.3) holds at time step $n+1$ we rewrite (2.14) using incremental coefficients:

$$Z_j^{n+1} = Z_j^n + C_{j+1/2}^n \Delta_+ Z_j^n - D_{j-1/2}^n \Delta_- Z_j^n, \tag{3.8}$$

where

$$C_{j+1/2}^n = \frac{1}{2}\left(q - \mu \hat{U}_j^n\right), \quad D_{j-1/2}^n = \frac{1}{2}\left(q + \mu \hat{U}_j^n\right). \tag{3.9}$$

Using $\hat{U}_j^n = \hat{Z}_j^n - \sum_{k=1}^K \lambda_k \hat{W}_{k,j}^n$, and $\hat{W}_{k,j}^n \in [0,1]$ we see that $C_{j+1/2}^n \ge 0$, $D_{j-1/2}^n \ge 0$ due to (3.7). At the same time $C_{j+1/2}^n + D_{j-1/2}^n = q \le 1/2$. Next we rewrite (3.8):

$$Z_j^{n+1} = (1 - C_{j+1/2}^n - D_{j-1/2}^n) Z_j^n + C_{j+1/2}^n Z_{j+1}^n + D_{j-1/2}^n Z_{j-1}^n. \tag{3.10}$$

From (3.10) it is clear that Z_j^{n+1} is a convex combination of Z_{j+1}^n, Z_j^n, Z_{j-1}^n, implying that $Z_{\min}^n \le Z_j^{n+1} \le Z_{\max}^n$. Invoking the induction hypothesis then completes the proof of (3.3) for $n+1$.

Next we prove that (3.5) holds for $n+1$. We rewrite the second equation of (2.6):

$$W_{k,j}^{n+1} = (1 - \alpha_k^n - \beta_k^n) W_{k,j}^n + \alpha_k^n W_{k,j+1}^n + \beta_k^n W_{k,j-1}^n, \tag{3.11}$$

where

$$\alpha_k^n = \frac{1}{2}(q - \mu c_k^n), \quad \beta_k^n = \frac{1}{2}(q + \mu c_k^n). \tag{3.12}$$

By (3.7) we have $\alpha_k^n \ge 0$, $\beta_k^n \ge 0$, and (3.1) implies $\alpha_k^n + \beta_k^n = q \le 1/2$. Thus $W_{k,j}^{n+1}$ is a convex combination of $W_{k,j-1}^n, W_{k,j}^n, W_{k,j+1}^n$, implying that $W_{k,j}^{n+1} \in [0,1]$ after invoking the induction hypothesis. By differencing (3.11) we get

$$\Delta_+ W_{k,j}^{n+1} = (1 - \alpha_k^n - \beta_k^n) \Delta_+ W_{k,j}^n + \alpha_k^n \Delta_+ W_{k,j+1}^n + \beta_k^n \Delta_+ W_{k,j-1}^n. \tag{3.13}$$

Invoking the induction hypothesis again yields $\Delta_+ W_{k,j}^{n+1} \ge 0$. Finally, summing (3.13) over j and then applying the induction hypothesis yields $\sum_{j\in\mathbb{Z}} \Delta_+ W_{k,j}^{n+1} = 1$.

To prove (3.4) holds at $n+1$, we employ the result of the previous two paragraphs. Recalling (2.12), the proven bound on Z_j^{n+1} is equivalent to

$$z_{\min}^0 - \sum_{k=1}^K \lambda_k W_{k,j}^{n+1} \le U_j^{n+1} \le z_{\max}^0 - \sum_{k=1}^K \lambda_k W_{k,j}^{n+1}. \tag{3.14}$$

It is readily verified that $u_{\min}^0 \le z_{\min}^0$ and $z_{\max}^0 \le u_{\max}^0 + \sum_{k=1}^K \lambda_k$. Replacing $z_{\min}^0$ and $z_{\max}^0$ in (3.14), the result is

$$u_{\min}^0 - \sum_{k=1}^K \lambda_k W_{k,j}^{n+1} \le U_j^{n+1} \le u_{\max}^0 + \sum_{k=1}^K \lambda_k - \sum_{k=1}^K \lambda_k W_{k,j}^{n+1}. \tag{3.15}$$

Recalling that $\lambda_k > 0$ and $W_{k,j}^{n+1} \in [0,1]$, it is clear that (3.4) holds.

To verify that (3.6) holds for $n+1$, we start with the third formula of (2.6), from which it is evident that

$$c_k^{n+1} = \left(1 - \frac{\Delta t \lambda_k}{2m_k} \sum_{j\in\mathbb{Z}} \left(W_{k,j+1}^n - W_{k,j-1}^n\right)\right) c_k^n + \frac{\Delta t \lambda_k}{2m_k} \sum_{j\in\mathbb{Z}} \left(W_{k,j+1}^n - W_{k,j-1}^n\right) \hat{U}_j^n. \tag{3.16}$$

The induction hypothesis yields $\sum_{j\in\mathbb{Z}} \left(W_{k,j+1}^n - W_{k,j-1}^n\right) = 2$, and so after taking absolute values, and applying (3.2), equation (3.16) becomes

$$\begin{aligned}
\left|c_k^{n+1}\right| &\le \left(1 - \frac{\Delta t \lambda_k}{m_k}\right) \left|c_k^n\right| + \frac{\Delta t \lambda_k}{2m_k} \sum_{j\in\mathbb{Z}} \left(W_{k,j+1}^n - W_{k,j-1}^n\right) \left|\hat{U}_j^n\right| \\
&\le \left(1 - \frac{\Delta t \lambda_k}{m_k}\right) \left|c_k^n\right| + \frac{\Delta t \lambda_k}{2m_k} \sum_{j\in\mathbb{Z}} \left(W_{k,j+1}^n - W_{k,j-1}^n\right) \left(\|u_0\|_\infty + \sum_{k=1}^K \lambda_k\right) \\
&= \left(1 - \frac{\Delta t \lambda_k}{m_k}\right) \left|c_k^n\right| + \frac{\Delta t \lambda_k}{m_k} \left(\|u_0\|_\infty + \sum_{k=1}^K \lambda_k\right) \\
&\le \left(1 - \frac{\Delta t \lambda_k}{m_k}\right) \max\left(\left|c_k^0\right|, \|u_0\|_\infty + \sum_{k=1}^K \lambda_k\right) + \frac{\Delta t \lambda_k}{m_k} \left(\|u_0\|_\infty + \sum_{k=1}^K \lambda_k\right),
\end{aligned} \tag{3.17}$$

from which the desired inequality follows readily. □

Lemma 3.2. *U_j^n and Z_j^n satisfy spatial variation bounds:*

$$\sum_{j\in\mathbb{Z}} \left|\Delta_+ Z_j^n\right| \le \mathrm{TV}(u_0) + \sum_{k=1}^K \lambda_k, \tag{3.18}$$

and

$$\sum_{j\in\mathbb{Z}} \left|\Delta_+ U_j^n\right| \le \mathrm{TV}(u_0) + 2\sum_{k=1}^K \lambda_k. \tag{3.19}$$

Proof. We claim that the scheme is a so-called Total Variation Decreasing (TVD) scheme with respect to the variable Z_j^n, i.e.,

$$\sum_{j\in\mathbb{Z}} \left|\Delta_+ Z_j^{n+1}\right| \le \sum_{j\in\mathbb{Z}} \left|\Delta_+ Z_j^n\right|. \tag{3.20}$$

To prove the claim we use (3.8). We have shown that $C_{j+1/2}^n, D_{j+1/2}^n \ge 0$. It suffices by a standard result [12, p. 116] to show that $C_{j+1/2}^n + D_{j+1/2}^n \le 1$. Using (3.9) we find that

$$\begin{aligned}
C_{j+1/2}^n + D_{j+1/2}^n &= q - \frac{\mu}{4}\left(U_{j+1}^n + U_{j-1}^n\right) + \frac{\mu}{4}\left(U_{j+2}^n + U_j^n\right) \\
&\le q + \mu \|U^n\|_\infty \\
&\le q + \mu\left(\|u_0\|_\infty + \sum_{k=1}^K \lambda_k\right) \\
&\le 2q.
\end{aligned} \tag{3.21}$$

Here we have used (3.1) to get the last inequality. The desired bound then results by recalling that $q \leq 1/2$. Then by induction it follows from (3.18) that

$$\sum_{j\in\mathbb{Z}} \left|\Delta_+ Z_j^n\right| \leq \sum_{j\in\mathbb{Z}} \left|\Delta_+ Z_j^0\right| \leq \mathrm{TV}(z_0). \tag{3.22}$$

It is readily verified using (2.12) that

$$\sum_{j\in\mathbb{Z}} \left|U_{j+1}^n - U_j^n\right| - \sum_{k=1}^K \lambda_k \leq \sum_{j\in\mathbb{Z}} \left|Z_{j+1}^n - Z_j^n\right| \leq \sum_{j\in\mathbb{Z}} \left|U_{j+1}^n - U_j^n\right| + \sum_{k=1}^K \lambda_k. \tag{3.23}$$

Then (3.18) follows from (3.22) and the $n = 0$ version of (3.23), along with the fact that $\sum_{j\in\mathbb{Z}} \left|\Delta_+ U_j^0\right| \leq \mathrm{TV}(u_0)$. Finally, (3.19) results from (3.18) and (3.23). □

Lemma 3.3. *The following time continuity estimate holds:*

$$\sum_{j\in\mathbb{Z}} \left|U_j^{n+1} - U_j^n\right| \leq B, \tag{3.24}$$

where the constant B is independent of Δ.

Proof. Rearranging the first equation of (2.6), and using (2.17) to rewrite $\Delta_- \bar{f}_{j+1/2}^n$ yields

$$\begin{aligned} U_j^{n+1} - U_j^n = {} & \frac{1}{2}(q - \mu\hat{U}_j^n)\Delta_+ U_j^n - \frac{1}{2}(q + \mu\hat{U}_j^n)\Delta_- U_j^n \\ & + \frac{\mu}{2}\sum_{k=1}^K \lambda_k (c_k^n - \hat{U}_j^n)(W_{k,j+1}^n - W_{k,j-1}^n). \end{aligned} \tag{3.25}$$

After taking absolute values, applying the triangle inequality, then using the bounds on c_k^n and $\hat{U}_j^n$ provided by Lemma 3.1, we sum over $j \in \mathbb{Z}$. The result is

$$\sum_{j\in\mathbb{Z}} \left|U_j^{n+1} - U_j^n\right| \leq B_1 \sum_{j\in\mathbb{Z}} \left|\Delta_+ U_j^n\right| + B_2 \sum_{k=1}^K \sum_{j\in\mathbb{Z}} \left|W_{k,j+1}^n - W_{k,j-1}^n\right|, \tag{3.26}$$

where B_1 and B_2 are Δ-independent constants. The proof is completed by invoking Lemma 3.2, along with the observation that $\sum_{j\in\mathbb{Z}} \left|W_{k,j+1}^n - W_{k,j-1}^n\right| = 2$, which follows from (3.5). □

Lemma 3.4. *The particle velocity approximations satisfy the following bound:*

$$\left|c_k^{n+1} - c_k^n\right| \leq \frac{\lambda_k \Delta t}{m_k}\left(\max\left(\left|c_k^0\right|, \|u_0\|_\infty + \sum_{k=1}^K \lambda_k\right) + \|u_0\|_\infty + \sum_{k=1}^K \lambda_k\right). \tag{3.27}$$

Proof. We start with the third formula of (2.6). Subtracting c_k^n from both sides, taking absolute values, and then using the triangle inequality, the result is

$$\begin{aligned} \left|c_k^{n+1} - c_k^n\right| &\leq \frac{1}{m_k}\sum_{j\in\mathbb{Z}} \frac{\Delta t \lambda_k}{2}\left|c_k^n - \hat{U}_j^n\right|\left(W_{k,j+1}^n - W_{k,j-1}^n\right) \\ &\leq \frac{1}{m_k}\sum_{j\in\mathbb{Z}} \frac{\Delta t \lambda_k}{2}\left(|c_k^n| + \|U^n\|_\infty\right)\left(W_{k,j+1}^n - W_{k,j-1}^n\right) \\ &= \frac{\Delta t \lambda_k}{m_k}\left(|c_k^n| + \|U^n\|_\infty\right). \end{aligned} \tag{3.28}$$

The proof of (3.27) is completed using (3.4) and (3.6). □

Lemma 3.5. *The approximations u^Δ converge boundedly a.e. and in $L^1_{\rm loc}(\Pi_T)$ as $\Delta \to 0$, along a subsequence, to some $u \in L^\infty(\Pi_T) \cap C([0,T]; L^1_{\rm loc}(\mathbb{R}))$. For each $k \in \{1,\dots,K\}$ the sequence h_k^Δ converges (along the same subsequence) in $W^{1,\infty}([0,T])$ to some $h_k \in W^{2,\infty}([0,T])$, and c_k^Δ converges (also along the same subsequence) to h_k' in $L^1_{\rm loc}((0,T))$.*

Proof. The proof is a standard argument (e.g., the proof of Proposition 2.4 of [1]) using Lemmas 3.1, 3.2, and 3.3 for the u portion, and Lemmas 3.1 and 3.4 for the h_k portion. □

Remark 6. In Sections 5 and 6 we will assume that the particle trajectories do not intersect except possibly on a subset of $(0,T)$ having Lebesgue measure zero. The convergence result above holds without any assumptions about particle path intersections.

In what follows $(u,\vec{h})$ refers to a fixed subsequential limit of the type whose existence is guaranteed by Lemma 3.5. When taking the limit as $\Delta \to 0$ it is understood to be along this fixed subsequence.

4. Convergence of w_k^Δ and z^Δ.

Lemma 4.1. *$W_{k,j}^n$ satisfies a spatial variation bound and a time continuity estimate for each $k \in \{1,\dots,K\}$:*

$$\sum_{j\in\mathbb{Z}} \left|\Delta_+ W_{k,j}^n\right| = 1, \quad \sum_{j\in\mathbb{Z}} \left|W_{k,j}^{n+1} - W_{k,j}^n\right| \le 1/2. \tag{4.1}$$

Proof. The first part of (4.1) is evident from (3.5). For the second part of (4.1), we write (3.11) in the form

$$W_{k,j}^{n+1} - W_{k,j}^n = \alpha_k^n \Delta_+ W_{k,j}^n - \beta_k^n \Delta_- W_{k,j}^n. \tag{4.2}$$

Taking absolute values, and recalling from the proof of Lemma 3.1 that $\alpha_k^n, \beta_k^n \in [0,1]$ yields

$$\left|W_{k,j}^{n+1} - W_j^n\right| \le \alpha_k^n \left|\Delta_+ W_{k,j}^n\right| + \beta_k^n \left|\Delta_- W_{k,j}^n\right|. \tag{4.3}$$

Then summing over $j \in \mathbb{Z}$ and using $\sum_{j\in\mathbb{Z}} \left|\Delta_+ W_{k,j}^n\right| = 1$, $\alpha_k^n + \beta_k^n \le 1/2$, gives the second part of (4.1) □

Lemma 4.2. *As $\Delta \to 0$, $w_k^\Delta(x,t) \to H(x - h_k(t))$ boundedly a.e. and in $L^1_{\rm loc}(\Pi_T)$ for each $k \in \{1,\dots,K\}$.*

Proof. Lemma 4.1 along with $W_{k,j}^n \in [0,1]$ (Lemma 3.1) guarantees that w_k^Δ converges along a subsequence in $L^1_{\rm loc}(\mathbb{R}_+ \times \mathbb{R})$ and boundedly a.e. to some $w_k \in L^\infty(\Pi_T) \cap C([0,T]; L^1_{\rm loc}(\mathbb{R}))$.

A standard Lax-Wendroff calculation [9] proves that w_k is a weak solution of

$$\partial_t w_k + h_k'(t)\partial_x w_k = 0, \quad w_k(x,0) = H(x - h_k(0)). \tag{4.4}$$

One such weak solution is $w_k(x,t) = H(x - h_k(t))$. We will show that this is the only weak solution and the proof will be complete. Assume that w_k and $\tilde{w}_k$ are both weak solutions of (4.4). This implies that for every $\phi \in C_0^\infty(\mathbb{R} \times [0,T])$,

$$\int_0^T \int_{\mathbb{R}} (\tilde{w}_k - w_k) \left\{\phi_t + h_k'(t)\phi_x\right\} dx\, dt = \int_0^T (\tilde{w}_k - w_k)\, \phi(x,T)\, dt. \tag{4.5}$$

Fix $\psi \in C_0^\infty(\mathbb{R} \times [0,T])$. Let

$$\phi(x,t) = \int_T^t \psi\left(x - h_k(t) + h_k(\sigma), \sigma\right) d\sigma. \tag{4.6}$$

It is readily verified that $\phi_t + h_k'(t)\phi_x = \psi$, $\phi(\cdot, T) = 0$. Substituting into (4.5), we have

$$\int_0^T \int_{\mathbb{R}} (\tilde{w}_k - w_k)\, \psi(x,t)\, dx\, dt = 0. \tag{4.7}$$

Since (4.7) holds for any $\psi \in C_0^\infty(\mathbb{R} \times [0,T])$, we conclude that $w = \tilde{w}$ a.e. □

The following lemma is a direct consequence of (2.12), Lemma 3.5, and Lemma 4.2.

Lemma 4.3. *Define $z(x,t) = u(x,t) + \sum_{k=1}^K \lambda_k H(x - h_k(t))$. As $\Delta \to 0$, $z^\Delta(x,t) \to z(x,t)$ boundedly a.e. and in $L^1_{\mathrm{loc}}(\Pi_T)$.*

5. **Jump and entropy conditions for u.** In this section we verify that the subsequential limit u is a Kružkov entropy solution in $\Pi_T \setminus \Gamma$ and satisfies the jump condition (1.3).

Here and in Section 6 we will employ the test function $0 \le \psi_\delta(x) \in \mathcal{C}_0^\infty(\mathbb{R})$, $\delta > 0$, such that $\psi_\delta(0) = 1$, $\mathrm{supp}(\psi_\delta) = [-\delta, \delta]$, and

$$\psi_\delta'(x) = \begin{cases} \eta_\delta(x + \delta/2), & x \le 0, \\ -\eta_\delta(x - \delta/2), & x \ge 0, \end{cases} \tag{5.1}$$

where η_δ denotes the standard $C^\infty(\mathbb{R})$ mollifier:

$$\mathrm{supp}(\eta_\delta) = [-\delta/2, \delta/2], \quad \eta_\delta(x) \ge 0\ \forall x \in \mathbb{R}, \quad \int_{\mathbb{R}} \eta_\delta(x)\, dx = 1. \tag{5.2}$$

Assumption 5.1. Assume that the particle trajectories do not intersect except possibly on a subset $F \subset (0,T)$ having Lebesgue measure zero.

Remark 7. The set F has the form $F = \cup_{i \ne j} F_{i,j}$, where

$$F_{i,j} := \{t \in (0,T) | h_i(t) = h_j(t)\}.$$

Since each of the particle paths $t \mapsto h_k(t)$ is continuous, each $F_{i,j}$ is closed, and thus F is also a closed subset of $(0,T)$. There are no particle path intersections in the open set $E := (0,T) \setminus F$. E is a countable disjoint union of open intervals, $E = \cup_{\mathsf{m}=1}^{\mathsf{M}} (a_{\mathsf{m}}, b_{\mathsf{m}})$, where $1 \le \mathsf{M} \le \infty$ and each $(a_{\mathsf{m}}, b_{\mathsf{m}}) \subseteq (0,T)$. By Assumption 5.1, E is of full measure, $\mathrm{meas}((0,T) \setminus E) = 0$.

Lemma 5.1. *Define $\mathcal{U} = [u^0_{\min} - \sum_{k=1}^K \lambda_k, u^0_{\max} + \sum_{k=1}^K \lambda_k]$. Referring to (2.6), let $G(U^n_{j+1}, U^n_j, U^n_{j-1}) = U^n_j - \mu \Delta_- \bar{f}^n_{j+1/2}$. Then G is nondecreasing with respect to each of $U^n_{j+1}, U^n_j, U^n_{j-1}$ if $U^n_{j+1}, U^n_j, U^n_{j-1} \in \mathcal{U}$. Referring to (2.13), Z^{n+1}_j is nondecreasing with respect to each of $Z^n_{j+1}, Z^n_j, Z^n_{j-1}$ if $Z^n_{j+1}, Z^n_j, Z^n_{j-1} \in [z^0_{\min}, z^0_{\max}]$.*

Proof. The partial derivatives of G are

$$\frac{\partial G}{\partial U^n_j} = 1 - q, \quad \frac{\partial G}{\partial U^n_{j+1}} = -\frac{\mu}{2} U^n_{j+1} + \frac{q}{2}, \quad \frac{\partial G}{\partial U^n_{j-1}} = \frac{\mu}{2} U^n_{j-1} + \frac{q}{2}. \tag{5.3}$$

Clearly $\partial G / \partial U^n_j \ge 0$ since $q \le 1/2$. For $\partial U^{n+1}_j / \partial U^n_{j \pm 1}$,

$$\frac{\partial G}{\partial U^n_{j \pm 1}} \ge \frac{1}{2}\left(q - \mu \left\|U^n\right\|_\infty\right) \ge \frac{1}{2}\left(q - \mu\left(\left\|u_0\right\|_\infty + \sum_{k=1}^K \lambda_k\right)\right). \tag{5.4}$$

In view of (5.4) and (3.1) it is clear that $\partial G/\partial U^n_{j\pm 1} \geq 0$.

For Z^{n+1}_j we use (2.13) to compute

$$\frac{\partial Z^{n+1}_j}{\partial Z^n_j} = 1 - q,$$
$$\frac{\partial Z^{n+1}_j}{\partial Z^n_{j+1}} = \frac{q}{2} - \frac{\mu}{2} Z^n_{j+1} + \frac{\mu}{2}\sum_{k=1}^K \lambda_k \hat{W}^n_{k,j}, \quad \frac{\partial Z^{n+1}_j}{\partial Z^n_{j-1}} = \frac{q}{2} + \frac{\mu}{2} Z^n_{j-1} - \frac{\mu}{2}\sum_{k=1}^K \lambda_k \hat{W}^n_{k,j}. \tag{5.5}$$

It is readily verified that each of these partial derivatives is nonnegative using (3.1) and the fact that $\hat{W}^n_{k,j} \in [0,1]$. □

The following lemma is a straightforward consequence of (3.5) and Lemma 4.2.

Lemma 5.2. *Define*

$$S^n_j = \sum_{k=1}^K \lambda_k \hat{W}^n_{k,j}, \quad S^\Delta(x,t) = \sum_{n=0}^N \sum_{j\in\mathbb{Z}} \chi_j(x)\chi^n(t) S^n_j. \tag{5.6}$$

S^n_j has the following properties:

$$0 \leq S^n_j \leq \sum_{k=1}^K \lambda_k, \quad \Delta_+ S^n_j \geq 0, \quad \sum_{j\in\mathbb{Z}} \Delta_+ S^n_j = \sum_{k=1}^K \lambda_k, \tag{5.7}$$

and as $\Delta \to 0$, $S^\Delta(x,t) \to \sum_{k=1}^K \lambda_k H(x - h_k(t))$ boundedly a.e. and in $L^1_{\mathrm{loc}}(\Pi_T)$.

Lemma 5.3. *The following discrete entropy inequalities hold for all $\kappa \in [z^0_{\min}, z^0_{\max}]$:*

$$\begin{aligned} Z^{n+1}_j \vee \kappa &\leq Z^n_j \vee \kappa - \mu\Delta_- \bar{f}(Z^n_{j+1}\vee\kappa, Z^n_j \vee \kappa) + \frac{\mu}{2} S^n_j \left(Z^n_{j+1}\vee\kappa - Z^n_{j-1}\vee\kappa\right), \\ Z^{n+1}_j \wedge \kappa &\geq Z^n_j \wedge \kappa - \mu\Delta_- \bar{f}(Z^n_{j+1}\wedge\kappa, Z^n_j \wedge \kappa) + \frac{\mu}{2} S^n_j \left(Z^n_{j+1}\wedge\kappa - Z^n_{j-1}\wedge\kappa\right). \end{aligned} \tag{5.8}$$

Proof. Writing (2.13) in the form $Z^{n+1}_j = P(Z^n_{j+1}, Z^n_j, Z^n_{j-1})$, it is readily apparent that $P(\kappa,\kappa,\kappa) = \kappa$. Using this observation the proof is a standard calculation [8, 9], using the fact that P is a nondecreasing function of all three arguments (Lemma 5.1). □

Lemma 5.4. *The limit solution u satisfies the jump condition (1.3) for a.e. $t \in (0,T)$ and each $k \in 1,\ldots,K$.*

Proof. We start with the first inequality in (5.8), and use the identity

$$A_j\left(B_{j+1} - B_{j-1}\right) = \Delta_+\left(A_j B_j\right) - B_{j+1}\Delta_+ A_j + \Delta_-\left(A_j B_j\right) - B_{j-1}\Delta_- A_j. \tag{5.9}$$

This results in

$$\begin{aligned} Z^{n+1}_j \vee \kappa \leq{}& Z^n_j \vee \kappa \\ &- \mu\Delta_-\Big(\bar{f}(Z^n_{j+1}\vee\kappa, Z^n_j\vee\kappa) - \frac{1}{2}S^n_{j+1}(Z^n_{j+1}\vee\kappa) - \frac{1}{2}S^n_j(Z^n_j\vee\kappa)\Big) \\ &- \frac{\mu}{2}\left((Z^n_{j+1}\vee\kappa)\Delta_+ S^n_j + (Z^n_{j-1}\vee\kappa)\Delta_- S^n_j\right). \end{aligned} \tag{5.10}$$

Since $\Delta_\pm S^n_j \geq 0$, we have

$$(Z^n_{j+1}\vee\kappa)\Delta_+ S^n_j \geq \kappa\Delta_+ S^n_j, \quad (Z^n_{j-1}\vee\kappa)\Delta_- S^n_j \geq \kappa\Delta_- S^n_j, \tag{5.11}$$

and so we can replace (5.10) by

$$\begin{aligned} Z_j^{n+1} \vee \kappa \leq Z_j^n \vee \kappa \\ &- \mu \Delta_- \Big(\bar{f}(Z_{j+1}^n \vee \kappa, Z_j^n \vee \kappa) - \frac{1}{2} S_{j+1}^n (Z_{j+1}^n \vee \kappa) - \frac{1}{2} S_j^n (Z_j^n \vee \kappa) \Big) \\ &- \frac{\mu\kappa}{2} \left(S_{j+1}^n - S_{j-1}^n \right). \end{aligned} \tag{5.12}$$

Following the proof of the Lax-Wendroff theorem [9], let ϕ be a nonnegative test function with $\phi(x,0) = 0$, and $\phi_j^n := \phi(x_j, t^n)$. We multiply (5.12) by $\phi_j^n \Delta x$, and then sum over $j \in \mathbb{Z}$, $n \geq 0$. After summation by parts the result is

$$\begin{aligned} &\Delta x \Delta t \sum_{j\in\mathbb{Z}} \sum_{n\geq 0} (Z_j^{n+1} \vee \kappa) \frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} \\ &+ \Delta x \Delta t \sum_{j\in\mathbb{Z}} \sum_{n\geq 0} \Big(\bar{f}(Z_{j+1}^n \vee \kappa, Z_j^n \vee \kappa) - \frac{1}{2} S_j^n (Z_j^n \vee \kappa) - \frac{1}{2} S_{j+1}^n (Z_{j+1}^n \vee \kappa) \Big) \frac{\Delta_+ \phi_j^n}{\Delta x} \\ &+ \Delta x \Delta t \kappa \sum_{j\in\mathbb{Z}} \sum_{n\geq 0} S_j^n \frac{\Delta_+ \phi_j^n}{\Delta x} \geq 0. \end{aligned} \tag{5.13}$$

Letting $\Delta \downarrow 0$ and recalling $z^\Delta \to z$, $S^\Delta \to \sum_{k=1}^K \lambda_k H(x - h_k(t))$ yields

$$\begin{aligned} \int_0^T \int_{\mathbb{R}} (z \vee \kappa) \phi_t \, dx \, dt + \int_0^T \int_{\mathbb{R}} \Big(f(z \vee \kappa) - \sum_{l=1}^K \lambda_l H(x - h_l(t))(z \vee \kappa) \Big) \phi_x \, dx \, dt \\ + \kappa \int_0^T \int_{\mathbb{R}} \sum_{l=1}^K \lambda_l H(x - h_l(t)) \phi_x \, dx \, dt \geq 0. \end{aligned} \tag{5.14}$$

After simplifying the last integral the result is

$$\begin{aligned} \int_0^T \int_{\mathbb{R}} (z \vee \kappa) \phi_t \, dx \, dt + \int_0^T \int_{\mathbb{R}} \Big(f(z \vee \kappa) - \sum_{l=1}^K \lambda_l H(x - h_l(t))(z \vee \kappa) \Big) \phi_x \, dx \, dt \\ - \kappa \sum_{l=1}^K \lambda_l \int_0^T \phi(h_l(t), t) \, dt \geq 0. \end{aligned} \tag{5.15}$$

A similar calculation starting from the second inequality of (5.8) yields

$$\begin{aligned} \int_0^T \int_{\mathbb{R}} (z \wedge \kappa) \phi_t \, dx \, dt + \int_0^T \int_{\mathbb{R}} \Big(f(z \wedge \kappa) - \sum_{l=1}^K \lambda_l H(x - h_l(t))(z \wedge \kappa) \Big) \phi_x \, dx \, dt \\ - \kappa \sum_{l=1}^K \lambda_l \int_0^T \phi(h_l(t), t) \, dt \leq 0. \end{aligned} \tag{5.16}$$

Recalling Assumption 5.1 and Remark 7, fix an interval $\mathcal{I}_\mathsf{m} := (a_\mathsf{m}, b_\mathsf{m}) \subseteq (0, T)$ where there are no path intersections, and fix a particle path, indexed by k. For this calculation we will use the abbreviations $z^\pm(t) = z(h_k(t)^\pm, t)$ and $c_k(t) = h_k'(t)$. The

ordering of the particles does not change in $\mathcal{I}_{\mathsf{m}}$, so we can assume that the particles are labeled so that

$$h_1(t) < h_2(t) < \cdots < h_k(t) < \cdots < h_K(t), \quad t \in \mathcal{I}_{\mathsf{m}}. \tag{5.17}$$

Let $\phi(x,t) = \psi_\delta(x - h_k(t))\rho(t)$, where $0 \le \rho \in C_0^\infty(\mathcal{I}_{\mathsf{m}})$. Letting $\delta \downarrow 0$ in (5.15) yields

$$\begin{aligned}\int_{\mathcal{I}_{\mathsf{m}}} \{f(z^- \vee \kappa) - c_k(z^- \vee \kappa) - \big(f(z^+ \vee \kappa) - c_k(z^+ \vee \kappa) - \lambda_k(z^+ \vee \kappa)\big) \\ - \gamma_k\left(z^- \vee \kappa - z^+ \vee \kappa\right) - \lambda_k\kappa\}\rho(t)\,dt \ge 0,\end{aligned} \tag{5.18}$$

where $\gamma_k = \sum_{l<k} \lambda_l$, and we have abbreviated $z^\pm = z^\pm(t)$, $c_k = c_k(t)$. Another such test function calculation, this time with (5.16) results in

$$\begin{aligned}\int_{\mathcal{I}_{\mathsf{m}}} \{f(z^- \wedge \kappa) - c_k(z^- \wedge \kappa) - \big(f(z^+ \wedge \kappa) - c_k(z^+ \wedge \kappa) - \lambda_k(z^+ \wedge \kappa)\big) \\ - \gamma_k\left(z^- \wedge \kappa - z^+ \wedge \kappa\right) - \lambda_k\kappa\}\rho(t)\,dt \le 0.\end{aligned} \tag{5.19}$$

Continuing with the abbreviation $z^\pm = z^\pm(t)$, $c_k = c_k(t)$, for a.e. $t \in \mathcal{I}_{\mathsf{m}}$ we have

$$\begin{aligned}f(z^- \vee \kappa) - c_k(z^- \vee \kappa) - \big(f(z^+ \vee \kappa) - c_k(z^+ \vee \kappa) - \lambda_k(z^+ \vee \kappa)\big) \\ - \gamma_k\left(z^- \vee \kappa - z^+ \vee \kappa\right) - \lambda_k\kappa \ge 0,\end{aligned} \tag{5.20}$$

$$\begin{aligned}f(z^- \wedge \kappa) - c_k(z^- \wedge \kappa) - \big(f(z^+ \wedge \kappa) - c_k(z^+ \wedge \kappa) - \lambda_k(z^+ \wedge \kappa)\big) \\ - \gamma_k\left(z^- \wedge \kappa - z^+ \wedge \kappa\right) - \lambda_k\kappa \le 0.\end{aligned} \tag{5.21}$$

Fix a time $t \in \mathcal{I}_{\mathsf{m}}$ where (5.20), (5.21) hold. If $z^- = z^+$ then (5.20) and (5.21) are satisfied. So assume for now that $z^- \ne z^+$. Substituting $z^- \le \kappa \le z^+$ into (5.20) and then (5.21) gives

$$z^- \le \kappa \le z^+ \implies \begin{cases} f(z^+) - f(\kappa) \le (\lambda_k + \tilde{c}_k)(z^+ - \kappa), \\ f(z^-) - f(\kappa) \le \tilde{c}_k(z^- - \kappa). \end{cases} \tag{5.22}$$

where $\tilde{c}_k = c_k + \gamma_k$. Repeating this calculation with $z^+ \le \kappa \le z^-$, we find that

$$z^+ \le \kappa \le z^- \implies \begin{cases} f(z^+) - f(\kappa) \ge (\lambda_k + \tilde{c}_k)(z^+ - \kappa), \\ f(z^-) - f(\kappa) \ge \tilde{c}_k(z^- - \kappa). \end{cases} \tag{5.23}$$

Plugging $\kappa = z^-$ into the first inequality of (5.22) and then into the first inequality of (5.23), and recalling $f(z) = z^2/2$, yields

$$z^+ + z^- \le 2(\lambda_k + \tilde{c}_k). \tag{5.24}$$

The second inequality of (5.22) (for $z^- < z^+$) or the second inequality of (5.23) (for $z^- > z^+$) implies that in either case

$$z^- \ge \tilde{c}_k. \tag{5.25}$$

Substituting $\kappa = z^+$ into the second inequalities of (5.22) and (5.23) yields

$$z^+ + z^- \ge 2\tilde{c}_k. \tag{5.26}$$

Finally, with $\epsilon > 0$, we substitute $\kappa = z^+ - \epsilon$ into the first inequality of (5.22), and $\kappa = z^+ + \epsilon$ into the first inequality of (5.23). Sending $\epsilon \downarrow 0$ results in

$$z^+ \le \lambda_k + \tilde{c}_k. \tag{5.27}$$

Thus either $z^+ = z^-$ or all of (5.24), (5.25), (5.26), (5.27) hold. Let $u^\pm = u(h_k(t)^\pm, t)$. Substituting $z^- = u^- + \gamma_k$, $z^+ = u^+ + \gamma_k + \lambda_k$ into these relationships we have shown that either

$$u^+ - c_k = u^- - c_k - \lambda_k, \tag{5.28}$$

or

$$u^- - c_k \geq 0, \quad u^+ - c_k \leq 0, \quad -\lambda_k \leq (u^- - c_k) + (u^+ - c_k) \leq \lambda_k. \tag{5.29}$$

Recalling Definition 1.1, and that $c_k = h_k'(t)$, it is evident from (5.28), (5.29) that

$$(u^-, u^+) \in \mathcal{G}(\lambda_k, c_k) = \mathcal{G}(\lambda_k, h_k'(t)), \tag{5.30}$$

and this holds for a.e. $t \in \mathcal{I}_{\mathsf{m}}$. The proof is completed by repeating this argument for each $k \in \{1, \ldots, K\}$ and each $\mathsf{m} \in \{1, \ldots, \mathsf{M}\}$. □

Lemma 5.5. *The following discrete entropy inequality holds for each $\kappa \in \mathbb{R}$:*

$$\begin{aligned} \left|U_j^{n+1} - \kappa\right| \leq \left|U_j^n - \kappa\right| - \mu \Delta_- \bar{F}\left(U_{j+1}^n, U_j^n\right) \\ + \frac{\mu}{2} \sum_{k=1}^K \lambda_k \left|c_k^n - \hat{U}_j^n\right| \left(W_{k,j+1}^n - W_{k,j-1}^n\right), \end{aligned} \tag{5.31}$$

where $\bar{F}\left(U_{j+1}^n, U_j^n\right) = \bar{f}(U_{j+1}^n \vee \kappa, U_j^n \vee \kappa) - \bar{f}(U_{j+1}^n \wedge \kappa, U_j^n \wedge \kappa)$.

Proof. First assume that $\kappa \in \mathcal{U} = [u_{\min}^0 - \sum_{k=1}^K \lambda_k, u_{\max}^0 + \sum_{k=1}^K \lambda_k]$. We write the first equation of (2.6) in the form

$$U_j^{n+1} = G(U_{j+1}^n, U_j^n, U_{j-1}^n) + Q_j^n, \tag{5.32}$$

where

$$\begin{aligned} V_j^{n+1} &:= G(U_{j+1}^n, U_j^n, U_{j-1}^n) = U_j^n - \mu \Delta_- \bar{f}_{j+1/2}^n, \\ Q_j^n &= \frac{\mu}{2} \sum_{k=1}^K \lambda_k \left(c_k^n - \hat{U}_j^n\right) \left(W_{k,j+1}^n - W_{k,j-1}^n\right). \end{aligned} \tag{5.33}$$

Invoking the monotonicity of G (Lemma 5.1), a standard calculation [8, 9] yields

$$\left|V_j^{n+1} - \kappa\right| \leq \left|U_j^n - \kappa\right| - \mu \Delta_- \bar{F}\left(U_{j+1}^n, U_j^n\right), \tag{5.34}$$

for $\kappa \in \mathcal{U}$. Substituting $V_j^{n+1} = U_j^{n+1} - Q_j^n$, and using the triangle inequality yields (5.31), assuming $\kappa \in \mathcal{U}$.

Now take the case where $\kappa \notin \mathcal{U}$, say $\kappa < u_{\min}^0 - \sum_{k=1}^K \lambda_k$. In that case (5.31) reduces to

$$U_j^{n+1} \leq U_j^n - \mu \Delta_- \bar{f}_{j+1/2}^n + \left|Q_j^n\right|. \tag{5.35}$$

which, recalling the first equation of (2.6), is clearly satisfied. The case where $\kappa > u_{\max}^0 + \sum_{k=1}^K \lambda_k$ is handled similarly. □

Lemma 5.6. *The limit u is a Kružkov entropy solution in $\Pi_T \setminus \Gamma$ of the Burgers equation with initial data u_0.*

Proof. Define $F(a, b) = f(a \vee b) - f(a \wedge b) = \operatorname{sgn}(a - b)(a^2/2 - b^2/2)$. We must show that u satisfies

$$\int_0^T \int_{\mathbb{R}} \left(|u - \kappa| \, \phi_t + F(u, \kappa) \phi_x\right) dx \, dt + \int_R |u_0 - \kappa| \, \phi(x, 0) \, dx \geq 0 \tag{5.36}$$

for every $\kappa \in \mathbb{R}$ and every nonnegative test function $\phi \in C_0^\infty(\mathbb{R} \times [0, T) \setminus \Gamma)$.

The proof is based on the discrete entropy inequality (5.31). Due to the bounds on U_j^n and c_k^n (Lemma 3.1), we have for some $B > 0$ which independent of Δ,

$$\frac{\mu}{2}\sum_{k=1}^{K}\lambda_k\left|c_k^n-\hat{U}_j^n\right|\left(W_{k,j+1}^n-W_{k,j-1}^n\right)\leq\frac{\mu}{2}B\sum_{k=1}^{K}\lambda_k\left(W_{k,j+1}^n-W_{k,j-1}^n\right). \tag{5.37}$$

Substituting into (5.31) the result is

$$\left|U_j^{n+1}-\kappa\right|\leq\left|U_j^n-\kappa\right|-\mu\Delta_-\bar{F}\left(U_{j+1}^n,U_j^n\right)+\frac{\mu}{2}B\sum_{k=1}^{K}\lambda_k\left(W_{k,j+1}^n-W_{k,j-1}^n\right). \tag{5.38}$$

Multiplying by $\phi_j^n=\phi(x_j,t^n)$ and then summing by parts we find that

$$\begin{aligned}&\Delta x\Delta t\sum_{n=0}^{N}\sum_{j\in\mathbb{Z}}\left\{\left|U_j^{n+1}-\kappa\right|(\phi_j^{n+1}-\phi_j^n)/\Delta t+\bar{F}\left(U_{j+1}^n,U_j^n\right)(\phi_{j+1}^n-\phi_j^n)/\Delta x\right\}\\&-B\sum_{k=1}^{K}\lambda_k\Delta x\Delta t\sum_{n=0}^{N}\sum_{j\in\mathbb{Z}}W_{k,j}^n\frac{1}{2}(\phi_{j+1}^n-\phi_{j-1}^n)/\Delta x+\Delta x\sum_{j\in\mathbb{Z}}\left|U_j^0-\kappa\right|\phi_j^0\,dx\geq 0.\end{aligned} \tag{5.39}$$

Letting $\Delta\to 0$, and using $u^\Delta\to u$, $w_k^\Delta\to H(x-h_k(t))$, results in

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}}\left(|u-\kappa|\,\phi_t+F(u,\kappa)\phi_x\right)\,dx\,dt-B\sum_{k=1}^{K}\lambda_k\int_0^T\int_{\mathbb{R}}H(x-h_k(t))\phi_x\,dx\,dt\\&+\int_{\mathbb{R}}|u_0(x)-\kappa|\,dx\geq 0.\end{aligned} \tag{5.40}$$

The proof is finished by observing that $\int_{\mathbb{R}}H(x-h_k(t))\phi_x\,dx=0$, since $\phi(h_k(t),t)=0$. □

6. **Differential equation for h_k and proof of the main theorem.** In this section we prove that the limit h_k satisfies the differential equation (1.4). This section also contains the proof of Theorem 1.3. Assumption 5.1 (restriction on particle intersections) remains in effect in this section.

Lemma 6.1. *The limit $h_k(t)$ satisfies the differential equation (1.4) for each $k\in 1,\dots,K$ and a.e. $t\in(0,T)$. Also, $(h_k(0),h_k'(0))=(h_{k,0},v_{k,0})$.*

Proof. Fix a particle with index k, $1\leq k\leq K$. Let $a_k^n=(c_k^{n+1}-c_k^n)/\Delta t$. The third equation of (2.6) yields

$$m_ka_k^n=-\sum_{j\in\mathbb{Z}}\frac{\lambda_k}{2}\left(c_k^n-\hat{U}_j^n\right)\left(W_{k,j+1}^n-W_{k,j-1}^n\right). \tag{6.1}$$

Define $\psi_j^n=\psi_\delta(x_j-h_k(t^n))$, where ψ_δ is defined by (5.1). Let $\xi(t)\in\mathcal{C}_0^\infty((0,T))$ and define $\xi^n=\xi(t^n)$. We re-write (6.1) in the form

$$\begin{aligned}m_ka_k^n=&-\frac{\lambda_k}{2}\sum_{j\in\mathbb{Z}}\left(c_k^n-\hat{U}_j^n\right)\left(W_{k,j+1}^n-W_{k,j-1}^n\right)\psi_j^n\\&-\frac{\lambda_k}{2}\sum_{j\in\mathbb{Z}}\left(c_k^n-\hat{U}_j^n\right)\left(W_{k,j+1}^n-W_{k,j-1}^n\right)(1-\psi_j^n).\end{aligned} \tag{6.2}$$

Next we multiply by $\xi^n \Delta t$ and sum over n :

$$\begin{aligned} m_k \Delta t \sum_{n\geq 0} a_k^n \xi^n = &-\frac{\lambda_k}{2}\Delta t \sum_{n\geq 0}\sum_{j\in\mathbb{Z}} \left(c_k^n - \hat{U}_j^n\right)\left(W_{k,j+1}^n - W_{k,j-1}^n\right)\psi_j^n \xi^n \\ &- \frac{\lambda_k}{2}\Delta t \sum_{n\geq 0}\sum_{j\in\mathbb{Z}} \left(c_k^n - \hat{U}_j^n\right)\left(W_{k,j+1}^n - W_{k,j-1}^n\right)(1-\psi_j^n)\xi^n. \end{aligned} \tag{6.3}$$

We solve for $\left(c_k^n - \hat{U}_j^n\right)\left(W_{k,j+1}^n - W_{k,j-1}^n\right)$ in the first equation of (2.6),

$$\begin{aligned} \left(c_k^n - \hat{U}_j^n\right)\left(W_{k,j+1}^n - W_{k,j-1}^n\right) = &\frac{2}{\lambda_k \mu}\left(U_j^{n+1} - U_j^n + \mu \Delta_- \bar{f}_{j+1/2}^n\right) \\ &- \frac{1}{\lambda_k}\sum_{l\neq k}\lambda_l \left(c_l^n - \hat{U}_j^n\right)\left(W_{l,j+1}^n - W_{l,j-1}^n\right), \end{aligned} \tag{6.4}$$

and substitute into the first sum on the right side of (6.3). The result is

$$\begin{aligned} m_k \Delta t \sum_{n\geq 0} a_k^n \xi^n = &\underbrace{-\Delta x \sum_{n\geq 0}\sum_{j\in\mathbb{Z}} \left(U_j^{n+1} - U_j^n + \mu\Delta_- \bar{f}_{j+1/2}^n\right)\psi_j^n \xi^n}_{\mathcal{S}_1} \\ &\underbrace{+\frac{1}{2}\Delta t \sum_{n\geq 0}\sum_{j\in\mathbb{Z}}\sum_{l\neq k}\lambda_l \left(c_l^n - \hat{U}_j^n\right)\left(W_{l,j+1}^n - W_{l,j-1}^n\right)\psi_j^n \xi^n}_{\mathcal{S}_2} \\ &\underbrace{-\frac{\lambda_k}{2}\Delta t \sum_{n\geq 0}\sum_{j\in\mathbb{Z}} \left(c_k^n - \hat{U}_j^n\right)\left(W_{k,j+1}^n - W_{k,j-1}^n\right)(1-\psi_j^n)\xi^n}_{\mathcal{S}_3}. \end{aligned} \tag{6.5}$$

Summing the left side of (6.5) by parts, we find that

$$m_k \Delta t \sum_{n\geq 0} a_k^n \xi^n = -m_k \Delta t \sum_{n\geq 0} c_k^{n+1} \frac{\xi^{n+1} - \xi^n}{\Delta t}. \tag{6.6}$$

Letting $\Delta \downarrow 0$ in (6.6), and using $c_k^\Delta \to h_k'$, the result is

$$m_k \Delta t \sum_{n\geq 0} a_k^n \xi^n \to -m_k \int_0^T h_k'(t)\xi'(t)\,dt, \tag{6.7}$$

and for $\mathcal{S}_1$, summation by parts followed by sending $\Delta \to 0$ yields

$$\mathcal{S}_1 \to \int_0^T \int_{\mathbb{R}} \left\{u\,\partial_t\left(\psi_\delta(x-h_k(t))\xi(t)\right) + f(u)\,\partial_x\left(\psi_\delta(x-h_k(t))\xi(t)\right)\right\} dx\,dt. \tag{6.8}$$

We next estimate $\mathcal{S}_2$. Fix $l \neq k$. It suffices to estimate $\mathcal{S}_{2,l}$, where

$$\mathcal{S}_{2,l} = \frac{1}{2}\Delta t \sum_{n\geq 0}\sum_{j\in\mathbb{Z}} \lambda_l \left(c_l^n - \hat{U}_j^n\right)\left(W_{l,j+1}^n - W_{l,j-1}^n\right)\psi_j^n \xi^n. \tag{6.9}$$

Since c_l^n and $\hat{U}_j^n$ are bounded (Lemma 3.1), and $\left(W_{l,j+1}^n - W_{l,j-1}^n\right) \geq 0$, $\psi_j^n \geq 0$,

$$|\mathcal{S}_{2,l}| \leq B\Delta t \sum_{n\geq 0} |\xi^n| \sum_{j\in\mathbb{Z}} \left(W_{l,j+1}^n - W_{l,j-1}^n\right)\psi_j^n \tag{6.10}$$

where B is some positive number independent of δ and Δ. Summation by parts yields

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\left(W^n_{l,j+1}-W^n_{l,j-1}\right)\psi^n_j &= \sum_{j\in\mathbb{Z}}\left(W^n_{l,j+1}\psi^n_{j+1}-W^n_{l,j-1}\psi^n_{j-1}\right)\\ &\quad-\sum_{j\in\mathbb{Z}}\left(W^n_{l,j+1}+W^n_{l,j}\right)\Delta_+\psi^n_j.\end{aligned}\tag{6.11}$$

The first sum on the right is telescoping and is equal to zero. Thus, referring back to (6.10) we have

$$\begin{aligned}|\mathcal{S}_{2,l}| &\le -B\Delta t\sum_{n\ge0}|\xi^n|\sum_{j\in\mathbb{Z}}\left(W_{l,j+1}+W_{l,j}\right)\Delta_+\psi^n_j\\ &= -2B\Delta x\Delta t\sum_{n\ge0}|\xi^n|\sum_{j\in\mathbb{Z}}\frac{1}{2}\left(W_{l,j+1}+W_{l,j}\right)\Delta_+\psi^n_j/\Delta x.\end{aligned}\tag{6.12}$$

Letting $\Delta\to0$ yields

$$\limsup_{\Delta\to0}|\mathcal{S}_{2,l}|\le -2B\int_0^T|\xi(t)|\int_{\mathbb{R}}w_l(x,t)\partial_x\psi_\delta(x-h_k(t))\,dx\,dt.\tag{6.13}$$

Recalling that $w_l(x,t)=H(x-h_l(t))$, we find that

$$\int_{\mathbb{R}}w_l(x,t)\partial_x\psi_\delta(x-h_k(t))\,dx=\int_{x=h_l(t)}^{\infty}\partial_x\psi_\delta(x-h_k(t))\,dx=-\psi_\delta(h_l(t)-h_k(t)).\tag{6.14}$$

Substituting into (6.13) yields the desired estimate of $\mathcal{S}_{2,l}$:

$$\limsup_{\Delta\to0}|\mathcal{S}_{2,l}|\le 2B\int_0^T|\xi(t)|\,\psi_\delta(h_l(t)-h_k(t))\,dt.\tag{6.15}$$

We claim that $\mathcal{S}_3\to0$. Since c^n_k and $\hat{U}^n_j$ are bounded (Lemma 3.1), and $\psi^n_j\le1$, $W^n_{k,j+1}-W^n_{k,j-1}\ge0$,

$$|\mathcal{S}_3|\le B\Delta t\sum_{n\ge0}|\xi^n|\sum_{j\in\mathbb{Z}}\left(W^n_{k,j+1}-W^n_{k,j-1}\right)(1-\psi^n_j).\tag{6.16}$$

where B is some positive number independent of the mesh size Δ. Using the formula (6.11) with $1-\psi^n_j$ replacing ψ^n_j,

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\left(W^n_{k,j+1}-W^n_{k,j-1}\right)(1-\psi^n_j) &= \sum_{j\in\mathbb{Z}}\left(W^n_{k,j+1}(1-\psi^n_{j+1})-W^n_{k,j-1}(1-\psi^n_{j-1})\right)\\ &\quad+\sum_{j\in\mathbb{Z}}\left(W^n_{k,j+1}+W^n_{k,j}\right)\Delta_+\psi^n_j.\end{aligned}\tag{6.17}$$

In the second term on the right side we have used $\Delta_+(1-\psi^n_j)=-\Delta_+\psi^n_j$. The first sum on the right is telescoping and is equal to 2. Thus, referring back to (6.16) we have

$$\begin{aligned}|\mathcal{S}_3| &\le 2B\Delta t\sum_{n\ge0}|\xi^n|+B\Delta t\sum_{n\ge0}|\xi^n|\sum_{j\in\mathbb{Z}}\left(W_{k,j+1}+W_{k,j}\right)\Delta_+\psi^n_j\\ &= 2B\Delta t\sum_{n\ge0}|\xi^n|+2B\Delta t\Delta x\sum_{n\ge0}|\xi^n|\sum_{j\in\mathbb{Z}}\frac{1}{2}\left(W_{k,j+1}+W_{k,j}\right)\Delta_+\psi^n_j/\Delta x.\end{aligned}\tag{6.18}$$

Letting $\Delta \to 0$ yields

$$\limsup_{\Delta\to 0} |\mathcal{S}_3| \le 2B \int_0^T |\xi(t)|\, dt + 2B \int_0^T |\xi(t)| \int_{\mathbb{R}} w_k(x,t)\partial_x \psi_\delta(x - h_k(t)\, dx\, dt. \tag{6.19}$$

Substituting $w_k(x,t) = H(x - h_k(t))$, and using $\psi_\delta(0) = 1$, the result is

$$\int_{\mathbb{R}} w_k(x,t)\partial_x \psi_\delta(x - h_k(t))\, dx = \int_{x=h_k(t)}^{\infty} \partial_x \psi_\delta(x - h_k(t))\, dx = -1. \tag{6.20}$$

Plugging (6.20) into (6.19) completes the proof of the claim.

Combining $\mathcal{S}_3 \to 0$ with (6.7), (6.8), and (6.15) we have

$$\begin{aligned} &- m_k \int_0^T h_k'(t)\xi'(t)\, dt = \\ &\int_0^T \int_{\mathbb{R}} \{u(\psi_\delta(x - h_k(t))\xi(t))_t + f(u)(\psi_\delta(x - h_k(t))\xi(t))_x\}\, dx\, dt \\ &+ \int_{\mathbb{R}} u_0(x)\psi_\delta(x - h_k(0))\xi(0)\, dx + R_k, \end{aligned} \tag{6.21}$$

where

$$|R_k| \le 2B \sum_{l\ne k} \int_0^T |\xi(t)|\, \psi_\delta(h_l(t) - h_k(t))\, dt. \tag{6.22}$$

Next we consider the limit when $\delta \to 0$ in (6.21), (6.22). Due to Assumption 5.1 (restriction on particle intersections), if $l \ne k$ then $|h_l(t) - h_k(t)| > 0$ for a.e. $t \in (0,T)$ and thus

$$\psi_\delta(h_l(t) - h_k(t)) \to 0 \text{ for a.e. } t \in (0,T), \tag{6.23}$$

with the result that $R_k \to 0$. Let

$$\begin{aligned} [u(h_k(t),t)] &= u(h_k(t)^+,t) - u(h_k(t)^-,t), \\ [f(u(h_k(t),t))] &= f(u(h_k(t)^+,t)) - f(u(h_k(t)^-,t)). \end{aligned} \tag{6.24}$$

A straightforward calculation using (5.1), (5.2) gives

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}} \{u(\psi_\delta(x - h_k(t))\xi(t))_t + f(u)(\psi_\delta(x - h_k(t))\xi(t))_x\}\, dx\, dt \\ &\qquad \to \int_0^T \{h_k'(t)[u(h_k(t),t)] - [f(u(h_k(t),t))]\}\, \xi(t)\, dt, \end{aligned} \tag{6.25}$$

and

$$\int_{\mathbb{R}} u_0(x)\psi_\delta(x - h_k(0))\xi(0)\, dx \to 0. \tag{6.26}$$

The result is that (6.21) becomes

$$\begin{aligned} &- m_k \int_0^T h_k'(t)\xi'(t)\, dt \\ &\qquad = \int_0^T \{h_k'(t)[u(h_k(t),t)] - [f(u(h_k(t),t))]\}\, \xi(t)\, dt. \end{aligned} \tag{6.27}$$

After integrating the left side by parts the result is

$$\int_0^T \{m_k h_k''(t) - [u(h_k(t),t)]h_k'(t) - [f(u(h_k(t),t))]\}\, \xi(t)\, dt = 0, \tag{6.28}$$

implying that (1.4) holds for a.e. $t \in [0, T]$.

The observation that for all $\Delta > 0$, $h_k^\Delta(0) = h_{k,0}$ and $c_k^\Delta(0) = v_{k,0}$ proves the assertion that $(h_k(0), h_k'(0)) = (h_{k,0}, v_{k,0})$. □

Proof of the main theorem.

Proof. Lemma 3.5 provides the convergence portion of Theorem 1.3. That the limit $(u, \vec{h})$ is an entropy solution results from Lemmas 5.4, 5.6, and 6.1. □

Remark 8. For the single-particle case, Theorem 8 of [6] states that Definition 1.2 is sufficient for uniqueness. Thus if $K = 1$, the Lax-Friedrichs approximations (u^Δ, h_1^Δ) converge to the unique entropy solution, and convergence is along the entire computed sequence, not just a subsequence.

7. **Improved resolution via MUSCL processing.** It is possible to somewhat reduce the excessively diffusive nature of Lax-Friedrichs differencing without adding too much complexity by using the MUSCL approach. Our incorporation of MUSCL processing is standard [12]. Let $\mathcal{M}(\cdot, \cdot)$ denote the minmod function:

$$\mathcal{M}(a, b) = \frac{1}{2}\left(\operatorname{sgn}(a) + \operatorname{sgn}(b)\right) \min(|a|, |b|). \tag{7.1}$$

We replace the numerical fluxes $\bar{f}_{j+1/2}^n$, $\bar{g}_{k,j+1/2}^n$ in (2.7) by

$$\begin{aligned} \tilde{f}_{j+1/2}^n &= \frac{1}{2}\left(\left(U_{j+1}^{n,-}\right)^2/2 + \left(U_j^{n,+}\right)^2/2\right) - \frac{q}{2\mu}\left(U_{j+1}^{n,-} - U_j^{n,+}\right), \\ \tilde{g}_{k,j+1/2}^n &= \frac{1}{2}\left(c_k^n W_{k,j+1}^{n,-} + c_k^n W_{k,j}^{n,+}\right) - \frac{q}{2\mu}\left(W_{k,j+1}^{n,-} - W_{k,j}^{n,+}\right), \end{aligned} \tag{7.2}$$

where

$$\begin{aligned} U_j^{n,\pm} &= U_j^n \pm \frac{1}{2}\mathcal{M}(\Delta_+ U_j^n, \Delta_- U_j^n), \\ W_{k,j}^{n,\pm} &= W_{k,j}^n \pm \frac{1}{2}\mathcal{M}(\Delta_+ W_{k,j}^n, \Delta_- W_{k,j}^n). \end{aligned} \tag{7.3}$$

We do not presently have any convergence results or even stability estimates for the resulting scheme with MUSCL processing incorporated. A moderate amount of numerical experience indicates that the algorithm produces approximations that converge to the same solution as the basic algorithm of Section 2.

8. **Numerical examples.** Following are a few numerical examples. We refer to the scheme of Section 2 as the basic scheme, and the modified scheme of Section 7 as the MUSCL scheme. We used $q = 1/2$ in all examples.

Example 8.1. This is a single-particle Riemann problem, with

$$(u_L, u_R) = (.15, -.15), \quad (h(0), h'(0)) = (0, .65), \quad \lambda = .5, \quad m = 2. \tag{8.1}$$

The exact solution is available for comparison, using the results of [11]. See Figure 1. The approximations appear to improve when the mesh size is halved, as expected. It is also apparent that the MUSCL scheme is more accurate than the basic one.

The sharp transition at $x \approx 0.8$ is a shock that is collocated with the particle. With our Lax-Friedrichs scheme there is some smearing of the shock. We must rely on a very small mesh size, even with the MUSCL version, to obtain a very sharp transition. The schemes of [1], [5], and [14] resolve this type of shock (i.e., the shock is collocated with the particle) with no smearing.

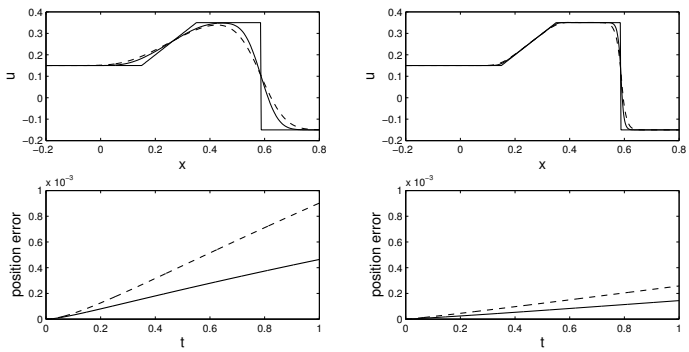

FIGURE 1. Example 8.1. Top: Fluid velocity u at $t = 1$. Exact solution is solid line, with sharp corners. Bottom: Particle position error vs. time. Basic scheme (left plots) and MUSCL scheme (right plots). $\Delta x = .0025$ (dashed line), and $\Delta x = .00125$ (solid line). Both approximations used $\mu = .25$.

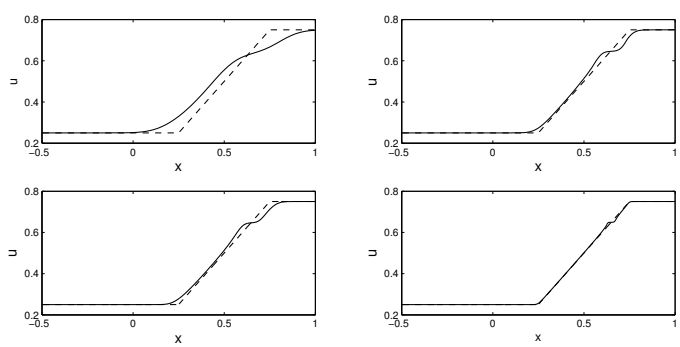

FIGURE 2. Example 8.2. Fluid velocity u at $t = 1$. Basic scheme (left plots) and MUSCL scheme (right plots). Exact solution (dashed line) and approximate solution (solid line). Top plots used $\Delta x = .005$, bottom plots used $\Delta x = .000625$. All approximations used $\mu = .25$. A spurious kink is visible. Its magnitude diminishes with grid refinement.

Example 8.2. This is another single-particle Riemann problem with

$$(u_L, u_R) = (.25, .75), \quad (h(0), h'(0)) = (0, .65), \quad \lambda = .5, \quad m = 1. \tag{8.2}$$

As in the previous example the exact solution is available via [11]. This example displays a spurious kink, see Figure 2, that appears in some cases where a particle's velocity $h_k'(t)$ lies between $u(h_k^-(t), t)$ and $u(h_k^+(t), t)$. The kink is probably due to the large numerical viscosity of the Lax-Friedrichs scheme. The size of the kink diminishes, as expected, when the mesh shrinks. Also the MUSCL approximation has a smaller kink than the basic approximation.

Example 8.3. This is a two-particle example with $z(x,t) = \text{constant} = \hat{z}$. It is possible to explicitly solve this type of problem. With $z(x,t) = \hat{z}$, we have $u(x,t) = \hat{z} - \lambda_1 H(x - h_1(t)) - \lambda_2 H(x - h_2(t))$. Thus the problem reduces to determining the particle paths $h_1(t)$ and $h_2(t)$. This can be accomplished using the differential equations (1.4), which become

$$h_k'' + \frac{\lambda_k}{m_k} h_k' = \sigma_k(t), \quad k = 1, 2. \tag{8.3}$$

Here

$$\sigma_k(t) = \frac{\lambda_k \hat{z} - \lambda_k^2/2}{m_k} + p_k(t), \tag{8.4}$$

where

$$p_1(t) = \begin{cases} 0, & h_1(t) < h_2(t), \\ -\frac{\lambda_1 \lambda_2}{m_1}, & h_1(t) > h_2(t), \end{cases} \quad p_2(t) = \begin{cases} -\frac{\lambda_1 \lambda_2}{m_2}, & h_1(t) < h_2(t), \\ 0, & h_1(t) > h_2(t). \end{cases} \tag{8.5}$$

Assume that the particle trajectories do not intersect except for a finite set of times τ_ν with $0 < \tau_1 < \ldots < \tau_M < T$. Define $\tau_0 = 0$, $\tau_{M+1} = T$, and let $r_k = \lambda_k / m_k$. The solution of (8.3), (8.4), (8.5) can be expressed piecewise. For $t \in (\tau_\nu, \tau_{\nu+1})$ the solution is

$$h_k(t) = h_k(\tau_\nu) + \frac{h_k'(\tau_\nu)}{r_k}(1 - \exp(-r_k(t - \tau_\nu))) - \frac{\sigma_k}{r_k^2}(1 - \exp(-r_k(t - \tau_\nu))) + \frac{\sigma_k}{r_k}(t - \tau_\nu). \tag{8.6}$$

The parameters used in this example are

$$\begin{aligned} &m_1 = .025, m_2 = .02, (h_1(0), h_1'(0)) = (.2, 1.2), (h_2(0), h_2'(0)) = (.3, 0.9), \\ &\lambda_1 = .75, \lambda_2 = .5, \hat{z} = .5. \end{aligned} \tag{8.7}$$

See Figures 3 and 4. From Figure 3 it appears that the MUSCL scheme is more accurate than the basic scheme, as expected. We also see that the discrete L^1 error in u decreases as we decrease the mesh size. Figure 4 shows the approximate and exact particle trajectories. At the level of discretization shown, the particle trajectories produced by the basic scheme do not quite agree with the exact trajectories. This discrepancy diminishes when the mesh size is decreased (not shown), but convergence is slow. For the MUSCL scheme the resolution is better; the exact and computed trajectories are not visually distinguishable at this level of grid refinement.

Example 8.4. This is another two-particle example. This time the particles are initially heading toward each other, and the fluid is initially at rest. Unlike the previous example the true solution is not known. In Figure 5 we show the particle trajectories at three levels of grid refinement. It appears that the particle trajectories are converging as the mesh size is refined. The MUSCL scheme is better able to resolve the fine details of the trajectory, especially after the first crossing of trajectories.

The initial fluid velocity is zero, $u_0(x) = 0$. The other parameters of the problem are

$$m_1 = .04, m_2 = .02, (h_1(0), h_1'(0)) = (.1, -2), (h_2(0), h_2'(0)) = (-.1, 4), \lambda_1 = \lambda_2 = 1. \tag{8.8}$$

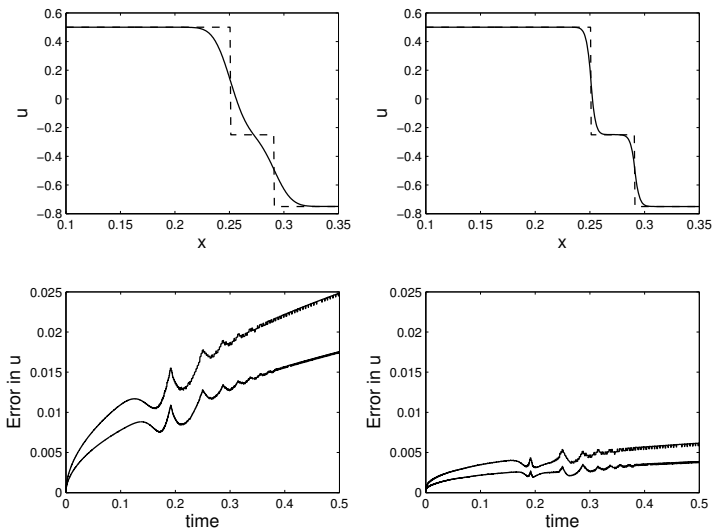

FIGURE 3. Example 8.3. Solution u using basic scheme at $t = .125$ (upper left), and using MUSCL (upper right). True solution (dashed line) and approximate solution (solid line). Both upper plots computed with $\Delta x = .00325$, $\mu = .25$. The lower plots show the error in u in discrete L^1 norm as a function of time using the basic scheme (lower left) and MUSCL scheme (lower right). Uses $\Delta x = .00325$ and $\Delta x = .001625$, $\mu = .25$.

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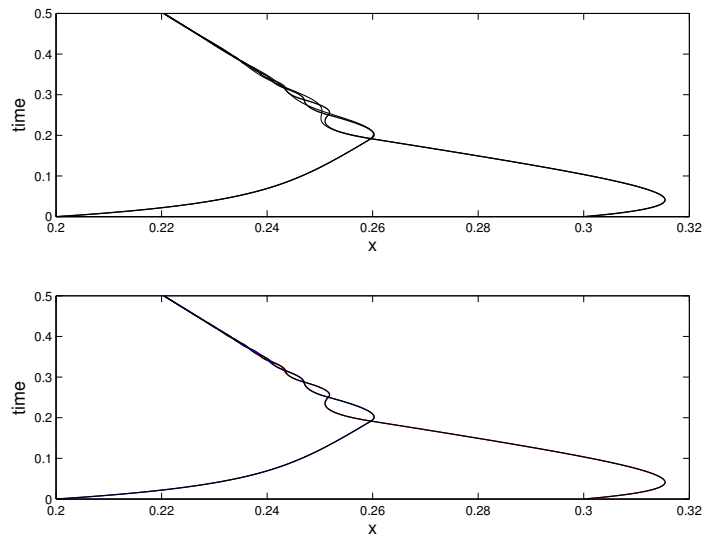

FIGURE 4. Example 8.3. Particle trajectories using basic scheme (upper plot) and MUSCL (lower plot). Both the true (thick line) and approximate (thin line) trajectories are plotted. For the MUSCL scheme (lower plot) the true and approximate trajectories are visually indistinguishable at this level of discretization. $\Delta x = 1.95 \times 10^{-5}$, $\mu = .25$, 102401 time steps.

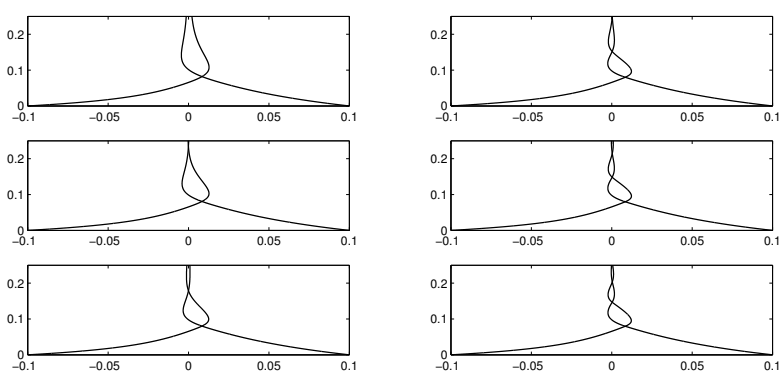

FIGURE 5. Example 8.4. Basic scheme (left) and MUSCL (right). The horizontal axis represents x, and the vertical axis represents t. Top level plots: $\Delta x_1 = 3.75 \times 10^{-4}$. Middle level plots: $\Delta x_2 = \frac{1}{2}\Delta x_1$. Bottom level plots: $\Delta x_3 = \frac{1}{4}\Delta x_1$. $\mu = .125$ for all plots.

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