GREEN'S FUNCTION FOR ELLIPTIC SYSTEMS: MOMENT BOUNDS

PETER BELLA

Institute of Mathematics
Leipzig University
Augustusplatz 10
04109 Leipzig, Germany

ARIANNA GIUNTI

Institute for Applied Mathematics
University of Bonn
Endenicher Allee 60
53115 Bonn, Germany

(Communicated by Leonid Berlyand)

ABSTRACT. We study estimates of the Green's function in $\mathbb{R}^d$ with $d \geq 2$, for the linear second order elliptic equation in divergence form with variable uniformly elliptic coefficients. In the case $d \geq 3$, we obtain estimates on the Green's function, its gradient, and the second mixed derivatives which scale optimally in space, in terms of the "minimal radius" r_* introduced in [Gloria, Neukamm, and Otto: A regularity theory for random elliptic operators; ArXiv e-prints (2014)]. As an application, our result implies optimal stochastic Gaussian bounds on the Green's function and its derivatives in the realm of homogenization of equations with random coefficient fields with finite range of dependence. In two dimensions, where in general the Green's function does not exist, we construct its gradient and show the corresponding estimates on the gradient and mixed second derivatives. Since we do not use any scalar methods in the argument, the result holds in the case of uniformly elliptic systems as well.

1. **Introduction.** This paper is a contribution to the recently very active area of quantitative stochastic homogenization of second order uniformly elliptic operators, the main goal of which is to quantify how close is the large scale behavior of the heterogeneous operator $(-\nabla \cdot A(x)\nabla)^{-1}$ to the behavior of the constant-coefficient solution operator $(-\nabla \cdot A_{\text{hom}}\nabla)^{-1}$. Here $A(x)$ stands for the non-constant (random) coefficient field defined on $\mathbb{R}^d$ and A_{hom} is called the matrix of homogenized coefficients.

As originally realized in the seminal papers by Papanicolaou and Varadhan [23] and, independently, by Kozlov [20], the central object in the homogenization of

---

elliptic operators with random coefficients is the *corrector* ϕ_ξ, defined for each direction $\xi \in \mathbb{R}^d$ as a solution of the following elliptic problem

$$-\nabla_x \cdot (A(x)\nabla_x(x \cdot \xi + \phi_\xi(A, x))) = 0$$

in the whole space $\mathbb{R}^d$. The function ϕ is called corrector since it corrects the linear function $x \cdot \xi$, which is clearly solution to the constant-coefficient equation, to be a solution of the equation with heterogeneous coefficients. Since ϕ serves as a correction of a linear function, it should naturally be smaller, i.e., sublinear. Assuming the distribution of random coefficient fields A is stationary (meaning it is shift-invariant in the sense that A and $A(\cdot + x)$ have the same law for any $x \in \mathbb{R}^d$) and ergodic (meaning any shift-invariant random variable is almost surely constant, a property encoding decorrelation of coefficient fields over large scales), they showed that correctors are almost surely sublinear and can be used to define the homogenized coefficient

$$A_{\text{hom}}e_i := \langle A(e_i + \nabla\phi_{e_i}) \rangle.$$

Since the problem is linear, it clearly suffices to study the d correctors $\phi_i := \phi_{e_i}$ for $i = 1, \ldots, d$. Secondly, borrowing notation from the statistical physics, $\langle \cdot \rangle$ stands for the ensemble average (expected value) with respect to a probability distribution on the space of coefficient fields A. Here and also later, we will often drop the argument A in random quantities like the corrector as well as the argument x related to the spatial dependence of quantities like the coefficient field A or the corrector ϕ.

Both mentioned works [20, 23] were purely qualitative in the sense that they showed the sublinearity of the corrector in the limit of large scales without any rate. Assuming that the correlation of the coefficient fields decays with a specific rate (either encoded by some functional inequality like the Spectral Gap estimate or the Logarithmic Sobolev Inequality, or by some mixing conditions or even assuming finite range of dependence), one goal of quantitative theory is to *quantify* the sublinearity (smallness) of the corrector and consequences thereof.

Though the present result is purely deterministic in the sense that it translates the fact that the energy of any A-harmonic function satisfies a "mean value property" from some scale on (a fact that follows from the sublinearity of the corrector) into estimates on Green's function and its derivatives, we will first mention some recent results related to sublinearity of the corrector without going too much into details.

A central assumption in our result involves a *minimal radius*, a notion introduced by Gloria, Neukamm, and Otto [16]: for given fixed $\delta = \delta(d, \lambda) > 0$ (here λ denotes the ellipticity contrast), the random variable $r_* = r_*(A)$ is defined as

$$r_* := \inf\left\{r \geq 1 : \forall R \geq r : \frac{1}{R^2}\fint_{B_R}\left|(\phi, \sigma) - \fint_{B_R}(\phi, \sigma)\right|^2 \leq \delta\right\}. \tag{1}$$

Here (ϕ, σ) stands for the augmented corrector, where the new element σ (called vector potential) can be used for obtaining good error estimates and was originally introduced for the periodic homogenization (see, e.g., [5]). Here and in what follows B_R stands for a ball of radius R centered at the *origin* and $\fint$ denotes the average integral.

The introduction of the minimal radius r_* allows to split the arguments nicely into a deterministic and a stochastic part. For example, Gloria, Neukamm, and Otto [16] showed that the sublinearity of the corrector, as encoded in the definition of the random variable r_*, implies the mean value property (large-scale $C^{1,0}$-regularity)

for A-harmonic functions. This means that for $R \geq r \geq r_*$ and for any A-harmonic function u on B_R (i.e., a solution of $-\nabla \cdot A\nabla u = 0$ in B_R) one has

$$\fint_{B_r} |\nabla u|^2 \leq C \fint_{B_R} |\nabla u|^2.$$

The idea that the (large-scale) regularity theory of A-harmonic functions is closely related to the error estimates in stochastic homogenization was first realized by Armstrong and Smart [4]. Building on the work of Avellaneda and Lin [5] in the periodic setting, Armstrong and Smart studied homogenization of uniformly convex integral functionals under an independence assumption. For a quantity similar to r_* (which they denoted $\mathcal{Y}$), assuming finite range of dependence of the coefficient fields they obtained almost Gaussian bounds of the form $\langle \exp(\mathcal{Y})^s \rangle < \infty$ for any $s < d$ (see also [1, 2, 3] for further developments). While $\mathcal{Y}$ and r_* are quite related, the definition of r_* is more intuitive and explicit, and as such it is easier to work with.

Assuming that the ensemble on the coefficient fields satisfies a coarsened version of the Logarithmic Sobolev Inequality, Gloria, Neukamm, and Otto [16] showed that the minimal radius r_* has stretched exponential moments

$$\left\langle \exp\left(\tfrac{1}{C} r_*^{d(1-\beta)}\right) \right\rangle \leq C,$$

where $0 \leq \beta < 1$ appearing in the exponent is related to the coarsening rate in the Logarithmic Sobolev Inequality. Observe that in the case $\beta = 0$, i.e., the case when we consider the Logarithmic Sobolev Inequality without coarsening, this is the optimal Gaussian bound.

Recently, reviving the parabolic approach used in the discrete setting [17], which has the benefit of conveniently disintegrating contributions to the corrector from different scales, Gloria and Otto [18] obtained a similar results assuming the coefficient fields have finite range of dependence. As a by-product, assuming finite range of dependence Gloria and Otto got the estimates for the minimal radius r_* with optimal stochastic integrability of the form

$$\left\langle \exp\left(\tfrac{1}{C} r_*^{d(1-\epsilon)}\right) \right\rangle < \infty, \quad \forall \epsilon > 0.$$

As already said, using completely different methods, such almost Gaussian bounds for a related quantity $\mathcal{Y}$ in the case of scalar but possibly nonlinear equations were obtained before by Armstrong and Smart [4].

Finally, on the other side of the spectrum, Fischer and Otto [14] combined Meyer's estimate together with sensitivity analysis to show that for strongly correlated coefficient fields (more precisely, they consider coefficient fields which are 1-Lipschitz images of a stationary Gaussian field with correlations bounded by $|x|^{-\beta}$, where $\beta > 0$ has to be much smaller that 1, so that Meyer's estimate holds for $p = 2 + 2\beta$) it holds

$$\left\langle \exp\left(\tfrac{1}{C} r_*^{\beta}\right) \right\rangle \leq C.$$

In the present paper we will obtain deterministic estimates for the Green's function based on the minimal radii r_* at different points. More precisely, fixing two points $x_0, y_0 \in \mathbb{R}^d$, we take as the input the coefficient field A and the corresponding minimal radii $r_*(x_0), r_*(y_0)$ (here and in what follows $r_*(x_0)$ stays for the minimal radius r_* of the shifted coefficient field $A(\cdot - x_0)$), and produce estimates on the Green's function G and its derivatives $\nabla_x G, \nabla_y G, \nabla_x \nabla_y G$, averaged over small scale

around the points x_0 and y_0. This averaging is necessary since we do not assume any smoothness of the coefficient fields.

An obvious advantage of the present approach is that it clearly separates the random effects, described by r_*, from the average large-scale behavior of the Green's function. Indeed, our result is modular in the sense that it can be combined with estimates on stochastic moments of r_*, as for example obtained under different ergodicity assumptions on the ensemble in [16, 18], to yield estimates on the Green's function and its derivatives for different models of random coefficients.

Our only goal in this paper is to obtain bounds, and not to show existence (or other properties) of the Green's function. In fact, a well known counterexample of De Giorgi [11] shows that there are uniformly elliptic coefficient fields for which the Green's function does not exist. Nevertheless, as recently shown in [10] by Conlon, Otto, and the second author, this is not a generic behavior. More precisely, in [10] they show that for *any* uniformly elliptic coefficient field A the Green's function $G = G(A; x, y)$ exists at *almost every* point $y \in \mathbb{R}^d$, provided the dimension $d \geq 3$. Therefore, in the case $d \geq 3$, we will assume that the Green's function $G(A; \cdot, y) \in L^1_{\text{loc}}(\mathbb{R}^d)$ exists, at least in the almost everywhere sense (i.e., for a.e. $y \in \mathbb{R}^d$), and focus solely on the estimates. Since in $\mathbb{R}^2$ the Green's function does not have to exist, but its "gradient" can possibly exists, using a reduction from $3D$ (where the Green's function exists) in Section 4 we construct and estimate ∇G.

There are several works studying estimates on the Green's function in the context of uniformly elliptic equations with random coefficients. Using De Giorgi-Nash-Moser approach for a parabolic equation (which is naturally restricted to the scalar case), Delmotte and Deuschel [12] obtained annealed estimates on the first and second gradient of the Green's function, in L^2 and L^1 in probability respectively, under mere assumption of stationarity of the ensemble (see also [22] for a different approach). Using different methods, Conlon, Otto, and the second author [10] recently obtained similar estimates, together with other properties of the Green's function, but without the restriction to the scalar case. Compared to that work, our more quantitative assumption in terms of r_* implies *local* and *quenched* estimates, and not only estimates in average. In the discrete setting, assuming that the spatial correlation of the coefficient fields decays sufficiently fast to the effect that the Logarithmic Sobolev Inequality is satisfied, Marahrens and Otto [21] upgraded the Delmotte-Deuschel bounds to any stochastic moments. Recently, again only in the scalar case this work was extended by Gloria and Marahrens [15] into the continuum setting.

Both works [15, 21] used De Giorgi-Nash-Moser-type argument, and as such were restricted to a single equation. In contrast, our result is not restricted to the scalar case, a reason why we had to develop different techniques to obtain the estimates.

Before we state the main result, let us mention other works relating the smallness of the corrector and the properties of solutions to the heterogeneous equation. Together with Otto [8], we compare the finite energy solution u of

$$- \nabla \cdot A \nabla u = \nabla \cdot g,$$

with $g \in L^2(\mathbb{R}^d; \mathbb{R}^d)$ being supported in a unit ball around the origin, with *twice* corrected solution u_{hom} of the homogenized equation

$$- \nabla \cdot A_{\text{hom}} \nabla u_{\text{hom}} = \nabla \cdot \tilde{g}.$$

Here by twice corrected we mean that first the right-hand side g from the heterogeneous equation is replaced by $\tilde{g} = g(\text{Id} + \nabla \phi)$ in the constant-coefficient equation,

and second, we compare u with $(\mathrm{Id} + \phi_i \partial_i) v$ at the level of gradients. Using duality argument together with a compactness lemma, this gives an estimate of the difference between $\nabla_x \nabla_y G(x, y)$ and $\partial_i \partial_j G_{\mathrm{hom}}(e_i + \nabla \phi_i(x)) \otimes (e_j + \nabla \phi_j(y))$ (averaged over small balls both in x and y). In order to get such estimates, it is not enough to assume that the corrector is at most linear with small slope (as in (1)), but rather we need to assume that for some $\beta \in (0, 1)$, it grows in the L^2-sense at most like $|x|^{1-\beta}$:

$$\frac{1}{R^2} \fint_{B_R} \left| (\phi, \sigma) - \fint_{B_R} (\phi, \sigma) \right|^2 \leq C R^{-2\beta}, \quad \forall R \geq r_{*,\beta}. \tag{2}$$

Compared to the condition (1) which we use in the present paper, the above condition (2) is obviously stronger. Indeed, while for example r_* is almost surely finite under pure assumptions of stationarity and ergodicity of the ensemble, this is not true for $r_{*,\beta}$. Moreover, while one would believe that the stochastic estimates on r_* (e.g. those in [16]) can be with some farther work modified to estimates on $r_{*,\beta}$, we are not aware of any such results.

Hence, in comparison with the present work, in [8] we get a stronger statement (since we estimate the difference between the heterogeneous Green's function and corrected constant-coefficient Green's function while in the present paper we only control the heterogeneous Green's function alone), at the expense of stronger assumptions on the smallness of the corrector and a more involved proof. More precisely, here we show that the second mixed derivative of the Green's function $\nabla_x \nabla_y G$ behaves like $C|x - y|^{-d}$ (clearly this estimate is sharp in scaling since it agrees with the behavior of the constant-coefficient Green's function), while in [8] we show that the homogenization error, i.e., the difference between $\nabla_x \nabla_y G$ and twice corrected mixed second derivative of the constant-coefficient Green's function, is estimated by $C|x - y|^{-(d+\beta)}$ - that means we gain a factor of $|x - y|^{-\beta}$, where the exponent $\beta \in (0, 1)$ is the one appearing in (2).

Since we are dealing with linear equations, we make use of a duality argument, first introduced by Avellaneda and Lin in [5]. This allows us to obtain estimates on the L^2-norm (in one variable) of the averages (in the other variable) for G, ∇G and $\nabla_x \nabla_y G$. In particular, the main pivotal estimate is the one on the L^2-norm in both space variables for the mixed derivatives $\nabla_x \nabla_y G$. Since we do not assume any smoothness on the coefficient field A, and therefore we may not appeal to Schauder estimates, we obtain an off-diagonal L^2-estimate in both variables out of estimates on averaged quantities by what we call a compactness lemma for A-harmonic functions. This ingredient was first introduced in [8] and later exploited and extended also in [10]. In this work, we make use of the same result and provide a different proof which is based on a refined iteration of Caccioppoli's inequalities. In our opinion, this new argument admits an easier extension outside of the L^2-framework, as it is required in the case of non-uniformly elliptic coefficients as considered in [6]. We plan to investigate this question in the future. Besides, we remark that a simplified version of the previous compactness argument appears already in the works by Otto and the first author [9] and in [19]: There, the quantities involved are stationary (in fact they are related to the gradient of the corrector) and, under this assumption, the proof of such compactness result turns out to be much simpler. In our case, as well as in [8] and [10], one deals with Green's functions and it is no more possible to appeal to stationarity in one space variable, since a translation in space for these objects involves also a translation of the singularity.

Last, let us mention the work of Otto and the authors [7], where we push far-
ther the results of [8] using higher order correctors. The second and higher order
correctors were introduced into the stochastic homogenization setup by Fischer and
Otto [13], in order to extend the $C^{1,\alpha}$ regularity estimates on large scales [16] to
$C^{2,\alpha}$ estimates and $C^{k,\alpha}$ estimates respectively. In [7], under the assumption of
smallness of the corrector, we obtain two results about A-harmonic function in ex-
terior domains: first, for any integer k we construct a finite dimensional space of
functions such that the distance between any A-harmonic function in the exterior
domain and this space is bounded by $C|x - y|^{-(d+k)}$ (this statement can be seen
as an analogue of Liouville statements for finite energy solutions in the exterior
domain). Second, assuming smallness of both the first and second order augmented
correctors (i.e., also including the second order vector potential, which we had to
introduce), compared with [8] we improve by 1 the exponent in the estimate be-
tween the solution of the heterogeneous equation in the exterior domain and some
corrected solution of the constant-coefficient equation.

The paper is organized as follows: In the next section we will state our assump-
tions together with the main result, Theorem 1, and its corollaries, Corollary 1,
Corollary 2, and Corollary 3. In Section 3 we prove Theorem 1 and in Section 4
we give the argument for Corollary 1, which is the only corollary which does not
immediately follow from the theorem.

Notation. Throughout the article, we denote by C a positive generic constant
which is allowed to depend on the dimension d and the ellipticity contrast λ, and
which may be different from line to line of the same estimate. By $\lesssim$ we will mean
$\leq C$. Finally, the integrals without specified domain of integration are meant as
integrals over the whole space $\mathbb{R}^d$.

2. **The main result.** We fix a coefficient field $A \in L^\infty(\mathbb{R}^d; \mathbb{R}^{d \times d})$, which we assume
to be uniformly elliptic in the sense that

$$\int_{\mathbb{R}^d} \nabla\varphi \cdot A(x)\nabla\varphi \, dx \geq \lambda \int_{\mathbb{R}^d} |\nabla\varphi|^2, \qquad \forall\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d),$$
$$|A(x)\xi| \leq |\xi|, \qquad\qquad \forall \text{a.e. } x \in \mathbb{R}^d, \forall\xi \in \mathbb{R}^d, \tag{3}$$

where $\lambda \in (0, 1)$ is fixed throughout the paper. Then we have the following result:

Theorem 1. *Let $d \geq 3$, let A be a uniformly elliptic coefficient field on $\mathbb{R}^d$ in
the sense of (3), and let $x_0, y_0 \in \mathbb{R}^d$ with $|x_0 - y_0| \geq 10$. For a point $x \in \mathbb{R}^d$,
let $r_*(x) = r_*(A, x) \geq 1$ denote a radius such that for $R \geq r \geq r_*(x)$ and any
A-harmonic function u in $B_R(x)$ we have*

$$\fint_{B_r(x)} |\nabla u|^2 \leq C(d, \lambda) \fint_{B_R(x)} |\nabla u|^2. \tag{4}$$

Let $G = G(A; x, y)$ be the Green's function defined through

$$- \nabla_x \cdot A\nabla_x G(A; \cdot, y) = \delta(\cdot - y),$$

assuming it exists for a.e. $y \in \mathbb{R}^d$. Then we have

$$\int_{B_1(x_0)} \int_{B_1(y_0)} |\nabla_x \nabla_y G(A; x, y)|^2 \, dx \, dy \leq C(d, \lambda) \left(\frac{r_*(x_0) r_*'(y_0)}{|x_0 - y_0|^2} \right)^d, \tag{5}$$

$$\int_{B_1(x_0)} \int_{B_1(y_0)} |\nabla_y G(A;x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y \leq C(d,\lambda)|x_0-y_0|^2 \left(\frac{r_*(x_0)r'_*(y_0)}{|x_0-y_0|^2} \right)^d, \quad (6)$$

$$\int_{B_1(x_0)} \int_{B_1(y_0)} |\nabla_x G(A;x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y \leq C(d,\lambda)|x_0-y_0|^2 \left(\frac{r'_*(x_0)r_*(y_0)}{|x_0-y_0|^2} \right)^d, \quad (7)$$

$$\int_{B_1(x_0)} \int_{B_1(y_0)} |G(A;x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y$$
$$\leq C(d,\lambda)|x_0-y_0|^4 \left(\frac{r_*(x_0)r'_*(y_0) + r'_*(x_0)r_*(y_0)}{|x_0-y_0|^2} \right)^d. \quad (8)$$

where $r'_*(y) = r_*(A^t,y) \geq 1$ denotes the minimal radius for the adjoint coefficient field A^t at a point y.

Though the Green's function does not have to exist in $2D$, with the help of the Green's function in $3D$ we can at least define and estimate "its gradient & second mixed derivatives":

Corollary 1. *Let $d = 2$. Let A be a uniformly elliptic coefficient field on $\mathbb{R}^2$ in the sense of* (3), *such that for its extension into $\mathbb{R}^3$ of the form*

$$\bar{A}(x,x_3) := \begin{pmatrix} A(x) & 0 \\ 0 & 1 \end{pmatrix} \quad (9)$$

there exists two points $\bar{X}, \bar{Y} \in \mathbb{R}^3$ so that the minimal radii for $\bar{A}^t$ and $\bar{A}$ at those points are finite, respectively (i.e., $r_*(\bar{A}^t,\bar{X}) < \infty$, $r_*(\bar{A},\bar{Y}) < \infty$).

Then for a.e. $y \in \mathbb{R}^2$ there exists a function on $\mathbb{R}^2$, which we denote $\nabla G(A;\cdot,y)$, so that it satisfies in a weak sense

$$-\nabla_x \cdot A\nabla G(A;\cdot,y) = \delta(\cdot-y).$$

Moreover, given $x_0, y_0 \in \mathbb{R}^2$ with $|x_0-y_0| \geq 10$, we have estimates for ∇G as well as for $\nabla_y \nabla G$:

$$\int_{B_1(x_0)} \int_{B_1(y_0)} |\nabla_y \nabla G(A;x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y \leq C(\lambda) \frac{\left(r_*(A,x_0)r_*(A^t,y_0)\right)^2}{|x_0-y_0|^4}, \quad (10)$$

$$\int_{B_1(x_0)} \int_{B_1(y_0)} |\nabla G(A;x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y \leq C(\lambda) \frac{\left(r_*(\bar{A}^t,(x_0,0))r_*(\bar{A},(y_0,0))\right)^2}{|x_0-y_0|^2}. \quad (11)$$

Remark 1. Assuming that the coefficient field A in the statement of Corollary 1 is chosen at random with respect to a stationary and ergodic ensemble, by the standard ergodic argument (see, e.g., [16]), applied in $3D$ to the ensemble obtained as a push-forward of the $2D$ ensemble through (9), the assumption on the finiteness of the minimal radii is almost surely satisfied (say with $\bar{X} = \bar{Y} = (0,0,0)$).

Remark 2. It is clear from the proof of Theorem 1 that all the above estimates, i.e. (5)-(11), are true also if the domains of integration $B_1(x_0)$ and $B_1(y_0)$ are replaced by larger balls with the corresponding radii r_*. Moreover, the radii of the balls could be even larger than the minimal radii (as long as these new radii are not larger than one third of a distance between centers of those balls), in which case we need to replace the minimal radii on the right-hand sides of those estimates with the actual radii of the balls.

Remark 3. The appearance of different minimal radii in (10) and (11) (in (10) the minimal radii are related to the equation in $2D$, while in (11) they are the minimal

radii for the equation in $3D$) is not a typo. The reason is that while (10) is proved directly in $2D$, the proof of (11) passes through $3D$ - hence the need to consider the minimal radii in $3D$. In view of the relation $r_*(A, x) \leq r_*(\bar{A}, \bar{x})$, which easily follows from the fact that any A-harmonic function in $B_R \subset \mathbb{R}^2$ can be trivially extended to an $\bar{A}$-harmonic function in $\bar{B}_R \subset \mathbb{R}^3$, the estimate (11) seems to be less optimal.

For notational convenience we state the result for a single equation. Since in the proof of Theorem 1 we do not use any *scalar* methods (like for example De Giorgi-Nash-Moser iteration), the result holds also in the case of elliptic systems - for that one just considers that u has values in some finite-dimensional Hilbert space. Naturally, in that case all the constants will depend on the dimension of this Hilbert space.

Using the Gaussian bounds on r_* for the case of coefficient fields with finite range of dependence, which were obtained recently in [18], Theorem 1 implies the following bounds:

Corollary 2. *Suppose $\langle \cdot \rangle$ is an ensemble of λ-uniformly elliptic coefficient fields which is stationary and of unit range of dependence, and let $d \geq 2$. Then there exist $C(d, \lambda)$ such that for every two points $x_0, y_0 \in \mathbb{R}^d$, $|x_0 - y_0| \geq 10$, and every $\epsilon > 0$ we have*

$$\left\langle \exp\left(\left(C|x_0 - y_0|^{2d} \int_{B_1(x_0)} \int_{B_1(y_0)} |\nabla_x \nabla_y G(A; x, y)|^2 \, \mathrm{d}x \, \mathrm{d}y \right)^{d(1-\epsilon)}\right) \right\rangle < \infty,$$

$$\left\langle \exp\left(\left(C|x_0 - y_0|^{2d-2} \int_{B_1(x_0)} \int_{B_1(y_0)} |(\nabla_x, \nabla_y) G(A; x, y)|^2 \, \mathrm{d}x \, \mathrm{d}y \right)^{d(1-\epsilon)}\right) \right\rangle < \infty,$$

and in $d \geq 3$ also

$$\left\langle \exp\left(\left(C|x_0 - y_0|^{2d-4} \int_{B_1(x_0)} \int_{B_1(y_0)} |G(A; x, y)|^2 \, \mathrm{d}x \, \mathrm{d}y \right)^{d(1-\epsilon)}\right) \right\rangle < \infty.$$

In the case of coefficient fields with stronger correlations we can use the result from [16]:

Corollary 3. *Suppose $d \geq 2$, and that the ensemble $\langle \cdot \rangle$ is stationary and satisfies a logarithmic Sobolev inequality of the following type: There exists a partition $\{D\}$ of $\mathbb{R}^d$ not too coarse in the sense that for some $0 \leq \beta < 1$ it satisfies*

$$diam(D) \leq (dist(D) + 1)^\beta \leq C(d) \, diam(D).$$

Moreover, let us assume that there is $0 < \rho \leq 1$ such that for all random variables F

$$\left\langle F^2 \log F^2 \right\rangle - \left\langle F^2 \right\rangle \log \left\langle F^2 \right\rangle \leq \frac{1}{\rho} \left\langle \left\| \frac{\partial F}{\partial A} \right\|^2 \right\rangle,$$

where the carré-du-champ of the Malliavin derivative is defined as

$$\left\| \frac{\partial F}{\partial A} \right\|^2 := \sum_D \left(\int_D \left| \frac{\partial F}{\partial A} \right|^2 \right).$$

Then there exists a constant $0 < C < \infty$, depending only on d, λ, ρ, β, such that

$$\left\langle \exp\left(\left(C|x_0 - y_0|^{2d} \int_{B_1(x_0)} \int_{B_1(y_0)} |\nabla_x \nabla_y G(A; x, y)|^2 \, \mathrm{d}x \, \mathrm{d}y\right)^{d(1-\beta)}\right)\right\rangle < \infty,$$

$$\left\langle \exp\left(\left(C|x_0 - y_0|^{2d-2} \int_{B_1(x_0)} \int_{B_1(y_0)} |(\nabla_x, \nabla_y) G(A; x, y)|^2 \, \mathrm{d}x \, \mathrm{d}y\right)^{d(1-\beta)}\right)\right\rangle < \infty,$$

and in $d \geq 3$ also

$$\left\langle \exp\left(\left(C|x_0 - y_0|^{2d-4} \int_{B_1(x_0)} \int_{B_1(y_0)} |G(A; x, y)|^2 \, \mathrm{d}x \, \mathrm{d}y\right)^{d(1-\beta)}\right)\right\rangle < \infty.$$

3. **Proof of Theorem 1.** The proof is inspired by a duality argument of Avellaneda and Lin [5, Theorem 13], which they used to obtain Green's function estimates in the periodic homogenization. After stating and proving two auxiliary lemmas, we first prove the estimate on the second mixed derivative (5). Then, (6) will follow from (5) using Poincaré inequality and one additional estimate. Next we observe that (7) can be obtained from (6) by replacing the role of x and y, which can be done by considering the adjoint A^t instead of A. Finally, (8) will follow from (7) in a similar way as (6) follows from (5).

We thus start with the following two auxiliary lemmas. The first one is very standard:

Lemma 1 (Caccioppoli inequality). *Let $\rho > 0$, $\delta > 0$, and let u be a solution of a uniformly elliptic equation $-\nabla \cdot A\nabla u = 0$ in $B_{(1+\delta)\rho}$. Then*

$$\int_{B_\rho} |\nabla u|^2 \leq \frac{C(d)}{\lambda \rho^2 \delta^2} \int_{B_{(1+\delta)\rho}} |u - c|^2 \tag{12}$$

for any $c \in \mathbb{R}$.

Proof. By considering $u - c$ instead of u, it is enough to show estimate (12) with $c = 0$. We test the equation for u with $\eta^2 u$, where η is a smooth cut-off function for B_ρ in $B_{(1+\delta)\rho}$ with $|\nabla\eta| \lesssim (\delta\rho)^{-1}$, use (3) and Young's inequality to get

$$\int_{\mathbb{R}^d} |\nabla(\eta u)|^2 \leq \frac{C(d)}{\lambda} \int |\nabla\eta|^2 \, u^2.$$

Since $|\nabla\eta| \leq \frac{C}{\rho\delta}$, (12) immediately follows. $\qquad\square$

Lemma 2. *Let $R_0 \geq r_*(0)$, and let u be an A-harmonic function in B_{R_0}. Then we have*

$$\fint_{B_{r_*(0)}} |u|^2 \leq C(d, \lambda) \fint_{B_{R_0}} |u|^2. \tag{13}$$

Proof. Throughout the proof we write r_* instead of $r_*(0)$. We assume that $2r_* < R_0$, since otherwise (13) is trivial. For $r \in [r_*, R_0]$ we denote $u_r := \fint_{B_r} u$. We have

$$\fint_{B_{r_*}} |u - u_{r_*}|^2 \overset{\text{Poincaré}}{\lesssim} r_*^2 \fint_{B_{r_*}} |\nabla u|^2 \overset{(4)}{\lesssim} r_*^2 \fint_{B_{R_0/2}} |\nabla u|^2$$

$$\overset{(12)}{\lesssim} \left(\frac{r_*}{R_0}\right)^2 \fint_{B_{R_0}} |u|^2 \leq \fint_{B_{R_0}} |u|^2.$$

Hence, to prove (4) it is enough to show

$$|u_{r_*}|^2 = \left| \fint_{B_{r_*}} u \right|^2 \lesssim \fint_{B_{R_0}} |u|^2 . \tag{14}$$

To prove it, we use the following estimate

$$\left| u_r - u_{2r} \right| \lesssim r \left(\fint_{B_{2r}} |\nabla u|^2 \right)^{\frac{1}{2}} , \tag{15}$$

which in fact holds for any function $u \in W^{1,2}(B_{2r})$.

We first argue how to obtain (14) thanks to estimate (15): Let $n \geq 0$ be the largest integer that satisfies $2^n r_* \leq R_0/2$; using (15) multiple times we get

$$|u_{r_*} - u_{2^n r_*}| \leq \sum_{k=0}^{n-1} |u_{2^k r_*} - u_{2^{k+1} r_*}| \overset{(15)}{\lesssim} \sum_{k=0}^{n-1} 2^k r_* \left(\fint_{B_{2^{k+1} r_*}} |\nabla u|^2 \right)^{\frac{1}{2}}$$

$$\overset{(4)}{\lesssim} \left(\fint_{B_{R_0/2}} |\nabla u|^2 \right)^{\frac{1}{2}} \sum_{k=0}^{n-1} 2^k r_* \overset{(12)}{\lesssim} R_0 \left(\frac{1}{R_0^2} \fint_{B_{R_0}} |u|^2 \right)^{\frac{1}{2}}$$

$$= \left(\fint_{B_{R_0}} |u|^2 \right)^{\frac{1}{2}} .$$

Using Jensen's inequality and the fact that $R_0 \leq 2^{n+2} r_*$ we get

$$|u_{2^n r_*}| = \left| \fint_{B_{2^n r_*}} u \right| \leq \left(\fint_{B_{2^n r_*}} |u|^2 \right)^{\frac{1}{2}} \lesssim \left(\fint_{B_{R_0}} |u|^2 \right)^{\frac{1}{2}} .$$

Combination of the two previous estimates then gives (14).

It remains to prove (15). Using Jensen's and Poincaré's inequalities we get

$$|u_r - u_{2r}| = \left| \fint_{B_r} (u - u_r) - (u - u_{2r}) \right| \lesssim \fint_{B_r} |u - u_r| + \fint_{B_{2r}} |u - u_{2r}|$$

$$\lesssim \left(\fint_{B_r} |u - u_r|^2 \right)^{\frac{1}{2}} + \left(\fint_{B_{2r}} |u - u_{2r}|^2 \right)^{\frac{1}{2}} \lesssim r \left(\fint_{B_{2r}} |\nabla u|^2 \right)^{\frac{1}{2}} .$$

$\square$

3.1. **Proof of** (5). We denote $R_0 := |x_0 - y_0|/3$. We split the proof of (5) into 4 steps. In the first step we show that

$$\int_{B_1(y_0)} |F_\rho (\nabla_x \nabla_y G(\cdot, y))|^2 \, \mathrm{d}y \lesssim \left(\frac{r_*(x_0) r_*'(y_0)}{R_0^2} \right)^d \tag{16}$$

for any $\rho \in [1,2]$ and any functional F_ρ on $L^2(B_\rho(x_0))$ which satisfies

$$|F_\rho(\nabla v)|^2 \leq \int_{B_\rho(x_0)} |\nabla v|^2 , \tag{17}$$

for any $v \in W^{1,2}(B_\rho(x_0))$. In the second step, using Neumann eigenfunctions on a ball, we define a family of functionals F_k satisfying (17), which will play the role of Fourier coefficients. Using these F_k we then estimate $\int |v|^2$ with a sum of N terms $|F_k(v)|^2$ plus a residuum in the form $\frac{1}{\lambda_N} \int |\nabla v|^2$. Here, λ_N denotes the Nth Neumann eigenvalue of Laplacian on a ball. Using (17) together with the second

step we get an estimate on $\int |\nabla_x \nabla_y G|^2$ in terms of a (good) term and a small prefactor times $\int |\nabla_x \nabla_y G|^2$, integrated over a slightly larger ball. In the last step, we apply iteratively this estimate.

Step 1. Proof of (16) (inspired by the duality argument of Avellaneda and Lin [5]).

Let $f \in L^2(B_{R_0}(y_0); \mathbb{R}^d)$, and let u be the finite energy solution of

$$- \nabla \cdot A\nabla u = -\nabla \cdot f$$

in $\mathbb{R}^d$, for which holds the energy estimate

$$\int_{\mathbb{R}^d} |\nabla u|^2 \lesssim \int_{\mathbb{R}^d} |f|^2 . \tag{18}$$

Then on the one hand, the Green's function representation formula yields

$$\nabla u(x) = \int_{B_{R_0}(y_0)} \nabla_x \nabla_y G(x,y) f(y) \, \mathrm{d}y. \tag{19}$$

If $r_*(x_0) \leq R_0$ (w. l. o. g. we assume $r_*(x_0) \geq \rho$), we use this in (4) to get

$$|F_\rho(\nabla u)|^2 \leq \int_{B_\rho(x_0)} |\nabla u|^2 \, \mathrm{d}x \leq \int_{B_{r_*(x_0)}(x_0)} |\nabla u|^2 \lesssim \left(\frac{r_*(x_0)}{R_0} \right)^d \int_{B_{R_0}(x_0)} |\nabla u|^2$$

$$\overset{(18)}{\lesssim} \left(\frac{r_*(x_0)}{R_0} \right)^d \int_{\mathbb{R}^d} |f|^2 .$$

If $r_*(x_0) \geq R_0$, we simply have

$$|F_\rho(\nabla u)|^2 \leq \int_{B_\rho(x_0)} |\nabla u|^2 \, \mathrm{d}x \overset{(18)}{\lesssim} \int_{\mathbb{R}^d} |f|^2 \, \mathrm{d}x \leq \left(\frac{r_*(x_0)}{R_0} \right)^d \int_{\mathbb{R}^d} |f|^2 .$$

Since F_ρ is linear, using (19) we have

$$F_\rho(\nabla u) = \int_{B_{R_0}(y_0)} F_\rho \left(\nabla_x \nabla_y G(\cdot, y) \right) f(y) \, \mathrm{d}y,$$

where the dot means that F_ρ acts on the first variable. The previous relations then give

$$\left| \int_{B_{R_0}(y_0)} F_\rho \left(\nabla_x \nabla_y G(\cdot, y) \right) f(y) \, \mathrm{d}y \right|^2 \lesssim \left(\frac{r_*(x_0)}{R_0} \right)^d \int_{B_{R_0}(x_0)} |f|^2 .$$

Using definition of the norm $L^2(B_{R_0}(y_0))$ by duality we get

$$\int_{B_{R_0}(y_0)} |F_\rho \left(\nabla_x \nabla_y G(\cdot, y) \right)|^2 \, \mathrm{d}y \lesssim \left(\frac{r_*(x_0)}{R_0} \right)^d . \tag{20}$$

Let $r'_*(y_0)$ play the same role as $r_*(x_0)$ but for the adjoint equation. In the case $r'_*(y_0) \leq R_0$, since the mapping $y \mapsto F_\rho(\nabla_x G(\cdot, y))$ solves the adjoint equation $-\nabla \cdot A^t \nabla F_\rho(\nabla_x G(\cdot, y)) = 0$ in $B_{R_0}(y_0)$, arguing similarly as we did after (19), an analogue of (4) solutions of the adjoint equation implies

$$\int_{B_1(y_0)} |F_\rho(\nabla_x \nabla_y G(\cdot, y))|^2 \, \mathrm{d}y \leq \int_{B_{r'_*(y_0)}(y_0)} |F_\rho(\nabla_x \nabla_y G(\cdot, y))|^2 \, \mathrm{d}y$$

$$\lesssim \left(\frac{r'_*(y_0)}{R_0} \right)^d \int_{B_{R_0}(y_0)} |F_\rho(\nabla_x \nabla_y G(\cdot, y))|^2 \, \mathrm{d}y \overset{(20)}{\lesssim} \left(\frac{r_*(x_0) r'_*(y_0)}{R_0^2} \right)^d . \tag{21}$$

If $r'_*(y_0) \geq R_0$, we get the same conclusion for free:

$$\int_{B_1(y_0)} |F_\rho(\nabla_x \nabla_y G(\cdot, y))|^2 \, \mathrm{d}y \leq \int_{B_{R_0}(y_0)} |F_\rho(\nabla_x \nabla_y G(\cdot, y))|^2 \, \mathrm{d}y$$
$$\overset{(20)}{\lesssim} \left(\frac{r_*(x_0)}{R_0}\right)^d \leq \left(\frac{r_*(x_0) r'_*(y_0)}{R_0^2}\right)^d.$$

Step 2. Let $\rho \in [1, 2)$ and $\delta > 0$ be fixed such that $(1 + \delta)\rho \leq 2$. For $n \geq 0$, let F_n denote the functional on $L^2(B_{(1+\delta)\rho})$, defined as an inner product of the nth eigenfunction of Neumann Laplacian with v, and let λ_n be the associated eigenvalue. By writing any $v \in W^{1,2}(B_{(1+\delta)\rho})$ with zero mean in terms of Neumann eigenfunctions (which can be done since they form an orthonormal basis in L^2) we see that for any $N \geq 1$

$$\int_{B_{(1+\delta)\rho}} |v|^2 = \sum_{k=1}^{\infty} |F_k(v)|^2 \text{ and } \int_{B_{(1+\delta)\rho}} |\nabla v|^2 = \sum_{k=1}^{\infty} \lambda_k |F_k(v)|^2 \geq \lambda_N \sum_{k=N}^{\infty} |F_k(v)|^2,$$

where the last inequality follows from the monotonicity of λ_k. Hence we see

$$\int_{B_{(1+\delta)\rho}} |v|^2 = \sum_{k=1}^{N-1} |F_k(v)|^2 + \sum_{k=N}^{\infty} |F_k(v)|^2 \leq \sum_{k=1}^{N-1} |F_k(\nabla v)|^2 + \frac{1}{\lambda_N} \int_{B_{(1+\delta)\rho}} |\nabla v|^2,$$
$$(22)$$

where we used that $\lambda_k \geq 1$.

Step 3. Combination of Step 1 and Step 2 (applied to $\nabla_y G(\cdot, y)$) and use of (21) yields

$$\int_{B_1(y_0)} \int_{B_\rho(x_0)} |\nabla_x \nabla_y G(x, y)|^2 \, \mathrm{d}x \, \mathrm{d}y$$

$$\overset{(12)}{\lesssim} \frac{1}{\delta^2} \int_{B_1(y_0)} \int_{B_{(1+\delta)\rho}(x_0)} |\nabla_y G(x, y) - \mathcal{G}_{\mathrm{avg}}(y)|^2 \, \mathrm{d}x \, \mathrm{d}y$$

$$\overset{(22)}{\lesssim} \frac{1}{\delta^2} \left(\sum_{k=0}^{N-1} \int_{B_1(y_0)} |F_k(\nabla_x \nabla_y G(\cdot, y))|^2 + \frac{1}{\lambda_N} \int_{B_1(y_0)} \int_{B_{(1+\delta)\rho}(x_0)} |\nabla_x \nabla_y G(x, y)|^2 \mathrm{d}x \, \mathrm{d}y \right)$$

$$\overset{(21)}{\lesssim} \frac{1}{\delta^2} \left(N \left(\frac{r_*(x_0) r'_*(y_0)}{R_0^2}\right)^d + \frac{1}{\lambda_N} \int_{B_1(y_0)} \int_{B_{(1+\delta)\rho}(x_0)} |\nabla_x \nabla_y G(x, y)|^2 \, \mathrm{d}x \, \mathrm{d}y \right),$$
$$(23)$$

where we defined $\mathcal{G}_{\mathrm{avg}}(y) = \fint_{B_{(1+\delta)\rho}(x_0)} \nabla_y G(x, y) \, \mathrm{d}x$.

Step 4. For a given sequence $\delta_k > 0$ such that $\rho \Pi_{k=1}^{\infty}(1 + \delta_k) \leq 2$ we consider the following iteration procedure. Let $\rho_0 := 1$, and for $k \geq 1$ set $\rho_k := (1 + \delta_k)\rho_{k-1}$. We denote

$$M_k := \left(\frac{r_*(x_0) r'_*(y_0)}{R_0^2}\right)^{-d} \int_{B_1(y_0)} \int_{B_{\rho_k}(x_0)} |\nabla_x \nabla_y G(x, y))|^2 \, \mathrm{d}x \, \mathrm{d}y.$$

For any $N_k \geq 1$, estimate (23) in Step 3 yields

$$M_k \leq \frac{C}{\delta_k^2} N_k + \frac{C}{\delta_k^2} \frac{1}{\lambda_N} M_{k+1}, \tag{24}$$

where the values of δ_k and N_k are at our disposal. We choose $\delta_k := (2k)^{-2}$ and $N_k := \alpha k^{2d} 2^d$. Since $\Pi_{k=1}^{\infty}(1 + \frac{1}{4k^2}) \sim 1.46 \leq 2$, for this choice of δ_k for all $k \geq 1$

we have $\rho_k \in [1,2]$. Using lower bound on the Neumann eigenvalues for the ball in the form $\lambda_k \geq Ck^{\frac{2}{d}}$ (in the case of a cube one can use trigonometric functions to explicitly write down the formula for eigenfunctions and eigenvalues; for the ball, one uses the monotonicity of the eigenvalues with respect to the domain, which follows from the variational formulation of the eigenvalues), we can find large enough α such that the prefactor in front of M_{k+1} above satisfies

$$\frac{C}{\delta_k^2}\frac{1}{\lambda_N} \leq C'k^4(\alpha k^{2d}2^d)^{-\frac{2}{d}} = \frac{C'}{\alpha^{\frac{2}{d}}}\frac{1}{4} \leq \frac{1}{4}.$$

For this choice (24) turns into

$$M_k \leq C\alpha k^4 k^{2d}2^d + \frac{1}{4}M_{k+1}.$$

Iterating this we get

$$M_1 \leq C\alpha \sum_{k=1}^{K} 4^{-k}k^4 k^{2d}2^d + \left(\frac{1}{4}\right)^K M_{K+1}.$$

Assuming we have $\sup_k M_k < \infty$, we send $K \to \infty$ to get

$$M_1 \leq C\alpha 2^d \sum_{k=1}^{\infty} 4^{-k}k^{4+2d}.$$

Since the sum on the right-hand side is summable, we get that $M_1 \lesssim 1$.

It remains to justify the assumption $\sup_k M_k < \infty$. For any $\Lambda \geq 1$, let $\chi_\Lambda(y)$ be the characteristic function of the set $\left\{y \in B_1(y_0) : \int_{B_2(x_0)} |\nabla_x\nabla_y G(x,y)|^2 \leq \Lambda\right\}$. Using the previous arguments, applied to $\nabla_x\nabla_y G(x,y)\chi_\Lambda(y)$, we get that

$$\left(\frac{r_*(x_0)r_*'(y_0)}{R_0^2}\right)^{-d} \int_{B_1(y_0)} \left(\int_{B_1(x_0)} |\nabla_x\nabla_y G(x,y))|^2 \, \mathrm{d}x\right)\chi_\Lambda(y) \, \mathrm{d}y \leq C,$$

where the right-hand side does not depend on Λ. Now we send $\Lambda \to \infty$, and get

$$M_1 = \left(\frac{r_*(x_0)r_*'(y_0)}{R_0^2}\right)^{-d} \int_{B_1(y_0)} \int_{B_1(x_0)} |\nabla_x\nabla_y G(x,y))|^2 \, \mathrm{d}x \, \mathrm{d}y \leq C$$

by the Monotone Convergence Theorem. This completes the proof of (5).

3.2. **Proof of** (6). We first observe that using Poincaré's inequality we can control the difference between $\nabla_y G$ and its averages over $B_1(x_0)$ by the L^2-norm of $\nabla_x\nabla_y G$, which we already control by (5). Hence, to obtain (6) it is enough to estimate averages $\fint_{B_1(x_0)} \nabla_y G(x,y) \, \mathrm{d}x$. Such estimate will follow from an analogue of (16) applied to one particular functional F. Since in this setting we need to work with $\int |u|^2$ and not with previously used $\int |\nabla u|^2$, we will need to use Lemma 2.

Step 1. By Poincaré inequality in the x-variable we have

$$\int_{B_1(y_0)} \left(\int_{B_1(x_0)} \left|\nabla_y G(x,y) - \left(\fint_{B_1(x_0)} \nabla_y G(x',y) \, \mathrm{d}x'\right)\right|^2 \mathrm{d}x\right) \mathrm{d}y \qquad (25)$$

$$\lesssim \int_{B_1(y_0)} \int_{B_1(x_0)} |\nabla_x\nabla_y G(x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y$$

$$\overset{(5)}{\lesssim} \left(\frac{r_*(x_0)r_*'(y_0)}{R_0^2}\right)^d.$$

By the triangle inequality we have

$$\int_{B_1(y_0)} \int_{B_1(x_0)} |\nabla_y G(x,y)|^2 \, dx \, dy$$

$$\lesssim \int_{B_1(y_0)} \left(\int_{B_1(x_0)} \left| \nabla_y G(x,y) - \left(\fint_{B_1(x_0)} \nabla_y G(x',y) \, dx' \right) \right|^2 dx \right) dy$$

$$+ |B_1| \int_{B_1(y_0)} \left(\fint_{B_1(x_0)} \nabla_y G(x,y) \, dx \right)^2 dy,$$

and so (6) follows from (25) provided we show

$$\int_{B_1(y_0)} \left(\fint_{B_1(x_0)} \nabla_y G(x,y) \, dx \right)^2 dy \lesssim \frac{(r_*(x_0) r_*'(y_0))^d}{R_0^{2d-2}}. \tag{26}$$

Step 2. Proof of (26). Similarly as for (5), consider arbitrary $f \in L^2(B_{R_0}(y_0); \mathbb{R}^d)$ and the finite energy solution u of

$$- \nabla \cdot A \nabla u = - \nabla \cdot f$$

in $\mathbb{R}^d$, which satisfies the energy estimate

$$\int_{\mathbb{R}^d} |\nabla u|^2 \lesssim \int_{\mathbb{R}^d} |f|^2. \tag{27}$$

Let F be a linear functional on $L^2(B_1(x_0))$ such that $|F(v)|^2 \leq \int_{B_1(x_0)} |v|^2$ for any $v \in L^2(B_1(x_0))$. Then, if $r_*(x_0) \leq R_0$

$$|F(u)|^2 \quad \leq \quad \int_{B_1(x_0)} |u|^2 \leq \int_{B_{r_*(x_0)}(x_0)} |u|^2 \overset{\text{Lemma 2}}{\lesssim} r_*^d(x_0) \fint_{B_{R_0}(x_0)} |u|^2$$

$$\overset{\text{Jensen}}{\leq} r_*^d(x_0) \left(\fint_{B_{R_0}(x_0)} |u|^{\frac{2d}{d-2}} \right)^{\frac{d-2}{d}} \lesssim \frac{r_*^d(x_0)}{R_0^{d-2}} \left(\int_{\mathbb{R}^d} |u|^{\frac{2d}{d-2}} \right)^{\frac{d-2}{d}}$$

$$\overset{\text{Sobolev}}{\lesssim} \frac{r_*^d(x_0)}{R_0^{d-2}} \int_{\mathbb{R}^d} |\nabla u|^2 \overset{(27)}{\lesssim} \frac{r_*^d(x_0)}{R_0^{d-2}} \int_{\mathbb{R}^d} |f|^2.$$

If otherwise $r_*(x_0) > R_0$, we do not need anymore to appeal to Lemma 2 and may directly bound

$$|F(u)|^2 \quad \leq \quad \int_{B_1(x_0)} |u|^2 \leq \int_{B_{R_0}(x_0)} |u|^2 \lesssim r_*^d(x_0) \fint_{B_{R_0}(x_0)} |u|^2$$

and proceed as in the previous inequality. As before, we use linearity of F and write

$$|F(u)| = \left| \int_{B_{R_0}(y_0)} F(\nabla_y G(\cdot, y)) f(y) \, dy \right|.$$

Since $f \in L^2(B_{R_0}(y_0); \mathbb{R}^d)$ was arbitrary, combination of the two previous estimates yields

$$\int_{B_{R_0}(y_0)} |F(\nabla_y G(\cdot, y))|^2 \, dy \lesssim \frac{r_*^d(x_0)}{R_0^{d-2}}. \tag{28}$$

As before, it remains to argue that by going from $\int_{B_{R_0}(y_0)}$ to $\int_{B_1(y_0)}$ we gain a factor R_0^{-d}. We define $v(y) := F(G(\cdot, y))$, and observe that $-\nabla \cdot A^t \nabla v = 0$ in $B_{R_0}(y_0)$,

where A^t denotes the adjoint coefficient field. Then by definition of v estimate (28) implies

$$\int_{B_1(y_0)} |F(\nabla_y G(\cdot, y))|^2 \, \mathrm{d}y = \int_{B_1(y_0)} |\nabla v|^2 \, \mathrm{d}y \leq \int_{B_{r'_*(y_0)}(y_0)} |\nabla v|^2 \, \mathrm{d}y$$

$$\lesssim \left(\frac{r'_*(y_0)}{R_0}\right)^d \int_{B_{R_0}(y_0)} |\nabla v|^2 \lesssim \frac{(r_*(x_0) r'_*(y_0))^d}{R_0^{2d-2}}. \tag{29}$$

For the choice $F(v) = \fint_{B_1(x_0)} v$ (29) is exactly (26).

3.3. **Proof of** (8). Similarly to the proof of (6), we use Poincaré's inequality (Step 1) to show that (8) follows from (7) provided we control averages of G (Step 2).

Step 1. By Poincaré's inequality in the x-variable we have

$$\int_{B_1(y_0)} \left(\fint_{B_1(x_0)} \left| G(x, y) - \left(\fint_{B_1(x_0)} G(x', y) \, \mathrm{d}x'\right) \right|^2 \mathrm{d}x\right) \mathrm{d}y$$

$$\lesssim \int_{B_1(y_0)} \int_{B_1(x_0)} |\nabla_x G(x, y)|^2 \, \mathrm{d}x \, \mathrm{d}y$$

$$\overset{(7)}{\lesssim} R_0^2 \left(\frac{r'_*(x_0) r_*(y_0)}{R_0^2}\right)^d.$$

Then by the triangle inequality we have

$$\int_{B_1(y_0)} \int_{B_1(x_0)} |G(x, y)|^2 \, \mathrm{d}x \, \mathrm{d}y$$

$$\lesssim \int_{B_1(y_0)} \left(\fint_{B_1(x_0)} \left| G(x, y) - \left(\fint_{B_1(x_0)} G(x', y) \, \mathrm{d}x'\right) \right|^2 \mathrm{d}x\right) 6 \, \mathrm{d}y$$

$$+ |B_1| \int_{B_1(y_0)} \left(\fint_{B_1(x_0)} G(x, y) \, \mathrm{d}x\right)^2 \mathrm{d}y,$$

and so (8) follows provided we show

$$\int_{B_1(y_0)} \left(\fint_{B_1(x_0)} G(x, y) \, \mathrm{d}x\right)^2 \mathrm{d}y \lesssim \frac{(r_*(x_0) r'_*(y_0))^d}{R_0^{2d-4}}. \tag{30}$$

Step 2. Proof of (30). Similarly as for (6), consider arbitrary $f \in L^2(B_{R_0}(y_0))$, but this time u being a finite energy solution of

$$-\nabla \cdot A \nabla u = f$$

in $\mathbb{R}^d$. In order to get the energy estimate, we test the equation with u to obtain:

$$\lambda \int_{\mathbb{R}^d} |\nabla u|^2 \quad \leq \quad \int_{B_{R_0}(y_0)} f u \leq R_0^{\frac{d}{2}} \left(\fint_{B_{R_0}(y_0)} |f|^2\right)^{\frac{1}{2}} \left(\fint_{B_{R_0}(y_0)} |u|^2\right)^{\frac{1}{2}}$$

$$\overset{\text{Jensen}, d \geq 3}{\leq} R_0^{\frac{d}{2}} \left(\fint_{B_{R_0}(y_0)} |f|^2\right)^{\frac{1}{2}} \left(\fint_{B_{R_0}(y_0)} |u|^{\frac{2d}{d-2}}\right)^{\frac{d-2}{2d}}$$

$$= \quad R_0 \left(\int_{B_{R_0}(y_0)} |f|^2\right)^{\frac{1}{2}} \left(\int_{B_{R_0}(y_0)} |u|^{\frac{2d}{d-2}}\right)^{\frac{d-2}{2d}}$$

$$\overset{\text{Sobolev}}{\lesssim} \quad R_0 \left(\int_{B_{R_0}(y_0)} |f|^2 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} |\nabla u|^2 \right)^{\frac{1}{2}},$$

and so

$$\int_{\mathbb{R}^d} |\nabla u|^2 \lesssim R_0^2 \int_{B_{R_0}(y_0)} |f|^2 . \tag{31}$$

We point out that compared to the proof of (5) or (6), we got additional R_0^2 due to the right-hand side of the equation being f and not $\nabla \cdot f$.

Let F be a linear functional on $L^2(B_1(x_0))$ such that $|F(v)|^2 \leq \int_{B_1(x_0)} |v|^2$. If $r_*(x_0) \leq R_0$, then

$$|F(u)|^2 \quad \leq \quad \int_{B_1(x_0)} |u|^2 \leq \int_{B_{r_*(x_0)}(x_0)} |u|^2 \overset{\text{Lemma } 2}{\lesssim} r_*^d(x_0) \fint_{B_{R_0}(x_0)} |u|^2$$

$$\overset{\text{Jensen,}d\geq 3}{\leq} r_*^d(x_0) \left(\fint_{B_{R_0}(x_0)} |u|^{\frac{2d}{d-2}} \right)^{\frac{d-2}{d}} \lesssim \frac{r_*^d(x_0)}{R_0^{d-2}} \left(\int_{\mathbb{R}^d} |u|^{\frac{2d}{d-2}} \right)^{\frac{d-2}{d}}$$

$$\overset{\text{Sobolev}}{\lesssim} \frac{r_*^d(x_0)}{R_0^{d-2}} \int_{\mathbb{R}^d} |\nabla u|^2 \overset{(31)}{\lesssim} \frac{r_*^d(x_0)}{R_0^{d-4}} \int_{\mathbb{R}^d} |f|^2 .$$

If otherwise $r_*(x_0) > R_0$, then we directly bound

$$|F(u)|^2 \quad \leq \quad \int_{B_1(x_0)} |u|^2 \leq \int_{B_{R_0}(x_0)} |u|^2 \lesssim r_*^d(x_0) \fint_{B_{R_0}(x_0)} |u|^2$$

and proceed analogously to the other case. Using the Green's function representation formula we have $u(x) = \int_{B_{R_0}(y_0)} G(x,y) f(y) \, dy$, and thus the linearity of F yields

$$|F(u)| = \left| \int_{B_{R_0}(y_0)} F(G(\cdot, y)) f(y) \, dy \right|.$$

Since $f \in L^2(B_{R_0}(y_0))$ was arbitrary, we may combine the two previous estimates and conclude

$$\int_{B_{R_0}(y_0)} |F(G(\cdot, y))|^2 \, dy \lesssim \frac{r_*^d(x_0)}{R_0^{d-4}} . \tag{32}$$

As before, it remains to argue that by going from $\int_{B_{R_0}(y_0)}$ to $\int_{B_1(y_0)}$ we gain a factor R_0^{-d}. We define $v(y) := F(G(\cdot, y))$, and observe that $-\nabla A^t \nabla v = 0$ in $B_{R_0}(y_0)$.

Now we use Lemma 2 with v to get

$$\int_{B_1(x_0)} |v|^2 \leq \int_{B_{r_*'(y_0)}} |v|^2 \overset{\text{Lemma } 2}{\lesssim} \left(\frac{r_*'(y_0)}{R_0} \right)^d \int_{B_{R_0}(y_0)} |v|^2 \overset{(32)}{\lesssim} \frac{(r_*(x_0) r_*'(y_0))^d}{R_0^{2d-4}} . \tag{33}$$

For the choice $F(v) = \fint_{B_1(x_0)} v$, relation (33) is exactly (30).

4. **Proof of Corollary 1.** We provide a generalization of (6)-(7) in the two-dimensional case. When $d = 2$, the Green's function for the whole space $\mathbb{R}^2$ does not have to exist; nevertheless, we may give a definition for ∇G via the Green's function on $\mathbb{R}^3$. To this purpose we introduce the following notation: If $\bar{x} \in \mathbb{R}^3$, we write $\bar{x} = (x, x_3) \in \mathbb{R}^2 \times \mathbb{R}$ and denote by $\bar{B}_r \subset \mathbb{R}^3$ and $B_r \subset \mathbb{R}^2$ the balls of radius r

and centered at the origin. For a given bounded and uniformly elliptic coefficient field A in $\mathbb{R}^2$, recall that its trivial extension $\bar{A}$ to $\mathbb{R}^3$ was defined in (9) by

$$\bar{A}(x, x_3) := \begin{pmatrix} A(x) & 0 \\ 0 & 1 \end{pmatrix},$$

and the three-dimensional Green's function $\bar{G} = \bar{G}(\bar{A}; \bar{x}, \bar{y})$ is defined as a solution of

$$-\nabla_{\bar{x}} \cdot \bar{A} \nabla_{\bar{x}} \bar{G}(\bar{A}; \cdot, \bar{y}) = \delta(\cdot - \bar{y}).$$

It will become clear below that the argument for the representation formula for ∇G through $\nabla_x \bar{G}$ calls for the notion of pointwise existence in $\bar{y} \in \mathbb{R}^3$ of the Green's function $\bar{G}(\bar{A}; \cdot, \bar{y})$. As mentioned in Section 1, in the case of systems we may only rely on a definition of the Green's function for almost every singularity point $\bar{y}$. Therefore, differently from the previous sections, we need to bear in mind this weaker notion of existence of $\bar{G}$.

Step 1. We argue that for almost every $y \in \mathbb{R}^2$ the function ∇G (since G does not exist, ∇G should be understood as a symbol for a function and not as a gradient of some function G), defined through

$$\nabla G(A; \cdot, y) := \int_{\mathbb{R}} \nabla_x \bar{G}(\bar{A}; (\cdot, x_3), (y, y_3)) \, \mathrm{d}x_3, \tag{34}$$

satisfies for every $\zeta \in C_0^\infty(\mathbb{R}^2)$

$$\int \nabla_x \zeta(x) \cdot A(x) \nabla G(A; x, y) \, \mathrm{d}x = \zeta(y), \tag{35}$$

i.e., in a weak sense it solves $-\nabla_x \cdot A \nabla G(A; \cdot, y) = \delta(\cdot - y)$.

By definition of $\bar{G}(\bar{A}; \cdot, \cdot)$, we have for almost every $\bar{y} \in \mathbb{R}^3$ and every $\bar{\zeta} \in C_0^\infty(\mathbb{R}^3)$

$$\int \nabla_{\bar{x}} \bar{\zeta}(\bar{x}) \cdot \bar{A} \nabla_{\bar{x}} \bar{G}(\bar{A}; \bar{x}, \bar{y}) \, \mathrm{d}\bar{x} = \bar{\zeta}(\bar{y}).$$

Thus, for any $\bar{\rho} \in C_0^\infty(\mathbb{R}^3)$ this yields

$$\int \bar{\rho}(\bar{y}) \int \nabla_{\bar{x}} \bar{\zeta}(\bar{x}) \cdot \bar{A} \nabla_{\bar{x}} \bar{G}(A; \bar{x}, \bar{y}) \, \mathrm{d}\bar{x} \, \mathrm{d}\bar{y} = \int \bar{\rho}(y) \bar{\zeta}(\bar{y}) \, \mathrm{d}\bar{y}.$$

We now choose a sequence $\{\bar{\zeta}_n\}_{n \in \mathbb{N}}$ of test functions $\bar{\zeta}_n = \eta_n \zeta$, with $\zeta = \zeta(x) \in C_0^\infty(\mathbb{R}^2)$ and $\eta_n = \eta_n(x_3)$ smooth cut-off function for $\{|x_3| < n\}$ in $\{|x_3| < n+1\}$: From the previous identity and definition (9) it follows

$$\int \bar{\rho}(\bar{y}) \int \zeta(x) \eta_n'(x_3) \partial_{x_3} \bar{G}(\bar{A}; \bar{x}, \bar{y}) \, \mathrm{d}\bar{x} \, \mathrm{d}\bar{y}$$

$$+ \int \bar{\rho}(\bar{y}) \int \eta_n(x_3) \nabla \zeta(x) \cdot A \nabla \bar{G}(\bar{A}; \bar{x}, \bar{y}) \, \mathrm{d}\bar{x} \, \mathrm{d}\bar{y} = \int \bar{\rho}(y) \zeta(y) \, \mathrm{d}\bar{y}.$$

We now want to send $n \to +\infty$ in the previous identity : By our assumptions on $\bar{\rho}$ and $\bar{\zeta}_n$, if we show that

$$\int_{\mathrm{supp}(\bar{\rho})} \int_{\mathrm{supp}(\zeta) \times \mathbb{R}} |\nabla_{\bar{x}} \bar{G}(\bar{A}; \bar{x}, \bar{y})| \, \mathrm{d}\bar{x} \, \mathrm{d}\bar{y} < +\infty, \tag{36}$$

then by the Dominated Convergence Theorem we may conclude that

$$\int \bar{\rho}(\bar{y}) \int \nabla \zeta(x) \cdot A \left(\int_{\mathbb{R}} \nabla \bar{G}(\bar{A}; \bar{x}, \bar{y}) \, \mathrm{d}x_3 \right) \mathrm{d}x \, \mathrm{d}\bar{y} = \int \bar{\rho}(\bar{y}) \zeta(y) \, \mathrm{d}\bar{y},$$

and thus (35) by the arbitrariness of the test function $\bar{\rho}$ and the separability of $C_0^\infty(\mathbb{R}^2)$.

To argue inequality (36) we proceed as follows: We define a finite radius M such that

$$M \geq \max(r_*(\bar{A}^t, \bar{X}), r_*(A, \bar{Y})) \quad \text{and} \quad \operatorname{supp}(\bar{\rho}) \subset \bar{B}_M(\bar{Y}), \ \operatorname{supp}(\zeta) \subset B_{M/2}(X),$$

and observe that inequality (36) is implied by

$$\int_{\bar{B}_M(\bar{Y})} \int_{B_{M/2}(X) \times \mathbb{R}} |\nabla_{\bar{x}} \bar{G}| \, \mathrm{d}\bar{x} \, \mathrm{d}\bar{y} < +\infty. \tag{37}$$

Since $\bar{A}$ is translational invariant in x_3, the minimal radius $r_*(\bar{A}^t, \cdot)$ is independent of x_3. Then, by the definition of M and Remark 2 we have

$$\int_{\bar{B}_M(\bar{Y})} \int_{\bar{B}_M((X,X_3))} |\bar{\nabla}_x \bar{G}(\bar{A}; \bar{x}, \bar{y})|^2 \, \mathrm{d}\bar{x} \, \mathrm{d}\bar{y} \lesssim \frac{M^6}{|Y - (X, X_3)|^4} \leq \frac{M^6}{|Y_3 - X_3|^4} \tag{38}$$

provided $|X_3 - Y_3| \geq 3M$.

We now cover the cylinder $B_{M/2}(X) \times \mathbb{R}$ with countably many balls of radius M centered at the points $(X, \pm Mn) \in \mathbb{R}^3$. By translational invariance we can w. l. o. g. assume that $Y_3 = 0$. We thus bound the integral in (37) by

$$\int_{\bar{B}_M(\bar{Y})} \int_{B_{M/2}(X) \times \mathbb{R}} |\nabla_{\bar{x}} \bar{G}| \, \mathrm{d}\bar{x} \, \mathrm{d}\bar{y} \leq \sum_{n=0}^{+\infty} \int_{\bar{B}_M(\bar{Y})} \int_{\bar{B}_M(X, \pm Mn)} |\nabla_{\bar{x}} \bar{G}| \, \mathrm{d}\bar{x} \, \mathrm{d}\bar{y}$$

$$\lesssim \int_{\bar{B}_M(\bar{Y})} \int_{\bar{B}_{4M}((X,0))} |\nabla_{\bar{x}} \bar{G}| \, \mathrm{d}\bar{x} \, \mathrm{d}\bar{y} + \sum_{n>4} \int_{\bar{B}_M(\bar{Y})} \int_{\bar{B}_M(X, \pm Mn)} |\nabla_{\bar{x}} \bar{G}| \, \mathrm{d}\bar{x} \, \mathrm{d}\bar{y}. \tag{39}$$

We claim that $\nabla_{\bar{x}} \bar{G}(\bar{A}; \cdot, \cdot) \in L^1_{loc}(\mathbb{R}^3 \times \mathbb{R}^3)$, and so the first integral on the r. h. s. of the previous identity is finite.

Here we only sketch the idea why $\bar{\nabla}_{\bar{x}} \bar{G} \in L^1_{loc}(\mathbb{R}^3 \times \mathbb{R}^3)$; for the proof with all the details we refer to the proof of [10, Theorem 1]. To show that $\bar{\nabla}_{\bar{x}} \bar{G} \in L^1_{loc}$ it suffices to show that $\int_{\bar{B}_R(0)} \int_{\bar{B}_R(0)} |\bar{\nabla}_{\bar{x}} \bar{G}| < \infty$. In order to do that we observe that for given two distinct points $\bar{x}, \bar{y} \in \mathbb{R}^3$, the proof of Theorem 1 (without the use of r_* to go to smaller scales; see also Remark 2) implies in $3D$

$$\left(\int_{\bar{B}_r(\bar{x})} \int_{\bar{B}_r(\bar{y})} |\bar{\nabla}_{\bar{x}} \bar{G}|^2 \right)^{\frac{1}{2}} \lesssim \frac{|\bar{B}_r|}{r^2},$$

where $r = |\bar{x} - \bar{y}|/3$, which by Hölder's inequality turns into

$$\int_{\bar{B}_r(\bar{x})} \int_{\bar{B}_r(\bar{y})} |\bar{\nabla}_{\bar{x}} \bar{G}| \lesssim \frac{|\bar{B}_r|^2}{r^2}.$$

Using a simple covering argument, the above estimate holds also in the case when the balls are replaced by cubes. Since $\bar{B}_R(0) \times \bar{B}_R(0)$ can be written as a null-set plus a countable union of pairs of open cubes $\bar{Q}_{r_n}(\bar{x}_n) \times \bar{Q}_{r_n}(\bar{y}_n)$, each with size $r_n := |\bar{x}_n - \bar{y}_n|/3$ and such that each pair of points $(\bar{x}, \bar{y}) \in \bar{B}_R(0) \times \bar{B}_R(0)$ belongs to at most one such pair of cubes, we conclude

$$\int_{\bar{B}_R(0)} \int_{\bar{B}_R(0)} |\bar{\nabla}_{\bar{x}} \bar{G}| \lesssim \int_{\bar{B}_{2R}(0)} \int_{\bar{B}_{2R}(0)} |\bar{x} - \bar{y}|^{-2} \, \mathrm{d}\bar{x} \, \mathrm{d}\bar{y} < \infty,$$

where we used that for $(\bar{x}, \bar{y}) \in \bar{Q}_{r_n}(\bar{x}_n) \times \bar{Q}_{r_n}(\bar{y}_n)$ we have $|\bar{x} - \bar{y}| \sim r_n$.

Going back to the second term on the right-hand side of (39), an application of Hölder's inequality in both variables $\bar{x}$ and $\bar{y}$ yields for the the sum over n

$$\sum_{n>4} \int_{\bar{B}_M(\bar{Y})} \int_{\bar{B}_M(X,\pm Mn)} |\nabla_{\bar{x}}\bar{G}| \lesssim M^3 \sum_{n>4} \left(\int_{\bar{B}_M(\bar{Y})} \int_{\bar{B}_M(X,\pm Mn)} |\bar{\nabla}_{\bar{x}}\bar{G}|^2 \right)^{\frac{1}{2}}.$$

We now may apply to the r.h.s. the bound (38) and thus obtain

$$\sum_{n>4} \int_{\bar{B}_M(\bar{Y})} \int_{\bar{B}_M(X,\pm Mn)} |\nabla_{\bar{x}}\bar{G}| \lesssim M^6 \sum_{n>4} (Mn)^{-2} \lesssim M^4 < \infty.$$

We have established (36).

Before concluding Step 1, we show that the representation formula (34) does not depend on the choice of the coordinate $y_3 \in \mathbb{R}$, namely that for almost every two values $y_{0,3}, y_{1,3} \in \mathbb{R}$, for almost every $y_0, x_0 \in \mathbb{R}^2$

$$\int_{\mathbb{R}} \nabla_x \bar{G}(\bar{A}; (x_0, x_3), (y_0, y_{0,3})) \, \mathrm{d}x_3 = \int_{\mathbb{R}} \nabla_x \bar{G}(\bar{A}; (x_0, x_3), (y_0, y_{1,3})) \, \mathrm{d}x_3.$$

Without loss of generality we assume $y_{0,3} = 0$: Since by the uniqueness of $\bar{G}(\bar{A}; \cdot, \cdot)$, for every $\bar{z} \in \mathbb{R}^3$ and almost every $\bar{x}, \bar{y} \in \mathbb{R}^3$

$$\bar{G}(\bar{A}; \bar{x} + \bar{z}, \bar{y} + \bar{z}) = \bar{G}(\bar{A}(\cdot + \bar{z}); \bar{x}, \bar{y}),$$

by choosing $\bar{z} = (0, z_3)$ and using definition (9) for $\bar{A}$, we get

$$\bar{G}(\bar{A}; \bar{x} + \bar{z}, \bar{y} + \bar{z}) = \bar{G}(\bar{A}; \bar{x}, \bar{y}). \tag{40}$$

Let $x_0, y_0 \in \mathbb{R}^2$ and $y_{1,3} \in \mathbb{R}^3$ be fixed: For every $\delta > 0$ we may write

$$\fint_{B_\delta(x_0)} \fint_{\bar{B}_\delta((y_0, y_{1,3}))} \int_{\mathbb{R}} \nabla_x \bar{G}(\bar{A}; \bar{x}, \bar{y}) \, \mathrm{d}\bar{x} \, \mathrm{d}\bar{y}$$

$$= \fint_{B_\delta(x_0)} \fint_{\bar{B}_\delta((y_0, y_{1,3}))} \int_{\mathbb{R}} \nabla_x \bar{G}(\bar{A}; (x, x_3 - y_{1,3} + y_{1,3}), (y, y_3 - y_{1,3} + y_{1,3})) \, \mathrm{d}\bar{x} \, \mathrm{d}\bar{y},$$

and use (40) with $\bar{z} = (0, y_{1,3})$ to get

$$\fint_{B_\delta(x_0)} \fint_{\bar{B}_\delta((y_0, y_{1,3}))} \int_{\mathbb{R}} \nabla_x \bar{G}(\bar{A}; \bar{x}, \bar{y}) \, \mathrm{d}\bar{x} \, \mathrm{d}\bar{y} = \fint_{B_\delta(x_0)} \fint_{\bar{B}_\delta((y_0, 0))} \int_{\mathbb{R}} \nabla_x \bar{G}(\bar{A}; \bar{x}, \bar{y}) \, \mathrm{d}\bar{x} \, \mathrm{d}\bar{y}.$$

We now appeal to Lebesgue's theorem and conclude (38).

Step 2. Proof of (11). For this part we denote $r_x := r_*(\bar{A}^t, (x_0, 0))$ and $r_y := r_*(\bar{A}, (y_0, 0))$. By translational invariance of $\bar{A}$ and $\bar{A}^t$ we have $r_x = r_*(\bar{A}^t, (x_0, x_3))$ and $r_y = r_*(\bar{A}, (y_0, y_3))$ for any $x_3, y_3 \in \mathbb{R}$. Denoting $\mathcal{B} := B_1(y_0) \times (-r_y/2, r_y/2)$, the independence of (34) from y_3 yields

$$\int_{\mathcal{B}} \int_{B_1(x_0)} \left| \int_{\mathbb{R}} \nabla_{\bar{x}} \bar{G}(\bar{x}, \bar{y}) \, \mathrm{d}x_3 \right|^2 \, \mathrm{d}x \, \mathrm{d}\bar{y}$$

$$= r_y \int_{B_1(y_0)} \int_{B_1(x_0)} \left| \int_{\mathbb{R}} \nabla_{\bar{x}} \bar{G}(\bar{x}, (y, 0)) \, \mathrm{d}x_3 \right|^2 \, \mathrm{d}x \, \mathrm{d}y$$

$$\overset{(34)}{=} r_y \int_{B_1(y_0)} \int_{B_1(x_0)} |\nabla G(A; x, y)|^2 \, \mathrm{d}x \, \mathrm{d}y.$$

Since $\mathcal{B} \subset \bar{B}_{r_y}((y_0, 0))$, the previous identity implies

$$r_y \int_{B_1(x_0)} \int_{B_1(y_0)} |\nabla_x G(A; x, y)|^2 \, \mathrm{d}x \, \mathrm{d}y = \int_{\mathcal{B}} \int_{B_1(x_0)} \left| \int_{\mathbb{R}} \nabla_x \bar{G}(\bar{A}; \bar{x}, \bar{y}) \, \mathrm{d}x_3 \right|^2 \, \mathrm{d}x \, \mathrm{d}\bar{y}$$

$$\lesssim \int_{\bar{B}_{r_y}((y_0, 0))} \int_{B_1(x_0)} \left| \int_{\mathbb{R}} \nabla_x \bar{G}(\bar{A}; \bar{x}, \bar{y}) \, \mathrm{d}x_3 \right|^2 \, \mathrm{d}x \, \mathrm{d}\bar{y}$$

$$\leq \int_{\bar{B}_{r_y}((y_0, 0))} \int_{B_1(x_0)} \left(\sum_{n=-\infty}^{\infty} \int_{nr_x}^{(n+1)r_x} |\nabla_x \bar{G}(\bar{A}; \bar{x}, \bar{y})| \, \mathrm{d}x_3 \right)^2 \, \mathrm{d}x \, \mathrm{d}\bar{y}.$$

We define a sequence

$$a_n := \frac{(r_x r_y)^{\frac{3}{4}}}{(|x_0 - y_0|^2 + n^2 (r_x)^2)^{\frac{1}{2}}}$$

and observe that

$$\left(\sum_{n=-\infty}^{\infty} \int_{nr_x}^{(n+1)r_x} |\nabla_x \bar{G}(\bar{A}; \bar{x}, \bar{y})| \, \mathrm{d}x_3 \right)^2$$

$$= \left(\sum_{n=-\infty}^{\infty} a_n \frac{r_x}{a_n} \fint_{nr_x}^{(n+1)r_x} |\nabla_x \bar{G}(\bar{A}; \bar{x}, \bar{y})| \, \mathrm{d}x_3 \right)^2$$

$$\overset{\text{Hölder}}{\leq} \left(\sum_{n=-\infty}^{\infty} a_n^2 \right) \left(\sum_{n=-\infty}^{\infty} \frac{(r_x)^2}{a_n^2} \left(\fint_{nr_x}^{(n+1)r_x} |\nabla_x \bar{G}(\bar{A}; \bar{x}, \bar{y})| \, \mathrm{d}x_3 \right)^2 \right)$$

$$\overset{\text{Jensen}}{\leq} \left(\sum_{n=-\infty}^{\infty} a_n^2 \right) \left(\sum_{n=-\infty}^{\infty} \frac{r_x}{a_n^2} \int_{nr_x}^{(n+1)r_x} |\nabla_x \bar{G}(\bar{A}; \bar{x}, \bar{y})|^2 \, \mathrm{d}x_3 \right).$$

Since

$$\sum_{n=-\infty}^{\infty} a_n^2 \lesssim \frac{(r_x r_y)^{\frac{3}{2}}}{|x_0 - y_0| r_x}, \tag{41}$$

where for simplicity we assumed $|x_0 - y_0| \geq r_x$, we combine the three above relations to infer

$$r_y \int_{B_1(x_0)} \int_{B_1(y_0)} |\nabla_x G(A; x, y)|^2 \, \mathrm{d}x \, \mathrm{d}y$$

$$\lesssim \frac{(r_x r_y)^{\frac{3}{2}}}{|x_0 - y_0| r_x} \sum_n \frac{r_x}{a_n^2}$$

$$\times \int_{\bar{B}_{r_y}((y_0, 0))} \int_{\bar{B}_{r_x}(x_0, (n+1/2)r_x)} |\nabla_x \bar{G}(\bar{A}; \bar{x}, \bar{y})|^2 \, \mathrm{d}\bar{x} \, \mathrm{d}\bar{y}$$

$$\overset{(7), d=3}{\lesssim} \frac{(r_x r_y)^{\frac{3}{2}}}{|x_0 - y_0| r_x)} \sum_n \frac{r_x}{a_n^2} a_n^4 \overset{(41)}{\lesssim} \frac{(r_x r_y)^3}{|x_0 - y_0|^2 r_x},$$

which is exactly (11).

Concerning (10), there are two possible ways how to proceed. For the first we observe that (35) implies for every test function $\phi \in C_c^{\infty}(\mathbb{R}^2)$

$$\int \nabla \phi(x) \cdot A(x) \left(\int \nabla_y \nabla G(x, y) \cdot f(y) \, \mathrm{d}y \right) \mathrm{d}x = \int \nabla \phi \cdot f = \int \nabla \phi \cdot A \nabla u,$$

where $f \in L^2(\mathbb{R}^2; \mathbb{R}^2)$ and u is a solution of $-\nabla \cdot A\nabla u = -\nabla \cdot f$. Therefore we have that

$$\nabla u(x) = \int \nabla_y \nabla G(x, y) \cdot f(y) \, \mathrm{d}y,$$

and the proof of (5) applies verbatim. A different way would be to mimic the argument for (11), i.e., to define $\nabla_y \nabla G$ as an integral of the second mixed derivative of the Green's function in three dimension. Unfortunately, this way we would obtain the estimate where the minimal radii in $2D$ appearing on the right-hand side of (10) would need to be replaced (with possibly larger) minimal radii for $3D$.

REFERENCES

[1] S. Armstrong, T. Kuusi and J.-C. Mourrat, Mesoscopic higher regularity and subadditivity in elliptic homogenization, *Comm. Math. Phys.*, **347** (2016), 315–361.

[2] _____, The additive structure of elliptic homogenization, *Invent. Math.*, **208** (2017), 999–1154.

[3] S. N. Armstrong and J.-C. Mourrat, Lipschitz regularity for elliptic equations with random coefficients, *Arch. Ration. Mech. Anal.*, **219** (2016), 255–348.

[4] S. N. Armstrong and C. K. Smart, Quantitative stochastic homogenization of convex integral functionals, *Ann. Sci. Éc. Norm. Supér. (4)*, **49** (2016), 423–481.

[5] M. Avellaneda and F.-H. Lin, Compactness methods in the theory of homogenization, *Comm. Pure Appl. Math.*, **40** (1987), 803–847.

[6] P. Bella, B. Fehrman and F. Otto, A Liouville theorem for elliptic systems with degenerate ergodic coefficients, *To appear in Annals of App. Probabiliy,* ArXiv e-prints (2016).

[7] P. Bella, A. Giunti and F. Otto, Effective multipoles in random media, ArXiv e-prints (2017).

[8] P. Bella, A. Giunti and F. Otto, Quantitative stochastic homogenization: Local control of homogenization error through corrector, *Mathematics and Materials, IAS/Park City Math. Ser., Amer. Math. Soc.,* Providence, RI, **23** (2017), 301–327.

[9] P. Bella and F. Otto, Corrector estimates for elliptic systems with random periodic coefficients, *Multiscale Model. Simul.*, **14** (2016), 1434–1462.

[10] J. G. Conlon, A. Giunti and F. Otto, Green's function for elliptic systems: Existence and Delmotte–Deuschel bounds, *Calc. Var. Partial Differential Equations,* **56** (2017), Art. 163, 51 pp.

[11] E. De Giorgi, Un esempio di estremali discontinue per un problema variazionale di tipo ellittico, *Boll. Un. Mat. Ital. (4),* **1** (1968), 135–137.

[12] T. Delmotte and J.-D. Deuschel, On estimating the derivatives of symmetric diffusions in stationary random environment, with applications to $\nabla\phi$ interface model, *Probab. Theory Related Fields*, **133** (2005), 358–390.

[13] J. Fischer and F. Otto, A higher-order large-scale regularity theory for random elliptic operators, *Comm. Partial Differential Equations*, **41** (2016), 1108–1148.

[14] _____, Sublinear growth of the corrector in stochastic homogenization: Optimal stochastic estimates for slowly decaying correlations, *Stoch. Partial Differ. Equ. Anal. Comput.*, **5** (2017), 220–255.

[15] A. Gloria and D. Marahrens, Annealed estimates on the green functions and uncertainty quantification, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **33** (2016), 1153–1197.

[16] A. Gloria, S. Neukamm and F. Otto, A regularity theory for random elliptic operators, ArXiv e-prints (2014).

[17] _____, Quantification of ergodicity in stochastic homogenization: optimal bounds via spectral gap on Glauber dynamics, *Invent. Math.*, **199** (2015), 455–515.

[18] A. Gloria and F. Otto, The corrector in stochastic homogenization: optimal rates, stochastic integrability, and fluctuations, ArXiv e-prints (2015).

[19] _____, Quantitative results on the corrector equation in stochastic homogenization, *J. Eur. Math. Soc. (JEMS)*, **19** (2017), 3489–3548.

[20] S. M. Kozlov, The averaging of random operators, *Mat. Sb. (N.S.),* **109** (1979), 188–202, 327.

[21] D. Marahrens and F. Otto, Annealed estimates on the Green function, *Probab. Theory Related Fields*, **163** (2015), 527–573.

[22] D. Marahrens and F. Otto, On annealed elliptic Green's function estimates, *Math. Bohem.,* **140** (2015), 489–506.

[23] G. C. Papanicolaou and S. R. S. Varadhan, Boundary value problems with rapidly oscillating random coefficients, *Random Fields, Vol. I, II (Esztergom, 1979), Colloq. Math. Soc. János Bolyai,* vol. 27, North-Holland, Amsterdam-New York, 1981, 835–873.

E-mail address: bella@math.uni-leipzig.de
E-mail address: agiunti@uni-bonn.de