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STATIONARY SOLUTIONS AND ASYMPTOTIC BEHAVIOUR FOR A CHEMOTAXIS HYPERBOLIC MODEL ON A NETWORK

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Abstract. This paper approaches the question of existence and uniqueness of stationary solutions to a semilinear hyperbolic-parabolic system and the study of the asymptotic behaviour of global solutions. The system is a model for some biological phenomena evolving on a network composed by a finite number of nodes and oriented arcs. The transmission conditions for the unknowns, set at each inner node, are crucial features of the model.

1. Introduction.

In this paper we consider a semilinear hyperbolic-parabolic system evolving on a finite planar network composed from nodes connected by m oriented arcs I_i,

$$\begin{cases} \partial_t u_i + \lambda_i \partial_x v_i = 0 \ , \\ \partial_t v_i + \lambda_i \partial_x u_i = u_i \partial_x \phi_i - \beta_i v_i \ , & t \geq 0 \, , x \in I_i \, , \ i = 1, ..., m; \\ \partial_t \phi_i = D_i \partial_{xx} \phi_i + a_i u_i - b_i \phi_i \ , \end{cases} \tag{1.1}$$

the system is complemented by initial, boundary and transmission conditions at the nodes (see Section 2).

We are interested in the study of stationary solutions and asymptotic behaviour of global solutions of the problem.

The above system has been proposed as a model for chemosensitive movements of bacteria or cells on an artificial scaffold [12]. The unknown u stands for the cells concentration, λv is the average flux and ϕ is the chemo-attractant concentration. In particular, the model turns out to be useful to describe the process of dermal wound healing, when the stem cells in charge of the reparation of dermal tissue (fibroblasts) create an extracellular matrix and move along it to fill the wound, driven by chemotaxis; tissue engineers use artificial scaffolds, constituted by a network of crossed polymeric threads, inserting them within the wound to accelerate the process (see [13, 20, 25]). In the above mathematical model, the arcs of the graph mimic the fibers of the scaffold; each of them is characterized by a tipical velocity λ_i, a friction coefficient β_i, a diffusion coefficient D_i, and a production rate

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a_i and a degradation one b_i; the functions u_i, ϕ_i are the densities of fibroblasts and chemoattractant on each arc.

Starting from the Keller-Segel paper [18] in 1970 until now, a lot of articles have been devoted to PDE models in domains of $\mathbb{R}^n$ for chemotaxis phenomena. The parabolic (or parabolic-elliptic) Patlak-Keller-Segel system is the most studied model [17, 23, 22]; in recent years, hyperbolic models have been introduced too, in order to avoid the unrealistic infinite speed of propagation of cells, occurring in parabolic models [8, 9, 14, 23, 24, 1, 15, 16, 11].

In [11] the Cauchy and the Neumann problems for the system in (1.1), respectively in $\mathbb{R}$ and in bounded intervals of $\mathbb{R}$, are studied, providing existence of global solutions and stability of constant states results.

Recently an interest in these mathematical models evolving on networks is arising, due to their applications in the study of biological phenomena and traffic flows, both in parabolic cases [2, 6, 21] and in hyperbolic ones [10, 7, 26, 12, 3].

We notice that the transmission conditions for the unknowns, at each inner node, which complement the equations on networks, are crucial characteristics of the model, since they are the coupling among the solution's components on each arc.

Most of the studies carried out until now, consider continuity conditions at each inner node for the density functions [7, 6, 21]; nevertheless, the eventuality of discontinuities at the nodes seems a more appropriate framework to decribe movements of individuals or traffic flows phenomena [5].

For these reason in [12], transmission conditions which link the values of the density functions at the nodes with the fluxes, without imposing any continuity, are introduced; these conditions guarantee the fluxes conservation at each inner node, and, at the same time, the m-dissipativity of the linear spatial differential operators, a crucial property in the proofs of existence of local and global solutions contained in that paper.

In this paper we focus our attention on stationary solutions to problem (1.1) complemented by null fluxes boundary conditions and by the same transmission conditions of [12] (see next section and Section 3 in [12] for details). We consider acyclic networks and we prove the existence and uniqueness of the stationary solution with fixed mass of cells $\displaystyle\sum_{i=1}^{m}\int_{I_i} u_i(x)dx$, under the assumption that the mass is suitably small. If the quantity $\frac{a_i}{b_i}$ does not vary with the index i, we easily show that such solution is a constant state on the whole network. We notice that, in the case of acyclic networks, although the transmission conditions do not set the continuity of the density u at the inner nodes, the fluxes conservation at those nodes and the boundary null fluxes conditions imply the absence of jumps discontinuities at the inner vertexes, for the component u of a stationary solution.

For general networks and the parameters a_i and b_i in the same range as above, it is easy to show that, for any fixed mass, a stationary solution constant on the whole network exists and the constant values of the densities are determined by the mass. In this case we also obtain a uniqueness result: in the set of stationary solutions with small density u in H^1-norm and fixed mass of cells, the constant state on the network is the unique element.

Finally we study the large time behaviour of global solutions on general networks, when the ratio between a_i and b_i is constant. We consider initial data with fixed small mass, which are small perturbations of the constant state on the network with the same mass, then we prove that such state is the asymptotic profile of the solutions corresponding to the data. So, we point out that, for small global solutions to our problem, the discontinuities at the inner nodes vanish when t goes to infinity, since their asymptotic profiles are continuous functions on the whole network.

The study of the asymptotic behaviour provide informations about the evolution of a small mass of individuals moving on a network driven by chemotaxis: suitable initial distributions of individuals and chemoattractant, for large time evolve towards constant distributions on the network, preserving the mass of individuals.

We recall that the stability of the constant solutions to this system, considered on bounded interval in $\mathbb{R}$, is studied in [11] and stationary solutions and asymptotic behaviour for a linear system of uncoupled conservation laws on network are studied in [19].

Finally, in [3] the authors introduce a numerical scheme to approximate the solutions to the problem (2.1); in that paper transmission conditions are set for the Riemann invariants of the hyperbolic part of the system, $w_i^{\pm} = \frac{1}{2}(u_i \pm v_i)$, and are equivalent to our ones for some choices of the transmission coefficients. The tests presented there, in the case of acyclic graph and dissipative transmission coefficients, show an asymptotic behaviour of the solutions which agrees with our theoretical results.

The paper is organized as follows. In Section 2 we give the statement of the problem and, in particular, we introduce the transmission conditions, while in Section 3 we prove the results about existence and uniqueness of stationary solutions. The last section is devoted to study the asymptotic behaviour of solutions; the results obtained in this section constitute the sequel and the development of the result of existence of global solutions in [12] and the proofs are based on the same techniques and use simple modifications of the a priori estimates obtained in [12].

2. **Statement of the evolution problem.**

We consider a finite connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ composed by a set $\mathcal{V}$ of n nodes (or vertexes) and a set $\mathcal{A}$ of m oriented arcs, $\mathcal{A} = \{I_i : i \in \mathcal{M} = \{1, 2, ..., m\}\}$.

Each node is a point of the plane and each oriented arc I_i is an oriented segment joining two nodes.

We use e_j, $j \in \mathcal{J}$, to indicate the external vertexes (or boundary vertexes) of the graph, i.e. the vertexes belonging to only one arc, and by $I_{i(j)}$ the external arc outgoing or incoming in the external vertex e_j.

Moreover, we use N_ν, $\nu \in \mathcal{P}$, to denote the inner nodes; for each of them we consider the set of incoming arcs $\mathcal{A}_{in}^{\nu} = \{I_i : i \in \mathcal{I}^\nu\}$ and the set of the outgoing ones $\mathcal{A}_{out}^{\nu} = \{I_i : i \in \mathcal{O}^\nu\}$; let $\mathcal{M}^\nu = \mathcal{I}^\nu \cup \mathcal{O}^\nu$.

In this paper, a *path* in the graph is a sequence of arcs, two by two adjacent, without taking into account orientations . Moreover, we call *acyclic* a graph which does not contain cycles: for each couple of nodes, there exists a unique path with no repeated arcs connecting them (an example of acyclic graph is in Fig. 1).

Each arc I_i is considered as a one dimensional interval $(0, L_i)$. A function f defined on $\mathcal{A}$ is a m-tuple of functions f_i, $i \in \mathcal{M}$, each one defined on I_i; $f_i(N_\nu)$ denotes $f_i(0)$ if N_ν is the initial point of the arc I_i and $f_i(L_i)$ if N_ν is the end point, and similarly for $f(e_j)$.

We set $L^p(\mathcal{A}) := \{f : f_i \in L^p(I_i), i \in \mathcal{M}\}$, $H^s(\mathcal{A}) := \{f : f_i \in H^s(I_i), i \in \mathcal{M}\}$ and

$$\|f\|_2 := \sum_{i \in \mathcal{M}} \|f_i\|_2 \ , \ \|f\|_{H^s} := \sum_{i \in \mathcal{M}} \|f_i\|_{H^s}.$$

We consider the evolution of the following one-dimensional problem on the graph $\mathcal{G}$

$$\begin{cases} \partial_t u_i + \lambda_i \partial_x v_i = 0 \ , \\ \\ \partial_t v_i + \lambda_i \partial_x u_i = u_i \partial_x \phi_i - \beta_i v_i \ , \qquad t \geq 0, \ x \in I_i, \ i \in \mathcal{M}, \\ \\ \partial_t \phi_i = D_i \partial_{xx} \phi_i + a_i u_i - b_i \phi_i \ , \end{cases} \tag{2.1}$$

where $a_i \geq 0$, λ_i $b_i, D_i, \beta_i > 0$.

We complement the system with the initial conditions

$$u_{i0}, v_{i0} \in H^1(I_i) \ , \ \phi_{i0} \in H^2(I_i) \ \text{ for } i \in \mathcal{M} \ ; \tag{2.2}$$

the boundary conditions at each outer point e_j are the null flux conditions

$$v_{i(j)}(e_j, t) = 0 \ , \qquad t > 0 \ , \ j \in \mathcal{J} \ , \tag{2.3}$$

$$\phi_{i(j)_x}(e_j, t) = 0 \qquad t > 0 \ , \ j \in \mathcal{J}. \tag{2.4}$$

In addition, at each inner node N_ν we impose the following transmission conditions for the unknown ϕ

$$\begin{cases} D_i \phi_{i_x}(N_\nu, t) = \displaystyle\sum_{j \in \mathcal{M}^\nu} \alpha_{ij}^\nu (\phi_j(N_\nu, t) - \phi_i(N_\nu, t)) \ , \quad i \in \mathcal{I}^\nu \ , \ t > 0 \ , \\ \\ -D_i \phi_{i_x}(N_\nu, t) = \displaystyle\sum_{j \in \mathcal{M}^\nu} \alpha_{ij}^\nu (\phi_j(N_\nu, t) - \phi_i(N_\nu, t)) \ , \ i \in \mathcal{O}^\nu \ , \ t > 0 \ , \\ \\ \alpha_{ij}^\nu \geq 0 \ , \ \alpha_{ij}^\nu = \alpha_{ji}^\nu \ \text{ for all } i, j \in \mathcal{M}^\nu \ , \end{cases} \tag{2.5}$$

which imply the continuity of the flux at each node, for all $t > 0$,

$$\sum_{i \in \mathcal{I}^\nu} D_i \phi_{i_x}(N_\nu, t) = \sum_{i \in \mathcal{O}^\nu} D_i \phi_{i_x}(N_\nu, t) \ .$$

For the unknonws v and u we impose the transmission conditions

$$\begin{cases} -\lambda_i v_i(N_\nu, t) = \displaystyle\sum_{j \in \mathcal{M}^\nu} K_{ij}^\nu \left(u_j(N_\nu, t) - u_i(N_\nu, t) \right) \ , \ i \in \mathcal{I}^\nu \ , \ t > 0 \ , \\ \\ \lambda_i v_i(N_\nu, t) = \displaystyle\sum_{j \in \mathcal{M}^\nu} K_{ij}^\nu \left(u_j(N_\nu, t) - u_i(N_\nu, t) \right) \ , \quad i \in \mathcal{O}^\nu \ , \ t > 0 \ , \\ \\ K_{ij}^\nu \geq 0 \ , \ K_{ij}^\nu = K_{ji}^\nu \ \text{ for all } i, j \in \mathcal{M}^\nu \ . \end{cases} \tag{2.6}$$

These conditions ensure the conservation of the flux of the density of cells at each node N_ν , for $t > 0$,

$$\sum_{i \in \mathcal{I}^\nu} \lambda_i v_i(N_\nu, t) = \sum_{i \in \mathcal{O}^\nu} \lambda_i v_i(N_\nu, t) \ ,$$

which corresponds to the conservation of the total mass

$$\sum_{i \in \mathcal{M}} \int_{I_i} u_i(x, t) \ dx = \sum_{i \in \mathcal{M}} \int_{I_i} u_{0i}(x) \ dx \ ,$$

i.e. no death nor birth of individuals occours during the observation.

Motivations for the constraints on the coefficients in the transmission conditions can be found in [12] .

Finally, we impose the following compatibility conditions

$$u_{i0}, v_{i0}, \phi_{i0} \text{ satisfy conditions (2.3)-(2.6) for all } i \in \mathcal{M} \ . \tag{2.7}$$

Existence and uniqueness of local solutions to problem (2.1)-(2.7),

$$u, v \in C([0,T]; H^1(\mathcal{A})) \cap C^1([0,T]; L^2(\mathcal{A})), \phi \in C([0,T]; H^2(\mathcal{A})) \cap C^1([0,T]; L^2(\mathcal{A}))$$

are achieved in [12] by means of the linear contraction semigroup theory coupled with the abstract theory of nonhomogeneous and semilinear evolution problems; in fact, the transmission conditions (2.5) and (2.6) allows us to prove that the linear differential operators in (2.1) are m-dissipative and then, to apply the Hille-Yosida-Phillips Theorem (see [4]) . The existence of global solutions when the initial data are small in $(H^1(\mathcal{A}))^2 \times H^2(\mathcal{A})$ norm is proved [12] too; this result holds under the further assumption

$$\text{for all } \nu \in \mathcal{P}, \text{ for some } k \in \mathcal{M}^\nu, K^\nu_{ik} \neq 0 \text{ for all } i \in \mathcal{M}^\nu, i \neq k \ . \tag{2.8}$$

3. **Non negative small stationary solutions on acyclic networks.** In this section we approach the question of existence and uniqueness of stationary solutions of problem (2.1)-(2.8), with fixed mass

$$\mu := \sum_{i \in \mathcal{M}} \int_{I_i} u_i(x) dx \geq 0 \ ,$$

in the case of an acyclic network (see Section 2). We look for stationary solutions $(u, v, \phi) \in (H^1(\mathcal{A}))^2 \times H^2(\mathcal{A})$.

Obviously, the flux v of a stationary solution has to be constant on each arc and has to be null on the external arcs; in the case of acyclic networks, the boundary and transmission conditions (2.3), (2.6) force it to be null on each arc. In order to prove this fact we consider an internal arc I_j and its initial node N_μ; we consider the set

$$\mathcal{Q} = \{\nu \in \mathcal{P} : N_\nu \text{ is linked to } N_\mu \text{ by a path not covering } I_j\}$$

(see Fig.1: if, for example, $j = 9$ then $\mu = 4$, $\mathcal{Q} = \{1, 2, 3, 5\}$ and the arcs in bold type form the path which links the nodes N_5 and N_4).

At each node the conservation of the flux of the density of cells, stated in Section 2, holds; then

$$\sum_{\nu \in \mathcal{Q} \cup \{\mu\}} \left(\sum_{i \in I^\nu} \lambda_i v_i(N_\nu) - \sum_{i \in O^\nu} \lambda_i v_i(N_\nu) \right) = 0 \ .$$

Since, for all $i \in \mathcal{M}$, $v_i(x)$ is constant on I_i and $v_i(x) = 0$ if I_i is an external arc, the above equality reduces to

$$v_j(N_\mu) = 0 \ ;$$

then $v_j(x) = 0$ for all $x \in I_j$.

The previous result implies that stationary solutions must have the form $(u, 0, \phi)$, where u and ϕ have to verify the system

$$\begin{cases} \lambda_i u_{i_x} = u_i \phi_{i_x}, \\ \\ -D_i \phi_{i_{xx}} + b_i \phi_i = a_i u_i, \end{cases} \quad x \in I_i, \ i \in \mathcal{M}, \ t > 0, \tag{3.1}$$

with the boundary condition at each outer point e_j, $j \in \mathcal{J}$,

$$\phi_{i(j)_x}(e_j, t) = 0 \qquad t > 0 \ , \tag{3.2}$$

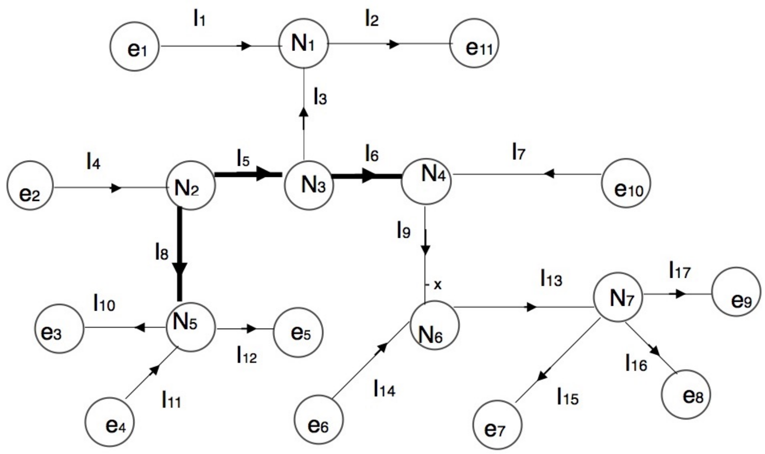

FIGURE 1. Example of acyclic network; the highlighted arcs form the path linking the nodes N_4 and N_5.

and the transmission conditions, at each inner node N_ν,

$$\sum_{j\in\mathcal{M}^\nu} K_{ij}^\nu (u_j(N_\nu) - u_i(N_\nu)) = 0, \qquad i \in \mathcal{M}^\nu \ , \tag{3.3}$$

$$\begin{aligned} D_i \phi_{ix}(N_\nu) &= \sum_{j\in\mathcal{M}^\nu} \alpha_{ij}^\nu (\phi_j(N_\nu) - \phi_i(N_\nu)), \qquad && i \in \mathcal{I}^\nu, \\ D_i \phi_{ix}(N_\nu) &= -\sum_{j\in\mathcal{M}^\nu} \alpha_{ij}^\nu (\phi_j(N_\nu) - \phi_i(N_\nu)), \qquad && i \in \mathcal{O}^\nu. \end{aligned} \tag{3.4}$$

For each fixed inner node N_ν, let $k \in \mathcal{M}^\nu$ be the index in condition (2.8) and let consider the transmission relations, for $i \in \mathcal{M}^\nu$, $i \neq k$,

$$\begin{aligned} 0 &= \sum_{j\in\mathcal{M}^\nu, j\neq i} K_{ij}^\nu (u_j(N_\nu) - u_i(N_\nu)) \\ &= \sum_{j\in\mathcal{M}^\nu, j\neq i,k} K_{ij}^\nu (u_j(N_\nu) - u_k(N_\nu)) - \left(\sum_{j\in\mathcal{M}^\nu, j\neq i} K_{ij}^\nu \right) (u_i(N_\nu) - u_k(N_\nu)) \ ; \end{aligned} \tag{3.5}$$

the assumptions on K_{kj}^ν in (2.8) ensure that the matrix of the coefficients of this linear system in the unknowns $(u_j(N_\nu) - u_k(N_\nu))$, $j \neq k$, is non singular (if $k = 1$ it is immediate to check that it has strictly dominant diagonal). Then the condition (3.3) can be rewritten as

$$u_j(N_\nu) = u_k(N_\nu) \qquad \text{for all } j \in \mathcal{M}^\nu.$$

Now we fix $\mu_0 \geq 0$ and we look for stationary solutions such that

$$\sum_{i\in\mathcal{M}} \int_{I_i} u_i(x)dx = \mu_0 \ ; \tag{3.6}$$

notice that for the evolution problem, the quantity $\displaystyle\sum_{i\in\mathcal{M}}\int_{I_i} u_i(x,t)dx$ is preserved for all $t \geq 0$, thanks to the transmission conditions (2.6).

Integrating the first equation in (3.1) we can rewrite problem (3.1)-(3.6) as the following elliptic problem on network:

Find $C = (C_1, C_2, ..., C_m)$ *and* $\phi \in H^2(\mathcal{A})$ *such that*

$$\begin{cases} -D_i\phi_{i\,xx} + b_i\phi_i = a_i u_i & x \in I_i\ , \quad i \in \mathcal{M}\ , \\ u_i(x) = C_i \exp\left(\frac{\phi_i(x)}{\lambda_i}\right) & x \in I_i\ , \quad i \in \mathcal{M}\ , \\ \phi_{i(j)\,x}(e_j) = 0\ , & j \in \mathcal{J}, \\ D_i\phi_{i\,x}(N_\nu) = \displaystyle\sum_{j\in\mathcal{M}^\nu} \alpha_{ij}^\nu(\phi_j(N_\nu) - \phi_i(N_\nu))\ , & i \in \mathcal{I}^\nu, \quad \nu \in \mathcal{P}\ , \\ D_i\phi_{i\,x}(N_\nu) = -\displaystyle\sum_{j\in\mathcal{M}^\nu} \alpha_{ij}^\nu(\phi_j(N_\nu) - \phi_i(N_\nu))\ , & i \in \mathcal{O}^\nu, \quad \nu \in \mathcal{P}\ , \\ C_j \exp\left(\frac{\phi_j(N_\nu)}{\lambda_j}\right) = C_i \exp\left(\frac{\phi_i(N_\nu)}{\lambda_i}\right)\ , & i, j \in \mathcal{M}^\nu, \quad \nu \in \mathcal{P}\ , \\ \displaystyle\sum_{i\in\mathcal{M}} C_i \int_{I_i} \exp\left(\frac{\phi_i(x)}{\lambda_i}\right) dx = \mu_0\ . \end{cases} \tag{3.7}$$

We consider the linear operator $A : D(A) \to L^2(\mathcal{A})$,

$$\begin{aligned} D(A) &= \left\{\phi \in H^2(\mathcal{A}) : (3.2), (3.4) \text{ hold }\right\}\ , \\ A(\phi) &= \{-D_i\phi_{i\,xx} + b_i\phi_i\}_{i\in\mathcal{M}}\ ; \end{aligned} \tag{3.8}$$

then the equation in (3.7) and the boundary and transmission conditions for ϕ can be written as

$$A\phi = F(\phi, C), \tag{3.9}$$

where, for $i \in \mathcal{M}$, $F_i(\phi(x), C) = a_i C_i \exp\left(\frac{\phi_i(x)}{\lambda_i}\right)$.

We are going to prove the existence and uniqueness of solutions to the problem (3.7) by using the Banach Fixed Point Theorem; in order to do this we need some preliminary results about the linear equation

$$A\phi = F(f, C^f), \tag{3.10}$$

where $f \in H^2(\mathcal{A})$ is a given function, $C^f = (C_1^f, ..., C_m^f)$ and C_i^f are non-negative given real constants.

The existence and uniqueness of the solution $\phi \in H^2(\mathcal{A})$ to the above problem (for a general $F \in L^2(\mathcal{A})$ and a general network) is showed in [12], by Lax-Milgram theorem, in the proof of Proposition 4.1; here, we need to prove some properties holding for the solution in the case of acyclic graphs, under suitable assumptions on f and C_i^f.

The transmission conditions (2.5) imply the following equality which will be useful in the next proofs:

$$
\begin{aligned}
&\sum_{i\in\mathcal{M}}\int_{I_i} D_i\left(\phi_i(x)\phi_{ix}(x)\right)_x\,dx\\
&=\sum_{\nu\in\mathcal{P}}\left(\sum_{i\in I^\nu} D_i\phi_i(N_\nu)\phi_{ix}(N_\nu)-\sum_{i\in O^\nu} D_i\phi_i(N_\nu))\phi_{ix}(N_\nu)\right)\\
&=\sum_{\nu\in\mathcal{P}}\sum_{ij\in\mathcal{M}^\nu}\alpha^\nu_{ij}\phi_i(N_\nu)(\phi_j(N_\nu)-\phi_i(N_\nu))\\
&=-\frac{1}{2}\sum_{\nu\in\mathcal{P}}\sum_{ij\in\mathcal{M}^\nu}\alpha^\nu_{ij}(\phi_j(N_\nu)-\phi_i(N_\nu))^2\ .
\end{aligned}
\tag{3.11}
$$

Let $|\mathcal{A}|:=\sum_{i\in\mathcal{M}}|I_i|$ and $\|g\|_\infty:=\max\{\|g_i\|_\infty, i\in\mathcal{M}\}$.

Lemma 3.1. *Let $\mathcal{G}=(\mathcal{V},\mathcal{A})$ be an acyclic network, let $f\in H^2(\mathcal{A})$ and let C_i^f be non-negative real numbers, for $i\in\mathcal{M}$. Then the solution ϕ to problem (3.8),(3.10) is non-negative. Moreover, if*

$$\sum_{i\in\mathcal{M}} C_i^f\int_{I_i}\exp\left(\frac{f_i(x)}{\lambda_i}\right)dx=\mu_0\ , \tag{3.12}$$

then

$$\|\phi_x\|_\infty\le\frac{2\max\{a_i\}_{i\in\mathcal{M}}}{\min\{D_i\}_{i\in\mathcal{M}}}\mu_0; \tag{3.13}$$

if (3.12) holds and

$$\|f_x\|_\infty\le\frac{2\max\{a_i\}_{i\in\mathcal{M}}}{\min\{D_i\}_{i\in\mathcal{M}}}\mu_0\ , \tag{3.14}$$

then there exists a quantity $K_{\mu_0}=K_{\mu_0}(a_i,b_i,D_i,\lambda_i,|\mathcal{A}|,\mu_0)$, depending only on the parameters appearing in brackets, infinitesimal when μ_0 goes to zero, such that

$$\|\phi\|_{W^{2,1}(\mathcal{A})},\|\phi\|_{H^2(\mathcal{A})}\le K_{\mu_0}\ . \tag{3.15}$$

Proof. Let consider a function $\Gamma\in C^1(\mathbb{R})$, strictly increasing in $(0,+\infty)$, and let $\Gamma(y)=0$ for $y\le 0$; following standard methods for the proofs of the maximum principle for elliptic equations and setting $F_i(x)=F_i(f(x),C^f)$, we obtain

$$
\begin{aligned}
&\sum_{i\in\mathcal{M}}\int_{I_i}\left(-D_i(\phi_{i_x}(x)\Gamma(-\phi_i(x)))_x-D_i\Gamma'(-\phi_i(x))\phi_{i_x}^2(x)\right.\\
&\left.+b_i\phi_i(x)\Gamma(-\phi_i(x))-F_i(x)\Gamma(-\phi_i(x))\right)\,dx=0\ .
\end{aligned}
$$

As regard to the first term, we can argue as in (3.11), taking into account the properties of Γ,

$$
\begin{aligned}
&\sum_{i\in\mathcal{M}}\int_{I_i} D_i\left(\Gamma(-\phi_i)\phi_{ix}\right)_x\\
&=-\frac{1}{2}\sum_{\nu\in\mathcal{P}}\sum_{ij\in\mathcal{M}^\nu}\alpha^\nu_{ij}(\phi_j(N_\nu)-\phi_i(N_\nu))(\Gamma(-\phi_j(N_\nu))-\Gamma(-\phi_i(N_\nu)))\ge 0\ ;
\end{aligned}
\tag{3.16}
$$

the above inequality and the non-negativity of F_i imply that

$$\sum_{i\in\mathcal{M}} b_i \int_{I_i} \phi_i(x)\Gamma(-\phi_i(x))dx \geq 0 \ ,$$

so that, thanks to the properties of Γ , we can conclude that $\phi_i(x) \geq 0$ for all $i \in \mathcal{M}$.

By integration of the equation (3.10), taking into account (3.4) and (3.2), we obtain

$$\sum_{i\in\mathcal{M}} b_i \int_{I_i} \phi_i(x)dx = \sum_{i\in\mathcal{M}} \int_{I_i} F_i(f(x), C^f)dx \tag{3.17}$$

which implies

$$\|\phi\|_1 \leq \frac{\max\{a_i\}}{\min\{b_i\}}\mu_0 \ . \tag{3.18}$$

In order to obtain (3.13), first we notice that, if I_j is an external arc, then the following inequality holds

$$|D_j{\phi_j}_x(x)| \leq \int_{I_j} D_j|{\phi_j}_{yy}(y)| \ dy \leq \int_{I_j} \left(b_j\phi_j(y) + C_j^f a_j \exp\left(\frac{f_j(y)}{\lambda_j}\right)\right) dy \ .$$

Then we consider an internal arc I_j and its initial node N_μ and the sets

$$\mathcal{Q} = \{\nu \in \mathcal{P} : N_\nu \text{ is linked to } N_\mu \text{ by a path not covering } I_j\} \ ,$$

$$\mathcal{S} = \{i \in \mathcal{M} : I_i \text{ is incident with } N_l \text{ for some } l \in \mathcal{Q}\}$$

(see Fig.1: if, for example, $j = 9$, then $\mu = 4$, $\mathcal{Q} = \{1, 2, 3, 5\}$, $\mathcal{S} = \{1, 2, 3, 4, 5, 6, 8, 10, 11, 12\}$); at each node the conservation of the flux, stated in Section 2 as a consequence of the transmission conditions, holds; then

$$\sum_{\nu\in\mathcal{Q}\cup\{\mu\}} \left(\sum_{i\in I^\nu} D_i{\phi_i}_x(N_\nu) - \sum_{i\in O^\nu} D_i{\phi_i}_x(N_\nu)\right) = 0 \ .$$

Let x be a point on the arc I_j (see Fig.1 for $j = 9$, $\mu = 4$) and I_j^x be the part of I_j which connects N_μ and x; then, using the above equality and the boundary conditions (2.4), we have

$$\begin{aligned}
|D_j{\phi_j}_x(x)| &= \left| D_j{\phi_j}_x(x) + \sum_{\nu\in\mathcal{Q}\cup\{\mu\}} \left(\sum_{i\in I^\nu} D_i{\phi_i}_x(N_\nu) - \sum_{i\in O^\nu} D_i{\phi_i}_x(N_\nu)\right)\right| \\
&= \left|\sum_{i\in\mathcal{S}} \int_{I_i} D_i{\phi_i}_{yy}(y)dy + \int_{I_j^x} D_j{\phi_j}_{yy}(y)dy\right| \\
&= \left|\sum_{i\in\mathcal{S}} \int_{I_i} \left(b_i\phi_i(y) - C_i^f a_i \exp\left(\frac{f_i(y)}{\lambda_i}\right)\right) dy \right. \\
&\quad \left. + \int_{I_j^x} \left(b_j\phi_j(y) - C_j^f a_j \exp\left(\frac{f_j(y)}{\lambda_j}\right)\right) dy\right| .
\end{aligned} \tag{3.19}$$

Then $D_j\|\phi_{j_x}\|_\infty \leq 2\mu_0 \max\{a_i\}_{i\in\mathcal{M}}$ for all $j \in \mathcal{M}$ and we obtain (3.13) which implies

$$\|\phi_x\|_1 \leq \frac{2\max\{a_i\}_{i\in\mathcal{M}}}{\min\{D_i\}_{i\in\mathcal{M}}}|\mathcal{A}|\mu_0 \, , \tag{3.20}$$

and

$$\|\phi_x\|_2 \le \frac{2\max\{a_i\}_{i\in\mathcal{M}}}{\min\{D_i\}_{i\in\mathcal{M}}}|\mathcal{A}|^{\frac{1}{2}}\mu_0 \ ; \tag{3.21}$$

moreover, by Sobolev embedding theorem, we obtain

$$\|\phi\|_\infty \le K_1\mu_0 \ , \tag{3.22}$$

where $K_1 = K_1(a_i, b_i, D_i, |\mathcal{A}|)$ is a suitable constant.

The estimates for the function ϕ_{xx} follow by using the equation (3.10); first, using (3.12) and (3.17), we obtain

$$\|\phi_{xx}\|_1 \le \frac{2\max\{a_i\}_{i\in\mathcal{M}}}{\min\{D_i\}_{i\in\mathcal{M}}}\mu_0 \ ; \tag{3.23}$$

then, using (3.11), we have

$$\sum_{i\in\mathcal{M}} \frac{D_i^2}{b_i}\int_{I_i} {\phi_i}_{xx}^2(x)\ dx \le \sum_{i\in\mathcal{M}} \frac{\|F_i\|_\infty}{b_i}\int_{I_i} F_i(x)\ dx \le \frac{\max\{a_i\|F_i\|_\infty\}}{\min\{b_i\}}\mu_0$$

and the embedding of $W^{1,1}(I_i)$ in $L^\infty(I_i)$ gives

$$\sum_{i\in\mathcal{M}} \int_{I_i} {\phi_i}_{xx}^2(x)\ dx \le K_2(1+\|f_x\|_\infty)\mu_0^2 \ , \tag{3.24}$$

where $K_2 = K_2(a_i, b_i, D_i, \lambda_i)$ is a suitable constant.

Finally, the inequalities (3.18), (3.20)-(3.24) imply the inequalities (3.15) in the claim. □

Now we can prove the following theorem.

Theorem 3.1. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ be an acyclic network. There exists $\epsilon > 0$ such that, if $0 \le \mu_0 \le \epsilon$, then problem (2.1)-(2.8) has a unique stationary solution satisfying (3.6); the solution has the form*

$$\left(C_i \exp\left(\frac{\phi_i(x)}{\lambda_i}\right), 0, \phi_i(x)\right) \quad i\in\mathcal{M},$$

where $\phi_i(x) \ge 0$ and C_i are nonnegative constants such that $u_j(N_\nu) = u_i(N_\nu)$ for all $\nu \in \mathcal{P}$, $i, j \in \mathcal{M}^\nu$.

Proof. First we notice that, if a stationary solution (u, v, ϕ) satisfying (3.6) exists, then u is non-negative, since the costants C_i in (3.7) must have the same sign, so that they have to be non-negative to satisfy the condition $\mu_0 \ge 0$; arguing as in the proof of Lemma 3.1 we prove that ϕ is non-negative too. If $\mu_0 > 0$ then u and ϕ are positive functions.

We are going to use a fixed point technique. Given $\phi^0 \in D(A)$, we want to define a function $u^0(x)$ on the network, such that, for $i \in \mathcal{M}$,

$$u_i^0(x) = C_i^{\phi^0} \exp\left(\frac{\phi_i^0(x)}{\lambda_i}\right) ,$$

where the constants $C_i^{\phi^0}$ satisfy the following linear system composed by the last conditions in (3.7)

$$C_j^{\phi^0} \exp\left(\frac{\phi_j^0(N_\nu)}{\lambda_j}\right) = C_i^{\phi^0} \exp\left(\frac{{\phi_i}^0(N_\nu)}{\lambda_i}\right) \ , \qquad i,j\in\mathcal{M}^\nu, \quad \nu\in\mathcal{P} \ , \tag{3.25}$$

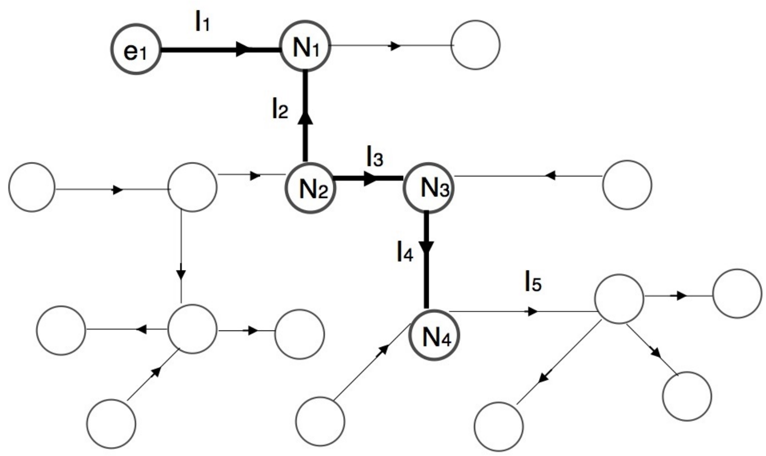

FIGURE 2. Example: the highlighted arcs form the path from the outer point e_1 to the inner node N_4 and I_5 is an arc incident with N_4, not belonghing to the path.

$$\sum_{i\in\mathcal{M}} C_i^{\phi^0} \int_{I_i} \exp\left(\frac{{\phi_i}^0(x)}{\lambda_i}\right) dx = \mu_0 \ . \tag{3.26}$$

The system (3.25),(3.26) has a unique solution; actually, since the network has no cycles, the system (3.25) has ∞^1 solutions $C^{\phi^0,\alpha} = (\alpha, \alpha\delta_2, \alpha\delta_3, ..., \alpha\delta_m)$, $\alpha \in \mathbb{R}$, where δ_i are suitable coefficients, and the condition (3.26) determines the value of α.

In order to give an explicit expression for the coefficients δ_i we consider an arc, I_1, and we define

$$u_1^0(x) := \alpha \exp\left(\frac{\phi_1^0(x)}{\lambda_1}\right) \ .$$

Let N_μ one of the extreme points of I_1, then we define the function u^0 on the other arcs which are incident with N_μ in such a way to verify the equalities in (3.25) for the node N_μ,

$$u_j^0(x) := \alpha \exp\left(\frac{\phi_1^0(N_\mu)}{\lambda_1}\right) \exp\left(-\frac{\phi_j^0(N_\mu)}{\lambda_j}\right) \exp\left(\frac{\phi_j^0(x)}{\lambda_j}\right) \quad \text{for all } j \in \mathcal{M}^\mu \, , j \neq 1;$$

i.e. we set $C_j^{\phi^0} = \alpha \exp\left(\frac{\phi_1^0(N_\mu)}{\lambda_1}\right) \exp\left(-\frac{\phi_j^0(N_\mu)}{\lambda_j}\right)$, $j \in \mathcal{M}^\mu \, , j \neq 1$.

This procedure can be iterated at each node reached by one of the arcs I_j, $j \in \mathcal{M}^\mu$, and at the other extreme point of I_1, if it is an internal arc, and so on, defining in this way the function u^0 on each arc of the network. Notice that this construction is possible since there are no cycles in the graph. The function u can be expressed, on each arc of the network, as follows (if it is the case, renumbering in suitable way the arcs and the nodes): let consider the path from the outer point e_1 to an inner node N_{h-1}, composed from the arcs I_i, $i = 1, ..., h-1$, (passing through the vertexes N_i, $i = 1, ..., h-1$), and let I_h be an arc incident with the node N_{h-1}, not belonging to the path (see Fig.2 where h=5 and the highlighted arcs form the path); following the procedure described before, after setting

$$\mathcal{E}_h(\phi^0) := \frac{\Pi_{i=1,\ldots,h-1}\exp\left(\frac{\phi_i^0(N_i)}{\lambda_i}\right)}{\Pi_{i=1,\ldots,h-1}\exp\left(\frac{\phi_{i+1}^0(N_i)}{\lambda_{i+1}}\right)} \ ,$$

we define

$$u_h^0(x) := \alpha\mathcal{E}_h(\phi^0)\exp\left(\frac{\phi_h^0(x)}{\lambda_h}\right) \ .$$

The quantity α is fixed in such a way to verify the last condition in (3.7),

$$\alpha\sum_{i\in\mathcal{M}}\mathcal{E}_i(\phi^0)\int_{I_i}\exp\left(\frac{{\phi_i}^0(x)}{\lambda_i}\right)dx = \mu_0 \ ,$$

so that, for all $i\in\mathcal{M}$,

$$u_i^0(x) = C_i^{\phi^0}\exp\left(\frac{\phi_i^0(x)}{\lambda_i}\right) \ , \quad C_i^{\phi^0} := \mu_0\frac{\mathcal{E}_i(\phi^0)}{\displaystyle\sum_{j\in\mathcal{M}}\mathcal{E}_j(\phi^0)\int_{I_j}\exp\left(\frac{\phi_j^0(x)}{\lambda_j}\right)dx} . \tag{3.27}$$

Let G be the operator defined in $D(A)$ such that $\phi^1 := G(\phi^0)$ is the solution of problem (3.10) where $f=\phi_0$ and $C_i^f = C_i^{\phi^0}$ for $i\in\mathcal{M}$,

$$A\phi^1 = F(C^{\phi^0},\phi^0) \ ;$$

let $K_{\mu_0} = K_{\mu_0}(a_i,b_i,D_i,\lambda_i,|\mathcal{A}|,\mu_0)$ be the quantity in Lemma 3.1, and let

$$B_{\mu_0} := \{\phi\in D(A) : \phi\geq 0 \ , \|\phi_x\|_\infty \leq \frac{2\max\{a_i\}_{i\in\mathcal{M}}}{\min\{D_i\}_{i\in\mathcal{M}}}\mu_0 \ , \|\phi\|_{H^2}\leq K_{\mu_0} \ \}$$

equipped with the distance d generated by norm of $H^2(\mathcal{A})$; (B_{μ_0},d) is a complete metric space. From the lemma we know that solutions to problem (3.8)-(3.12) have to belong to B_{μ_0}, then $G(B_{\mu_0})\subseteq B_{\mu_0}$; next we are proving that, if μ_0 is small enough, then G is a contraction in B_{μ_0}.

We consider $\phi^0,\overline{\phi}^0\in B_{\mu_0}$ and the corresponding $u^0,\overline{u}^0$ and $\phi^1,\overline{\phi}^1$; using the equation satisfied by ϕ^0 and $\overline{\phi}^0$, for all $i\in\mathcal{M}$ we can write

$$\begin{aligned} b_i&\int_{I_i}(\phi_i^1(x)-\overline{\phi}_i^1(x))^2\,dx + D_i\int_{I_i}(\phi_{ix}^1(x)-\overline{\phi}_{ix}^1(x))^2\,dx \\ &- D_i\int_{I_i}\Big((\phi_{ix}^1(x)-\overline{\phi}_{ix}^1(x))(\phi_i^1(x)-\overline{\phi}_i^1(x))\Big)_x\,dx \\ =\,& a_i\int_{I_i}(u_i^0(x)-\overline{u}_i^0(x))(\phi_i^1(x)-\overline{\phi}_i^1(x))\ \,dx \ ; \end{aligned} \tag{3.28}$$

using (3.11), from (3.28) we infer that

$$\sum_{i\in\mathcal{M}}\|{\phi_i}^1-\overline{\phi_i}^1\|_{H^2}\leq K(a_i,b_i,D_i)\sum_{i\in\mathcal{M}}\|u_i^0-\overline{u}_i^0\|_2 \ , \tag{3.29}$$

We set

$$J_i^{\phi^0} := \int_{I_i}\exp\left(\frac{{\phi_i}^0(x)}{\lambda_i}\right)dx\,, \qquad E_i^{\phi^0}(x) := \exp\left(\frac{{\phi_i}^0(x)}{\lambda_i}\right) \quad ;$$

we have

$$|u_i^0(x) - \overline{u}_i^0(x)| = \mu_0 \left| \frac{\mathcal{E}_i(\phi^0) E_i^{\phi^0}(x)}{\sum_{j\in\mathcal{M}} \mathcal{E}_j(\phi^0) J_j^{\phi^0}} - \frac{\mathcal{E}_i(\overline{\phi}^0) E_i^{\overline{\phi}^0}(x)}{\sum_{j\in\mathcal{M}} \mathcal{E}_j(\overline{\phi}^0) J_j^{\overline{\phi}^0}} \right| . \tag{3.30}$$

In order to treat the above quantity we have to consider that, for all $g \in B_{\mu_0}$, $E_i^g(x) \geq 1$, $J_i^g \geq |I_i|$ and there exists a constant $K_6 = K_6(K_{\mu_0}, \lambda_i)$, increasing with μ_0, such that, for all $i \in \mathcal{M}$

$$\max_{I_i} E_i^g(x) \leq K_6 \ , \quad J_i^g \leq K_6 |I_i| \ ,$$

$$\left| E_i^{\phi^0}(x) - E_i^{\overline{\phi}^0}(x) \right| \leq K_6 |\phi_i^0(x) - \overline{\phi}_i^0(x)| \ ,$$

$$|J_i^{\phi^0} - J_i^{\overline{\phi}^0}| \leq K_6 \int_{I_i} |\phi_i^0(x) - \overline{\phi}_i^0(x)| \, dx \ .$$

The above inequalities can be used in (3.30) so that (3.29) implies

$$\sum_{i\in\mathcal{M}} \|{\phi_i}^1 - \overline{\phi_i}^1\|_{H^2} \leq \mu_0 K_7(a_i, b_i, D_i, K_{\mu_0}, |\mathcal{A}|) \sum_{i\in\mathcal{M}} \|\phi_i^0 - \overline{\phi}_i^0\|_{H^1} \ , \tag{3.31}$$

where K_7 increases with μ_0; hence, for μ_0 small enough, G is a contraction on B_{μ_0} and we can use the Banach Fixed Point Theorem.

Let ϕ be the unique fixed point of G in B_{μ_0} and let $C^\phi = (C_1^\phi, C_2^\phi, ..., C_m^\phi)$ where C_i^ϕ, for $i \in \mathcal{M}$, are computed as in (3.27); then (ϕ, C^ϕ) is the unique solution to Problem (3.7) and the claim is proved. □

For any constant $U \geq 0$, the triple $(U, 0, \frac{a_i}{b_i}U)$ satisfies the equations in (2.1) on the arc I_i. Let Q be a real non-negative number; if $\frac{a_i}{b_i} = Q$ for all $i \in \mathcal{M}$, then the same triple $(U, 0, QU)$ satisfies the equations on each arc I_i and it is a stationary solution to the problem (2.1)-(2.8). Then, as a consequence of the previous theorem, we have the following proposition, with ϵ as in the theorem.

Proposition 3.1. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ be an acyclic network. If $\frac{a_i}{b_i} = Q$ for all $i \in \mathcal{M}$ and $0 \leq \mu_0 \leq \epsilon$, then the unique stationary solution to problem (2.1)-(2.8), (3.6) is the constant solution* $\left(\frac{\mu_0}{|\mathcal{A}|}, 0, Q\frac{\mu_0}{|\mathcal{A}|} \right)$.

Remark 3.1. For general networks, when the value of $\frac{a_i}{b_i} = Q$ on each arc, the stationary solution of Proposition 3.1 always exists. More precisely, if $\frac{a_i}{b_i} = Q$, in the class of the functions (u, v, ϕ) which are constant on each arc, the stationary solution $\left(\frac{\mu_0}{|\mathcal{A}|}, 0, Q\frac{\mu_0}{|\mathcal{A}|} \right)$ is the unique stationary solution with mass μ_0; this fact is true without any restrictions on the value of μ_0 and on the structure of the network. Actually, if we assume that u is constant on each arc, then, using the equations, we infer that, on each arc, $\phi_x(x)$ is constant too, hence $\phi_{xx} = 0$ and $\phi(x)$ is constant. Then $v(x) = 0$ on each arc; hence, arguing as at the beginning of this section, we obtain that u is continuous on the network.

In the next proposition we are going to prove that, in a set of *small* solutions, such stationary solution is the unique one with fixed mass μ_0.

Proposition 3.2. *Let $\frac{a_i}{b_i} = Q$ for all $i \in \mathcal{M}$ and let $(u, v, \phi) \in (H^1(\mathcal{A}))^2 \times H^2(\mathcal{A})$ be a stationary solution of problem (2.1)-(2.8),(3.6). There exists $\epsilon_0 > 0$, depending on $\lambda_i, a_i, b_i, D_i, \beta_i, |\mathcal{A}|$, such that, if $\|u\|_{H^1} \le \epsilon_0$, then $(u, v, \phi) = \left(\frac{\mu_0}{|\mathcal{A}|}, 0, Q\frac{\mu_0}{|\mathcal{A}|}\right)$.*

Proof. We set $H := \|u\|_{H^1}$. The transmission conditions (2.6) imply that

$$\sum_{\nu\in\mathcal{P}} \left(\sum_{i\in\mathcal{I}^\nu} \lambda_i u_i(N^\nu) v_i(N^\nu) - \sum_{i\in\mathcal{O}^\nu} \lambda_i u_i(N^\nu) v_i(N^\nu) \right) \ge 0 \ ,$$

so, by using the first two equations in (2.1), we obtain

$$2 \sum_{i\in\mathcal{M}} \beta_i \int_{I_i} v_i^2(x)dx \le \sum_{i\in\mathcal{M}} \|u_i\|_\infty \int_{I_i} (v_i^2(x) + \phi_{i_x}^2(x))\ dx$$

and

$$\sum_{i\in\mathcal{M}} \lambda_i \int_{I_i} u_{i_x}^2(x)\ dx \le \sum_{i\in\mathcal{M}} \|u_i\|_\infty \int_{I_i} (u_{i_x}^2(x) + \phi_{i_x}^2(x))\ dx\ + \sum_{i\in\mathcal{M}} \frac{\beta_i^2}{\lambda_i} \int_{I_i} v_i^2(x)\ dx\ ;$$

the above inequalities implies the following one

$$\|v\|_2^2 + \|u_x\|_2^2 \le K_0 H \left(\|\phi_x\|_2^2 + \|v\|_2^2 + \|u_x\|_2^2\right)\ , \tag{3.32}$$

where K_0 is a positive constant depending on the parameters λ_i, β_i and the Sobolev embedding costant.

The transmission conditions (2.5) imply that

$$-\sum_{\nu\in\mathcal{P}} \left(\sum_{i\in\mathcal{I}^\nu} D_i \phi_i(N^\nu) \phi_{i_x}(N^\nu) - \sum_{i\in\mathcal{O}^\nu} D_i \phi_i(N^\nu) \phi_{i_x}(N^\nu) \right) \ge 0\ ;$$

moreover, the assumption (2.8) imply that, for each $\nu \in \mathcal{P}$, for suitable coefficients θ_{ij}^ν and suitable $k \in \mathcal{M}^\nu$,

$$u_j(N_\nu) = u_k(N_\nu) + \sum_{i\in\mathcal{M}^\nu, i\ne k} \theta_{ij}^\nu v_i(N_\nu) \qquad \text{for all } j \in \mathcal{M}^\nu\ ,$$

(see Lemma 5.9 in [12]); then, by the last equation in (2.1), arguing as in the proof of Proposition 5.8 in [12], we obtain

$$\|\phi_x\|_2^2 + \|\phi_{xx}\|_2^2 \le K_1 \left(\|v\|_2^2 + \|u_x\|_2^2\right)\ , \tag{3.33}$$

where K_1 is a positive constant depending on the parameters $D_i, a_i, b_i, \theta_{ij}^\nu$.

By inequalities (3.32) and (3.33) we deduce the following one

$$\|v\|_2^2 + \|u_x\|_2^2 \le K_0(1 + K_1) H \left(\|v\|_2^2 + \|u_x\|_2^2\right)\ ,$$

which, for small H, implies $\|v\|_2, \|u_x\|_2 = 0$. □

In the cases when $\frac{a_i}{b_i}$ depends on the arc in consideration, stationary solutions with the component u constant on each arc, can be inadmissible. As we showed before, v should be zero, u should be constant on the whole network and ϕ should be constant on each arc,

$$u_i(x) = \frac{\mu_0}{|\mathcal{A}|}\ , \quad \phi_i(x) = \frac{a_i}{b_i} \frac{\mu_0}{|\mathcal{A}|}\ , \qquad i \in \mathcal{M}\ .$$

Therefore the transmission conditions, for each $\nu \in \mathcal{P}$,

$$\sum_{j\in\mathcal{M}^\nu} \alpha_{ij}^\nu \frac{\mu_0}{|\mathcal{A}|} \left(\frac{a_j}{b_j} - \frac{a_i}{b_i}\right) = 0\ , \qquad i \in \mathcal{M}^\nu\ ,$$

are constraints on the relations between the parameters of the problem which have to hold if the constant stationary solution exists.

For example, in the case of two arcs, if $\frac{b_2}{a_2} \neq \frac{b_1}{a_1}$ (and $0 < \mu_0 \leq \epsilon$), the stationary solution can not be constant on the arcs , since the trasmission condition at the node,

$$\alpha_{11} \frac{\mu_0}{|\mathcal{A}|} \left(\frac{b_2}{a_2} - \frac{b_1}{a_1} \right) = 0 \ ,$$

cannot be satisfied.

Hence, in the cases when $\frac{a_i}{b_i}$ depends on the arc in consideration, if (u, v, ϕ) is the stationary solution in Theorem 3.1, then u is a continuous function on all the network but it is not constant on each arc.

4. **Asymptotic behaviour.** In this section we are going to show that the constant stationary solutions previously introduced, provide the asymptotic profiles for a class of solutions to problem (2.1)-(2.8). We recall that existence and uniqueness of global solutions

$$\begin{aligned} &u, v \in C\left([0,+\infty); H^1(\mathcal{A})\right) \cap C^1\left([0,+\infty); L^2(\mathcal{A})\right), \\ &\phi \in C\left([0,+\infty); H^2(\mathcal{A})\right) \cap C^1\left([0,+\infty); L^2(\mathcal{A})\right), \phi_x \in H^1\left(\mathcal{A} \times (0,+\infty)\right), \end{aligned} \tag{4.1}$$

to such problem is proved in [12], when the initial data are sufficiently small in $(H^1(\mathcal{A}))^2 \times H^2(\mathcal{A})$ norm and the following condition holds

$$\frac{a_i}{b_i} = Q \quad \text{for all } i \in \mathcal{M} \ ; \tag{4.2}$$

in particular it is proved that the functional F defined by

$$\begin{aligned} F_T^2(u,v,\phi) := \sum_{i \in \mathcal{M}} &\left(\sup_{t\in[0,T]} \|u_i(t)\|_{H^1}^2 + \sup_{t\in[0,T]} \|v_i(t)\|_{H^1}^2 + \sup_{t\in[0,T]} \|\phi_{ix}(t)\|_{H^1}^2 \right) \\ &+ \int_0^T \left(\|u_x(t)\|_2^2 + \|v(t)\|_{H^1}^2 + \|v_t(t)\|_2^2 + \|\phi_x(t)\|_{H^1}^2 + \|\phi_{xt}(t)\|_2^2 \right) \, dt \end{aligned} \tag{4.3}$$

is uniformly bounded for $T > 0$.

Here and below we use the notations

$$\|f_i(t)\|_2 := \|f_i(\cdot,t)\|_{L^2(I_i)}, \quad \|f_i(t)\|_{H^s} := \|f_i(\cdot,t)\|_{H^s(I_i)} \ .$$

Now we assume (4.2), we fix $\overline{\mu} \geq 0$ and we consider the constant stationary solution, $(\overline{u}, 0, \overline{\phi})$, to problem (2.1)-(2.8), such that $\overline{u}|\mathcal{A}| = \overline{\mu}$; moreover let $(\tilde{u}_0, \tilde{v}_0, \tilde{\phi}_0) \in (H^1(\mathcal{A}))^2 \times H^2(\mathcal{A})$ be a small perturbation of $(\overline{u}, 0, \overline{\phi})$, i.e., setting $u_0 := \tilde{u}_0 - \overline{u}, v_0 := \tilde{v}_0, \phi_0 := \tilde{\phi}_0 - \overline{\phi}$, the $(H^1(\mathcal{A}))^2 \times H^2(\mathcal{A})$ norm of (u_0, v_0, ϕ_0) is bounded by a suitable small $\epsilon_0 > 0$.

If $(\tilde{u}, \tilde{v}, \tilde{\phi})$ is the solution to problem (2.1)-(2.8) with initial data $(\tilde{u}_0, \tilde{v}_0, \tilde{\phi}_0)$ and $u := \tilde{u} - \overline{u}, v := \tilde{v}, \phi := \tilde{\phi} - \overline{\phi}$, then (u, v, ϕ) is solution to the system

$$\begin{cases} \partial_t u_i + \lambda_i \partial_x v_i = 0 \\ \partial_t v_i + \lambda_i \partial_x u_i = (u_i + \overline{u}) \partial_x \phi_i - \beta_i v_i & x \in I_i \, , t \geq 0 \, , i \in \mathcal{M}, \\ \partial_t \phi_i = D_i \partial_{xx} \phi_i + a_i u_i - b_i \phi_i \ , \end{cases} \tag{4.4}$$

complemented with the conditions (2.2)-(2.8) and initial data (u_0, v_0, ϕ_0) defined above.

The existence and uniqueness of local solutions to this problem can be achieved by means of semigroup theory, following the method used in[12], with little modifications.

On the other hand, if we assume that $\overline{u}$ is suitably small, the method used in that paper to obtain the global existence result in the case of small initial data can be used here too, modifying the estimates in order to treat the further term in the second equation and then using the smallness of $\overline{u}$.

Below we list a priori estimates holding for the solutions to problem (4.4), (2.2)-(2.7); we don't give the proofs since they are equal to those in [12], in Section 5, except for easy added calculations to treat the term $\overline{u}\phi_{i_x}$.

Proposition 4.1. *Let (u, v, ϕ) be a local solution to problem (4.4),(2.2)-(2.7),*

$$u, v \in C\left([0,T]; H^1(\mathcal{A})\right) \cap C^1\left([0,T]; L^2(\mathcal{A})\right)\ ,$$
$$\phi \in C\left([0,T]; H^2(\mathcal{A})\right) \cap C^1\left([0,T]; L^2(\mathcal{A})\right)\ ,\ \phi_x \in H^1\left(\mathcal{A}. \times (0,T)\right)\ ;$$

then

a)

$$\begin{aligned}
&\sum_{i\in\mathcal{M}}\left(\sup_{[0,T]}\|u_i(t)\|_2^2 + \sup_{[0,T]}\|v_i(t)\|_2^2 + \beta_i\int_0^T \|v_i(t)\|_2^2 dt\right)\\
\le\ & C\sum_{i\in\mathcal{M}}\left(\|u_{0i}\|_2^2 + \|v_{0i}\|_2^2\right)\\
&+C\sum_{i\in\mathcal{M}}\left(\sup_{[0,T]}\|u_i(t)\|_{H^1} + \overline{u}\right)\int_0^T\left(\|\phi_{i_x}(t)\|_2^2 + \|v_i(t)\|_2^2\right)dt\ ;
\end{aligned}$$

b)

$$\begin{aligned}
&\sum_{i\in\mathcal{M}}\left(\sup_{[0,T]}\|v_{i_x}(t)\|_2^2 + \sup_{[0,T]}\|v_{i_t}(t)\|_2^2 + \int_0^T \|v_{i_t}(t)\|_2^2\,dt\right)\\
\le\ & C\left(\|v_0\|_{H^1}^2 + \|u_0\|_{H^1}^2\|\phi_0\|_{H^2}^2\right)\\
&+C\sum_{i\in\mathcal{M}}\left(\sup_{[0,T]}\|u_i(t)\|_{H^1} + \overline{u}\right)\int_0^T\left(\phi_{i_{xt}}(t)\|_2^2 + \|v_{i_t}(t)\|_2^2\right)\ dt\\
&+C\sum_{i\in\mathcal{M}}\sup_{[0,T]}\|\phi_x(t)\|_{H^1}\int_0^T\left(\|v_{i_t}(t)\|_2^2 + \|v_i(t)\|_{H^1}^2\right)\ dt\ ;
\end{aligned}$$

c)

$$\begin{aligned}
&\sum_{i\in\mathcal{M}}\sup_{[0,T]}\|u_{i_x}(t)\|_2^2 \le C\sum_{i\in\mathcal{M}}\left(\sup_{[0,T]}\|v_{i_t}(t)\|_2^2 + \sup_{[0,T]}\|v_i(t)\|_2^2\right)\\
&+C\sum_{i\in\mathcal{M}}\left(\sup_{[0,T]}\|u_i(t)\|_{H^1} + \overline{u}\right)\left(\sup_{[0,T]}\|u_{i_x}(t)\|_2^2 + \sup_{[0,T]}\|\phi_{i_x}(t)\|_2^2\right)\ ;
\end{aligned}$$

d)

$$\sum_{i\in\mathcal{M}}\int_0^T \|u_{ix}(t)\|_2^2\,dt \le C\sum_{i\in\mathcal{M}}\int_0^T \left(\|v_{it}(t)\|_2^2+\|v_i(t)\|_2^2\right)\,dt$$

$$+C\sum_{i\in\mathcal{M}}\left(\sup_{[0,T]}\|u_i(t)\|_{H^1}+\overline{u}\right)\int_0^T \left(\|u_{ix}(t)\|_2^2+\|\phi_{ix}(t)\|_2^2\right)\,dt\ ;$$

e)

$$\sum_{i\in\mathcal{M}}\int_0^T \|v_{ix}(t)\|_2^2\,dt \le C\sum_{i\in\mathcal{M}}\left(\|v_{0i}\|_2^2+\|u_{0i}\|_{H^1}^2\left(1+\|\phi_{0i}\|_{H^1}^2\right)\right)$$

$$+C\sum_{i\in\mathcal{M}}\left(\int_0^T \|v_{it}(t)\|_2^2\,dt+\sup_{[0,T]}\|v_{it}(t)\|_2^2\right)$$

$$+C\sum_{i\in\mathcal{M}}\left(\sup_{[0,T]}\|u_i(t)\|_{H^1}+\sup_{[0,T]}\|\phi_{ix}(t)\|_{H^1}+\overline{u}\right)$$

$$\times\int_0^T \left(\|v_i(t)\|_{H^1}^2+\|\phi_{ixt}(t)\|_2^2\right)\,dt\ ;$$

f)

$$\sum_{i\in\mathcal{M}}\left(\sup_{[0,T]}\|\phi_{it}(t)\|_2^2+\int_0^T \left(\|\phi_{it}(t)\|_2^2+\|\phi_{itx}(t)\|_2^2\right)\,dt\right)$$

$$\le\quad C\sum_{i\in\mathcal{M}}\left(\|\phi_{0i}\|_{H^2}^2+\|u_{0i}\|_2^2+\int_0^T \|u_{it}(t)\|_2^2\right)\ ;$$

g)

$$\sum_{i\in\mathcal{M}}\left(\sup_{[0,T]}\|\phi_{ixx}(t)\|_2^2+\sup_{[0,T]}\|\phi_{ix}(t)\|_2^2\right)$$

$$\le\quad C\sum_{i\in\mathcal{M}}\left(\sup_{[0,T]}\|\phi_{it}(t)\|_2^2+\sup_{[0,T]}\|u_i(t)\|_2^2\right)\ ;$$

h) *if (2.8) and (4.2) hold, then*

$$\sum_{i\in\mathcal{M}}\int_0^T \left(\|\phi_{ix}(t)\|_2^2+\|\phi_{ixx}(t)\|_2^2\right)\,dt$$

$$\le\quad C\sum_{i\in\mathcal{M}}\int_0^T \left(\|u_{ix}(t)\|_2^2+\|v_i(t)\|_{H^1}^2+\|\phi_{it}(t)\|_2^2\right)\,dt\ ,$$

for suitable costants C.

The estimates in the previous proposition allow to prove the following theorem about the existence of global solutions to problem (4.4),(2.2)-(2.8).

Let $F_T(u,v,\phi)$ be the functional defined in (4.3) .

Theorem 4.1. *Let (4.2) hold. There exists $\epsilon_0, \epsilon_1 > 0$ such that, if*

$$\overline{u} \leq \epsilon_1 \ , \quad \|u_0\|_{H^1}, \|v_0\|_{H^1}, \|\phi_0\|_{H^2} \leq \epsilon_0 \ ,$$

then there exists a unique global solution (u,v,ϕ) to problem (4.4),(2.2)-(2.8),

$$u, v \in C([0,+\infty); H^1(\mathcal{A})) \cap C^1([0,+\infty); L^2(\mathcal{A})) \ ,$$

$$\phi \in C([0,+\infty); H^2(\mathcal{A})) \cap C^1([0,+\infty); L^2(\mathcal{A})) \ , \ \phi_x \in H^1(\mathcal{A} \times (0,+\infty)) \ .$$

Moreover, $F_T(u,v,\phi)$ is bounded, uniformly in T.

Proof. It is sufficient to prove that the functional $F_T(u,v,\phi)$ is bounded, uniformly in T.

We notice that each term in $F_T^2(u,v,\phi)$ is in the left hand side of one of the estimates in Proposition 4.1, therefore, arranging all the estimates, we can prove the following inequality

$$F_T^2(u,v,\phi) \leq c_1 F_0^2(u,v,\phi) + c_2 \overline{u} F_T^2(u,v,\phi) + c_3 F_T^3(u,v,\phi) \ ,$$

taking into account also that, on the right hand side of the estimates, the quadratic terms (not involving initial data) which have not the coefficient $\overline{u}$, can be bounded by means of cubic ones.

If $\overline{u}$ is sufficiently small, the previous inequality implies

$$F_T^2(u,v,\phi) \leq c_4 F_0^2(u,v,\phi) + c_5 F_T^3(u,v,\phi)$$

for suitable positive constants c_i.

It is easy to verify that, if y_0 is a sufficiently small positive real number and $h(y) = c_5 y^3 - y^2 + c_4 y_0$ then there exists $0 < \overline{y} < \frac{2}{3c_5}$ such that $h(y) > 0$ in $[0, \overline{y})$ and $h(y) < 0$ in $(\overline{y}, \frac{2}{3c_5}]$.

Then we can conclude that, if F_0 is suitably small , then F_T remains bounded for all $T > 0$. □

The above result, in particular the uniform, in time, boundedness of the functional F_T, allow us to prove the theorem below.

Let (4.2) hold and let $(\overline{u}, 0, \overline{\phi})$ be the constant stationary solution to problem (2.1)-(2.8) such that $\overline{u}|\mathcal{A}| = \overline{\mu}$; moreover, let $\mathcal{C}(\mathcal{A})$ be the set of the funcions f such that $f_i \in \mathcal{C}(\overline{I}_i)$ for $i \in \mathcal{M}$.

Theorem 4.2. *Let (4.2) hold. There exist $\epsilon_0, \epsilon_2 > 0$ such that, if*

$$\overline{u} \leq \epsilon_2 \ , \quad \sum_{i \in \mathcal{M}} \int_{I_i} u_0(x) = \overline{\mu} \ , \quad \|(u_0 - \overline{u}, v_0, \phi_0 - \overline{\phi})\|_{(H^1)^2 \times H^2} \leq \epsilon_0 \ ,$$

then problem (2.1)-(2.8) has a unique global solution (u,v,ϕ) ,

$$u, v \in C([0,+\infty); H^1(\mathcal{A})) \cap C^1([0,+\infty); L^2(\mathcal{A})) \ ,$$

$$\phi \in C([0,+\infty); H^2(\mathcal{A})) \cap C^1([0,+\infty); L^2(\mathcal{A})) \ ,$$

and, for all $i \in \mathcal{M}$,

$$\lim_{t\to+\infty} \|u_i(\cdot,t) - \overline{u}\|_{C(\overline{I}_i)} \ , \lim_{t\to+\infty} \|v_i(\cdot,t)\|_{C(\overline{I}_i)} \ , \lim_{t\to+\infty} \|\phi_i(\cdot,t) - \overline{\phi}\|_{C^1(\overline{I}_i)} = 0 \ .$$

Proof. Let (u,v,ϕ) be the local solution to problem (2.1)-(2.8) and let

$$\hat{u} := u - \overline{u}, \hat{v} := v, \hat{\phi} := \phi - \overline{\phi};$$

we already noticed that $(\hat{u},\hat{v},\hat{\phi})$ is the local solution to system (4.4) complemented by the initial condition $(u_0 - \overline{u}, v_0, \phi_0 - \overline{\phi})$ and the same boundary and transmission condition given by (2.3)-(2.8) for system (2.1).

For suitable ϵ_0, ϵ_2 the assumptions of Theorem 4.1 are satisfied, then we obtain the uniform boundedness of the functional $F_T(\hat{u},\hat{v},\hat{\phi})$, for $T \in [0,+\infty)$. Hence the set $\{\hat{u}(t),\hat{v}(t),\hat{\phi}(t)\}_{t\in[0,+\infty)}$ is uniformly bounded in $(H^1(\mathcal{A}))^2 \times H^2(\mathcal{A})$; thus, if we call E_s the set of accumulation points of $\{\hat{u}(t),\hat{v}(t),\hat{\phi}(t)\}_{t\geq s}$ in $(C(\mathcal{A}))^2 \times C^1(\mathcal{A})$, then E_s is not empty and $E := \cap_{s\geq 0} E_s \neq \emptyset$.

Let $(U(x),V(x),\Phi(x))$ be such that, for a sequence $t_n \to +\infty$,

$$\begin{aligned}
&\lim_{n\to+\infty} \sum_{i\in\mathcal{M}} \|\hat{u}_i(\cdot,t_n) - U_i(\cdot)\|_{C(\overline{I}_i)} = 0 \ , \\
&\lim_{n\to+\infty} \sum_{i\in\mathcal{M}} \|\hat{v}_i(\cdot,t_n) - V_i(\cdot)\|_{C(\overline{I}_i)} = 0 \ , \\
&\lim_{n\to+\infty} \sum_{i\in\mathcal{M}} \|\hat{\phi}_i(\cdot,t_n) - \Phi_i(\cdot)\|_{C^1(\overline{I}_i)} = 0 \ .
\end{aligned} \tag{4.5}$$

In order to identify the limit functions we notice that $\displaystyle\sum_{i\in\mathcal{M}} \int_{I_i} U_i(x)dx = 0$, since $\displaystyle\sum_{i\in\mathcal{M}} \int_{I_i} \hat{u}(x,t_n)\,dx = 0$ for all t_n.

Moreover, since $\hat{v}_i \in H^1(I_i \times (0,+\infty))$ for all $i \in \mathcal{M}$, if we set

$$\omega_i(t) := \|\hat{v}_i(t,\cdot)\|_{L^2(I_i)}$$

then $\omega_i \in H^1((0,+\infty))$ and, as a consequence, $\lim_{t\to+\infty} \omega_i(t) = 0$.

As $\lim_{n\to+\infty} \|\hat{v}_i(\cdot,t_n)\|_2 = \|V_i(\cdot)\|_2$, we obtain $\|V\|_2 = 0$.

The same argument can be applied to the functions $\hat{\phi}_{i_x}$, for all $i \in \mathcal{M}$, since they belongs to $H^1(I_i \times (0,+\infty))$. Finally, it can be applied to the functions $a_i\hat{u}_i - b_i\hat{\phi}_i$ since $\hat{\phi}_{i_t}, \hat{\phi}_{i_{xx}}, \hat{u}_{i_x}, \hat{\phi}_{i_x} \in L^2(I_i \times (0,+\infty))$, thanks to the uniform boundedness of $F_T(\hat{u},\hat{v},\hat{\phi})$ and to estimate f) in Proposition 4.1.

As a consequence we have that

$$V_i(x) = 0, \quad a_iU_i(x) - b_i\Phi_i(x) = 0 \ , \qquad \Phi_i(x) = \overline{\Phi}_i, \qquad x \in I_i \ ,$$

where $\overline{\Phi}_i$ are real numbers, so that the limit function is given by $(\frac{b_i}{a_i}\Phi_i, 0, \Phi_i)$ in each interval I_i, for all $i \in \mathcal{M}$. It is easily seen that such function is a stationary solution to problem (2.1)-(2.8), which is constant in each arc I_i; in particular it verifies the transmission conditions since $(\hat{u},\hat{v},\hat{\phi})$ verifies them and the convergence result (4.5) holds.

The condition $\displaystyle\sum_{i\in\mathcal{M}} \int_{I_i} U_i(x)dx = 0$ and Remark 3.1 imply that $\Phi_i = 0$ for all $i \in \mathcal{M}$, so that we can conclude that the unique possible limit function is $(U(x),V(x),\Phi(x)) = (0,0,0)$; this fact proves the claimed convergence results . □

5. **Conclusions and perspectives for the future.** The main features of the present work are: *a)* the proof of the existence and uniqueness of stationary solutions with fixed small mass to problem (2.1)-(2.6) considered on acyclic networks; *b)* the proof of the stability of particular stationary solutions, the constant states on the whole network, when the their masses are small and the quantity $\frac{a_i}{b_i}$ does not vary with the index i, for general networks. We can conclude that, in this range of parameters, although the transmission conditions do not impose continuity of the densities at the internal nodes, for suitable initial data the asymptotic profiles of the solutions are continuous functions, constant on the network.

These results are useful in describing the large time behaviour of small masses of individuals moving on networks driven by chemotaxis.

For the future, our aim is approaching the same questions when the system (2.1) is complemented by non-null fluxes conditions at the boundaries, which provide models for different situations at the outer nodes, in order to describe the features of the behaviour of cells moving along the arcs searching food. We notice that the condition $v_i(x) = 0$ for $x \in I_i$, for all $i \in \mathcal{M}$, which prevents the presence of jumps for the density u at the inner nodes, in this case is no longer necessary for stationary solutions on acyclic networks.

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