

REMARKS ON THE SCHRÖDINGER-LOHE MODEL

HYUNGJIN HUH*

Department of Mathematics
Chung-Ang University
Seoul 156-756, Republic of Korea

(Communicated by Seung-Yeal Ha)

ABSTRACT. We study the Schrödinger-Lohe model. Making use of the principal fundamental matrix Y of linear ODEs with variable coefficients, the coupled nonlinear Schrödinger-Lohe system is transformed into the decoupled linear Schrödinger equations. The boundedness of Y is shown for the case of complete synchronization. We also study the cases where the principal fundamental matrices can be derived explicitly.

1. Introduction. We are interested in the quantum Schrödinger-Lohe model

$$\begin{aligned} i\partial_t\psi_i + \Delta\psi_i &= V\psi_i + \frac{iK}{N} \sum_{k=1}^N (\psi_k - \langle\psi_i, \psi_k\rangle\psi_i), \\ \psi_i(x, 0) &= \psi_i^0(x). \end{aligned} \tag{1}$$

Here ψ_i ($i = 1, 2, \dots, N$) are complex valued functions defined on $\mathbb{R}^d \times \mathbb{R}^+$ satisfying $\|\psi(\cdot, t)\|_{L^2(\mathbb{R}^d)} = 1$ and $V = V(x)$ is the real-valued one-body potential. We refer to section 2 for more precise assumptions on V . The standard L^2 inner product is defined by $\langle f, g \rangle(t) = \int_{\mathbb{R}^d} f(x, t) \bar{g}(x, t) dx$ and K is positive constant representing coupling strength. The Schrödinger-Lohe model (1) was first introduced in [10] as an infinite state generalization of the Lohe matrix model [11].

Quantum synchronization has received much attention from the physics community because of its possible applications in quantum computing and quantum information. The synchronous behaviors of (1) were partially treated in [5, 6, 7] for some restricted class of initial data and a large coupling strength regime. In particular, dynamical system approach to synchronization of the Schrödinger-Lohe model has been studied in [1, 7, 8]. Actually, they set

$$h_{ij}(t) = \langle\psi_i, \psi_j\rangle = \int_{\mathbb{R}^d} \psi_i(x, t) \bar{\psi}_j(x, t) dx, \tag{2}$$

2010 *Mathematics Subject Classification.* 82C10, 34E10, 35C05.

Key words and phrases. Schrödinger-Lohe model, principal fundamental matrix, explicit solution.

This research was supported by LG Yonam Foundation (of Korea) and Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education (2017R1D1A1B03028308).

* Corresponding author: Hyungjin Huh.

and showed that h_{ij} satisfies the system of ODEs:

$$\frac{dh_{ij}}{dt} = \frac{K}{N}(1 - h_{ij}) \left(2 + \sum_{k \neq i}^N h_{ik} + \sum_{k \neq j}^N h_{kj} \right), \quad 1 \leq i < j \leq N, \quad (3)$$

with the initial data $h_{ij}^0 := h_{ij}(0) = \int_{\mathbb{R}^d} (\psi_i^0 \bar{\psi}_j^0)(x) dx$.

The system (1) admits a unique global solution ψ_i which was proved in [1, 7]. We refer to section 2 for more precise results on the existence and uniqueness of a solution. From now on, we consider the solution of the system (1) with regularity

$$\psi_i \in C([0, \infty), H^m(\mathbb{R}^d)) \cap C^1([0, \infty), H^{m-2}(\mathbb{R}^d)),$$

where $m > 2 + \frac{d}{2}$.

Let us consider equations

$$\begin{aligned} i\partial_t \phi_i + \Delta \phi_i &= V \phi_i, & i &= 1, 2, \dots, N, \\ \phi_i(x, 0) &= \psi_i^0(x). \end{aligned} \quad (4)$$

Note that ϕ_i has the same initial data as ψ_i in (1). Let us denote $\psi = (\psi_1, \psi_2, \dots, \psi_N)^t$ and $\phi = (\phi_1, \phi_2, \dots, \phi_N)^t$. Our first result is to transform (1) into the decoupled linear Schrödinger equations.

Theorem 1.1. *The solution ψ_i of (1) is given by*

$$\psi(x, t) = Y(t)\phi(x, t). \quad (5)$$

Here ϕ is a solution of (4) and Y is the principal fundamental matrix solution of the ODE system

$$\frac{dy}{dt} = H(t)y,$$

where

$$H = \frac{K}{N} \begin{pmatrix} -\sum_{k \neq 1} h_{1k} & 1 & \dots & 1 \\ 1 & -\sum_{k \neq 2} h_{2k} & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 1 & \dots & 1 & -\sum_{k \neq N} h_{Nk} \end{pmatrix}, \quad (6)$$

and h_{ij} are solutions to (3). Note that $Y(0)$ is an $N \times N$ identity matrix.

The novelty of this result is that the coupled nonlinear PDEs in (1) are transformed into the decoupled linear Schrödinger equations (4) through the relation (5). The nonlinearity of (1) is hidden from view and appears in the coupled nonlinear ODEs (3). This observation can be used to study asymptotic behavior and the nonlinear stability problem for the Schrödinger Lohe model. We studied in [9] the stability of standing wave solutions for the Schrödinger Lohe model with a harmonic potential $V = |x|^2$. We will present another application of Theorem 1.1 in Theorem 1.2.

Taking Theorem 1.1 into account, analysis of the principal fundamental matrix Y is crucial for understanding the dynamics of the solution to (1). In section 4, we consider the case of $h_{ij}(t) \rightarrow 1$ exponentially as $t \rightarrow \infty$. Then we have

$$\lim_{t \rightarrow \infty} \|\psi_i(t) - \psi_j(t)\|_{L^2(\mathbb{R}^d)} = 0 \quad \text{for all } i, j,$$

which is complete synchronization. To state the next Theorem, we define

$$\zeta = \frac{1}{N} \sum_{k=1}^N \psi_k, \quad \tilde{r}_j = \operatorname{Re}\langle \zeta, \psi_j \rangle, \quad \tilde{s}_j = \operatorname{Im}\langle \zeta, \psi_j \rangle,$$

and

$$\mathcal{H}(t) = \max_{1 \leq i \leq N} \mathcal{H}_i(t), \quad \text{where} \quad \mathcal{H}_i(t) = \sum_{k=1}^N |1 - h_{ik}(t)|.$$

The following result is concerned with the boundedness of Y for the case of complete synchronization.

Theorem 1.2. *Let ψ_i be the solution of (1) with the initial data satisfying $\tilde{r}_j(0) > 0$ or $\mathcal{H}(0) < N$. Then the principal fundamental matrix Y of*

$$\frac{dy}{dt} = H(t)y,$$

is bounded. Here $H(t)$ is given by (6).

Remark 1. As an application of Theorem 1.2, we consider the solution ψ_i to (1) with $V = 0$. Then we have, for the initial data satisfying $\tilde{r}_j(0) > 0$ or $\mathcal{H}(0) < N$,

$$|\psi_i(x, t)| \leq Ct^{-d/2},$$

which is derived from the boundedness of Y and decay estimate $|\phi_i(x, t)| \leq Ct^{-d/2}$.

Consider linear differential equation

$$\frac{dy}{dt} = A(t)y(t),$$

where $A(t)$ is $N \times N$ matrix. In the general case there is no closed form solution y and an approximation method such as Magnus expansion [3, 4, 12] has been used. In section 5, we calculate some explicit principal fundamental matrices for special cases.

The rest of this paper is organized as follows. In section 2, we briefly review the basic properties of the Schrödinger Lohe model. In section 3, we prove Theorem 1.1 and present basic properties of the principal fundamental matrix $Y(t)$. In section 4, Theorem 1.2 is proved. In section 5, we study cases where the principal fundamental matrices can be derived explicitly.

2. Preliminaries. Let $V(x)$ be a given smooth real-valued potential function satisfying

$$\sum_{k=0}^m \|\nabla^k V\|_{L^\infty(\mathbb{R}^d)} \leq C_m < \infty \quad \text{for a positive integer } m. \quad (7)$$

Global existence of classical solution to the equation (1) satisfying the condition (7) was proved in [1, 7].

Theorem 2.1. *Let initial data $\psi_i^0 \in H^m(\mathbb{R}^d)$ for $m = 1, 2, \dots$. Then, for any $T \in (0, \infty)$, the Cauchy problem for (1) has a unique global solution ψ_i such that*

$$\psi_i \in C([0, \infty), H^m(\mathbb{R}^d)) \cap C^1([0, \infty), H^{m-2}(\mathbb{R}^d)).$$

By Sobolev embedding theorem, it is easy to see that for $m > 2 + \frac{d}{2}$, H^m solution is a classical solution. We consider initial data $\psi_i^0 \in H^m(\mathbb{R}^d)$ satisfying $\|\psi_i^0\|_{L^2(\mathbb{R}^d)} = 1$. Then the L^2 norm of ψ_i is constant along the evolution:

$$\|\psi_i(t)\|_{L^2(\mathbb{R}^d)} = \|\psi_i^0\|_{L^2(\mathbb{R}^d)} = 1.$$

This can be seen as follows. Multiplying (1) by $\bar{\psi}_i$, taking the imaginary part and integrating by parts, we obtain

$$\frac{d}{dt} (\|\psi_i\|_{L^2(\mathbb{R}^d)}^2 - 1) + \frac{K}{N} \left(\sum_{k=1}^N 2\operatorname{Re}\langle \psi_i, \psi_k \rangle \right) (\|\psi_i\|_{L^2(\mathbb{R}^d)}^2 - 1) = 0,$$

which is an ordinary differential equation for $\|\psi_i\|_{L^2}^2 - 1$. Since we have $\|\psi_i^0\|_{L^2}^2 - 1 = 0$, the uniqueness of solution to ODE says that $\|\psi_i(t)\|_{L^2}^2 = 1$.

Considering, for two wave functions ψ_i and ψ_j with unit L^2 norms, that

$$\|\psi_i(t) - \psi_j(t)\|_{L^2} = d_{ij} \iff \operatorname{Re}\langle \psi_i(t), \psi_j(t) \rangle = 1 - \frac{d_{ij}^2}{2}, \quad (8)$$

it is important to study the evolution of h_{ij} for the behaviors of d_{ij} . The dynamics of h_{ij} obeys the following ODEs which was proved in [7].

Proposition 1. *Let ψ_i be a solution to (1). Then h_{ij} satisfies the coupled system of ODEs:*

$$\frac{dh_{ij}}{dt} = \frac{K}{N} (1 - h_{ij}) \left(2 + \sum_{k \neq i} h_{ik} + \sum_{k \neq j} h_{kj} \right).$$

Taking (2) into account, we have

$$h_{ij} = \bar{h}_{ji}, \quad h_{ii} = \|\psi_i\|^2 = 1, \quad |h_{ij}| = \left| \int_{\mathbb{R}^d} \psi_i \bar{\psi}_j dx \right| \leq \|\psi_i\|_{L^2} \|\psi_j\|_{L^2} = 1.$$

The initial data should be restricted in the following admissible set

$$\mathcal{A}_N = \{(h_{ij})_{1 \leq i < j \leq N} \mid h_{ij} = \int_{\mathbb{R}^d} \psi_i \bar{\psi}_j dx \text{ with } \|\psi_i\|_{L^2} = 1 = \|\psi_j\|_{L^2}\}.$$

We can check that $(1, 1, 1) \in \mathcal{A}_3$ and $(-1, -1, -1) \notin \mathcal{A}_3$.

3. Proof of Theorem 1.1. In this section, we prove Theorem 1.1 and present basic properties of principal fundamental matrix Y .

Proof. The system (1) can be rewritten as

$$i\partial_t \psi + \Delta \psi - V\psi = iH\psi,$$

where $N \times N$ matrix H is given by (6). Note that H is a matrix of variable t . Let Y be the principal fundamental matrix solution of the ODE system

$$\frac{dy}{dt} = H(t)y. \quad (9)$$

Denoting $\psi(x, t) = Y(t)\phi(x, t)$, we have

$$\begin{aligned} 0 &= i\partial_t \psi + \Delta \psi - V\psi - iH\psi \\ &= Y(i\partial_t \phi + \Delta \phi - V\phi). \end{aligned}$$

Since the principal fundamental matrix Y is non-singular and $Y(0)$ is an identity matrix, we have

$$\begin{aligned} i\partial_t \phi + \Delta \phi - V\phi &= 0, \\ \phi(x, 0) &= Y^{-1}(0)\psi(x, 0) = \psi^0(x). \end{aligned}$$

This leads to the proof of Theorem 1.1. \square

Remark 2. We can derive the ODEs (9) in another way. Let us consider

$$\begin{aligned} i\partial_t u + \Delta u &= Vu, \\ u(x, 0) &= u_0(x), \end{aligned} \quad (10)$$

where u is a vector. Multiplying (1) by \bar{u} and (10) by $\bar{\psi}_i$, we have

$$(i\partial_t \psi_i + \Delta \psi_i) \bar{u} = V \psi_i \bar{u} + \frac{iK}{N} \sum_{k=1}^N (\psi_k \bar{u} - \langle \psi_i, \psi_k \rangle \psi_i \bar{u}), \quad (11)$$

$$(i\partial_t u + \Delta u) \bar{\psi}_i = Vu \bar{\psi}_i. \quad (12)$$

Then $\int_{\mathbb{R}^d} ((11) - \overline{(12)}) dx$ leads to

$$\int_{\mathbb{R}^d} i\partial_t (\psi_i \bar{u}) + \bar{u} \Delta \psi_i - \psi_i \Delta \bar{u} dx = \frac{iK}{N} \sum_{k=1}^N \int_{\mathbb{R}^d} (\psi_k \bar{u} - \langle \psi_i, \psi_k \rangle \psi_i \bar{u}) dx. \quad (13)$$

Integrating by parts gives us

$$\int_{\mathbb{R}^d} \bar{u} \Delta \psi_i - \psi_i \Delta \bar{u} dx = \int_{\mathbb{R}^d} \nabla \bar{u} \cdot \nabla \psi_i - \nabla \psi_i \cdot \nabla \bar{u} dx = 0.$$

Denoting $y_j = \int_{\mathbb{R}^d} \psi_j \bar{u} dx$ and considering $y_i - h_{ii} y_i = 0$, (13) leads us to

$$\frac{d}{dt} y_i = \frac{K}{N} \sum_{k=1}^N (y_k - h_{ik} y_i) = \frac{K}{N} \sum_{k \neq i} (y_k - h_{ik} y_i),$$

which is (9). Then y_i is a column vector of Y .

In the remaining part of the section, we present some properties of the principal fundamental matrix Y . Let us denote the elements of Y as $Y = (y_{ij})$. Then we have

$$\psi_i = \sum_{k=1}^N y_{ik} \phi_k. \quad (14)$$

• By the definition of h_{ij} , we have

$$\begin{aligned} h_{ij} &= \int_{\mathbb{R}^d} \psi_i \bar{\psi}_j dx = \int_{\mathbb{R}^d} \sum_{k=1}^N y_{ik} \phi_k \cdot \sum_{l=1}^N \overline{y_{jl} \phi_l} dx \\ &= \sum_{k=1}^N \sum_{l=1}^N y_{ik} \bar{y}_{jl} h_{kl}^0. \end{aligned} \quad (15)$$

Here we used the following fact

$$\int_{\mathbb{R}^d} (\phi_k \bar{\phi}_l)(x, t) dx = \int_{\mathbb{R}^d} (\phi_k \bar{\phi}_l)(x, 0) dx = \int_{\mathbb{R}^d} \psi_k^0 \bar{\psi}_l^0 dx = h_{kl}^0,$$

which can be derived from the following identity

$$i \frac{d}{dt} \int_{\mathbb{R}^d} \phi_k \bar{\phi}_l dx + \int_{\mathbb{R}^d} \bar{\phi}_l \Delta \phi_k - \phi_k \Delta \bar{\phi}_l dx = 0.$$

Integrating by parts, we have $\frac{d}{dt} \int_{\mathbb{R}^d} \phi_k \bar{\phi}_l dx = 0$. When $i = j$ in (15), we have

$$1 = \|\psi_i(t)\|_{L^2(\mathbb{R}^d)}^2 = \sum_{k=1}^N \sum_{j=1}^N y_{ik} \bar{y}_{ij} h_{kj}^0,$$

where we considered the conservation of L^2 norm of ψ . Higher order norm of ψ can be determined by the similar process. In fact, considering (14), we have $\nabla\psi_i = \sum_{k=1}^N y_{ik} \nabla\phi_k$ which implies

$$\|\nabla^\alpha\psi_i(t)\|_{L^2(\mathbb{R}^d)}^2 = \sum_{k=1}^N \sum_{j=1}^N y_{ik}(t)y_{ij}(t) \int_{\mathbb{R}^d} \nabla^\alpha\psi_k^0(x)\nabla^\alpha\bar{\psi}_j^0(x) dx,$$

where $\alpha = (\alpha_1, \dots, \alpha_d)$ is a multi-index.

• By Liouville’s Theorem, we have

$$\det(Y)(t) = \exp\left(\int_0^t \text{tr}(H(s)) ds\right).$$

4. Proof of Theorem 1.2. Here we show a boundedness of principal fundamental matrix Y for the case of complete synchronization. We first summarize the recent results in [1, 8] which show the emergence of the complete synchronization.

Theorem 4.1. [1] *Suppose that the initial data satisfy $\tilde{r}_j^0 > 0$ for $j = 1, \dots, N$. Then, for any solution ψ_i to (1), we have*

$$|1 - \tilde{r}_j(t)|^2 + |\tilde{s}_j(t)|^2 \leq Ce^{-Kt} \quad \text{as } t \rightarrow \infty.$$

Theorem 4.2. [8] *Suppose that initial data satisfy $\mathcal{H}(0) < N$. Then the solution h_{ij} to (3) converges to 1 exponentially fast for all i and j .*

Next we introduce Theorem 1 in Chapter 2 in [2] which concerns the boundedness of solutions of equations with almost constant coefficients.

Theorem 4.3. *Let A be a constant $n \times n$ matrix. If all solutions of $\frac{dy}{dt} = Ay$ are bounded as $t \rightarrow \infty$, then the same is true for the solution of*

$$\frac{dz}{dt} = (A + B(t))z,$$

provided that $\int_0^\infty \|B(t)\| dt < \infty$. Here $B(t)$ is $n \times n$ matrix with elements b_{ij} and the matrix norm is defined by $\|B\| = \sum_{i,j=1}^n |b_{ij}|$.

Now we are ready to prove Theorem 1.2. With the notations

$$y_j^1 := \text{Re}(y_j), \quad y_j^2 := \text{Im}(y_j), \quad r_{ij} := 1 - \text{Re}(h_{ij}), \quad s_{ij} := \text{Im}(h_{ij}),$$

we can rewrite $\frac{dy}{dt} = H(t)y$ as

$$\frac{d\tilde{y}}{dt} = \frac{K}{N} (A + B(t))\tilde{y}.$$

Here we denote $\tilde{y} = (y_1^1, y_1^2, \dots, y_N^1, y_N^2)^t$ and $2N \times 2N$ matrices

$$A = \begin{pmatrix} (1-N)E & E & \cdots & E \\ E & (1-N)E & \cdots & E \\ \vdots & \vdots & \ddots & \vdots \\ E & \cdots & (1-N)E & E \\ E & \cdots & E & (1-N)E \end{pmatrix}$$

and

$$B(t) = \begin{pmatrix} B_1 & O & \cdots & O \\ O & B_2 & \cdots & O \\ \vdots & \vdots & \vdots & \vdots \\ O & \cdots & B_{N-1} & O \\ O & \cdots & O & B_N \end{pmatrix},$$

with 2×2 blocks

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B_i = \begin{pmatrix} \sum_{k \neq i} r_{ik} & \sum_{k \neq i} s_{ik} \\ -\sum_{k \neq i} s_{ik} & \sum_{k \neq i} r_{ik} \end{pmatrix}.$$

When the initial data satisfy $\tilde{r}_j(0) > 0$ or $\mathcal{H}(0) < N$, Theorem 4.1 or 4.2 tells us that r_{ij} and s_{ij} decay exponentially to zero which implies that $\int_0^\infty \|B(t)\| dt < \infty$. To show the boundedness of solution to $\frac{d\tilde{y}}{dt} = \frac{K}{N}A\tilde{y}$, we check that the characteristic polynomial of A is $\lambda^2(\lambda + N)^{2N-2}$. In fact, we have

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ \cdot \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ \cdot \\ 0 \\ 1 \end{pmatrix} \quad \text{are eigenvectors corresponding to eigenvalue 0,}$$

and $2N - 2$ eigenvectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ -1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \cdot \\ \cdot \\ 0 \\ -1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

correspond to eigenvalues $-N$. Since we have non-positive eigenvalues and two independent eigenvectors corresponding to eigenvalue 0, all solutions of $\frac{d\tilde{y}}{dt} = \frac{K}{N}A\tilde{y}$ are bounded for $t \geq 0$. Note that if we have only one independent eigenvector corresponding to eigenvalue 0, then t factor appears from which boundedness of the solution cannot be induced. Applying Theorem 4.3, we can prove Theorem 1.2.

5. Explicit solutions. In this section, we study the case where the principal fundamental matrix can be solved explicitly. Consider linear differential equation

$$y'(t) = A(t)y(t). \tag{16}$$

If $A(t)$ commutes with $\int_0^t A(s)ds$, then the solution to (16) is given by

$$y(t) = e^{\int_0^t A(s)ds} y_0,$$

where y_0 is the initial value. But in the general case there is no closed form solution, and an approximation method such as Magnus expansion [3, 4, 12] may have to be used.

For the system (9) we can check, with the notation $S_j = \sum_{k \neq j} h_{jk}$,

$$\begin{aligned} & H(t) \int_0^t H(s) ds \\ &= \frac{K^2}{N^2} \begin{pmatrix} S_1 \int_0^t S_1 ds + (N-1)t & \cdots & -tS_1 - \int_0^t S_N ds + (N-2)t \\ \vdots & & \vdots \\ -tS_N - \int_0^t S_1 ds + (N-2)t & \cdots & S_N \int_0^t S_N ds + (N-1)t \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} & \int_0^t H(s) ds H(t) \\ &= \frac{K^2}{N^2} \begin{pmatrix} S_1 \int_0^t S_1 ds + (N-1)t & \cdots & -\int_0^t S_1 ds - tS_N + (N-2)t \\ \vdots & & \vdots \\ -\int_0^t S_N ds - tS_1 + (N-2)t & \cdots & S_N \int_0^t S_N ds + (N-1)t \end{pmatrix}. \end{aligned}$$

Therefore we have the principal fundamental matrix

$$Y(t) = e^{\int_0^t H(s) ds},$$

provided that

$$H(t) \int_0^t H(s) ds = \int_0^t H(s) ds H(t). \quad (17)$$

From the matrices presented above, we have

$$H(t) \int_0^t H(s) ds = \left(\int_0^t H(s) ds H(t) \right)^T,$$

where A^T stands for the transpose of a matrix A . To achieve the relation (17), the equality

$$tS_i + \int_0^t S_j ds = tS_j + \int_0^t S_i ds$$

has to be justified for all pairs (i, j) .

Recall that the solution ϕ_j to (4) with $V = 0$ is given by

$$\phi_j(x, t) = \frac{1}{(4\pi it)^{d/2}} \int_{\mathbb{R}^d} e^{\frac{i|x-y|^2}{4t}} \psi_j^0(y) dy.$$

Then Theorem 1.1 leads us to

$$\psi(x, t) = \frac{1}{(4\pi it)^{d/2}} \exp\left(\int_0^t H(s) ds\right) \int_{\mathbb{R}^d} e^{\frac{i|x-y|^2}{4t}} \psi^0(y) dy,$$

where $\psi^0 = (\psi_1^0, \dots, \psi_N^0)^t$. It would be interesting problem to study (4) with some other potentials like $V = |x|^2$.

5.1. A two-oscillator system. Here we consider a two-oscillator system. In this case, system (3) can be reduced to a single equation for $h := h_{12}$:

$$\frac{dh}{dt} = K(1 - h^2),$$

with the initial data $h(0) = h^0$. Then we have

$$h(t) = \frac{(1 + h^0)e^{2Kt} - (1 - h^0)}{(1 + h^0)e^{2Kt} + (1 - h^0)}.$$

Hence, all solutions with initial data $h^0 \neq -1$ will converge to 1 exponentially fast as $t \rightarrow \infty$. Considering

$$H = \frac{K}{2} \begin{pmatrix} -h & 1 \\ 1 & -\bar{h} \end{pmatrix},$$

we can check that (17) holds if

$$t h(t) + \int_0^t \bar{h}(s) ds = t \bar{h}(t) + \int_0^t h(s) ds. \quad (18)$$

For $h^0 \in \mathbb{R}$, we have $h(t) \in \mathbb{R}$ for which the equality (18) holds. Then we have

$$\begin{aligned} \int_0^t H(s) ds &= A + B \\ &:= \begin{pmatrix} -\frac{K}{2} \int_0^t h(s) ds & 0 \\ 0 & -\frac{K}{2} \int_0^t h(s) ds \end{pmatrix} + \begin{pmatrix} 0 & \frac{K}{2} t \\ \frac{K}{2} t & 0 \end{pmatrix}. \end{aligned}$$

We can check $AB = BA$ which implies $e^{A+B} = e^A e^B$. Then the principal fundamental matrix reads as

$$Y(t) = e^{\int_0^t H(s) ds} = e^A e^B.$$

The direct calculation shows

$$e^A = e^{-\frac{K}{2} \int_0^t h(s) ds} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $-\int_0^t h(s) ds = t + \frac{1}{K} \log \frac{2}{(1-h^0)+(1+h^0)e^{2Kt}}$ and

$$e^B = \frac{e^{\frac{K}{2}t} + e^{-\frac{K}{2}t}}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{e^{\frac{K}{2}t} - e^{-\frac{K}{2}t}}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Therefore we arrive at

$$Y(t) = \frac{1}{2} e^{\frac{1}{2} \log \frac{2}{(1-h^0)+(1+h^0)e^{2Kt}}} \left[(e^{Kt} + 1) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (e^{Kt} - 1) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right].$$

Note that $e^{\frac{1}{2} \log \frac{2}{(1-h^0)+(1+h^0)e^{2Kt}}} \cdot e^{Kt} \rightarrow \frac{\sqrt{2}}{\sqrt{1+h^0}}$ as $t \rightarrow \infty$ for $h^0 \neq -1$.

5.2. A three-oscillator system. Here we consider a three-oscillator system. Considering $h_{ij} = \overline{h_{ji}}$, system (3) becomes the following 3×3 system:

$$\begin{aligned} \frac{dh_{12}}{dt} &= \frac{K}{3} (2 + 2h_{12} + \overline{h_{23}} + \overline{h_{31}}) (1 - h_{12}), \\ \frac{dh_{23}}{dt} &= \frac{K}{3} (2 + 2h_{23} + \overline{h_{31}} + \overline{h_{12}}) (1 - h_{23}), \\ \frac{dh_{31}}{dt} &= \frac{K}{3} (2 + 2h_{31} + \overline{h_{12}} + \overline{h_{23}}) (1 - h_{31}). \end{aligned} \quad (19)$$

Let us consider the case of $h_{12} = h_{23} = h_{31}$. Actually, if $h_{12}(0) = h_{23}(0) = h_{31}(0)$ holds initially, the relation is preserved in time, i.e., $h_{12}(t) = h_{23}(t) = h_{31}(t)$ for $t > 0$. Then the system (19) reduces to a single equation:

$$\frac{dh_{12}}{dt} = \frac{K}{3} (2 + 2h_{12} + 2\overline{h_{12}}) (1 - h_{12}). \quad (20)$$

We can check that (17) is satisfied for this case. Actually we have, with the notation of $h = h_{12}$,

$$H(t) \int_0^t H(s) ds = \begin{pmatrix} A & B & B \\ B & A & B \\ B & B & A \end{pmatrix} = \int_0^t H(s) ds H(t),$$

where $A = (\frac{K}{3})^2 \left((h + \bar{h}) \int_0^t (h + \bar{h}) ds + 2t \right)$, $B = (\frac{K}{3})^2 \left(-t(h + \bar{h}) - \int_0^t (h + \bar{h}) ds + t \right)$.

Let us write

$$\begin{aligned} \int_0^t H(s) ds &= G(t)I + \frac{K}{3}tD \\ &= G(t) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{K}{3}t \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \end{aligned}$$

where $G(t) = -\frac{K}{3} \int_0^t (h + \bar{h})(s) ds$. Then the principal fundamental matrix reads as $e^{G(t)I} e^{\frac{Kt}{3}D}$. First of all, we have

$$e^{G(t)I} = \begin{pmatrix} e^{G(t)} & 0 & 0 \\ 0 & e^{G(t)} & 0 \\ 0 & 0 & e^{G(t)} \end{pmatrix}.$$

Taking into account

$$D = C \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} C^{-1},$$

where

$$C = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} \quad \text{and} \quad C^{-1} = \frac{1}{3} \begin{pmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{pmatrix},$$

we have

$$\exp\left(\frac{K}{3}tD\right) = C \begin{pmatrix} e^{-\frac{K}{3}t} & 0 & 0 \\ 0 & e^{-\frac{K}{3}t} & 0 \\ 0 & 0 & e^{\frac{2K}{3}t} \end{pmatrix} C^{-1}.$$

Let us find more precise form of $h + \bar{h}$ in G . With the notation $h_{12} = f + ig$, (20) is equivalent to the following two-dimensional system:

$$\frac{df}{dt} = \frac{2K}{3}(1+2f)(1-f), \quad \frac{dg}{dt} = -\frac{2K}{3}(1+2f)g. \quad (21)$$

Then the solution to the first equation in (21) is given by

$$f(t) = \frac{(1+2f^0)e^{2Kt} + f^0 - 1}{(1+2f^0)e^{2Kt} - 2(f^0 - 1)},$$

where $f^0 = f(0)$. Considering $h + \bar{h} = 2f$, the principal fundamental matrix is given by

$$e^{\int_0^t H(s)ds} = \begin{pmatrix} e^{G(t)} & 0 & 0 \\ 0 & e^{G(t)} & 0 \\ 0 & 0 & e^{G(t)} \end{pmatrix} C \begin{pmatrix} e^{-\frac{K}{3}t} & 0 & 0 \\ 0 & e^{-\frac{K}{3}t} & 0 \\ 0 & 0 & e^{\frac{2K}{3}t} \end{pmatrix} C^{-1},$$

where $G(t) = -\frac{K}{3} \int_0^t 2f(s)ds$.

REFERENCES

- [1] P. Antonelli and P. Marcati, [A model of synchronization over quantum networks](#), *J. Phys. A*, **50** (2017), 315101, 19 pp.
- [2] R. Bellman, *Stability Theory of Differential Equations*, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953.
- [3] S. Blanes, F. Casas, J. A. Oteo and J. Ros, [Magnus and Fer expansions for matrix differential equations: The convergence problem](#), *J. Phys. A*, **31** (1998), 259–268.
- [4] S. Blanes, F. Casas, J. A. Oteo and J. Ros, [The Magnus expansion and some of its applications](#), *Phys. Rep.*, **470** (2009), 151–238.
- [5] S.-H. Choi, J. Cho and S.-Y. Ha, [Practical quantum synchronization for the Schrödinger-Lohe system](#), *J. Phys. A*, **49** (2016), 205203, 17 pp.
- [6] S.-H. Choi and S.-Y. Ha, [Quantum synchronization of the Schrödinger-Lohe model](#), *J. Phys. A*, **47** (2014), 355104, 16 pp.
- [7] H. Huh and S.-Y. Ha, [Dynamical system approach to synchronization of the coupled Schrödinger-Lohe system](#), *Quart. Appl. Math.*, **75** (2017), 555–579.
- [8] H. Huh, S.-Y. Ha and D. Kim, [Emergent behaviors of the Schrödinger-Lohe model on cooperative-competitive networks](#), *J. Differential Equations*, **263** (2017), 8295–8321.
- [9] H. Huh, S.-Y. Ha and D. Kim, [Asymptotic behavior and stability for the Schrödinger-Lohe model](#), *J. Math. Phys.*, **59** (2018), 102701, 21 pp.
- [10] M. A. Lohe, [Quantum synchronization over quantum networks](#), *J. Phys. A*, **43** (2010), 465301, 20 pp.
- [11] M. A. Lohe, [Non-Abelian Kuramoto model and synchronization](#), *J. Phys. A*, **42** (2009), 395101, 25 pp.
- [12] W. Magnus, [On the exponential solution of differential equations for a linear operator](#), *Comm. Pure Appl. Math.*, **7** (1954), 649–673.

Received October 2018; revised July 2019.

E-mail address: huh@cau.ac.kr