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THE GENERAL NON-ABELIAN KURAMOTO MODEL ON THE 3-SPHERE

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Abstract. We introduce non-Abelian Kuramoto model on S^3 in *the most general form*. Following an analogy with the classical Kuramoto model (on the circle S^1), we study some interesting variations of the model on S^3 that are obtained for particular coupling functions. As a partial case, by choosing "standard" coupling function one obtains a previously known model, that is referred to as *Kuramoto-Lohe model* on S^3.

We briefly address two particular models: Kuramoto models on S^3 with frustration and with external forcing. These models on higher dimensional manifolds have not been studied so far. By choosing suitable values of parameters we observe different nontrivial dynamical regimes even in the simplest setup of globally coupled homogeneous population.

Although non-Abelian Kuramoto models can be introduced on various symmetric spaces, we restrict our analysis to the case when underlying manifold is the 3-sphere. Due to geometric and algebraic properties of this specific manifold, variations of this model are meaningful and geometrically well justified.

1. **Introduction.** The tendency of coupled oscillators to synchronize their oscillations is universal phenomenon with a great variety of manifestations in Nature [33]. However, synchronization is just one particular example of self-organization phenomena that are observed in populations of coupled oscillators.

In 1975, Kuramoto has proposed a paradigmatic model of this kind. Since then, Kuramoto model attracts a stable and enduring interest of researchers in Physics and other fields. Success of the Kuramoto model stems from its simplicity combined with nontriviality. Indeed, even in its original form, the Kuramoto model exhibits surprisingly rich variety of dynamical regimes. In addition, it is very tractable for the rigorous (and nontrivial) mathematical analysis.

In Kuramoto model, oscillators' states are described only by their phases $\varphi_l \in [0, 2\pi]$ while the amplitudes are neglected. The second crucial assumption is that oscillators are coupled through the first harmonics only (and not through higher harmonics). After adopting these simplifications, Kuramoto has written the system

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of ODE's describing the population of N coupled oscillators in the following form [19, 20]

$$\dot{\varphi}_l = \omega_l + \frac{K}{N}\sum_{m=1}^{N}\sin(\varphi_m - \varphi_l), \quad l = 1, \dots, N. \tag{1}$$

Here, $\varphi_l(t)$ and $\omega_l \in \mathbb{R}$ are respectively the phase and the intrinsic frequency of the l-th oscillator, and K is a global coupling strength.

Underline that the original setup introduced by Kuramoto assumed the population of nonidentical oscillators (as their intrinsic frequencies ω_l are different) with the global (all-to-all) coupling.

Kuramoto has also conducted mathematical analysis of his model and found analytically the critical coupling strength K_c depending on the width of the distribution of intrinsic frequencies, such that transition towards partial synchronization takes place for $K > K_c$, see [1, 20, 42].

Following seminal papers of Kuramoto, numerous generalizations and variations of his model have been studied, see survey [1] and references therein. The most commonly studied generalizations include phase shift [37] or time-delay [40] in the coupling function. In addition, the system (1) with an external periodic forcing is frequently studied, see [36, 41]. In some papers [18, 30], combination of these (and some other) effects has been studied.

In the present paper, we study generalization of the Kuramoto model in a completely different direction. In order to introduce such a generalization, emphasize that the state of each oscillator in (1) is represented by a point on the unit circle S^1, and the Kuramoto model is a dynamical system on the N-torus $T^N = S^1 \times \cdots \times S^1$.

Some authors have extended this setup by proposing systems of coupled generalized "oscillators" that evolve on higher dimensional symmetric spaces. In such a setup, each "oscillator" is described by a point on a certain symmetric space, such as a unitary matrix group, a sphere or Grassmannian. Such models are named *non-Abelian Kuramoto models* or *Kuramoto-Lohe models*. In particular, non-Abelian Kuramoto model on unitary matrix groups $U(n)$ is written as [9, 24]

$$i\dot{U}_l U_l^* = H_l - \frac{iK}{2N}\sum_{m=1}^{N}(U_l U_m^* - U_m U_l^*), \quad l = 1, \dots, N. \tag{2}$$

This is the system of matrix ODE's for complex matrices U_l, and H_l are given Hermitian matrices. The notion U_l^* stands for the conjugate matrix of U_l. Notice that (2) preserves $U(n)$, i.e. if the initial conditions satisfy $U_l(0) \in U(n)$ one has that $U_l(t) \in U(n)$ for any t.

From this point of view we can refer to the classical Kuramoto model on S^1 as *Abelian Kuramoto model*. This terminology underlines the most essential features of higher-dimensional models, since multiplication on the group $U(n)$ for $n > 1$ is not commutative. Still, (2) is an extension of the classical Kuramoto model, since for the case of $U(1)$ it reduces to (1).

Interest in various non-Abelian Kuramoto models is motivated by potential physical interpretations (so-called *quantum synchronization*, see for instance [13, 14, 16, 25]). In addition, non-Abelian Kuramoto models with zero frequencies ($H_l \equiv 0$) play an important role in distributed and cooperative control on non-Euclidean spaces. For instance, the system (2) with $H_l \equiv 0$ appears as the distributed consensus algorithm in [38, 39]. Similar systems on spheres S^{d-1} are written in real

coordinates (using unit vectors in $\mathbb{R}^d$ instead of matrices) as models of opinion dynamics [4] or controllable swarms on spheres [26, 31].

The main purpose of the present paper is to introduce non-Abelian Kuramoto model on S^3 *in the general form*. After that, various physical effects are analyzed in the framework of this general model by choosing coupling functions of different types. By including some of these effects one can observe different kinds of collective behavior and a variety of dynamical regimes. This might have interesting applications in Physics, but also in modeling opinion dynamics, or designing various communication protocols for swarms that evolve on S^3.

In the next section some well-known variations of the Kuramoto model on the circle S^1 are recalled. We start with the general Kuramoto model on S^1 and show how these variations are obtained for different coupling functions. In Section 3 we introduce general non-Abelian Kuramoto model on S^3 and obtain some known and unknown variations of this model. Kuramoto models on S^1 can be written using complex numbers in order to represent points on the unit circle. In analogy, non-Abelian Kuramoto model on S^3 is written using the algebra of quaternions, with points on S^3 represented by unit quaternions. In sections 4 and 5 we respectively examine some basic features of particular models with phase-shifted coupling (frustration) and external driving. Finally, Section 6 contains some concluding remarks.

2. **General Kuramoto model on the circle.** We start from the classical Kuramoto model on S^1 written in the general form

$$\dot{\varphi}_l = f e^{i\varphi_l} + \omega_l + \bar{f} e^{-i\varphi_l}, \quad l = 1, \dots, N, \tag{3}$$

where $f = f(t, \varphi_1, \dots, \varphi_N)$ is a global complex-valued coupling function .

By introducing the change of variables $z_l = e^{i\varphi_l}$ one can represent phases by unit complex numbers. In these variables (3) is written as

$$\dot{z}_l = i(f z_l^2 + \omega_l z_l + \bar{f}), \quad l = 1, \dots, N.$$

The first step in study of the Kuramoto model typically consists in defining the complex order parameter

$$\langle z \rangle = \frac{1}{N} \sum_{l=1}^{N} z_l = \frac{1}{N} \sum_{l=1}^{N} e^{i\varphi_l}$$

and the real order parameter: $r = |\langle z \rangle|$.

In the rest of this section we substitute some commonly used coupling functions f in (3) in order to recover several well-known variations of the Abelian Kuramoto model.

Example 1. (Standard Kuramoto model) Assume that the coupling strength is proportional to the complex order parameter and consider the coupling function of the following form

$$f = i \frac{K}{2N} \sum_{l=1}^{N} e^{-i\varphi_l} = \frac{iK}{2} \bar{\langle z \rangle}.$$

Plugging this coupling function in (3) yields the standard Kuramoto model (1).

One can consider the case $K > 0$ (attractive coupling) or $K < 0$ (repulsive coupling).

Example 2. (Kuramoto model with phase shift) Consider the following coupling function

$$f = i\frac{K}{2N}\sum_{l=1}^{N} e^{-i(\varphi_l-\beta)} = \frac{iK}{2}e^{i\beta}\bar{\langle z\rangle}.$$

This choice of f turns (3) into

$$\dot{\varphi}_l = \omega_l + \frac{K}{N}\sum_{m=1}^{N}\sin(\varphi_m - \varphi_l - \beta), \quad l = 1,\dots,N.$$

This is Kuramoto model with the phase shift β. It is also sometimes referred to as *Kuramoto model with frustration* or *Kuramoto-Sakaguchi model*. This model still draws attention of many researchers. Recent advances in study of different aspects of this model can be found in [11, 12, 15, 32].

Example 3. (Kuramoto model with time-delayed coupling) The third frequently studied model is obtained for the following coupling function

$$f = i\frac{K}{2N}\sum_{l=1}^{N} e^{-i\varphi_l(t-\tau)}. \tag{4}$$

Plugging (4) into (3) yields

$$\dot{\varphi}_l(t) = \omega_l + \frac{K}{N}\sum_{m=1}^{N}\sin(\varphi_m(t-\tau) - \varphi_l(t)), \quad l = 1,\dots,N.$$

This is the Kuramoto model with the global time delay τ in the coupling. Various aspects of Kuramoto model with the global time delay in coupling have been studied or applied in [27, 35, 44].

Example 4. (Kuramoto model with external forcing) In order to introduce the model with external forcing, we slightly modify (3) by including an additional term

$$\dot{\varphi}_l = fe^{i\varphi_l} + (\omega_l - \delta) + \bar{f}e^{-i\varphi_l}, \quad l = 1,\dots,N,$$

with $\delta \in \mathbb{R}$ and consider the following coupling function

$$f = \frac{i}{2}\left(\frac{K}{N}\sum_{l=1}^{N} e^{-i\varphi_l} - D\right) = \frac{i}{2}\left(K\bar{\langle z\rangle} - D\right),$$

with $D \in \mathbb{R}$. The substitution yields the system

$$\dot{\varphi}_l = (\omega_l - \delta) + \frac{K}{N}\sum_{m=1}^{N}\sin(\varphi_m - \varphi_l) + D\sin\varphi_l.$$

This is Kuramoto model with external driving. The frequency of an external signal is $\delta \in \mathbb{R}$, and $D \in \mathbb{R}$ is its intensity.

Kuramoto model with external driving has also been studied in a number of papers, see for instance [2, 7].

Clearly, a number of other effects can be studied as partial cases of (3). Here, we have singled out the most commonly studied ones. It is important to underline that (3) assumes the coupling depending on the first harmonic only, the presence of higher harmonics in the coupling function leads beyond the family of Kuramoto models. Another crucial assumption is that the coupling is *global* (or *mean field*

coupling). This assumption allows time-delayed or phase-shifted coupling, but excludes coupling with distributed time delays [21, 28, 29] or coupling through complex networks [3, 34].

At this point we end Section 2. It does not contain new results and all the examples of this Section are very well known. However, in the sequel we will introduce new non-Abelian Kuramoto models on S^3 following an analogy with the above considerations.

3. **General non-Abelian Kuramoto model on the three-sphere.** We start this Section with a short discussion about the concept of "generalized phase oscillator".

Classical phase oscillator is represented by a single phase (angle) $\varphi \in [0, 2\pi]$. Alternatively, by substituting $z = e^{i\varphi}$ one can represent states of the oscillator by unit complex numbers. Configuration space for the classical oscillator is the unit circle S^1.

In the absence of coupling, oscillator performs simple periodic motions given by the complex-valued ODE:

$$\dot{z} = i\omega z, \quad \omega \in \mathbb{R}. \tag{5}$$

The real number ω determining the velocity of these periodic motion is a natural (intrinsic) frequency of the oscillator.

There are several natural ways to extend this concept by introducing generalized oscillators whose states are represented by points on higher-dimensional manifolds. Recently, there is a growing interest in populations of coupled generalized oscillators. Such extensions yield generalized Kuramoto models on certain higher-dimensional manifolds.

One possible approach stems from the observation that S^1 is a Lie group. Hence, it is natural to consider oscillators whose states ("phases") are represented by points on a certain (compact) Lie group M. The "phase" m of such generalized oscillators evolves by the following ODE:

$$\frac{d}{dt}m = g(t) \cdot m + m \cdot h(t), \tag{6}$$

where $g(t)$ and $h(t)$ belong to the Lie algebra $\mathfrak{m}$ of the group M for each $t \geq 0$.

Classical result from the geometric theory of ODE's states that if $m(0) \in M$ and $g(t), h(t) \in \mathfrak{m}$ for each t, then motions (6) are restricted to M, i.e. $m(t) \in M$ for each $t > 0$.

In this context $g(t)$ and $h(t)$ are interpreted as intrinsic "frequencies" of the "oscillator" and $m(t)$ is the "phase" at the moment t.

Classical phase oscillators evolving by ODE (5) are recovered from this general scheme for the case $M = S^1$ with the corresponding Lie algebra $\mathfrak{m}$ isomorphic to $\mathbb{R}$. This means that the phase and the frequency of a classical oscillator can be described by a point on the unit circle and a real number, respectively.

Emphasize that the group M is in general non-Abelian and oscillators can have left and right intrinsic frequencies as in eq. (6).

By considering populations of generalized oscillators whose motions are described by (6) and introducing coupling between them, one obtains non-Abelian Kuramoto models, such as (2).

However, extension of the concept of oscillator does not necessarily requires manifold M to be a Lie group. In fact, Kuramoto model has been extended to symmetric spaces, notably on spheres, see [5, 6, 22, 23, 43]. In this approach, states ("phases")

of generalized oscillators are described by points on the $d-1$-sphere S^{d-1} and written as unit vectors in the d-dimensional real space. In this setup their intrinsic frequencies are given by $d \times d$ anti-symmetric matrices with real entries.

The common point of these two approaches is the non-Abelian Kuramoto model on S^3, as it is the only sphere (along with S^1) with the group property.

In the present paper we focus on this model due to its special relevance and potential applications, that are briefly mentioned in the present paper. In order to introduce the model we need coordinates on S^3. A convenient framework is provided by the algebra of quaternions.

Algebra of quaternions has originally been introduced by Hamilton [10] as an appropriate tool to work with rotations in 3-dimensional space. General quaternion is written as $q = q_1 + q_2 \cdot \mathbf{i} + q_3 \cdot \mathbf{j} + q_4 \cdot \mathbf{k}$, with three basis vectors (imaginary units) $\mathbf{i}, \mathbf{j}, \mathbf{k}$. Norm of quaternion q is defined as $|q| = \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2}$. If $|q| = 1$ we say that q is a unit quaternion. Real part of a quaternion q is defined as $\Re q = q_1$. We say that q is a pure quaternion, if $\Re q = 0$.

Manifold S^3 is parametrized by the set of unit quaternions in a pretty analogous way as S^1 is parametrized by the set of unit complex numbers. The set of unit quaternions is a group with the corresponding Lie algebra consisting of pure quaternions.

Applying the general scheme to S^3, we study oscillators whose "phases" are described by unit quaternions q_l, evolving by linear quaternion-valued ordinary differential equations (QODE's)

$$\dot{q}_l = w_l q_l + q_l u_l, \tag{7}$$

where w_l and u_l are pure quaternions, interpreted as intrinsic frequencies of the oscillator l. As explained above for the case of general Lie group, if $q_l(0)$ is a unit quaternion, the solution $q_l(t)$ of (7) will be a unit quaternion for any t. In other words, motions (7) are restricted to S^3.

Now, conceive a population of oscillators q_l that are coupled through a certain function f satisfying the following system of QODE's [17]

$$\dot{q}_l = q_l f q_l + w_l q_l + q_l u_l - \bar{f}, \quad l = 1, \ldots, N. \tag{8}$$

Here, $f = f(t, q_1, \ldots, q_N)$ is a quaternionic function. The notion $\bar{f}$ stands for the conjugate quaternion, i.e. if $f = f_1 + f_2 \cdot \mathbf{i} + f_3 \cdot \mathbf{j} + f_4 \cdot \mathbf{k}$, then $\bar{f} = f_1 - f_2 \cdot \mathbf{i} - f_3 \cdot \mathbf{j} - f_4 \cdot \mathbf{k}$.

Remark 1. Using the isomorphism between the group of unit quaternions and $SU(2)$ matrices, (8) can be treated as a system of matrix ODE's on the unitary group $SU(2)$. The corresponding Lie algebra $\mathfrak{su}(2)$ consists of skew-Hermitian 2×2 matrices with zero trace. It is isomorphic to the algebra of pure quaternions.

Multiplication rules and various ways to represent quaternions provide convenient ways to involve geometric concepts on S^3 in study of collective evolution of generalized coupled oscillators. Throughout this article, we will use Cayley-Dickson form of a quaternion q_l:

$$q_l = z_l + v_l \cdot \mathbf{j},$$

where z_l and v_l are complex numbers.

As q_l are unit quaternions, they can be further represented as

$$q_l = e^{i\varphi_l} \cos\theta_l + e^{i\psi_l} \sin\theta_l \cdot \mathbf{j}, \tag{9}$$

where $\varphi_l, \psi_l \in [0, 2\pi]$ and $\theta_l \in (0, \pi/2)$. This representation corresponds to Hopf (angular) coordinates that parametrize each point on S^3 by three angles.

In order to study (8) it is useful to introduce the quaternionic order parameter

$$\langle q\rangle = \frac{1}{N}\sum_{l=1}^{N} q_l$$

and the global real order parameter $r = |\langle q\rangle|$.

As the sphere in 4D has much richer geometry then the circle, we can also introduce some concepts that do not have analogues in the classical model. For instance, define two marginal complex order parameters in the following way

$$r_\varphi(t)e^{i\mu(t)} = \frac{1}{N}\sum_{l=1}^{N} e^{i\varphi_l(t)}, \quad r_\psi(t)e^{i\nu(t)} = \frac{1}{N}\sum_{l=1}^{N} e^{i\psi_l(t)},$$

then the real numbers

$$r_\varphi = \left|\frac{1}{N}\sum_{l=1}^{N} e^{i\varphi_l(t)}\right|, \quad r_\psi = \left|\frac{1}{N}\sum_{l=1}^{N} e^{i\psi_l(t)}\right|$$

are called *angular order parameters*.

Example 5. (Standard non-Abelian Kuramoto model on S^3) Standard non-Abelian Kuramoto model (Kuramoto-Lohe model) on the group $SU(2)$ with the group manifold S^3 is obtained from (8) for the coupling function of the form

$$f = -\frac{K}{2N}\sum_{l=1}^{N}\bar{q}_l = -\frac{K}{2}\bar{\langle q\rangle}, \tag{10}$$

and zero values of the right frequencies: $u_l = 0$.

Indeed, inserting (10) into (8) yields

$$\dot{q}_l = -\frac{K}{2N}\sum_{m=1}^{N} q_l\bar{q}_m q_l + w_l q_l + \frac{K}{2N}\sum_{m=1}^{N} q_m. \tag{11}$$

On the other hand, multiply all equations in (2) by $-i$ and by matrices U_l from the right. Than (2) is rearranged as

$$\dot{U}_l = -iH_lU_l + \frac{K}{2N}\sum_{m=1}^{N}(U_lU_m^*U_l - U_m), \quad l = 1,\dots,N, \tag{12}$$

where $-iH_l$ are zero-trace skew Hermitian matrices. The system (12) on $SU(2)$ is exactly the same as (11), where matrices $U_l \in SU(2)$ correspond to unit quaternions q_l and skew Hermitian matrices $-iH_l$ correspond to pure quaternions w_l.

The coupling (10) can be attractive or repulsive for positive and negative values of K respectively.

4. **Non-Abelian Kuramoto-Sakaguchi model on the three-sphere.** In this Section we introduce the Kuramoto model with frustration on S^3. In order to figure out how this model should look like, compare the coupling functions for the classical case in examples 1 and 2. It is easy to see that these functions differ by the multiplier $e^{i\beta}$, that is, model with frustration in Example 2 is obtained by rotating (conjugation of) the complex order parameter $\bar{\langle z\rangle}$ by an angle β in the complex plane.

Analogously, in order to introduce non-Abelian model with frustration on S^3, one should rotate $\bar{\langle q\rangle}$ in formula (10). Rotations in 4-dimensional space are realized

through multiplication of a quaternion by two unit quaternions from left and the right. More precisely, consider the point in 4-dimensional space that corresponds to a quaternion s. General $SO(4)$ rotation of this point is obtained as

$$\mathcal{R}(s) = a \cdot s \cdot b,$$

where a and b are unit quaternions.

Hence, we consider the model (8) with the coupling function

$$f = -\frac{K}{2N} a \cdot \left(\sum_{l=1}^{N} \bar{q}_l \right) \cdot b = -\frac{K}{2N} a \cdot \langle \bar{q} \rangle \cdot b,$$

with arbitrary unit quaternions a and b.

We can represent quaternion a by three angles

$$a = e^{i\alpha} \cos\gamma + e^{i\beta} \sin\gamma \cdot \mathbf{j}, \tag{13}$$

where $\alpha, \beta \in [0, 2\pi]$ and $\gamma \in \left(0, \frac{\pi}{2}\right)$.

In total, we have 6 real parameters in the model (3 angles for each quaternion a and b). For simplicity, set $b = 1$ and investigate the impact of the remaining three parameters α, β and γ on the dynamics. Than the coupling function f can be written as

$$\begin{aligned} f &= -\frac{K}{2N} a \cdot \langle \bar{q} \rangle \cdot b \\ &= -\frac{K}{2N} \left(e^{i\alpha} \cos\gamma + e^{i\beta} \sin\gamma \cdot \mathbf{j} \right) \cdot \left(\sum_{l=1}^{N} \left(e^{-i\varphi_l} \cos\theta_l - e^{i\psi_l} \sin\theta_l \cdot \mathbf{j} \right) \right) \cdot 1 \\ &= -\frac{K}{2N} \sum_{l=1}^{N} \left(e^{-i(\varphi_l - \alpha)} \cos\gamma \cos\theta_l + e^{-i(\psi_l - \beta)} \sin\gamma \sin\theta_l \right) \\ &\quad -\frac{K}{2N} \sum_{l=1}^{N} \left(e^{i(\varphi_l + \beta)} \sin\gamma \cos\theta_l - e^{i(\psi_l + \alpha)} \cos\gamma \sin\theta_l \right) \cdot \mathbf{j}. \end{aligned} \tag{14}$$

In all simulations of this Section we will assume that the population is homogeneous, i.e. that all oscillators have identical frequencies.[1]

Depending on parameters α, β, γ the model (8), (14) can exhibit three qualitatively different dynamical regimes: synchronization, desynchronization (convergence towards incoherent state), and oscillating order parameter. A closer look unveils that dynamics of the order parameter qualitatively depends only on the parameter α.

If $\alpha \in (0, \frac{\pi}{2})$ or $\alpha \in (\frac{3\pi}{2}, 2\pi)$ the system achieves a stable equilibrium where the KL oscillators are synchronized, see Figure 1. Figure 2 shows that the order parameter converges to zero when $\alpha \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

For $\alpha = \frac{\pi}{2}$ or $\alpha = \frac{3\pi}{2}$ the order parameter oscillates, as well as pairwise scalar product (angles) between KL oscillators, see Figure 3.

In order to understand why the phase shift α is crucial for the dynamics, notice that the quaternion $\langle \bar{q} \rangle$ corresponds to a point in 4-dimensional unit ball B^4, and a quaternionic multiplication $a \cdot \langle \bar{q} \rangle$ rotates this point in 4-dimensional space. The rate (velocity) of synchronization is determined by a cosine between vectors represented by quaternions $\langle \bar{q} \rangle$ and $a \cdot \langle \bar{q} \rangle$. Oscillators will synchronize if and only if this

[1]Numerical simulations of all models in the present paper are implemented using Wolfram Mathematica package and 4th order Runge-Kutta method for solving large systems of ODE's.

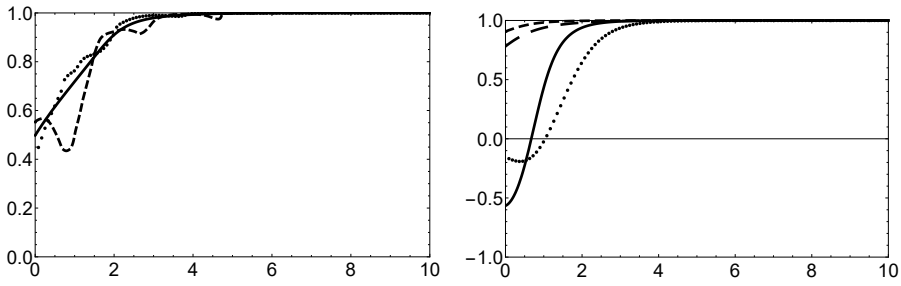

FIGURE 1. The population of $N = 50$ KL oscillators with intrinsic frequencies $w_l = (0.3, 1.3, 0.9)$, $u_l = (1.7, 0.5, 1.4)$ and phase shifts $\alpha = \frac{\pi}{3}$, $\beta = \frac{\pi}{4}$, $\gamma = \frac{\pi}{3}$ achieves coherent state: (a) evolution of the order parameters (thick line for the global order parameter r and dashed and dotted lines for r_φ and r_ψ respectively) and (b) cosines of the angles between some pairs of KL oscillators. Initial conditions are sampled from the von Mises-Fisher on S^3 with mean direction $\mu = (0.5, 0.5, 0.5, 0.5)$ and the concentration $\kappa = 2.5$.

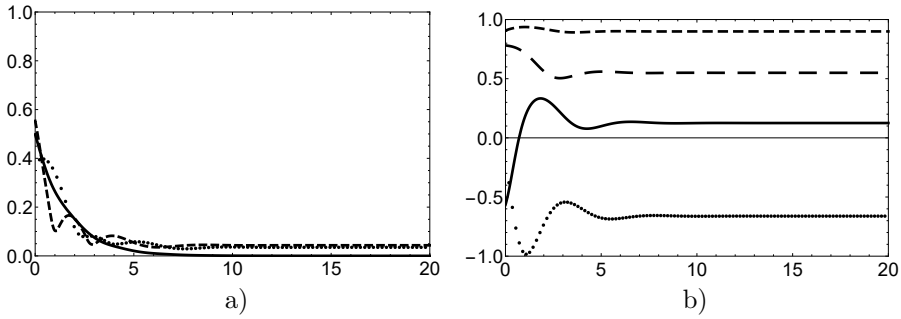

FIGURE 2. The population of $N = 50$ KL oscillators with intrinsic frequencies $w_l = (0.3, 1.3, 0.9)$, $u_l = (1.7, 0.5, 1.4)$ and phase shifts $\alpha = \frac{2\pi}{3}$, $\beta = \frac{\pi}{4}$, $\gamma = \frac{\pi}{3}$ achieves a fully incoherent state: (a) evolution of the order parameters (thick line for the global order parameter r and dashed and dotted lines for r_φ and r_ψ respectively) and (b) cosines of the angles between some pairs of KL oscillators. Initial conditions are sampled from the von Mises-Fisher on S^3 with mean direction $\mu = (0.5, 0.5, 0.5, 0.5)$ and the concentration $\kappa = 2.5$.

cosine is positive. Now, as quaternion a is represented by (13), one can check that multiplication by a rotates the 4-dimensional space by an angle whose cosine equals $\cos\alpha\cos\gamma$. Since $\gamma \in (0, \pi/2)$, this cosine is positive if and only if $\alpha \in (0, \pi/2)$ or $\alpha \in (3\pi/2, 2\pi)$. Hence, qualitative dynamics of r depends only on α. Angle γ affects the velocity of synchronization (or desynchronization), but does not have impact on asymptotic behavior of r. (Synchronization, or desynchronization, occurs at the maximal pace when $\gamma = 0$.) Parameter β in its turn affects the direction (radial axis) of synchronization, but has no influence on order parameter r.

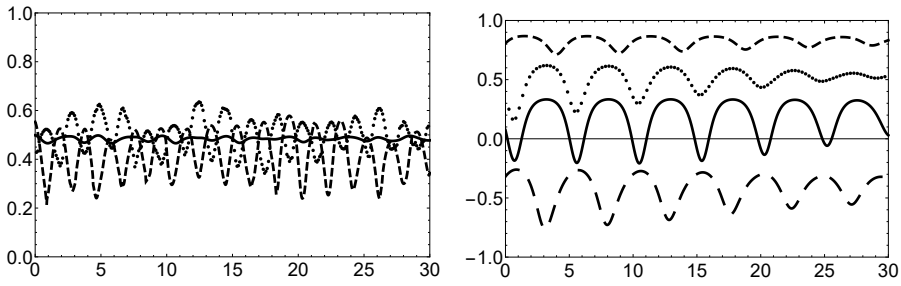

FIGURE 3. Oscillations of the system with $N = 50$ KL oscillators with intrinsic frequencies $w_l = (0.3, 1.3, 0.9)$, $u_l = (1.7, 0.5, 1.4)$ and phase shifts $\alpha = \frac{\pi}{2}$, $\beta = \frac{\pi}{4}$, $\gamma = \frac{\pi}{3}$: (a) evolution of the order parameters (thick line for the global order parameter r and dashed and dotted lines for r_φ and r_ψ respectively) and (b) cosines of the angles between some pairs of KL oscillators. Initial conditions are sampled from the von Mises-Fisher on S^3 with mean direction $\mu = (0.5, 0.5, 0.5, 0.5)$ and the concentration $\kappa = 2.5$.

In the boundary case when $\alpha = \pi/2$ or $\alpha = 3\pi/2$, the cosine between $\langle\bar{q}\rangle$ and $a \cdot \langle\bar{q}\rangle$ is zero and the dynamics of r is determined by higher order terms, leading to oscillatory dynamics, as shown in Figure 3.

5. **Non-Abelian Kuramoto model with external driving.** In this Section we include additional terms in (8) and consider the following system

$$\dot{q}_l = q_l f q_l + (w_l - v)q_l + q_l(u_l - p) - \bar{f}, \quad l = 1, \ldots, N, \tag{15}$$

where v and p are pure quaternions.

Further, suppose that the coupling function f is of the form

$$f = -\frac{1}{2}\left(\frac{K}{N}\sum_{l=1}^{N}\bar{q}_l - D_1 - D_2 \cdot \mathbf{j}\right) = -\frac{1}{2}\left(K\langle\bar{q}\rangle - D_1 - D_2 \cdot \mathbf{j}\right), \tag{16}$$

where $D_1, D_2 \in \mathbb{R}$.

The system of QODE's (15), (16) is the non-Abelian Kuramoto model on S^3 with an external forcing. Frequencies of an external signal are given by pure quaternions v and p, and $D = D_1 + D_2 \cdot \mathbf{j}$ is an intensity of this signal. (Notice that in general the signal intensity is given by two real numbers D_1 and D_2).

In order to get some feeling that this is really the model with external forcing, it is instructive to rewrite (15),(16) in angular coordinates. Representing q_l as (9) and writing pure quaternions p, v, w, u in the form $p = p_a \cdot \mathbf{i} + p_2 \cdot \mathbf{j}$, where $p_a \in \mathbb{R}$ and $p_2 \in \mathbb{C}$ (and similar for v, w_l, u_l), we plug (16) into (15) and obtain the following

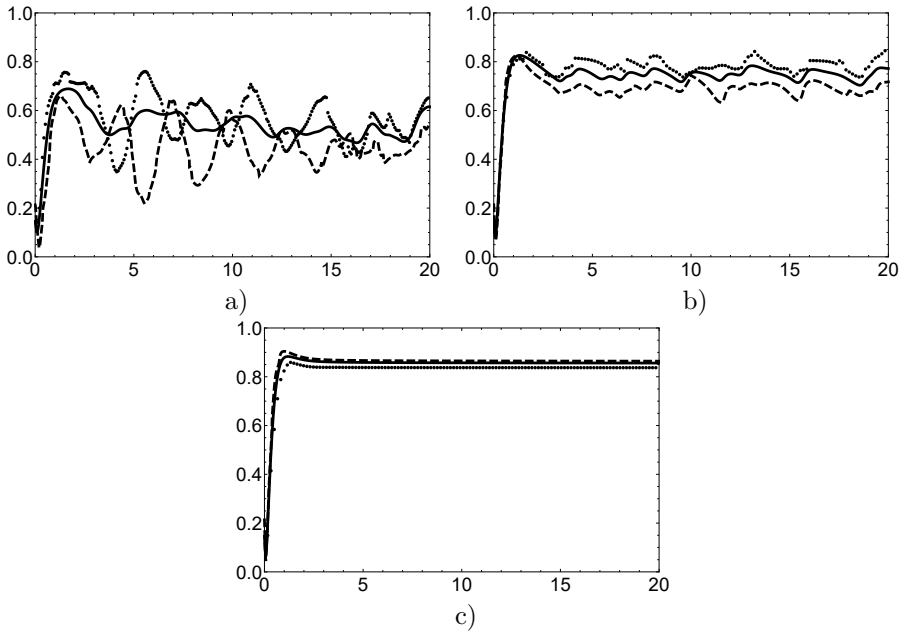

FIGURE 4. Evolution of the global and angular order parameters (thick line for the global order parameter r and dashed and dotted lines for r_φ and r_ψ respectively) in the population of $N = 50$ non-identical oscillators with the external forcing. The global coupling strength is set at $K = 1$, the frequencies of the external signal are $p = 0$, $v = (1.2, 2.3, 3.7)$ and the intensity of external forcing is (a) $D = 0.5 + 0.5\mathbf{j}$; (b) $D = 0.9 + 0.5\mathbf{j}$ and (c) $D = 1.2 + 0.5\mathbf{j}$. The right intrinsic frequencies u_l are zero, and the left intrinsic frequencies are sampled from the 3-dimensional Gaussian distribution with the expectation vector $(1, 2, 3)$ and covariance matrix $((2, -1/4, 1/3), (-1/4, 2/3, 1/5), (1/3, 1/5, 1/2))$.

system for $\varphi_l, \psi_l, \theta_l$

$$
\begin{cases}
\dot{\varphi}_l = w_{al} + u_{al} - v_a - p_a + \\
\frac{i}{2}[(\bar{u}_{2l} - \bar{p}_2)\tan(\theta_l)e^{-i(\varphi_l - \psi_l)} - (\bar{w}_{2l} - \bar{v}_2)\tan(\theta_l)e^{i(\varphi_l + \psi_l)} - \\
(u_{2l} - p_2)\tan(\theta_l)e^{i(\varphi_l - \psi_l)} + (w_{2l} - v_2)\tan(\theta_l)e^{-i(\varphi_l + \psi_l)}] + \\
\frac{K}{N\cos(\theta_l)}\sum_{m=1}^{N}(\sin(\varphi_m - \varphi_l + \theta_m) + \sin(\varphi_m - \varphi_l - \theta_m)) + D_1\frac{\sin(\varphi_l)}{\cos(\theta_l)}; \\
\\
\dot{\psi}_l = w_{al} - u_{al} - v_a + p_a + \\
\frac{i}{2}[(\bar{u}_{2l} - \bar{p}_2)\cot(\theta_l)e^{-i(\varphi_l - \psi_l)} + (\bar{w}_{2l} - \bar{v}_2)\cot(\theta_l)e^{i(\varphi_l + \psi_l)} - \\
(u_{2l} - p_2)\cot(\theta_l)e^{i(\varphi_l - \psi_l)} - (w_{2l} - v_2)\cot(\theta_l)e^{-i(\varphi_l + \psi_l)}] + \\
\frac{K}{N\sin(\theta_l)}\sum_{m=1}^{N}(\cos(\psi_m - \psi_l - \theta_m) - \cos(\psi_m - \psi_l + \theta_m)) - D_2\frac{\sin(\psi_l)}{\sin(\theta_l)}; \\
\\
\dot{\theta}_l = \frac{1}{2}[(\bar{u}_{2l} - \bar{p}_2)e^{-i(\varphi_l - \psi_l)} + \\
(\bar{w}_{2l} - \bar{v}_2)e^{i(\varphi_l + \psi_l)} + (u_{2l} - p_2)e^{i(\varphi_l - \psi_l)} + (w_{2l} - v_2)e^{-i(\varphi_l + \psi_l)}] - \\
\frac{K\sin(\theta_l)}{N}\sum_{m=1}^{N}(\cos(\varphi_m - \varphi_l - \theta_m) + \cos(\varphi_m - \varphi_l + \theta_m)) + \\
\frac{K\cos(\theta_l)}{N}\sum_{m=1}^{N}(\sin(\psi_m - \psi_l + \theta_m) - \sin(\psi_m - \psi_l - \theta_m)) + \\
D_1\sin(\theta_l)\cos(\varphi_l) + D_2\cos(\theta_l)\cos(\psi_l).
\end{cases}
$$

Here, as explained above, w_{al}, u_{al}, v_a, p_a are real numbers and w_{2l}, u_{2l}, v_2, p_2 are complex numbers. From the above system of ODE's we can see that D_1 acts as intensity of the forcing for angles φ_l, while D_2 is for angles ψ_l. However, φ_l and ψ_l are coupled through θ_l.

Heterogeneous population of KL oscillators can be entrained by an external signal to frequencies v and p. As in the case with classical Kuramoto oscillators on S^1, partial synchronization will occur for critical values of the signal strength. For sufficiently large D_1 and D_2 a portion of oscillators whose intrinsic frequencies w_l and u_l are close to v and p will be entrained by external signal.

Figure 4 shows simulation results for a population with zero right intrinsic frequencies u_l, while the left frequencies w_l are sampled from the Gaussian distribution on $\mathbb{R}^3$. (Here, we use the isomorphism of pure quaternions with the vector space $\mathbb{R}^3$). The Figure demonstrates that a larger portion of the population gets entrained as the external driving grows stronger.

6. **Conclusion.** We have introduced the general non-Abelian Kuramoto model on S^3. This model includes some important special cases that have not been studied so far. In particular, a widely studied model (2) arise as a partial case of (8) with the standard coupling function (10) and vanishing right intrinsic frequencies u_l. Very recently, DeVille [8] has studied some solutions and multistable regimes in non-Abelian Kuramoto models on certain Lie groups. In his paper a variation of non-Abelian model with frustration has been proposed, however, his approach differs significantly from ours.

We have restricted our attention to the model on one specific Lie group, namely the 3-sphere S^3. This case is of a special importance, due to the central role of S^3 in several group homomorphisms. In addition, algebraic and geometric properties of S^3 make the model natural, well justified and tractable for study. In particular, collective behavior is better understood through angular order parameters r_φ and r_ψ that can be introduced on S^3. The model on S^3 can be written either in the form of quaternion-valued ODE's or real-valued ODE's for three angles (Hopf coordinates on the three-sphere). In short, we believe that the Kuramoto models on S^3 and S^1 are the most important ones for physical interpretations and applications.

The models presented in sections 4 and 5 have natural interpretations and potential applications in different fields. For instance, the model with an external forcing (15)-(16) can be relevant in modeling opinion dynamics in Sociophysics. On the other hand, systems with phase shifts studied in Section 4 can be applied in cooperative control, as they provide receipts of how to design communication protocols in order to ensure formation keeping of the swarm.

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