NETWORKS AND HETEROGENEOUS MEDIA
doi:10.3934/nhm.2018006

Volume **13**, Number **1**, March **2018**
pp. **119–153**

FISHER-KPP EQUATIONS AND APPLICATIONS TO A MODEL IN MEDICAL SCIENCES

Benjamin Contri

Aix Marseille Université, CNRS, Centrale Marseille
Institut de Mathématiques de Marseille, UMR 7373, 13453 Marseille, France

(Communicated by Henri Berestycki)

Abstract. This paper is devoted to a class of reaction-diffusion equations with nonlinearities depending on time modeling a cancerous process with chemotherapy. We begin by considering nonlinearities periodic in time. For these functions, we investigate equilibrium states, and we deduce the large time behavior of the solutions, spreading properties and the existence of pulsating fronts. Next, we study nonlinearities asymptotically periodic in time with perturbation. We show that the large time behavior and the spreading properties can still be determined in this case.

1. **Framework and main results.**

We investigate equations of the form

$$u_t - u_{xx} = f^T(t,u),\ t \in \mathbb{R},\ x \in \mathbb{R}, \tag{1}$$

where $f^T : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is of the type

$$f^T(t,u) = g(u) - m^T(t)u, \tag{2}$$

and T is a positive parameter. We suppose that g is a KPP (for Kolmogorov, Petrovsky and Piskunov) function of class $\mathcal{C}^1(\mathbb{R}^+)$ with $\mathbb{R}^+ = [0,+\infty)$. More precisely, we have

$$g > 0 \text{ on } (0,1),\ g(0) = g(1) = 0,\ g'(0) > 0,\ g'(1) < 0, \tag{3}$$

and

$$u \mapsto \frac{g(u)}{u} \text{ decreasing on } (0,+\infty). \tag{4}$$

The previous hypotheses imply in particular that

$$g(u) \le g'(0)u,\ \forall u \in [0,+\infty), \tag{5}$$

and that

$$g < 0 \text{ on } (1,+\infty). \tag{6}$$

2010 *Mathematics Subject Classification.* Primary: 35K57, 92C50; Secondary: 35C07.

Key words and phrases. Reaction-diffusion equation, KPP nonlinearity, principal eigenvalue, pulsating front, spreading speed, medical models.

This work has been carried out in the framework of Archimedes LabEx (ANR-11-LABX-0033) and of the A*MIDEX project (ANR-11-IDEX-0001-02), funded by the "Investissements d'Avenir" French Government program managed by the French National Research Agency (ANR). The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP/2007-2013) / ERC Grant Agreement n.321186 - ReaDi - Reaction-Diffusion Equations, Propagation and Modelling, and from the ANR project NONLOCAL (ANR-14-CE25-0013).

In Sections 2 and 4, the function m^T is T-periodic, nonnegative and of class $\mathcal{C}^1(\mathbb{R})$. In this case, the function f^T is a T-periodic in time function of class $\mathcal{C}^1(\mathbb{R}\times\mathbb{R}^+)$ such that $f^T(\cdot,0)=0$ on $\mathbb{R}$. Furthermore, according to (6) and the nonnegativity of m^T, we have

$$f^T(t,u)<0,\ \forall (t,u)\in\mathbb{R}\times(1,+\infty). \tag{7}$$

In Section 3, the function m^T is asymptotically periodic in time. We give more details about this notion later in this introduction.

1.1. **Biological interpretation.** Equations of the type

$$u_t-u_{xx}=g(u)-m^T(t)u,\ t\in\mathbb{R},\ x\in\mathbb{R}, \tag{8}$$

are proposed to model the spatial evolution over time of a cancerous tumor in the presence of chemotherapy. The quantity $u(t,x)$ represents the density of cancer cells in the tumor at the position x and at the time t. We begin by considering, for $T>1$, a particular case of periodic function $m^T:\mathbb{R}^+\to\mathbb{R}$ of class $\mathcal{C}^1(\mathbb{R}^+)$ for which there exists a nontrivial function $\varphi:[0,1]\to[0,+\infty)$ with $\varphi(0)=\varphi(1)=0$ such that

$$\begin{cases} m^T=\varphi \text{ on } [0,1), \\ m^T=0 \text{ on } [1,T). \end{cases} \tag{9}$$

In the absence of treatment, cancer cells reproduce and spread in space. This reproduction is modeled by the reaction term of KPP type $g(u)$, which takes into account the fact that the resources of the environment of the tumor are not infinite and so, that there is a maximal size beyond which the tumor cannot grow anymore. To treat the patient, cycles of chemotherapy are given. Every cycle lasts a lapse of time T and is composed of two subcycles. The duration of the first one is equal to 1. During this time, the drug acts on the tumor. At every moment of the first subcycle, the death rate of the cancer cells due to the drug is equal to $\varphi(t)$. In this case, the total reaction term is $g(u)-\varphi(t)u$. There is a competition between the reproduction term and the death term. The chemotherapy has a toxic effect on the body because it destroys white blood cells. It is thus essential to take a break in the administration of the treatment. This break is the second subcycle of the cycle of chemotherapy. It lasts during a time equal to $T-1$. In this case, the reaction term is just $g(u)$, and thus, the tumor starts to grow again. To summarize, the term $m^T(t)$ defined in (9) represents the concentration of drug in the body of the patient at time t, and the integral $\int_0^T m_T(s)ds=\int_0^1\varphi(t)dt$ represents the total quantity of drug in the patient during a cycle of chemotherapy. Finally, we impose for this type of functions m^T that

$$g'(0)-\int_0^1\varphi(t)dt<0. \tag{10}$$

This inequality is not really restricting. Indeed, we shall see after that this hypothesis is in fact a condition so that the patient is cured in the case where there is no rest period in the cycles of chemotherapy (that is $T=1$).

We now refine the previous modelling. In fact, the concentration of drug in the patient's body is not a datum. We only know the concentration of drug injected to the patient. We denote $D^T(t)$ this concentration at time t, and we assume that the function $D^T:\mathbb{R}^+\to\mathbb{R}^+$ is T-periodic and satisfies

$$D^T(t)=\begin{cases} 1, & \forall t\in[0,1], \\ 0, & \forall t\in(1,T). \end{cases} \tag{11}$$

The concentration of drug m is then the Lipschitz-continuous and piecewise $\mathcal{C}^1$ solution $m : \mathbb{R}^+ \to \mathbb{R}$ of a Cauchy problem of the type

$$\begin{cases} m'(t) = D^T(t) - \frac{m(t)}{\tau}, \ \forall t \in \mathbb{R}^+, \\ m(0) = m_0 \geq 0. \end{cases} \tag{12}$$

The real number $\tau > 0$ is called clearance. It characterizes the ability of the patient's body to eliminate the drug. It is also possible to take into account that the patient does not necessarily take the treatment in an optimal way. It may happen to him/her, for example, to forget his/her medicine, or being forced to move a chemotherapy session if it is programmed on a holiday. So, we add to the nonlinearity a perturbative term of the type $\varepsilon p(t,u)$, where $\varepsilon \geq 0$ and $p : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$. It corresponds to study equations of the type

$$u_t - u_{xx} = g(u) - m(t)u + \varepsilon p(t,u), \ t \in \mathbb{R}, \ x \in \mathbb{R},$$

where m solves (12).

1.2. **Mathematical framework.** The mathematical study of reaction-diffusion equations began in the 1930's. Fisher [12] and Kolmogorov, Petrovsky and Piskunov [17] were interested in wave propagation in population genetics modeled by the homogeneous equation

$$u_t - u_{xx} = f(u), \ t \in \mathbb{R}, \ x \in \mathbb{R}. \tag{13}$$

In the 1970's, their results were generalized by Aronson and Weinberger [1] and Fife and McLeod [11]. In particular, if f is a KPP nonlinearity (that is, f satisfies (3) and (5)), there exists a unique (up to translation) planar fronts U_c of speed c, for any speed $c \geq c^* := 2\sqrt{f'(0)}$, that is, for any $c \geq c^*$, there exists a function u_c satisfying (13) and which can be written $u_c(t,x) = U_c(x - ct), \text{ with } 0 < U_c < 1, \ U_c(-\infty) = 1$ and $U_c(+\infty) = 0$. Furthermore, if $c < c^*$, there is no such front connecting 0 and 1. Another property for this type of nonlinearities is that if we start from a nonnegative compactly supported initial datum u_0 such that $u_0 \not\equiv 0$, then the solution u of (13) satisfies $u(t,x) \to 1$ as $t \to +\infty$. Aronson and Weinberger name this phenomenon the "hair trigger effect". Moreover the set where $u(t,x)$ is close to 1 expands at the speed c^*.

Freidlin and Gärtner in [13] were the first to study heterogeneous equations. More precisely, they generalized spreading properties for KPP type equations with periodic in space coefficients. Since this work, numerous papers have been devoted to the study of heterogeneous equations with KPP or other reaction terms. We can cite e.g. [2, 3, 4, 5, 6, 8, 10, 16, 19, 27, 28, 29] in the case of periodic in space environment, [14, 18, 19, 24, 25] in the case of periodic in time environment and [21, 22, 23] in the case of periodic in time and in space environment. The works of Nadin [21, 22] and Liang and Zhao [19] are the closest of our paper. We will compare later the contributions of our work with these references. We now give the main results of the paper.

When the nonlinearity is not homogeneous, there are no planar front solutions of (8) anymore. For equations with coefficients depending periodically on the space variable, Shigesada, Kawasaki and Teramoto [26] defined in 1986 a notion more general than the planar fronts, namely the pulsating fronts. This notion can be extended for time dependent periodic equations as follows.

Definition 1.1. For equation (1), assume that f^T is T-periodic and that (1) has a T-periodic solution $\theta : \mathbb{R} \to (0, +\infty), \ t \mapsto \theta(t)$. A pulsating front connecting 0 and $\theta(t)$ for equation (1) is a solution $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ such that there exists a real number c and a function $U : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ verifying

$$\begin{cases} u(t,x) = U(t, x - ct), \ \forall t \in \mathbb{R}, \ \forall x \in \mathbb{R}, \\ U(\cdot, -\infty) = \theta, \ U(\cdot, +\infty) = 0, \ \text{uniformly on } \mathbb{R}, \\ U(t+T, x) = U(t,x), \ \forall t \in \mathbb{R}, \ \forall x \in \mathbb{R}. \end{cases}$$

So, a pulsating front connecting 0 and θ for equation (1) is a couple $(c, U(t,\xi))$ solving the problem

$$\begin{cases} U_t - cU_\xi - U_{\xi\xi} - f^T(t,U) = 0, \ \forall (t,\xi) \in \mathbb{R} \times \mathbb{R}, \\ U(\cdot, -\infty) = \theta, \ U(\cdot, +\infty) = 0, \ \text{uniformly on } \mathbb{R}, \\ U(t+T, \xi) = U(t,\xi), \ \forall (t,\xi) \in \mathbb{R} \times \mathbb{R}. \end{cases}$$

In this definition, by standard parabolic estimates, the limiting state $\theta = U(\cdot, -\infty)$ solves the system

$$\begin{cases} y' = f^T(t,y) \text{ on } \mathbb{R}, \\ y(0) = y(T), \end{cases} \tag{14}$$

whose solutions are called equilibrium states of the equation (1).

If $\theta : \mathbb{R} \to \mathbb{R}$ is a solution of (14), let us now define λ_{θ,f^T} and $\Phi_{\theta,f^T} : \mathbb{R} \to \mathbb{R}$ as the unique real number and the unique function (up to multiplication by a constant) which satisfy

$$\begin{cases} (\Phi_{\theta,f^T})' = \big(f_u^T(t,\theta) + \lambda_{\theta,f^T}\big)\Phi_{\theta,f^T} \text{ on } \mathbb{R}, \\ \Phi_{\theta,f^T} > 0 \text{ on } \mathbb{R}, \\ \Phi_{\theta,f^T} \text{ is } T-\text{periodic}. \end{cases} \tag{15}$$

These quantities are called respectively principal eigenvalue and principal eigenfunction associated with f^T and the equilibrium state θ. Furthermore, if we divide the previous equation by Φ_{θ,f^T}, and if we integrate over $(0,T)$, we obtain an explicit formulation of the principal eigenvalue, namely

$$\lambda_{\theta,f^T} = -\frac{1}{T}\int_0^T f_u^T(s,\theta(s))ds.$$

We now recall the definition of the Poincaré map P^T associated with f^T. For any $\alpha \geq 0$, let $y_\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ be the solution of the Cauchy problem

$$\begin{cases} y' = f^T(t,y) \text{ on } \mathbb{R}, \\ y(0) = \alpha. \end{cases} \tag{16}$$

Definition 1.2. The Poincaré map associated with f^T is the function $P^T : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$P^T(\alpha) = y_\alpha(T).$$

We conclude, with the fact that each nonnegative solution of (14) is associated with a fixed point of P^T, and conversely. Furthermore, if $\alpha^T \geq 0$ is a fixed point of P_T we have the following equality

$$(P^T)'(\alpha^T) = e^{-T\lambda_{y_{\alpha^T},f^T}}. \tag{17}$$

We can find these results concerning the notions of principal eigenvalue and Poincaré map in [7], [9], [15] and [20].

1.3. **Nonlinearities periodic in time.** Let $T > 0$. In Section 2, we study (1) and (2) with functions m^T which are T-periodic in time. For these functions we assume there exists $T^* > 0$ such that

$$\lambda_{0,f^T} \begin{cases} > 0 \text{ if } T < T^*, \\ < 0 \text{ if } T > T^*, \\ = 0 \text{ if } T = T^*. \end{cases} \tag{18}$$

This is indeed the case if m^T is of the type (9) because

$$\lambda_{0,f^T} = -g'(0) + \frac{1}{T}\int_0^T m^T(s)ds = -g'(0) + \frac{1}{T}\int_0^1 \varphi(s)ds.$$

Furthermore, for this type of functions, hypothesis (10) implies that $\lambda_{0,f^{T=1}} > 0$. Hence, in this case $T^* > 1$. The existence and uniqueness of positive solutions of (14) is summarized in the following result.

Proposition 1. *We consider the real number T^* defined in (18).*
(I) If $T \leq T^*$, there is no positive solution of (14).
(II) If $T > T^*$, there is a unique positive solution w^T of (14). Furthermore,

(i) For any $t \in \mathbb{R}$ we have $w^T(t) \in (0,1]$, and

$$\frac{1}{T}\int_0^T f_u^T(s, w^T(s))ds \leq 0.$$

(ii) If $T \mapsto m^T$ is continuous in $L^\infty_{loc}(\mathbb{R})$, then the function $T \in (T^*, +\infty) \mapsto w^T(0)$ is continuous and, if m^T is of type (9) with assumption (10), it is increasing.

(iii) If $T \mapsto m^T$ is continuous in $L^\infty_{loc}(\mathbb{R})$, then the function w^T converges uniformly to 0 on $\mathbb{R}$ as $T \to (T^*)^+$.

(iv) If m^T is of type (9) with assumption (10), then w^T converges on average to 1 as T tends to $+\infty$:

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T w^T(t)dt = 1.$$

The same result of existence and uniqueness (result of the type (*II*)) was proved for KPP nonlinearities depending periodically on space by Berestycki, Hamel and Roques in [5] and for KPP nonlinearities depending periodically on space and time by Nadin in [22]. We give here a proof using the Poincaré map associated with f^T. The last two points of the proposition are quite intuitive. Indeed, the limit as $T \to (T^*)^+$ is explained by the fact that for $T \leq T^*$, the only nonnegative equilibrium state is zero. The limit as $T \to +\infty$ is explained by the fact that in this case, the nonlinearity f^T is "almost" the KPP function g since the function m^T has an average close to 0 when T is large.

Let us now summarize a result in [22], which deals with the evolution of $u(t,x)$ as $t \to +\infty$.

Proposition 2. [22] *Let $u_0 : \mathbb{R} \to \mathbb{R}$ be a bounded and continuous function on $\mathbb{R}$ such that $u_0 \geq 0$ and $u_0 \not\equiv 0$. Under assumption (18), we consider the function $u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ satisfying*

$$\begin{cases} u_t - u_{xx} = f^T(t,u) \text{ on } (0,+\infty) \times \mathbb{R}, \\ u(0,\cdot) = u_0 \text{ on } \mathbb{R}. \end{cases} \tag{19}$$

If $T < T^*$, then there exists $M > 0$ depending only on u_0 and Φ_{0,f^T} such that

$$0 \leq u(t,x) \leq M\Phi_{0,f^T}(t)e^{-\lambda_{0,f^T}t}, \ \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}. \tag{20}$$

If $T = T^*$, then

$$\sup_{x\in\mathbb{R}} |u(t,x)| \xrightarrow{t\to+\infty} 0.$$

If $T > T^*$, then for every compact set $K \subset \mathbb{R}$, we have

$$\sup_{x\in K} |u(t,x) - w^T(t)| \xrightarrow{t\to+\infty} 0.$$

A similar result was proved for KPP nonlinearities depending periodically on space by Berestycki, Hamel and Roques in [5].

In the biological context with m^T satisfying (9), the treatment is effective (in the sense that $u(t,x) \to 0$ uniformly on $\mathbb{R}$ as $t \to +\infty$) if and only if the duration of cycles of chemotherapy is equal or less than T^*. In particular, since hypothesis (10) implies that $T^* > 1$, the treatment is effective if there is no rest period between two injections of drug, that is as $T = 1$. The result is interesting because it implies that $T^* - 1$ is the longest rest period for which the patient recovers. Inequality (20) refines the criterion of cure of the patient because according to the fact that the function $T \mapsto \lambda_{0,f^T}$ is decreasing and positive on $(0, T^*)$, the convergence rate of the density $u(t,x)$ to 0 as $t \to +\infty$ is all the faster as T is small. In other words, in the case of effective treatment, shorter the period between two injections, more quickly the patient will be cured. If the treatment is not effective, the equilibrium state w^T invades the whole space as $t \to +\infty$. In particular, the tumor can not grow indefinitely. Finally, Proposition 2 also allows to clarify the result (ii) of Proposition 1. The fact that $T \mapsto w^T(0)$ is increasing on $(T^*, +\infty)$ implies that in the case where the treatment is not effective (that is $w^T > 0$ invades the whole space as $t \to +\infty$), the longer the rest period between two injections, the denser the equilibrium state of the tumor.

We now study in more detail the case where the treatment is not effective, that is, the case where $T > T^*$. We know that then, the equilibrium state w^T invades the whole space as $t \to +\infty$. The purpose of this part is to give the invasion rate of the zero state by w^T. To answer this question, we quote two results. The first one is about the existence of pulsating fronts connecting 0 and w^T, in the sense of Definition 1.1, and the second one concerns spreading properties. They are proved in [18] and in [21].

Theorem 1.3. [18],[21] *Let $T > T^*$, where T^* is given in (18). (I) There exists a positive real number c_T^* such that pulsating fronts with speed c connecting 0 and w^T exist if and only if $c \geq c_T^*$.*

(II) We denote $u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ the solution of the Cauchy problem

$$\begin{cases} u_t - u_{xx} = f^T(t,u) \text{ on } (0,+\infty) \times \mathbb{R}, \\ u(0,\cdot) = u_0 \text{ on } \mathbb{R}. \end{cases}$$

If u_0 is a bounded continuous function such that $u_0 \geq 0$ and $u_0 \not\equiv 0$, then

$$\forall c \in (0, c_T^*), \ \lim_{t\to+\infty} \sup_{|x|<ct} \left|u(t,x) - w^T(t)\right| = 0.$$

If u_0 is a continuous compactly supported function such that $u_0 \geq 0$, then

$$\forall c > c_T^*, \ \lim_{t\to+\infty} \sup_{|x|>ct} u(t,x) = 0.$$

In his paper [21], Nadin considers in the first assertion of the spreading properties in Theorem 1.3 initial conditions which are more general. He assumes that u_0 is not necessarily compactly supported but that u_0 is of the form $O(e^{-\beta|x|})$ as $|x| \to +\infty$, where $\beta > 0$. The previous theorem completes Proposition 2. Indeed, we know that in the case where the treatment is not effective, the equilibrium state w^T invades the whole space as $t \to +\infty$. Theorem 1.3 states that this invasion takes place at the speed c_T^*.

We can now characterize the critical speed c_T^* with the principal eigenvalue λ_{0,f^T}. More precisely:

Proposition 3. *For every $T > T^*$, the critical speed c_T^* is given by*

$$c_T^* = 2\sqrt{-\lambda_{0,f^T}}. \tag{21}$$

Hence, if $T \mapsto \int_0^T m^T(s)ds$ is continuous, then the function $T \in (T^*, +\infty) \mapsto c_T^*$ is continuous and, if $\int_0^T m^T(s)ds$ does not depend on T, it is increasing. Furthermore, we have the two following limit cases:

$$\lim_{T\to(T^*)^+} c_T^* = 0,$$

and, if $\frac{1}{T}\int_0^T m^T(s)ds \xrightarrow{T\to+\infty} 0$, then

$$\lim_{T\to+\infty} c_T^* = 2\sqrt{g'(0)}.$$

In the case where the treatment is not effective, the invasion of space by the equilibrium state w^T is all the faster as the rest time between injections is long. The two limits cases $T \to (T^*)^+$ and $T \to +\infty$ are explained in the same manner as in Proposition 1. Let us note that in the case where m^T is of the type (9), then the previous properties concerning $\int_0^T m^T(s)ds$ are satisfied.

We end this section by stating the existence of pulsating fronts in the case of nonlinearities which are not of KPP type (that is hypotheses (4) and (5) are not necessarily verified, but we still assume (3), (6) and (18)). For these nonlinearities, there is still a positive solution to problem (14), but it may not be unique. According to Cauchy-Lipschitz theorem, solutions of (14) are ordered on $[0, T]$. For $T > T^*$, we can thus define $y^T : \mathbb{R} \to \mathbb{R}$ as the infimum of all positive solutions of (14). After showing that $y^T > 0$, we will prove there exists a critical speed $c_T^{**} > 0$ such that there is a pulsating front connecting 0 and y^T for speed $c \geq c_T^{**}$ and there is no pulsating front connecting 0 and y^T for $c < c_T^{**}$. In this case, c_T^{**} is not necessarily equal to $2\sqrt{-\lambda_{0,f^T}}$. For this type of nonlinearity, Nadin shows in [21] that there exist two critical speeds c_* and c^* for which there is a pulsating front for $c \geq c^*$ and there is no pulsating front for $c \leq c_*$. Nevertheless the case $c \in (c_*, c^*)$ is not treated in [21]. In [18], Liang and Zhao prove the result using a semiflow method. We give here an alternative proof. We begin by proving the existence of pulsating front $U(t, \xi)$ for domains of the type $\mathbb{R} \times [-a, a]$ which are bounded in ξ, then we pass in the limit as $a \to +\infty$. We state the result.

Proposition 4. *Let f^T satisfy assumptions (2), (3), (6) and (18), and $T > T^*$. There exists a positive real number c_T^{**} such that pulsating fronts $U(t,\xi)$ monotone in ξ connecting 0 and y^T exist if and only if $c \geq c_T^{**}$.*

1.4. **Nonlinearities asymptotically periodic in time with perturbation.** We are interested in the case of nonlinearities which are no more periodic in time, but which are the sum of a function which converges as $t \to +\infty$ to a time periodic nonlinearity and of a small perturbation. More precisely, for $\varepsilon \geq 0$, we consider equations of the type

$$u_t - u_{xx} = g(u) - m(t)u + \varepsilon p(t,u), \ t \in \mathbb{R}, \ x \in \mathbb{R}, \tag{22}$$

where m solves (12) with $T > 1$ and D^T defined in (11). We assume that $p : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a function of class $\mathcal{C}^1$ for which there exists $C > 0$ such that

$$\left|\frac{p(t,u)}{u}\right| \leq C, \ \forall (t,u) \in \mathbb{R}^+ \times (0,+\infty). \tag{23}$$

The function m is not periodic, but it is asymptotically T-periodic in time. More precisely, there exists a T-periodic positive function $m_\infty^T : \mathbb{R} \to (0,+\infty)$ such that

$$\lim_{t\to+\infty} |m(t) - m_\infty^T(t)| = 0. \tag{24}$$

Indeed, an elementary calculation implies that for any $n \in \mathbb{N}$, we have

$$m(t) = \begin{cases} \tau\Big[1 + \Big(\frac{(e^{\frac{1}{\tau}}-1)(e^{\frac{nT}{\tau}}-1)}{e^{\frac{T}{\tau}}-1} + \frac{m_0}{\tau} - e^{\frac{nT}{\tau}}\Big)e^{-\frac{t}{\tau}}\Big], \ \forall t \in [nT, nT+1), \\ \tau\Big[\frac{(e^{\frac{1}{\tau}}-1)(e^{\frac{(n+1)T}{\tau}}-1)}{e^{\frac{T}{\tau}}-1} + \frac{m_0}{\tau}\Big]e^{-\frac{t}{\tau}}, \ \ \forall t \in [nT+1, (n+1)T). \end{cases}$$

Consequently, if we define the positive T-periodic function $m_\infty^T : \mathbb{R} \to (0,+\infty)$ by

$$m_\infty^T(t) = \begin{cases} \tau\Big[1 + \Big(\frac{e^{\frac{1}{\tau}}-1}{e^{\frac{T}{\tau}}-1} - 1\Big)e^{-\frac{t}{\tau}}\Big], \ \forall\, t \in [0,1], \\ \tau\frac{e^{\frac{1}{\tau}}-1}{e^{\frac{T}{\tau}}-1}e^{\frac{T-t}{\tau}}, \ \ \forall\, t \in [1,T), \end{cases}$$

then the convergence result (24) holds. Furthermore, we have $\int_0^T m_\infty^T(t)dt = \tau$. Consequently the function $f^T : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ defined by $f^T(t,u) = g(u) - m_\infty^T(t)u$ satisfies (18) because $\lambda_{0,f^T} = -g'(0) + \tau/T$. We assume that $\tau > g'(0)$. We notice that m_∞^T is independent of m_0. It was predictable because m_∞^T is the unique positive T-periodic solution of $m' = D^T - m/\tau$ on $\mathbb{R}$. We define the nonlinearities $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and $f_\varepsilon : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by

$$f(t,u) = g(u) - m(t)u, \ \text{and} \ f_\varepsilon(t,u) = f(t,u) + \varepsilon p(t,u).$$

According to (24), we have

$$\sup_{u\in(0,+\infty)} \left|\frac{f(t,u) - f^T(t,u)}{u}\right| \xrightarrow{t\to+\infty} 0. \tag{25}$$

The function f^T is T-periodic and satisfies the general assumptions given in Section 1.3. We still denote T^* the critical time (notice that $T^* > 1$ because $\tau > g'(0)$), w^T the unique positive equilibrium state for $T > T^*$ and c_T^* the critical speed associated with f^T for $T > T^*$.

The aim of this section is to show that Proposition 2 and the spreading results of Theorem 1.3 hold true when we replace f^T by f_ε in the statements, for ε small enough. It is reasonable to hope so. Indeed, on the one hand, if ε is small, then the term εp is negligible compared to f, and on the other hand, these results deal with the large time behavior of the solutions, and precisely, hypothesis (25) implies that f "looks like" f^T as $t \to +\infty$. The first result is the generalization of Proposition 2.

Theorem 1.4. *Let $u_0 : \mathbb{R} \to \mathbb{R}$ be a bounded and continuous function such that $u_0 \geq 0$ and $u_0 \not\equiv 0$. For all $\varepsilon \geq 0$, we consider the function $u_\varepsilon : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ satisfying*

$$\begin{cases} u_t - u_{xx} = f_\varepsilon(t,u) \text{ on } (0,+\infty) \times \mathbb{R}, \\ u(0,\cdot) = u_0 \text{ on } \mathbb{R}. \end{cases} \tag{26}$$

If $T < T^*$, there exists $\varepsilon_T > 0$ such that for all $\varepsilon \in (0, \varepsilon_T)$ we have

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} |u_\varepsilon(t,x)| = 0.$$

If $T > T^*$ and if $\lambda_{w^T,f^T} > 0$, then there exist $\tilde{\varepsilon}_T > 0$ and $M_T > 0$ such that for all $\varepsilon \in (0, \tilde{\varepsilon}_T)$ and for all compact $K \subset \mathbb{R}$, we have

$$\limsup_{t\to+\infty} \sup_{x\in K} |u_\varepsilon(t,x) - w^T(t)| \leq M_T \varepsilon.$$

We saw in Proposition 1 that $\lambda_{w^T,f^T} \geq 0$. In the previous theorem, in case $T > T^*$, we impose that $\lambda_{w^T,f^T} > 0$. This property is not necessarily satisfied. Indeed, if we consider the function $h : \mathbb{R}^+ \to \mathbb{R}$ defined by $h(u) = u(1-u)^3$, then we have $h(0) = h(1) = 0$, $h > 0$ on $(0,1)$, $h < 0$ sur $(1,+\infty)$, $h(u)/u$ decreasing on $(0,+\infty)$ and $h'(1) = 0$. In the case where the function $f^T(t,\cdot)$ is concave for all $t \in \mathbb{R}^+$, the property $\lambda_{w^T,f^T} > 0$ is verified for any $T > T^*$. Indeed, if we define $F : [0,1] \to \mathbb{R}$ by

$$F(x) = -\frac{1}{T}\int_0^T \frac{f^T(s, xw^T(s))}{w^T(s)} ds,$$

then we have $F(0) = F(1) = 0$ and F is convex on $[0,1]$. Consequently, if $F'(1) = 0$, that is, if $\lambda_{w^T,f^T} = 0$, then we have $F' = 0$ on $[0,1]$. It is a contradiction because $F'(0) = \lambda_{0,f^T} < 0$.

Let us give a sketch of the proof. For $T > 0$ and $\varepsilon > 0$, we will frame f_ε by two T-periodic functions f_ε^T and $f_{-\varepsilon}^T$ for which the results of Proposition 2 will apply. In the case where $T < T^*$, if f_ε^T is the upper bound function, we will show that for $\varepsilon > 0$ small enough, we have $\lambda_{0,f_\varepsilon^T} > 0$. Hence, the solution of (26) with f_ε^T as nonlinearity is a supersolution of problem (26) and, according to Proposition 2, it converges to 0 as $t \to +\infty$. In the case where $T > T^*$, we will prove that for $\varepsilon > 0$ small enough, we have $\lambda_{0,f_\varepsilon^T} < 0$ and $\lambda_{0,f_{-\varepsilon}^T} < 0$. Consequently, there is a unique positive solution w_ε^T (resp. $w_{-\varepsilon}^T$) of system (14) with f_ε^T (resp. $f_{-\varepsilon}^T$) as nonlinearity (owing to Proposition 1). The solution of (26) with f_ε^T as nonlinearity is a supersolution of (26) and, according to Proposition 2, it converges to w_ε^T as $t \to +\infty$. In the same way, the solution of (26) with $f_{-\varepsilon}^T$ as nonlinearity is a subsolution of (26), and it converges to $w_{-\varepsilon}^T$ as $t \to +\infty$. We will conclude using the fact that w_ε^T and $w_{-\varepsilon}^T$ are close to w^T as ε is small enough.

Note that the case $T = T^*$ is not treated in Theorem 1.4. If $\varepsilon = 0$, the solution of the Cauchy problem (26) converges uniformly to 0 as $t \to +\infty$, whereas if $\varepsilon > 0$, the convergence to 0 may not hold. We summarize these results in the following proposition.

Proposition 5. *Let $T = T^*$ and $\varepsilon \geq 0$. We consider the function $u_\varepsilon : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ satisfying the Cauchy problem* (26).
(I) If $\varepsilon = 0$, then u_ε converges uniformly to 0 as $t \to +\infty$.
(II) If $\varepsilon > 0$, we can conclude in two cases.
(i) If $f(t,u) = f^{T^*}(t,u)$ and $p(t,u) = u$, then, for ε small enough, u_ε converges to a positive solution of (14) *with f_ε as nonlinearity as $t \to +\infty$.*
(ii) If $p(t,u) \leq 0$, then, u_ε converges uniformly to 0 as $t \to +\infty$.

Concerning the spreading results of Theorem 1.3, they remain true if we replace f^T by f_ε in the statement.

Theorem 1.5. *Let $T > T^*$. For any $\varepsilon \geq 0$, we consider $u_\varepsilon : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ satisfying*

$$\begin{cases} u_t - u_{xx} = f_\varepsilon(t,u) \text{ on } (0,+\infty) \times \mathbb{R}, \\ u(0,\cdot) = u_0 \text{ on } \mathbb{R}. \end{cases}$$

If u_0 is a continuous bounded function such that $u_0 \geq 0$ and $u_0 \not\equiv 0$, and if $\lambda_{w^T, f^T} > 0$, then for all $c \in (0, c_T^*)$, there exists $\hat{\varepsilon}_{c,T} > 0$ such that for all $\varepsilon \in (0, \hat{\varepsilon}_{c,T})$ we have

$$\limsup_{t \to +\infty} \sup_{|x| < ct} \left| u_\varepsilon(t,x) - w^T(t) \right| \leq M_T \varepsilon,$$

where M_T is defined in Theorem 1.4.

If u_0 is a continuous compactly supported function such that $u_0 \geq 0$, then, for all $c > c_T^*$, there exists $\overline{\varepsilon}_{c,T} > 0$ such that for all $\varepsilon \in (0, \overline{\varepsilon}_{c,T})$ we have

$$\lim_{t \to +\infty} \sup_{|x| > ct} u_\varepsilon(t,x) = 0.$$

The proof of this theorem uses the same ideas as the proof of Theorem 1.4. For $T > T^*$ and $\varepsilon > 0$, we will frame f_ε by two T-periodic functions f_ε^T and $f_{-\varepsilon}^T$ for which the results of Theorem 1.3 will apply. An important point of the demonstration will be to notice that for ε small enough, the critical speeds $c_{T,\varepsilon}^*$ and $c_{T,-\varepsilon}^*$ associated respectively with f_ε^T and $f_{-\varepsilon}^T$ are close to the critical speed c_T^* associated with f^T.

1.5. **Influence of the protocol of the treatment.** As in Section 1.1, we consider a $\mathcal{C}^1$ and T-periodic function m^T (with $T \geq 1$) of the type

$$\begin{cases} m^T = \varphi \text{ on } [0,1), \\ m^T = 0 \text{ on } [1,T), \end{cases}$$

where $\varphi : [0,1] \to [0,+\infty)$ satisfies $\varphi(0) = \varphi(1) = 0$. In this part, we are interested in equations of the type

$$u_t - u_{xx} = g(u) - m_\tau^T(t)u, \ t \in \mathbb{R}, \ x \in \mathbb{R}, \tag{27}$$

where $0 < \tau \leq T$. The function g satisfies hypotheses (3), (4) and (6). The function $m_\tau^T : \mathbb{R}^+ \to \mathbb{R}^+$ is T-periodic and defined by

$$\begin{cases} m_\tau^T(t) = \frac{1}{\tau}\varphi\big(\frac{t}{\tau}\big), \ \forall t \in [0,\tau), \\ m_\tau^T(t) = 0, \quad \forall t \in [\tau, T), \end{cases}$$

where the function φ is the same as in m^T. In these equations, the duration of the treatment is equal to τ. Furthermore, we have

$$\int_0^T m_\tau^T(t) \, dt = \frac{1}{\tau} \int_0^\tau \varphi\big(\frac{t}{\tau}\big) dt = \int_0^1 \varphi(t) dt. \tag{28}$$

So, it is clear that the quantity of drug administered during a cycle of chemotherapy is independent of the treatment duration τ. We will study the influence of the parameter τ with respect to the results of previous sections. We define the functions $f_\tau^T : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and $f_\tau^T : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by

$$f^T(t,u) = g(u) - m^T(t)u \text{ and } f_\tau^T(t,u) = g(u) - m_\tau^T(t)u.$$

The first proposition deals with the principal eigenvalue associated with f_τ^T and the equilibrium state 0.

Proposition 6. *Let $T > 0$ and $\tau \in (0,T]$. The real number λ_{0,f_τ^T} is independent of τ. Actually, we have*

$$\lambda_{0,f_\tau^T} = \lambda_{0,f^T} = -g'(0) + \frac{\int_0^1 \varphi(s)ds}{T}.$$

Consequently, if $T^* > 0$ denotes the critical time for the function f^T, then, for any $\tau \in (0,T^*)$, f_τ^T satisfies (18) for $T \in [\tau, +\infty)$, and the critical time T^* associated with f_τ^T is the same as the one associated with f^T. We are interested here in the solutions of the system

$$\begin{cases} y' = f_\tau^T(t,y) \text{ on } \mathbb{R}, \\ y(0) = y(T). \end{cases} \tag{29}$$

The same proof as in Proposition 1 implies that for any $\tau \in (0,T^*)$ and $T \in [\tau, T^*]$, there is no positive solution of (29), while for any $T > T^*$ and $\tau \in (0,T]$, there is a unique positive solution $w_\tau^T : \mathbb{R} \to (0,1]$ of (29). Furthermore, the same proof as in Proposition 2 implies that if $\tau \in (0,T^*)$ and $T \in [\tau, T^*]$, then the treatment is efficient, and if $T > T^*$ and $\tau \in (0,T]$, then the equilibrium state w_τ^T invades the whole space as $t \to +\infty$. More precisely, Proposition 2 remains true by replacing f^T by f_τ^T and w^T by w_τ^T. To summarize, the optimal duration of a chemotherapy cycle for which the treatment is efficient does not depend on how the drug is injected.

Let us now study the case where the treatment is not efficient, that is, $T > T^*$ and $\tau \in (0,T]$. Theorem 1.3 remains valid if we replace f^T by f_τ^T and w^T by w_τ^T, but with a critical speed $c_{T,\tau}^*$ depending a priori on τ. Nevertheless Propositions 3 and 6 imply that $c_{T,\tau}^* = 2\sqrt{-\lambda_{0,f_\tau^T}} = 2\sqrt{-\lambda_{0,f^T}} = c_T^*$, where c_T^* is the critical speed associated with f^T. Consequently, the invasion rate does not depend on how the drug is administered.

Finally, we are interested in the influence of the parameter τ on the equilibrium state w_τ^T.

Proposition 7. *Let $T > T^*$. The function*

$$\begin{cases} (0,T) & \to & (0,+\infty) \\ \tau & \mapsto & w_\tau^T(0) \end{cases}$$

is continuous and decreasing.

Consequently, in the case where the treatment is not efficient, the shorter the duration of the chemotherapy cycle, the larger the value of the equilibrium state $w_\tau^T(0)$. This means that it is better to administer the treatment over long periods.

Outline. Section 2 is devoted to the proof of Propositions 1, 3 and 4. Section 3 gathers the proof of Theorem 1.4, Proposition 5 and Theorem 1.5. Finally, we prove in Section 4 Propositions 6 and 7.

2. **Nonlinearities periodic in time.**

2.1. **Proof of Proposition 1.** We first investigate solutions of (14), showing Proposition 1. We begin with the case where $T \leq T^*$. We argue by way of contradiction, supposing there is a positive solution w^* of (14). Then

$$\frac{(w^*)'(t)}{w^*(t)} = \frac{g(w^*(t))}{w^*(t)} - m^T(t),\ \forall t \in [0,T].$$

We integrate this equation between 0 and T. We obtain

$$\int_0^T \Big(\frac{g(w^*(s))}{w^*(s)} - m^T(s)\Big) ds = 0. \tag{30}$$

Yet, as $w^* > 0$ on $[0,T]$ and according to (4) and (18), we have

$$\frac{1}{T}\int_0^T \Big(\frac{g(w^*(s))}{w^*(s)} - m^T(s)\Big) ds < -\lambda_{0,f^T} \leq 0,$$

which contradicts (30).

We now consider the case where $T > T^*$. To prove the existence of a positive solution of (14), we give two lemmas demonstrating the existence of a positive fixed point of the Poincaré map P^T defined in Definition 1.2.

Lemma 2.1. *There exists $\alpha_0 > 0$ such that for all $\alpha \in (0,\alpha_0]$ we have $P^T(\alpha) > \alpha$.*

Proof. Indeed, according to the fact that $f^T(\cdot,0) = 0$, we have $P^T(0) = 0$, and owing to (17) and the fact that $\lambda_{0,f^T} < 0$ we have $(P^T)'(0) > 1$. □

Lemma 2.2. *For all $\alpha > 1$, we have $P^T(\alpha) < \alpha$.*

Proof. Let $\alpha > 1$. We consider y_α solution of (16). Two cases can occur.
1^{st} **case**. If $y_\alpha(t) > 1$ for all $t \geq 0$, then, according to (7) , we have $y'_\alpha(t) = f^T(t, y_\alpha(t)) < 0$ for all $t \geq 0$. Consequently $y_\alpha(T) < y_\alpha(0)$, that is $P^T(\alpha) < \alpha$.
2^{nd} **case**. If there exists $t_0 \geq 0$ such that $y_\alpha(t_0) \leq 1$, then, owing to (7), we have $y_\alpha(t) \leq 1$ for all $t \geq t_0$. In particular, for $n_0 \in \mathbb{N}$ such that $n_0T \geq t_0$, we have $y_\alpha(n_0T) \leq 1 < y_\alpha(0)$. Yet, the sequence $(y_\alpha(nT))_n$ is constant or strictly monotone. So it is decreasing. Consequently we have $y_\alpha(T) < y_\alpha(0)$, that is $P^T(\alpha) < \alpha$. □

Lemma 2.1 and Lemma 2.2 imply that there exists $\alpha^* \in (\alpha_0, 1]$ such that $P^T(\alpha^*) = \alpha^*$. Consequently, the solution of (16) with $\alpha = \alpha^*$ is a positive solution of (14). We prove now the uniqueness of such a solution. Let $w_1 : \mathbb{R} \to \mathbb{R}$ and $w_2 : \mathbb{R} \to \mathbb{R}$ two positive solutions of (14). There exists $\rho > 1$ such that $w_1 \leq \rho w_2$ on $[0,T]$. We can define

$$\rho^* = \inf\big\{\rho \geq 1 \mid w_1(t) \leq \rho w_2(t),\ \forall t \in [0,T]\big\}.$$

We have

$$w_1(t) \leq \rho^* w_2(t),\ \forall t \in [0,T]. \tag{31}$$

Moreover there exists $t^* \in [0,T]$ such that

$$w_1(t^*) = \rho^* w_2(t^*). \tag{32}$$

We are going to show that $\rho^* = 1$. We argue by way of contradiction supposing that $\rho^* > 1$. So

$$w_1'(t) = f^T(t, w_1(t)),\ \forall t \in [0,T]. \tag{33}$$

Furthermore

$$(\rho^* w_2)'(t) > f^T(t, \rho^* w_2(t)),\ \forall t \in [0,T]. \tag{34}$$

Indeed, for all $t \in [0,T]$,

$$\begin{aligned}(\rho^* w_2)'(t) &= \rho^* w_2'(t)\\ &= \rho^* w_2(t)\big(\tfrac{g(w_2(t))}{w_2(t)} - m^T(t)\big)\\ &> \rho^* w_2(t)\big(\tfrac{g(\rho^* w_2(t))}{\rho^* w_2(t)} - m^T(t)\big) \text{ (according to (4) since } \rho^* > 1)\\ &= f^T(t, \rho^* w_2(t)).\end{aligned}$$

According to (31), (32), (33), (34) and the T-periodicity of w_1 and w_2, we have

$$w_1(t) = \rho^* w_2(t),\ \forall t \in [0,T].$$

It is a contradiction because w_1 is a solution of $y' = f^T(t,y)$ whereas $\rho^* w_2$ is a strict supersolution. So $\rho^* = 1$. Consequently, by the symmetry of the roles played by w_1 and w_2, we have $w_1 \equiv w_2$ on $[0,T]$, and then on $\mathbb{R}$ by periodicity.

We denote w^T the positive solution of (14). We now show the properties of w^T. The previous proof implies that $(P^T)'(w^T(0)) \leq 1$. Hence, according to (17), it follows that $\lambda_{w^T,f^T} \geq 0$. We also saw that $w^T(0) \in (0,1]$. Consequently, owing to (7) and the fact that $f^T(\cdot,0) = 0$ on $\mathbb{R}$, we have $w^T(t) \in (0,1]$ for any $t \in \mathbb{R}$.

We now study the function $T \in (T^*, +\infty) \mapsto w^T(0)$. We show the monotonicity of $T \mapsto w^T(0)$ if m^T is of type (9), with assumption (10) (in this case $T^* > 1$). We consider two real numbers T_1 and T_2 such that $T^* < T_1 < T_2$. For $i \in \{1,2\}$, the Poincaré map P^{T_i} associated with f^{T_i} is defined on $\mathbb{R}^+$ by

$$P^{T_i}(\alpha) = y_\alpha^{T_i}(T_i),\ \forall \alpha \geq 0,$$

where $y_\alpha^{T_i}$ is the solution of the Cauchy problem

$$\begin{cases} y' = f^{T_i}(t,y) \text{ on } \mathbb{R},\\ y(0) = \alpha. \end{cases} \tag{35}$$

We saw in (*II*) that the function P^{T_i} has a unique positive fixed point α^{T_i}. Furthermore $\alpha^{T_i} \in (0,1]$. The unique equilibrium state $w^{T_i} : \mathbb{R} \to (0,1]$ associated with f^{T_i} is the solution of the Cauchy problem (35) with $\alpha = \alpha^{T_i}$. Consequently, if we prove that $P^{T_1} < P^{T_2}$ on $(0,1]$, then we will deduce that $\alpha^{T_1} < \alpha^{T_2}$, that is $w^{T_1}(0) < w^{T_2}(0)$. Let $\alpha \in (0,1]$. The functions $y_\alpha^{T_1}$ and $y_\alpha^{T_2}$ are solutions on $[0,T_1]$ of the equation

$$y' = f^{T_1}(t,y).$$

Consequently, since $y_\alpha^{T_1}(0) = y_\alpha^{T_2}(0) = \alpha$, we have

$$y_\alpha^{T_1} \equiv y_\alpha^{T_2} \text{ on } [0,T_1].$$

Furthermore, from (3), (7) and the fact that φ in (9) is nonnegative and nontrivial, there holds

$$0 < y_\alpha^{T_1}(T_1) = y_\alpha^{T_2}(T_1) < 1.$$

On $[T_1,T_2]$, $y_\alpha^{T_2}$ is a solution of $y' = g(y)$. Consequently, according to (3), we have $y_\alpha^{T_2}(T_1) < y_\alpha^{T_2}(T_2)$. Finally, it follows that

$$y_\alpha^{T_1}(T_1) = y_\alpha^{T_2}(T_1) < y_\alpha^{T_2}(T_2).$$

In other terms

$$P^{T_1}(\alpha) < P^{T_2}(\alpha).$$

Finally, we have necessarily $\alpha^{T_1} < \alpha^{T_2}$, that is $w^{T_1}(0) < w^{T_2}(0)$.

We show now the continuity property. Let $\tilde{T} \in (T^*, +\infty)$ and $(T_n)_n$ be a sequence of $(T^*, +\infty)$ such that $T_n \xrightarrow{n\to+\infty} \tilde{T}$. We fixe $T^- \in (T^*, \tilde{T})$. There exists $n^- \in \mathbb{N}$ and $T^+ > T^*$ such that

$$T^* < T^- < T_n < T^+, \ \forall n \geq n^-. \tag{36}$$

We will demonstrate that $w^{T_n}(0) \xrightarrow{n\to+\infty} w^{\tilde{T}}(0)$. Since $0 < w^{T_n} \leq 1$ and $T \mapsto m^T$ is continuous in $L^\infty_{loc}(\mathbb{R})$, the sequence $(w^{T_n})_n$ converges up to extraction of a subsequence to a function $\tilde{w}$ in $\mathcal{C}^{0,\delta}([0, T^+])$ for any $\delta \in (0, 1)$. The equilibrium state w^{T_n} satisfies

$$\begin{cases} w^{T_n}(t) = w^{T_n}(0) + \int_0^t f^{T_n}(s, w^{T_n}(s))ds, \ \forall t \in [0, T^+], \\ w^{T_n}(0) = w^{T_n}(T_n). \end{cases}$$

Passing to the limit as $n \to +\infty$, we obtain

$$\begin{cases} \tilde{w}(t) = \tilde{w}(0) + \int_0^t f^{\tilde{T}}(s, \tilde{w}(s))ds, \ \forall t \in [0, \tilde{T}] \subset [0, T^+], \\ \tilde{w}(0) = \tilde{w}(\tilde{T}). \end{cases}$$

The function $t \mapsto \int_0^t f^{\tilde{T}}(s, \tilde{w}(s))ds$ is of class $\mathcal{C}^1([0, \tilde{T}])$. Consequently $\tilde{w}$ is of class $\mathcal{C}^1([0, \tilde{T}])$ and it satisfies

$$\begin{cases} \tilde{w}' = f^{\tilde{T}}(t, \tilde{w}) \text{ on } [0, \tilde{T}], \\ \tilde{w}(0) = \tilde{w}(\tilde{T}), \end{cases}$$

and $0 \leq \tilde{w} \leq 1$ in $[0, \tilde{T}]$. Owing to (*II*), it follows that $\tilde{w} \equiv 0$, or $\tilde{w} \equiv w^{\tilde{T}}$. If $\tilde{w} = 0$, then $w^{T_n} \to 0$ as $n \to +\infty$ uniformly on $[0, T^+]$. For any $n \in \mathbb{N}$, we have

$$\frac{(w^{T_n})'(t)}{w^{T_n}(t)} = \frac{f^{T_n}(t, w^{T_n}(t))}{w^{T_n}(t)}, \ \forall t \in [0, T_n].$$

We integrate the previous equation over $[0, T_n]$, then we pass to the limit as $n \to +\infty$. We obtain $-\tilde{T}\lambda_{0, f^{\tilde{T}}} = 0$. It is a contradiction because $\lambda_{0, f^{\tilde{T}}} < 0$, as $\tilde{T} > T^*$. Hence, we have necessarily $\tilde{w} \equiv w^{\tilde{T}}$. The uniqueness of the accumulation point of $(w^{T_n})_n$ implies that the convergence holds for the whole sequence. In particular, $w^{T_n}(0) \xrightarrow{n\to+\infty} w^{\tilde{T}}(0)$, and consequently, the function $T \mapsto w^T(0)$ is continuous on $(T^*, +\infty)$.

We study now the behavior of the equilibrium state w^T for the limit cases where $T \to (T^*)^+$ and $T \to +\infty$. We begin by showing that the function w^T converges uniformly to 0 on $\mathbb{R}$ as $T \to (T^*)^+$. Let $(T_n)_n$ be a sequence such that $T_n \xrightarrow{n\to+\infty} T^*$ and $T_n > T^*$ for any $n \in \mathbb{N}$. Since $(T_n)_n$ is bounded, there exists $T^+ > T^*$ such that for any $n \in \mathbb{N}$ we have $T_n \in (T^*, T^+)$. Up to extraction of a subsequence, $(w^{T_n})_n$ converges to a function w^* in $\mathcal{C}^{0,\delta}([0, T^+])$ for any $\delta \in (0, 1)$. The equilibrium state w^{T_n} satisfies

$$\begin{cases} w^{T_n}(t) = w^{\tau_n}(0) + \int_0^t f^{T_n}(s, w^{T_n}(s))ds, \ \forall t \in [0, T^+], \\ w^{T_n}(0) = w^{T_n}(T_n). \end{cases}$$

Passing to the limit as $n \to +\infty$, we obtain

$$\begin{cases} w^*(t) = w^*(0) + \int_0^t f^{T^*}(s, w^*(s))ds, \ \forall t \in [0, T^*] \subset [0, T^+], \\ w^*(0) = w^*(T^*). \end{cases}$$

The function $t \mapsto \int_0^t f^{T^*}(s, w^*(s))ds$ is of class $\mathcal{C}^1([0, T^*])$. Consequently w^* is of class $\mathcal{C}^1([0, T^*])$ and it satisfies

$$\begin{cases} (w^*)' = f^{T^*}(t, w^*) \text{ on } [0, T^*], \\ w^*(0) = w^*(T^*), \end{cases}$$

and $0 \leq w^* \leq 1$ on $[0, T^*]$. According to (II), $w^* \equiv 0$. The uniqueness of accumulation point of $(w^{T_n})_n$ implies that the convergence holds for the whole sequence. Furthermore since $[0, T_n] \subset [0, T^+]$ for any $n \in \mathbb{N}$, by T_n-periodicity of w^{T_n}, it occurs that

$$\sup_{\mathbb{R}} |w_{T_n}| = \sup_{[0,T_n]} |w_{T_n}| \leq \sup_{[0,T^+]} |w_{T_n}| \xrightarrow{n \to +\infty} 0,$$

which completes the proof of this point.

We study now the case where $T \to +\infty$ under assumptions (9) and (10). The function w^T converges on average to 1 as T tends to $+\infty$. We give a technical lemma.

Lemma 2.3. *Under assumptions* (9) *and* (10)*, the real number* δ *defined by*

$$\delta := \inf \left\{ w^T(1) \mid T \geq T^* + 1 \right\}$$

is positive. Furthermore, $\delta < 1$.

Proof. We argue by way of contradiction. Let us suppose there exists a sequence $(T_n)_n$ such that $T_n \xrightarrow{n \to +\infty} +\infty$ and $w^{T_n}(1) \xrightarrow{n \to +\infty} 0$. We fix $T^+ > T^*$. There exists $n^+ \in \mathbb{N}$ such that for any $n \geq n^+$, we have $T_n \in [T^+, +\infty)$. According to the monotonicity of $T \mapsto w^T(0)$, it follows that

$$0 < w^{T^+}(0) < w^{T_n}(0), \ \forall n \geq n^+.$$

Up to extraction of a subsequence, $(w^{T_n})_n$ converges to a function w^* in $\mathcal{C}^{0,\beta}([0,1])$ for any $\beta \in (0, 1)$. Passing to the limit as $n \to +\infty$ in the previous inequalities implies that

$$0 < w^{T^+}(0) \leq w^*(0). \tag{37}$$

The same reasoning as previously implies that the function w^* is of class $\mathcal{C}^1([0, 1])$ and satisfies the Cauchy problem

$$\begin{cases} (w^*)' = g(w^*) - \varphi(t) w^* \text{ on } [0, 1], \\ w^*(1) = 0. \end{cases}$$

By uniqueness, we have necessarily $w^* \equiv 0$, that is, w^{T_n} converges uniformly to 0 on $[0, 1]$, which contradicts (37). Lastly, each function w^T ranges in $(0, 1]$, and due to (7) and the nontriviality of φ in (9), one has $w^T < 1$ on $\mathbb{R}$. Hence, we have $\delta < 1$. □

We return to the proof of the last point of Proposition 1. We consider y_δ the solution of the Cauchy problem

$$\begin{cases} y' = g(y) \text{ on } (1, +\infty), \\ y(1) = \delta, \end{cases}$$

where $\delta \in (0, 1)$ is defined in Lemma 2.3. Let $\varepsilon > 0$ be such that $\delta < 1 - \varepsilon < 1$. Since $y_\delta(t) \xrightarrow{t \to +\infty} 1$, there exists $l_\varepsilon > 1$ such that $y_\delta(l_\varepsilon) = 1 - \varepsilon/2$. We define

$T_\varepsilon = 4l_\varepsilon/\varepsilon$ ($> l_\varepsilon$), and we consider $T \geq T_\varepsilon$. The function w^T is a solution of

$$\begin{cases} y' = g(y) \text{ on } (1,T), \\ y(1) = w^T(1). \end{cases}$$

Since $w^T(1) \geq \delta$, we have $w^T \geq y_\delta$ on $[1,T)$. In particular $w^T(l_\varepsilon) \geq 1 - \varepsilon/2$, and since w^T is increasing on (l_ε, T), we have

$$1 - \frac{\varepsilon}{2} \leq w^T(t) < 1, \ \forall t \in (l_\varepsilon, T).$$

Furthermore

$$|\frac{1}{T}\int_0^T w^T(t)dt - 1| \leq \frac{1}{T}\Big(\int_0^{l_\varepsilon} |w^T(t) - 1|dt + \int_{l_\varepsilon}^T |w^T(t) - 1|dt\Big).$$

Yet,

$$\frac{1}{T}\int_0^{l_\varepsilon} |w^T(t) - 1|dt \leq \frac{2l_\varepsilon}{T} \leq \frac{2l_\varepsilon}{T_\varepsilon} = \frac{\varepsilon}{2}.$$

and

$$\frac{1}{T}\int_{l_\varepsilon}^T |w^T(t) - 1|dt \leq \frac{T - l_\varepsilon}{T}\frac{\varepsilon}{2} \leq \frac{\varepsilon}{2}.$$

So $|\frac{1}{T}\int_0^T w^T(t)dt - 1| \leq \varepsilon$, and the proof of Proposition 1 is complete.

2.2. **Proof of Proposition 3.** We begin by showing the characterization of c_T^* with the principal eigenvalue λ_{0,f^T}. Let $\mu \in \mathbb{R}$. We denote λ_μ the principal eigenvalue and Φ_μ the principal eigenfunction associated with the operator $L_\mu : \mathcal{C}^1(\mathbb{R}) \to \mathcal{C}^0(\mathbb{R})$ defined by $L_\mu\Psi = \Psi_t - \big(\mu^2 + f_u^T(t,0)\big)\Psi$. Consequently, we have

$$(\Phi_\mu)_t = \big(\mu^2 + f_u^T(t,0) + \lambda_\mu\big)\Phi_\mu \text{ on } \mathbb{R}.$$

We divide the previous equation by Φ_μ, then we integrate between 0 and T. According to the fact that Φ_μ is T-periodic, we obtain $\lambda_\mu = -\mu^2 + \lambda_{0,f^T}$. In [21], Nadin gives the following characterization of the critical speed c_T^*:

$$c_T^* = \inf\big\{c \in \mathbb{R} \mid \text{there exists } \mu > 0 \text{ such that } \lambda_\mu + \mu c = 0\big\}.$$

Consequently, we have

$$c_T^* = \inf\big\{c \in \mathbb{R} \mid \text{there exists } \mu > 0 \text{ such that } \mu^2 - \mu c - \lambda_{0,f^T} = 0\big\}.$$

We thus look for the smallest real number c for which the equation $\mu^2 - \mu c - \lambda_{0,f^T} = 0$ of the variable μ admits a positive solution. An elementary calculation leads to $c_T^* = 2\sqrt{-\lambda_{0,f^T}}$. Consequently, we have

$$c_T^* = 2\sqrt{g'(0) - \frac{1}{T}\int_0^T m^T(t)dt}.$$

Hence the function $T \in (T^*, +\infty) \mapsto c_T^*$ is continuous, increasing if $\int_0^T m^T(t)dt$ does not depend on T, and we have the two limits cases

$$\lim_{T\to+\infty} c_T^* = 2\sqrt{g'(0)} \text{ if } \frac{1}{T}\int_0^T m^T(t)dt \xrightarrow{T\to+\infty} 0, \text{ and } \lim_{T\to(T^*)^+} c_T^* = 0,$$

which concludes the proof of Proposition 3.

2.3. **Proof of Proposition 4.** Let $\alpha \in [0,1]$. We recall that if $y_\alpha : \mathbb{R} \to \mathbb{R}$ is the solution of the Cauchy problem

$$\begin{cases} y' = f^T(t,y) \text{ on } \mathbb{R}, \\ y(0) = \alpha, \end{cases}$$

then we denote $P^T : \alpha \in [0,1] \mapsto P^T(\alpha) = y_\alpha(T)$ the Poincaré map associated to the function f^T. According to the proof of Proposition 1, there exists a fixed point of P^T in $(0,1]$. Nevertheless, since hypothesis (4) is not satisfied here, this fixed point is not necessarily unique. We define

$$\alpha_0 = \inf \{ \alpha \in (0,1] \mid P^T(\alpha) = \alpha \}.$$

To simplify the notations, we denote $y^T : \mathbb{R} \to \mathbb{R}$ the function $y^T = y_{\alpha_0}$. We begin by proving that this infimum is not equal to zero.

Lemma 2.4. *We have* $\alpha_0 > 0$.

Proof. We assume that $\alpha_0 = 0$. So, there exists a sequence $(\alpha_n)_n \subset (0,1]^{\mathbb{N}}$ such that $P^T(\alpha_n) = \alpha_n$ and $\alpha_n \xrightarrow{n\to+\infty} 0$. We divide the equation $y'_{\alpha_n} = f^T(t, y_{\alpha_n})$ by y_{α_n}, then we intregrate between 0 and T. We obtain

$$\int_0^T \frac{f^T(s, y_{\alpha_n}(s)) - f^T(s,0)}{y_{\alpha_n}(s)} ds = 0.$$

Passing to the limit as $n \to +\infty$, since $y_{\alpha_n} \to 0$ uniformly on $[0,T]$ as $n \to +\infty$ by Cauchy-Lipschitz theorem, we have

$$\int_0^T f_u^T(s,0) ds = 0,$$

which contradicts the fact that $\lambda_{0,f^T} < 0$. Consequently $\alpha_0 > 0$. Notice also that, by continuity of P^T, there holds $P^T(\alpha_0) = \alpha_0$, and $y^T = y_{\alpha_0}$ solves (14). Furthermore $0 < y^T \le 1$ on $\mathbb{R}$. □

Since f^T is of class $\mathcal{C}^1(\mathbb{R} \times [0,1], \mathbb{R})$ and T-periodic, there exists $\varepsilon_0 \in (0,1)$ such that for all $\varepsilon \in (0, \varepsilon_0]$ and for all $t \in \mathbb{R}$ we have

$$|f^T(t, \varepsilon\Phi_{0,f^T}(t)) - \varepsilon\Phi_{0,f^T}(t) f_u^T(t,0)| \le \frac{|\lambda_{0,f^T}|}{2} \varepsilon \Phi_{0,f^T}(t), \tag{38}$$

where Φ_{0,f^T} is the principal eigenfunction associated with f^T and 0. Since $\lambda_{0,f^T} < 0$ and y^T is the smallest positive solution of system (14), we can apply Theorem 2.3 of the Nadin's paper [21]. Consequently, there exists a couple (c_0, U_0), where $U_0 : \mathbb{R} \times \mathbb{R} \to [0,1]$, $(t,\xi) \mapsto U_0(t,\xi)$ is of class $\mathcal{C}^{1,2}(\mathbb{R}^2)$ and solves

$$\begin{cases} (U_0)_t - (U_0)_{\xi\xi} - c_0 (U_0)_\xi = f^T(t, U_0) \text{ on } \mathbb{R} \times \mathbb{R}, \\ U_0(\cdot,\cdot) = U_0(\cdot + T, \cdot) \text{ on } \mathbb{R} \times \mathbb{R}, \\ U_0(\cdot, -\infty) = y^T \ , \ U_0(\cdot, +\infty) = 0 \text{ uniformly on } \mathbb{R}. \end{cases} \tag{39}$$

Necessarily $c_0 > 0$ because Nadin shows in [21] that for $c < 2\sqrt{-\lambda_{0,f^T}}$, which is a positive real number, there is no pulsating front of sped c connecting 0 and y^T. Furthermore, we have

$$\partial_\xi U_0(t,\xi) < 0, \ \forall (t,\xi) \in \mathbb{R} \times \mathbb{R}.$$

Let $c_1 > 0$ be a real number such that there exists a pulsating front U_1 with speed c_1 such that $\partial_\xi U_1 < 0$ on $\mathbb{R} \times \mathbb{R}$, and let $c_2 > c_1$. We are going to prove the existence of a pulsating front U_2 such that (c_2, U_2) solves (39) and $\partial_\xi U_2 < 0$ on $\mathbb{R}^2$. Yet, by [21], the set

$$\mathcal{C} = \{c \in \mathbb{R} \mid \text{there exists a pulsating front } U \text{ of speed } c \text{ such that } \partial_\xi U < 0 \text{ on } \mathbb{R}\times\mathbb{R}\}$$

is closed and included in $[2\sqrt{-\lambda_{0,f^T}}, +\infty)$. This will conclude the proof of Proposition 4 by denoting $c_T^{**} = \inf \mathcal{C}$.

Given $c_1 < c_2$ as above, let $a > 0$ and $r \in \mathbb{R}$. We define

$$\varepsilon_{a,r} = \min\Big\{ \min_{[0,T]\times[-a,a]} \frac{U_1(\cdot,\cdot+r)}{2\Phi_{0,f^T}(\cdot)}, \varepsilon_0, \frac{y^T(0)}{\Phi_{0,f^T}(0)} \Big\}.$$

We consider the problem

$$\begin{cases} U_t - U_{\xi\xi} - c_2 U_\xi = f^T(t,U) \text{ on } (0,T)\times(-a,a), \\ U(0,\cdot) = U(T,\cdot) \text{ on } [-a,a]. \\ U(\cdot,-a) = U_1(\cdot,-a+r)\ ,\ U(\cdot,a) = \varepsilon_{a,r}\Phi_{0,f^T} \text{ on } [0,T]. \end{cases} \tag{40}$$

We begin by showing that the previous problem has a solution.

Proposition 8. *There exists a solution to problem (40).*

Proof. We consider the problem

$$\begin{cases} U_t - U_{\xi\xi} - c_2 U_\xi = f^T(t,U) \text{ on } (0,+\infty)\times(-a,a), \\ U(\cdot,-a) = U_1(\cdot,-a+r)\ ,\ U(\cdot,a) = \varepsilon_{a,r}\Phi_{0,f^T} \text{ on } [0,+\infty), \\ U(0,\cdot) = \psi \text{ on } [-a,a], \end{cases}$$

where $\psi \in \mathcal{C}^0([-a,a],[0,1])$. This Cauchy problem admits a solution U_ψ defined on $\mathbb{R}^+\times[-a,a]$. Furthermore, $0 \le U_\psi \le 1$ in $\mathbb{R}^+\times[-a,a]$ from the maximum principle and the definition of $\varepsilon_{a,r}$. We define the closed convex set

$$C = \{\psi \in \mathcal{C}^0([-a,a],[0,1]) \mid \varepsilon_{a,r}\Phi_{0,f^T}(0) \le \psi \le U_1(0,\cdot+r) \text{ on } [-a,a]\}.$$

Note that this set is not empty since $\Phi_{0,f^T} > 0$, $U_1 \le 1$ and $\varepsilon_{a,r}\Phi_{0,f^T}(0) \le U_1(0,\cdot+r)$ on $[-a,a]$ according to the definition of $\varepsilon_{a,r}$. We start by proving that if $\psi \in C$, then $U_\psi(T,\cdot) \in C$ using a comparison lemma.

Lemma 2.5. *Let $\psi \in C$. Then we have*

$$\varepsilon_{a,r}\Phi_{0,f^T}(t) < U_\psi(t,\xi) < U_1(t,\xi+r)\ \forall\ (t,\xi) \in (0,+\infty)\times(-a,a). \tag{41}$$

Proof. Since $\partial_\xi U_1 < 0$ on $\mathbb{R}\times\mathbb{R}$ and $c_1 < c_2$, the function $U_1(\cdot,\cdot+r)$ satisfies on $[0,+\infty)\times(-a,a)$,

$$\big(U_1(\cdot+r)\big)_t - \big(U_1(\cdot+r)\big)_{\xi\xi} - c_2\big(U_1(\cdot+r)\big)_\xi - f^T\big(t, U_1(\cdot+r)\big) = (c_1-c_2)\big(U_1(\cdot+r)\big)_\xi > 0.$$

Moreover, since $\psi \in C$, we have $U_1(0,\cdot+r) \ge \psi$ on $[-a,a]$ and, according to the definition of $\varepsilon_{a,r}$ and the T-periodicity of U_1 and Φ_{0,f^T}, we have $U_1(\cdot,a+r) \ge \varepsilon_{a,r}\Phi_{0,f^T}$ on $[0,+\infty)$. Consequently, we can apply a comparison principle, and we obtain

$$U_\psi(t,\xi) \le U_1(t,\xi+r)\ \forall (t,\xi) \in [0,+\infty)\times[-a,a].$$

In the same way, since $\varepsilon_{a,r} \leq \varepsilon_0$, and according (38) and the negativity of λ_{0,f^T}, we have on $[0,+\infty) \times (-a,a)$

$$\begin{aligned}
&(\varepsilon_{a,r}\Phi_{0,f^T})_t - (\varepsilon_{a,r}\Phi_{0,f^T})_{\xi\xi} - c_2(\varepsilon_{a,r}\Phi_{0,f^T})_\xi - f^T(t, \varepsilon_{a,r}\Phi_{0,f^T}) \\
&= \varepsilon_{a,r}\Phi_{0,f^T}(\lambda_{0,f^T} + f_u^T(t,0)) - f^T(t,\varepsilon_{a,r}\Phi_{0,f^T}) \\
&= \varepsilon_{a,r}\lambda_{0,f^T}\Phi_{0,f^T} - \big(f^T(t,\varepsilon_{a,r}\Phi_{0,f^T}) - \varepsilon_{a,r}\Phi_{0,f^T} f_u^T(t,0)\big) \\
&\leq \varepsilon_{a,r}\lambda_{0,f^T}\Phi_{0,f^T} - \varepsilon_{a,r}\frac{\lambda_{0,f^T}}{2}\Phi_{0,f^T} < 0.
\end{aligned}$$

Furthermore since $\psi \in C$, we have $\varepsilon_{a,r}\Phi_{0,f^T}(0) \leq \psi$ on $[-a,a]$ and, according to the definition of $\varepsilon_{a,r}$ and the T-periodicity of U_1 and Φ_{0,f^T}, we have $\varepsilon_{a,r}\Phi_{0,f^T} \leq U_1(\cdot, -a+r)$ on $[0,+\infty)$. Consequently, we can apply a comparison principle and we conclude that

$$\varepsilon_{a,r}\Phi_{0,f^T}(t) \leq U_\psi(t,\xi) \ \forall\ (t,\xi) \in [0,T] \times [-a,a],$$

The fact that the inequalities in (41) are strict is a consequence of the strong maximum principle. □

We return to the proof of Proposition 8. We consider

$$\begin{array}{cccc} \mathcal{T} & : & C & \rightarrow & C \\ & & \psi & \mapsto & U_\psi(T,\cdot) \end{array}$$

Owing to (41) and the T-periodicity of Φ_{0,f^T} and U_1, $\mathcal{T}$ is well defined. We are going to demonstrate using the Schauder's fixed point theorem that the function $\mathcal{T}$ has a fixed point in the closed convex set C. We show now that $\mathcal{T}$ is continuous. In fact we show that $\mathcal{T}$ is a Lipschitz-continuous function. Let ψ and φ in C. We have on $(0,T] \times [-a,a]$

$$(U_\psi - U_\varphi)_t - (U_\psi - U_\varphi)_{\xi\xi} - c_2(U_\psi - U_\varphi)_\xi = \beta(t,\xi)(U_\psi - U_\varphi),$$

where $\beta : (0,T] \times [-a,a] \to \mathbb{R}$ is defined by

$$\beta(t,\xi) = \begin{cases} \frac{f^T\left(t,U_\psi(t,\xi)\right) - f^T\left(t,U_\varphi(t,\xi)\right)}{U_\psi(t,\xi) - U_\varphi(t,\xi)}, & \text{if } U_\psi(t,\xi) \neq U_\varphi(t,\xi), \\ f_u^T\big(t, U_\psi(t,\xi)\big), & \text{if } U_\psi(t,\xi) = U_\varphi(t,\xi). \end{cases}$$

Since $|\beta| \leq \|f_u^T\|_{L^\infty([0,T]\times[0,1])}$ on $(0,T]\times[-a,a]$, and $U_\psi - U_\varphi = 0$ on $[0,T]\times\{-a,a\}$, the maximum principle yields for any $(t,\xi) \in [0,T] \times [-a,a]$

$$|U_\psi(t,\xi) - U_\varphi(t,\xi)| \leq \|\psi - \varphi\|_{L^\infty([-a,a])} e^{\|f_u^T\|_{L^\infty([0,T]\times[0,1])} t}.$$

If we take $t = T$, we obtain

$$\|U_\psi(T,\cdot) - U_\varphi(T,\cdot)\|_{L^\infty([-a,a])} \leq e^{\|f_u^T\|_{L^\infty([0,T]\times[0,1])} T} \|\psi - \varphi\|_{L^\infty([-a,a])}.$$

So $\mathcal{T}$ is a Lipschitz-continuous function.

We prove now that $\mathcal{T}(C)$ is compact. Let $(\psi_n)_n$ be a sequence of C. By standard parabolic estimates, the sequence $(U_{\psi_n}(T,\cdot))_n$ is bounded in $\mathcal{C}^{2,\alpha}([-a,a],[0,1])$ for any $\alpha \in (0,1)$. Since $\mathcal{C}^{2,\alpha}([-a,a],[0,1])$ embeds compactly into $\mathcal{C}^0([-a,a],[0,1])$, $(U_{\psi_n}(T,\cdot))_n$ converges up to extraction of a subsequence in C.

So, according to Shauder's fixed point theorem, there exists $\psi_{a,r} \in \mathcal{C}([-a,a],[0,1])$ such that $\mathcal{T}(\psi_{a,r}) = \psi_{a,r}$, that is $U_{\psi_{a,r}}(T,\cdot) = U_{\psi_{a,r}}(0,\cdot)$. Actually, the function $U_{\psi_{a,r}}$ is solution of (40). By uniqueness and T-periodicity of f^T, $U_{\psi_{a,r}}$ can be extended as a T-periodic solution of (40) in $\mathbb{R} \times [-a,a]$. □

To simplify the notations, we denote now $U_{a,r}$ instead of $U_{\psi_{a,r}}$. Owing to Lemma 2.5 and the T-periodicity of $U_{a,r}$, we have the following inequalities

$$\varepsilon_{a,r}\Phi_{0,f^T}(t) < U_{a,r}(t,\xi) < U_1(t,\xi+r) \ \forall\ (t,\xi) \in [0,T]\times(-a,a). \tag{42}$$

We are now going to use a sliding method and we first give a comparison lemma.

Lemma 2.6. *Let U and V be two T-periodic functions solving problem (40). Let $h \in [0,2a]$. We define $V_h(t,\xi) = V(t,\xi+h)$ for any $(t,\xi) \in [0,T]\times[-a,a-h]$. Then, we have*

$$V_h \leq U \text{ on } [0,T]\times[-a,a-h].$$

Proof. We denote $I_h = [-a,a-h]$. For $h = 2a$, we have $I_h = \{-a\}$. Since $U(\cdot,-a) = U_1(\cdot,-a+r)$, $V_{2a}(\cdot,-a) = V(\cdot,a) = \varepsilon_{a,r}\Phi_{0,f^T}$ and

$$\varepsilon_{a,r}\Phi_{0,f^T} \leq \tfrac{U_1(\cdot,-a+r)}{2} < U_1(\cdot,-a+r) \text{ on } [0,T],$$

it occurs that $V_{2a} < U$ on $[0,T]\times I_{2a}$. Furthermore, $V_h \leq U$ on $[0,T]\times I_h$ for all $h \in [0,2a]$ sufficiently close to $2a$, by continuity of U and V. Consequently, we can define

$$h^* = \inf\left\{\underline{h} \geq 0 \mid \forall h \in [\underline{h},2a],\ V_h \leq U \text{ on } [0,T]\times I_h\right\}.$$

We have $0 \leq h^* < 2a$. We are going to show by way of contradiction that $h^* = 0$. Thus let us suppose that $h^* > 0$. By continuity and T-periodicity of U and V_h^*, the definition of h^* implies that

$$V_{h^*} \leq U \text{ on } \mathbb{R}\times I_{h^*}. \tag{43}$$

Furthermore, if we define the bounded function $\eta : \mathbb{R}\times I_{h^*} \to \mathbb{R}$ by

$$\eta(t,\xi) = \begin{cases} \frac{f^T\left(t,U(t,\xi)\right)-f^T\left(t,V_{h^*}(t,\xi)\right)}{U(t,\xi)-V_{h^*}(t,\xi)}, & \text{if } U(t,\xi) \neq V_{h^*}(t,\xi),\\ f_u^T\left(t,U(t,\xi)\right), & \text{if } U(t,\xi) = V_{h^*}(t,\xi), \end{cases}$$

then, we have on $\mathbb{R}\times I_{h^*}$

$$(U-V_{h^*})_t - c_2(U-V_{h^*})_\xi - (U-V_{h^*})_{\xi\xi} = \eta(t,\xi)(U-V_{h^*}). \tag{44}$$

Consequently, according to (43) and (44), if there exists $(t^*,\xi^*) \in \mathbb{R}\times(-a,a-h^*)$ such that $U(t^*,\xi^*) = V_{h^*}(t^*,\xi^*)$, then, by the strong maximum principle, the continuity and the T-periodicity of U and V_{h^*}, we have

$$V_{h^*} = U \text{ on } \mathbb{R}\times I_{h^*}. \tag{45}$$

Yet, according to (42) (which is automatically fulfilled from the arguments used in Lemma 2.5), and since $\partial_\xi U_1 < 0$ on $\mathbb{R}\times\mathbb{R}$, we have for any $t \in \mathbb{R}$,

$$V_{h^*}(t,-a) = V(t,-a+h^*) < U_1(t,-a+h^*+r) < U_1(t,-a+r) = U(t,-a).$$

Consequently, $V_{h^*} < U$ on $\mathbb{R}\times[-a,a-h^*)$. Furthermore, according to (42), for any $t \in \mathbb{R}$, we also have

$$V_{h^*}(t,a-h^*) = V(t,a) = \varepsilon_{a,r}\Phi_{0,f^T}(t) < U(t,a-h^*).$$

So, it occurs that

$$V_{h^*} < U \text{ on } \mathbb{R}\times I_{h^*}.$$

Since $[0,T]\times I_{h^*}$ is a compact set, and both U and V are continuous on $[0,T]\times[-a,a]$, there exists $h_0 \in (0,h^*)$ such that for any $\eta \in (0,h_0)$, we have $V_{h^*-\eta} < U$ on $[0,T]\times I_{h^*-\eta}$. This contradicts the definition of h^*. Consequently we have $h^* = 0$ and the proof of Lemma 2.6 is complete. □

Corollary 1. *There exists a unique function $U_{a,r}$ solving (40).*

Proof. We apply the conclusion of Lemma 2.6 with $h = 0$ and reverse the roles of U and V. □

Corollary 2. *The function* $r \in \mathbb{R} \mapsto U_{a,r} \in \mathcal{C}^0\big([0,T]\times[-a,a],[0,1]\big)$ *is continuous.*

Proof. Let $r^* \in \mathbb{R}$ and $(r_n)_n$ be a sequence of real numbers such that $r_n \xrightarrow{n\to\infty} r^*$. According to standard parabolic estimates and the T-periodicity of each function U_{a,r_n}, there exists U^* such that, up to extraction of a subsequence, $U_{a,r_n} \xrightarrow{n\to\infty} U^*$ in $\mathcal{C}^{1,\frac{\alpha}{2}}$ in t and in $\mathcal{C}^{2,\alpha}$ in ξ, for any $\alpha \in (0,1)$. Consequently,

$$\begin{cases} (U^*)_t - (U^*)_{\xi\xi} - c_2(U^*)_\xi = f^T(t,U^*) \text{ on } \mathbb{R}\times(-a,a), \\ U^*(0,\cdot) = U^*(T,\cdot) \text{ on } [-a,a], \\ U^*(\cdot,-a) = U_1(\cdot,-a+r^*)\ ,\ U^*(\cdot,a) = \varepsilon_{a,r^*}\Phi_{0,f^T} \text{ on } [0,T]. \end{cases}$$

The uniqueness of the solution of the previous problem (Corollary 1) implies that we have $U^* = U_{a,r^*}$, and that the whole sequence (U_{a,r_n}) converges to U^*. □

Corollary 3. *For any* $t \in [0,T]$ *and* $\xi \in (-a,a)$*, we have*

$$\partial_\xi U_{a,r}(t,\xi) < 0.$$

Proof. We apply Lemma 2.6 with $U = V = U_{a,r}$. The strict inequality is a consequence of the maximum principle applied to $\partial_\xi U_{a,r}$. □

Proposition 9. *There exist* $\varepsilon_a \in (0,\varepsilon_0]$ *and* $r_a \in \mathbb{R}$ *such that* $U_{a,r_a}(0,0) = \frac{\varepsilon_a\Phi_{0,f^T}(0)}{2}$.

Proof. There exists $(t_{a,r},\xi_{a,r}) \in [0,T]\times[-a,a]$ such that

$$\varepsilon_{a,r} = \min\left\{\frac{U_1(t_{a,r},\xi_{a,r}+r)}{2\Phi_{0,f^T}(t_{a,r})}, \varepsilon_0, \frac{y^T(0)}{\Phi_{0,f^T}(0)}\right\}$$

Let $(r_n)_n$ be a sequence of real numbers such that $r_n \xrightarrow{n\to+\infty} -\infty$. There exists a function $U_{a,-\infty}$ such that up to extraction of a subsequence, $U_{a,r_n} \xrightarrow{n\to+\infty} U_{a,-\infty}$ in $\mathcal{C}^{0,\alpha}\big([0,T]\times[-a,a]\big)$ for any $\alpha \in (0,1)$. Since $(t_{a,r_n})_n$ is bounded, there exists $t_a \in [0,T]$ such that up to extraction of a subsequence, we have $t_{a,r_n} \xrightarrow{n\to+\infty} t_a$. So, according to the fact that (ξ_{a,r_n}) is also bounded (because a is fixed here), it follows that

$$\varepsilon_{a,r_n} \xrightarrow{n\to+\infty} \varepsilon_a := \min\left\{\frac{y^T(t_a)}{2\Phi_{0,f^T}(t_a)}, \varepsilon_0, , \frac{y^T(0)}{\Phi_{0,f^T}(0)}\right\}$$

We thus have $U_{a,-\infty}(\cdot,a) = \varepsilon_a\Phi_{0,f^T}$ on $[0,T]$. Consequently, since $\partial_\xi U_{a,-\infty} \leq 0$ on $[0,T]\times[-a,a]$, it occurs that $U_{a,-\infty}(0,0) \geq \varepsilon_a\Phi_{0,f^T}(0)$. So there exists $n_0 \in \mathbb{N}$ such that $r_{n_0} < 0$ and

$$U_{a,r_{n_0}}(0,0) \geq \frac{3}{4}\varepsilon_a\Phi_{0,f^T}(0).$$

Let now $(\tilde{r}_n)_n$ be a sequence of real numbers such that $\tilde{r}_n \xrightarrow{n\to+\infty} +\infty$.There exists a function $U_{a,+\infty}$ such that up to extraction of a subsequence $U_{a,\tilde{r}_n} \xrightarrow{n\to+\infty} U_{a,+\infty}$ in $\mathcal{C}^{0,\alpha}([0,T]\times[-a,a])$ for any $\alpha \in (0,1)$. Furthermore, for any $t \in [0,T]$, we have $U_{a,\tilde{r}_n}(t,-a) = U_1(t,-a+\tilde{r}_n) \xrightarrow{n\to+\infty} 0$. Consequently, since $\partial_\xi U_{a,+\infty} \leq 0$ and $U_{a,+\infty} \geq 0$ on $[0,T]\times[-a,a]$, it occurs that $U_{a,+\infty} \equiv 0$. So, there exists $n_1 \in \mathbb{N}$ such that $\tilde{r}_{n_1} > 0$ and

$$U_{a,\tilde{r}_{n_1}}(0,0) \leq \frac{1}{4}\varepsilon_a\Phi_{0,f^T}(0).$$

According to Corollary 2, there exists $r_a \in (r_{n_0}, \tilde{r}_{n_1})$ such that

$$U_{a,r_a}(0,0) = \frac{1}{2}\varepsilon_a \Phi_{0,f^T}(0),$$

which completes the proof. □

Proposition 10. *There exists a sequence* $a_n \xrightarrow{n\to+\infty} +\infty$ *such that* $U_{a_n,r_{a_n}}$ *converges on any compact set in* $\mathcal{C}^{1,\frac{\alpha}{2}}$ *in* t *and in* $\mathcal{C}^{2,\alpha}$ *in* ξ*, for any* $\alpha \in (0,1)$*, to a function* U_2 *solving* (39) *with* $c = c_2$*, and such that* $(U_2)_\xi < 0$ *on* $\mathbb{R}^2$*.*

Proof. Since t_a is bounded, there exist $t^* \in [0,T]$ and a sequence $a_n \xrightarrow{n\to+\infty} +\infty$ such that $t_{a_n} \xrightarrow{n\to+\infty} t^*$. Consequently,

$$\varepsilon_{a_n} \xrightarrow{n\to+\infty} \varepsilon^* := \min\Big\{\frac{y^T(t^*)}{2\Phi_{0,f^T}(t^*)}, \varepsilon_0, , \frac{y^T(0)}{\Phi_{0,f^T}(0)}\Big\} > 0.$$

According the standard parabolic estimates, up to extraction of a subsequence, $U_{a_n,r_{a_n}}$ converges on any compact set to a function U_2 in $\mathcal{C}^{1,\frac{\alpha}{2}}$ in t and in $\mathcal{C}^{2,\alpha}$ in ξ, for any $\alpha \in (0,1)$. The function U_2 satisfies

$$\begin{cases} (U_2)_t - (U_2)_{\xi\xi} - c_2(U_2)_\xi = f^T(t,U_2) \text{ on } [0,T]\times\mathbb{R}, \\ U_2(0,\cdot) = U_2(T,\cdot) \text{ on } \mathbb{R}, \\ U_2(0,0) = \frac{1}{2}\varepsilon^*\Phi_{0,f^T}(0), \\ (U_2)_\xi \le 0 \text{ on } [0,T]\times\mathbb{R}. \end{cases}$$

Since $(\varepsilon^*\Phi_{0,f^T})' \le f^T(t,\varepsilon^*\Phi_{0,f^T})$ and $(y^T)' = f^T(t,y^T)$ on $[0,T]$ and since we have $\varepsilon^*\Phi_{0,f^T}(0) \le y^T(0)$, it occurs that $\varepsilon^*\Phi_{0,f^T} \le y^T$ on $[0,T]$. Consequently

$$U_2(0,0) \in \Big(0, \frac{y^T(0)}{2}\Big].$$

The functions $U_2(\cdot,-\infty)$ and $U_2(\cdot,+\infty)$ solve the equation $y' = f(t,y)$ on $[0,T]$. Furthermore, $U_2(t,\xi) \le y^T(t)$ for all $t \in [0,T]$ and all $\xi \in \mathbb{R}$, since this inequality holds for U_1 and since each function $U_{a,r}$ satisfies (42). Consequently, since $(U_2)_\xi \le 0$ on $[0,T]\times\mathbb{R}$, we have necessarily $U_2(\cdot,-\infty) = y^T$ and $U_2(\cdot,+\infty) = 0$. Finally we apply the strong maximum principle to the equation satisfied by $(U_2)_\xi$ and obtain $(U_2)_\xi < 0$ on $\mathbb{R}^2$ (otherwise $(U_2)_\xi$ would be identically equal to zero, which is impossible since $U_2(\cdot,-\infty) = y^T$ and $U_2(\cdot,+\infty) = 0$). □

3. Nonlinearities asymptotically periodic in time with perturbation.

3.1. **Proof of Theorem 1.4.** Let $T > 0$ with $T \neq T^*$ (that is $\lambda_{0,f^T} \neq 0$). We define

$$\varepsilon_T = \frac{1}{C+1}\min\Big\{|\lambda_{0,f^T}|, -\frac{g(2)}{2}\Big\} > 0,$$

where C is defined in (23). Let $\varepsilon \in (0,\varepsilon_T)$. According to (23) and (25), there exists $n_\varepsilon \in \mathbb{N}^*$ such that for all $t \ge n_\varepsilon T$ and for all $u \ge 0$ we have

$$f^T(t,u) - (C+1)\varepsilon u \le f_\varepsilon(t,u) \le f^T(t,u) + (C+1)\varepsilon u. \tag{46}$$

We define the $T-$periodic functions $f^T_{-\varepsilon} : \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}$ and $f^T_\varepsilon : \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}$ by

$$f^T_{-\varepsilon}(t,u) = f^T(t,u) - (C+1)\varepsilon u, \text{ and } f^T_\varepsilon(t,u) = f^T(t,u) + (C+1)\varepsilon u. \tag{47}$$

According to (7), it occurs that

$$\begin{cases} f^T_{-\varepsilon}(t,u) \leq 0, \ \forall (t,u) \in \mathbb{R} \times [2,+\infty), \\ f^T(t,u) \leq 0, \ \forall (t,u) \in \mathbb{R} \times [2,+\infty). \end{cases} \tag{48}$$

Furthermore, according to (4) and (6), for any $u \in [2,+\infty)$, we have $g(u)/u \leq g(2)/2 < 0$. Consequently, since $\varepsilon \in (0, -\frac{1}{C+1}\frac{g(2)}{2})$, the following inequality is true

$$f^T_{\varepsilon}(t,u) \leq 0, \ \forall (t,u) \in \mathbb{R} \times [2,+\infty), \tag{49}$$

Concerning the principal eigenvalues associated with the equilibrium 0 and functions f^T, $f^T_{-\varepsilon}$ and f^T_{ε}, the following relations hold

$$\begin{cases} \lambda_{0,f^T_{\varepsilon}} = \lambda_{0,f^T} - (C+1)\varepsilon, \\ \lambda_{0,f^T_{-\varepsilon}} = \lambda_{0,f^T} + (C+1)\varepsilon. \end{cases} \tag{50}$$

We begin by handling the case where $T < T^*$. Owing to (50), the fact that $\lambda_{0,f^T} > 0$ and since $\varepsilon \in (0, \frac{\lambda_{0,f^T}}{C+1})$, we have

$$\lambda_{0,f^T_{\varepsilon}} > 0. \tag{51}$$

We consider $v_\varepsilon : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ the solution of the Cauchy problem

$$\begin{cases} (v_\varepsilon)_t - (v_\varepsilon)_{xx} = f^T_{\varepsilon}(t, v_\varepsilon) \text{ on } (0,+\infty) \times \mathbb{R}, \\ v_\varepsilon(0,\cdot) = u_\varepsilon(n_\varepsilon T, \cdot) \text{ on } \mathbb{R}. \end{cases}$$

Owing to (46) and the $T-$periodicity of f^T_{ε}, the function $u_\varepsilon(\cdot + n_\varepsilon T, \cdot)$ satisfies on $(0,+\infty) \times \mathbb{R}$

$$\big(u_\varepsilon(\cdot+n_\varepsilon T,\cdot)\big)_t - \big(u_\varepsilon(\cdot+n_\varepsilon T,\cdot)\big)_{xx} = f_\varepsilon\big(t+n_\varepsilon T, u_\varepsilon(\cdot+n_\varepsilon T,\cdot)\big) \leq f^T_\varepsilon\big(t, u_\varepsilon(\cdot+n_\varepsilon T,\cdot)\big).$$

So, applying a comparison principle, we obtain

$$0 \leq u_\varepsilon(t+n_\varepsilon T, x) \leq v_\varepsilon(t,x), \ \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}. \tag{52}$$

According to (51), Proposition 2 applied with the T-periodic nonlinearity f^T_ε implies that

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} v_\varepsilon(t,x) = 0.$$

Hence, owing to (52),

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} u_\varepsilon(t,x) = 0,$$

which concludes the proof of the first part of Theorem 1.4.

We now consider the case where $T > T^*$. Since $\lambda_{w^T,f^T} > 0$, there exists $\mu_T > 0$ such that for all $\mu \in (0,\mu_T)$ and for all $(t,u,v) \in \mathbb{R} \times [0,2]^2$, we have

$$|u-v| \leq \mu \Rightarrow |f^T(t,v) - f^T(t,u) - f^T_u(t,u)(v-u)| \leq \frac{\lambda_{w^T,f^T}}{2}|v-u|. \tag{53}$$

We define the two positive real numbers $\tilde{M}_T$ and $\tilde{\varepsilon}_T$ by

$$\tilde{M}_T = \frac{8(C+1)}{\lambda_{w^T,f^T}} \frac{\sup_{[0,T]} w^T}{\inf_{[0,T]} \Phi_{w^T,f^T}} > 0,$$

and

$$\tilde{\varepsilon}_T = \min\left\{\varepsilon_T, \frac{\lambda_{w^T,f^T}}{4(C+1)}, \frac{\inf\limits_{[0,T]} w^T}{2\tilde{M}_T \sup\limits_{[0,T]} \Phi_{w^T,f^T}}, \frac{\min\{\mu_T, 1\}}{\tilde{M}_T \sup\limits_{[0,T]} \Phi_{w^T,f^T}}\right\} > 0, \tag{54}$$

where Φ_{w^T,f^T} is the principal eigenfunction associated with f^T and the equilibrium state w^T. Let $\varepsilon \in (0, \tilde{\varepsilon}_T)$. According to (50), the fact that $\lambda_{0,f^T} < 0$ and since $\varepsilon \in (0, -\frac{\lambda_{0,f^T}}{C+1})$, we have

$$\lambda_{0,f^T_{-\varepsilon}} < 0, \ \lambda_{0,f^T} < 0, \text{ and } \lambda_{0,f^T_\varepsilon} < 0. \tag{55}$$

Owing to (48), (49) and (55), the same proof as in Proposition 1 implies that there exists a unique T-periodic positive equilibrium state w^T_ε (resp. $w^T_{-\varepsilon}$) associated with f^T_ε (resp. $f^T_{-\varepsilon}$). Furthermore, for any $t \in \mathbb{R}$, we have $w^T_\varepsilon(t) \in (0, 2]$ (resp. $w^T_{-\varepsilon}(t) \in (0, 2]$).

Lemma 3.1. *There exists $M_T > 0$ independent of ε such that*

$$\begin{cases} \sup_{t\in[0,T]} |w^T_\varepsilon(t) - w^T(t)| \le M_T\varepsilon, \\ \sup_{t\in[0,T]} |w^T_{-\varepsilon}(t) - w^T(t)| \le M_T\varepsilon. \end{cases} \tag{56}$$

Proof. We begin by proving the first inequality. We define the function $\overline{v}_\varepsilon : \mathbb{R} \to \mathbb{R}$ by

$$\overline{v}_\varepsilon(t) = w^T(t) + \tilde{M}_T\varepsilon\Phi_{w^T,f^T}(t).$$

We are interested in the problem

$$\begin{cases} y' = f^T_\varepsilon(t, y) \text{ on } \mathbb{R}, \\ y(0) = y(T). \end{cases} \tag{57}$$

We will show that $\overline{v}_\varepsilon$ is a strict supersolution and w^T is a strict subsolution of (57). Let $t \in \mathbb{R}$. We have

$$\begin{aligned} &(\overline{v}_\varepsilon)'(t) - f^T(t, \overline{v}_\varepsilon(t)) - (C+1)\varepsilon\overline{v}_\varepsilon(t) \\ &\quad = f^T(t, w^T(t)) + \tilde{M}_T\varepsilon\Phi_{w^T,f^T}(t) f^T_u(t, w^T(t)) - f^T(t, \overline{v}_\varepsilon(t)) \\ &\qquad\qquad + \tilde{M}_T\varepsilon\Phi_{w^T,f^T}(t)\lambda_{w^T,f^T} - (C+1)\varepsilon\overline{v}_\varepsilon(t). \end{aligned}$$

Since $\varepsilon \in (0, \frac{\mu_T}{\tilde{M}_T \sup\limits_{[0,T]} \Phi_{w^T,f^T}})$, we have $|\overline{v}_\varepsilon(t) - w^T(t)| \le \mu_T$. Furthermore, $w^T(t) \in [0, 1]$, and since $\varepsilon \in (0, \frac{1}{\tilde{M}_T \sup\limits_{[0,T]} \Phi_{w^T,f^T}})$, the definition of $\overline{v}_\varepsilon$ implies that $\overline{v}_\varepsilon(t) \in [0, 2]$. Consequently, it follows from (53) that

$$f^T(t, w^T(t)) + \tilde{M}_T\varepsilon\Phi_{w^T,f^T}(t) f^T_u(t, w^T(t)) - f^T(t, \overline{v}_\varepsilon(t)) \ge -\frac{\lambda_{w^T,f^T}}{2}\tilde{M}_T\varepsilon\Phi_{w^T,f^T}(t).$$

Consequently,

$$\begin{aligned} &(\overline{v}_\varepsilon)'(t) - f^T(t, \overline{v}_\varepsilon(t)) - (C+1)\varepsilon\overline{v}_\varepsilon(t) \\ \ge& \frac{\lambda_{w^T,f^T}}{2}\tilde{M}_T\varepsilon\Phi_{w^T,f^T}(t) - (C+1)\varepsilon\overline{v}_\varepsilon(t) \\ =& \tilde{M}_T\varepsilon\Phi_{w^T,f^T}(t)\Big(\frac{\lambda_{w^T,f^T}}{2} - (C+1)\varepsilon\Big) - (C+1)\varepsilon w^T(t). \end{aligned}$$

Yet $\varepsilon \in (0, \frac{\lambda_{w^T,f^T}}{4(C+1)})$. So

$$\frac{\lambda_{w^T,f^T}}{2} - (C+1)\varepsilon \geq \frac{\lambda_{w^T,f^T}}{4}.$$

Hence

$$\begin{aligned}(\overline{v}_\varepsilon)'(t) - f^T(t, \overline{v}_\varepsilon(t)) - (C+1)\varepsilon\overline{v}_\varepsilon(t) &\geq \tilde{M}_T\varepsilon\Phi_{w^T,f^T}(t)\frac{\lambda_{w^T,f^T}}{4} - (C+1)\varepsilon w^T(t) \\ &= \varepsilon\big(\frac{\lambda_{w^T,f^T}}{4}\tilde{M}_T\Phi_{w^T,f^T}(t) - (C+1)w^T(t)\big).\end{aligned}$$

Consequently, according to the definition of $\tilde{M}_T$, it follows that

$$\frac{\lambda_{w^T,f^T}}{4}\tilde{M}_T\Phi_{w^T,f^T}(t) - (C+1)w^T(t) = \big(2\frac{\Phi_{w^T,f^T}(t)}{\inf\limits_{[0,T]}\Phi_{w^T,f^T}}\sup_{[0,T]} w^T - w^T(t)\big)(C+1) > 0.$$

Finally, $\overline{v}_\varepsilon$ is a strict supersolution of (57).

We now show that w^T is a strict subsolution of this problem. Let $t \in \mathbb{R}$. We have

$$(w^T)'(t) - f^T(t, w^T(t)) - (C+1)\varepsilon w^T(t) = -(C+1)\varepsilon w^T(t) < 0.$$

According to Lemma 3.1 of [22], there exists a solution $\tilde{w}_\varepsilon$ of (57), and one has

$$w^T(t) < \tilde{w}_\varepsilon(t) < w^T(t) + \tilde{M}_T\varepsilon\Phi_{w^T,f^T}(t), \ \forall t \in \mathbb{R}. \tag{58}$$

In particular, $\tilde{w}_\varepsilon$ is a positive solution of (57). So, by uniqueness, we have $\tilde{w}_\varepsilon = w^T_\varepsilon$. Finally, inequalities (58) rewrite

$$\sup_{t\in[0,T]} |w^T(t) - w^T_\varepsilon(t)| \leq \varepsilon M_T,$$

where M_T is defined by $M_T = \tilde{M}_T \sup_{[0,T]} \Phi_{w^T,f^T}$.

We now give a sketch of the proof of the second inequality of Lemma 3.1. We define the function $\underline{v}_\varepsilon : \mathbb{R} \to \mathbb{R}$ by

$$\underline{v}_\varepsilon(t) = w^T(t) - \tilde{M}_T\varepsilon\Phi_{w^T,f^T}.$$

We are interested in the problem

$$\begin{cases} y' = f^T_{-\varepsilon}(t, y) \text{ on } \mathbb{R}, \\ y(0) = y(T). \end{cases} \tag{59}$$

We can show in the same way as previously that $\underline{v}_\varepsilon$ is a strict subsolution and that w^T is a strict supersolution of (59). According to Lemma 3.1 of [22], there exists a solution $\hat{w}_\varepsilon$ of (59), and one has

$$w^T(t) - \tilde{M}_T\varepsilon\Phi_{w^T,f^T}(t) < \hat{w}_\varepsilon(t) < w^T(t), \ \forall t \in \mathbb{R}. \tag{60}$$

Yet $\varepsilon \in (0, \frac{\inf\limits_{[0,T]} w^T}{2\tilde{M}_T \sup\limits_{[0,T]} \Phi_{w^T,f^T}})$. So for any $t \in \mathbb{R}$

$$w^T(t) - \tilde{M}_T\varepsilon\Phi_{w^T,f^T}(t) \geq w^T(t) - \frac{1}{2}\frac{\Phi_{w^T,f^T}}{\sup\limits_{[0,T]} \Phi_{w^T,f^T}} \inf_{[0,T]} w^T > 0.$$

Consequently $\hat{w}_\varepsilon$ is a positive solution of (59). So, by uniqueness, we have $\hat{w}_\varepsilon = w^T_{-\varepsilon}$. Finally, inequalities (60) rewrite

$$\sup_{t\in[0,T]} |w^T(t) - w^T_{-\varepsilon}(t)| \leq \varepsilon M_T,$$

which completes the proof of Lemma 3.1. ☐

Let us now complete the proof of Theorem 1.4. We recall that $\varepsilon \in (0, \tilde{\varepsilon}_T)$, where $\tilde{\varepsilon}_T$ is defined in (54). Let $K \subset \mathbb{R}$ be a compact set and let $\eta > 0$. We consider $\tilde{u}_\varepsilon : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and $\tilde{u}_{-\varepsilon} : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ solving respectively

$$\begin{cases} (\tilde{u}_\varepsilon)_t - (\tilde{u}_\varepsilon)_{xx} = f_\varepsilon^T(t, \tilde{u}_\varepsilon) \text{ on } (0, +\infty) \times \mathbb{R}, \\ \tilde{u}_\varepsilon(0, \cdot) = u_\varepsilon(n_\varepsilon T, \cdot) \text{ on } \mathbb{R}, \end{cases}$$

and

$$\begin{cases} (\tilde{u}_{-\varepsilon})_t - (\tilde{u}_{-\varepsilon})_{xx} = f_{-\varepsilon}^T(t, \tilde{u}_{-\varepsilon}) \text{ on } (0, +\infty) \times \mathbb{R}, \\ \tilde{u}_{-\varepsilon}(0, \cdot) = u_\varepsilon(n_\varepsilon T, \cdot) \text{ on } \mathbb{R}, \end{cases}$$

where $n_\varepsilon \in \mathbb{N}$ is such that (46) holds for all $(t, u) \in [n_\varepsilon T, +\infty) \times \mathbb{R}^+$, and u_ε solves (26). The function $v_\varepsilon : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}, (t, x) \mapsto u_\varepsilon(t + n_\varepsilon T, x)$ satisfies

$$\begin{cases} (v_\varepsilon)_t - (v_\varepsilon)_{xx} = f_\varepsilon(t + n_\varepsilon T, v_\varepsilon) \text{ on } \mathbb{R}^+ \times \mathbb{R}, \\ v_\varepsilon(0, \cdot) = u_\varepsilon(n_\varepsilon T, \cdot) \text{ on } \mathbb{R}. \end{cases}$$

Owing to (46) and the T-periodicity of f_ε^T, it occurs that on $\mathbb{R}^+ \times \mathbb{R}$

$$(v_\varepsilon)_t - (v_\varepsilon)_{xx} = f_\varepsilon(t + n_\varepsilon T, v_\varepsilon) \leq f_\varepsilon^T(t + n_\varepsilon T, v_\varepsilon) = f_\varepsilon^T(t, v_\varepsilon)$$

Consequently, since $v_\varepsilon(0, \cdot) = u_\varepsilon(n_\varepsilon T, \cdot) = \tilde{u}_\varepsilon(0, \cdot)$ on $\mathbb{R}$, applying a comparison principle, we obtain

$$v_\varepsilon(t, x) \leq \tilde{u}_\varepsilon(t, x), \ \forall (t, x) \in \mathbb{R}^+ \times \mathbb{R}.$$

In other words

$$u_\varepsilon(t + n_\varepsilon T, x) \leq \tilde{u}_\varepsilon(t, x), \ \forall (t, x) \in \mathbb{R}^+ \times \mathbb{R}.$$

Actually, we can show in the same way that

$$\tilde{u}_{-\varepsilon}(t, x) \leq u_\varepsilon(t + n_\varepsilon T, x) \leq \tilde{u}_\varepsilon(t, x), \ \forall (t, x) \in \mathbb{R}^+ \times \mathbb{R}.$$

According to the T-periodicity of w^T, we have $w^T = w^T(\cdot + n_\varepsilon T)$ on $\mathbb{R}$. Hence, for any $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$

$$\tilde{u}_{-\varepsilon}(t, x) - w^T(t) \leq u_\varepsilon(t + n_\varepsilon T, x) - w^T(t + n_\varepsilon T) \leq \tilde{u}_\varepsilon(t, x) - w^T(t). \tag{61}$$

Therefore, for any $(t, x) \in \mathbb{R}^+ \times K$,

$$\begin{cases} \tilde{u}_{-\varepsilon}(t, x) - w^T(t) \geq - \sup_{x \in K} |\tilde{u}_{-\varepsilon}(t, x) - w_{-\varepsilon}^T(t)| - \sup_{t \in [0,T]} |w_{-\varepsilon}^T(t) - w^T(t)|, \\ \tilde{u}_\varepsilon(t, x) - w^T(t) \leq \sup_{x \in K} |\tilde{u}_\varepsilon(t, x) - w_\varepsilon^T(t)| + \sup_{t \in [0,T]} |w_\varepsilon^T(t) - w^T(t)|. \end{cases}$$

On the other hand, owing to Proposition 2, there exists $t_{\varepsilon,K,\eta} > 0$ such that for any $t \geq t_{\varepsilon,K,\eta}$

$$\sup_{x \in K} |\tilde{u}_{-\varepsilon}(t, x) - w_{-\varepsilon}^T(t)| + \sup_{x \in K} |\tilde{u}_\varepsilon(t, x) - w_\varepsilon^T(t)| \leq \eta. \tag{62}$$

According to Lemma 3.1, (61) and (62), we thus have, for any $(t, x) \in [t_{\varepsilon,K,\eta}, +\infty) \times K$

$$|u_\varepsilon(t + n_\varepsilon T, x) - w^T(t + n_\varepsilon T)| \leq \eta + M_T \varepsilon.$$

In other words, for any $t \geq t_{\varepsilon,K,\eta} + n_\varepsilon T$ we obtain

$$\sup_{x \in K} |u_\varepsilon(t, x) - w^T(t)| \leq \eta + M_T \varepsilon,$$

That is

$$\limsup_{t\to+\infty}\sup_{x\in K}|u_\varepsilon(t,x)-w^T(t)|\leq M_T\varepsilon,$$

which completes the proof of Theorem 1.4.

3.2. **Proof of Proposition 5.** We begin by proving (I). According to (25), there exists $t_0\geq 0$ such that

$$f(t,u)\leq f^{T^*}(t,u)-\frac{g(2)}{2}u,\ \forall t\in[t_0,+\infty),\forall u\in[0,+\infty), \tag{63}$$

where we recall that $g(2)<0$. According to (4) and (6), for any $u\in[2,+\infty)$, we have $g(u)/u\leq g(2)/2<0$. Consequently, (63) implies that

$$f(t,u)\leq 0,\ \forall t\in[t_0,+\infty),\forall u\in[2,+\infty), \tag{64}$$

We define

$$M=\max\{2,\sup_{\mathbb{R}} u_0\}.$$

The real number M is a supersolution of (26). Furthermore, 0 is solution of (26) and $0\leq u(0,\cdot)\leq M$ on $\mathbb{R}$. Consequently, according to the maximum principle we have

$$0\leq u(t,x)\leq M,\ \forall t\in\mathbb{R},\ \forall x\in\mathbb{R}. \tag{65}$$

We denote $v:\mathbb{R}^+\to\mathbb{R}$ the function satisfying

$$\begin{cases} v'=f(t,v) \text{ on } \mathbb{R}^+,\\ v(0)=M.\end{cases}$$

Owing to (65), we have $0\leq u(t_0,\cdot)\leq M$ on $\mathbb{R}$. It follows from the comparison principle that

$$0\leq u(t+t_0,x)\leq v(t),\ \forall t\geq 0,\ \forall x\in\mathbb{R}.$$

Furthemore, since $2\leq M$, it follows from (64) that

$$v(t)\leq M,\ \forall t\geq 0.$$

To summarize

$$0\leq u(t,+t_0,x)\leq v(t)\leq M,\ \forall t\geq 0,\ \forall x\in\mathbb{R}. \tag{66}$$

We will show that $v(t)\xrightarrow{t\to+\infty}0$. We argue by way of contradiction assuming there exists a real number $\delta_0>0$ and a sequence $t_n\xrightarrow{n\to+\infty}+\infty$ such that

$$v(t_n)>\delta_0,\ \forall n\in\mathbb{N}.$$

For any $n\in\mathbb{N}$, we write $t_n=\tilde{t}_n+k_nT^*$, where $\tilde{t}_n\in[0,T^*)$ and $k_n\in\mathbb{N}$, and we define the function $v_n:[-k_nT^*,+\infty)\to\mathbb{R}$ by $v_n(t)=v(t+k_nT^*)$. The function v_n satisfies

$$\begin{cases} v_n'(t)=f\big(t+k_nT^*,v_n(t)\big)\ \forall t\in[-k_nT^*,+\infty),\\ v_n(\tilde{t}_n)=v(t_n)>\delta_0.\end{cases}$$

Up to extraction of a subsequence, $\tilde{t}_n\xrightarrow{n\to+\infty}t^*\in[0,T^*]$. Consequently, according to (25) and the Arzela-Ascoli theorem, there exists $v^*:\mathbb{R}\to\mathbb{R}$ such that $v_n\xrightarrow{n\to+\infty}v^*$ locally uniformly on $\mathbb{R}$ and which satisfies

$$\begin{cases} (v^*)'=f^{T^*}(t,v^*) \text{ on } \mathbb{R},\\ v^*(t^*)\geq\delta_0.\end{cases} \tag{67}$$

Furthermore, owing to (66), we have

$$0 \le v^*(t) \le M, \ \forall t \in \mathbb{R}. \tag{68}$$

We consider $\sigma : \mathbb{R}^+ \to \mathbb{R}$ such that

$$\begin{cases} \sigma' = f^{T^*}(t, \sigma) \text{ on } \mathbb{R}^+, \\ \sigma(0) = M. \end{cases}$$

Owing to (7) and the fact that $M \ge 1$, we have $\sigma(0) \ge \sigma(T^*)$. Consequently, the sequence $(\sigma(nT^*))_n$ is nonincreasing. Furthermore, it is bounded below by 0. Hence, it converges up to extraction of a subsequence to a real number $l \ge 0$. For any $n \in \mathbb{N}$, we define the function $\sigma_n : \mathbb{R}^+ \to \mathbb{R}$ by $\sigma_n(t) = \sigma(t+nT^*)$. The sequence $(\sigma_n)_n$ converges up to extraction of a subsequence in $\mathcal{C}^1([0,T^*])$ to a function σ^* satisfying

$$\begin{cases} (\sigma^*)' = f^{T^*}(t, \sigma^*) \text{ on } [0, T^*], \\ \sigma^*(0) = \sigma^*(T^*) = l. \end{cases}$$

According to Proposition 1, we have necessarily $\sigma^* = 0$, and thus, the convergence holds for all the sequence. Owing to (68), for any $n \in \mathbb{N}$, we have $v^*(-nT^*) \le M$. Consequently, since f^{T^*} is T^*-periodic, we can apply a comparison principle and we obtain

$$v^*(-nT^* + t) \le \sigma(t), \ \forall t \in \mathbb{R}^+, \ \forall n \in \mathbb{N}.$$

In particular

$$v^*(t^*) \le \sigma_n(t^*), \ \forall n \in \mathbb{N}.$$

Passing to the limit as $n \to +\infty$, we obtain

$$v^*(t^*) \le \sigma^*(t^*) = 0,$$

which is a contradiction with (67). Consequently $v(t) \xrightarrow{t \to +\infty} 0$ and thus, we conclude the proof of (I) using (66).

We now prove (II). We begin by considering the case where $f(t,u) = f^{T^*}(t,u)$ and $p(t,u) = u$ for any $(t,u) \in \mathbb{R}^+ \times \mathbb{R}^+$. In this case, we have

$$f_\varepsilon(t,u) = f^{T^*}(t,u) + \varepsilon u, \ \forall (t,u) \in \mathbb{R} \times \mathbb{R}^+.$$

Let $\varepsilon \in (0, -g(2)/2)$. The function f_ε is T^*-periodic, and we have

$$f_\varepsilon(t,u) \le 0, \ \forall t \in \mathbb{R}, \ \forall u \in [2, +\infty).$$

Furthermore $\lambda_{0,f_\varepsilon} = \lambda_{0,f^{T^*}} - \varepsilon = -\varepsilon < 0$. Consequently, owing to Theorem 1, there exists $w_\varepsilon^T : \mathbb{R} \to (0, +\infty)$ solving (14) with f_ε as nonlinearity. According to Proposition 2, for all compact set $K \subset \mathbb{R}$, we have

$$\sup_{x \in K} |u_\varepsilon(t,x) - w_\varepsilon^T(t)| \xrightarrow{t \to +\infty} 0.$$

We now consider the case where $p(t,u) \le 0$ for any $(t,u) \in \mathbb{R} \times \mathbb{R}^+$. In this case

$$f_\varepsilon(t,u) \le f^{T^*}(t,u), \ \forall (t,u) \in \mathbb{R}^+ \times \mathbb{R}^+.$$

We denote u the solution of the Cauchy problem

$$\begin{cases} u_t - u_{xx} = f^{T^*}(t,u) \text{ on } (0, +\infty) \times \mathbb{R}, \\ u(0, \cdot) = u_0 \text{ on } \mathbb{R}. \end{cases}$$

From the comparison principle, it occurs that

$$0 \le u_\varepsilon(t,x) \le u(t,x) \ \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}. \tag{69}$$

According to (I), we have $\sup_{x\in\mathbb{R}} u(t,x) = 0$. Consequently $\sup_{x\in\mathbb{R}} u_\varepsilon(t,x) = 0$, which concludes the proof.

3.3. **Proof of Theorem 1.5.**

Proof. Let $T > T^*$ and $c \in (0, c_T^*)$, where c_T^* is the critical speed associated with f^T defined in Proposition 1.3. We recall that for $\varepsilon \in (0, \tilde{\varepsilon}_T)$, where $\tilde{\varepsilon}_T$ is defined in (54), inequalities (46), (48), (49) and (55) are satisfied. Furthermore, the critical speeds associated with nonlinearities f_ε^T and $f_{-\varepsilon}^T$ are respectively defined by

$$c_{T,\varepsilon}^* = 2\sqrt{|\lambda_{0,f_\varepsilon^T}|} = 2\sqrt{-\lambda_{0,f^T} + (C+1)\varepsilon},$$

and

$$c_{T,-\varepsilon}^* = 2\sqrt{|\lambda_{0,f_{-\varepsilon}^T}|} = 2\sqrt{-\lambda_{0,f^T} - (C+1)\varepsilon}.$$

In particular, since $c_T^* = 2\sqrt{|\lambda_{0,f^T}|} = 2\sqrt{-\lambda_{0,f^T}}$, there exists $\varepsilon_{c,T} > 0$ such that for all $\varepsilon \in (0, \varepsilon_{c,T})$ we have

$$c \in (0, c_{T,-\varepsilon}^*) \cap (0, c_{T,\varepsilon}^*). \tag{70}$$

We define

$$\hat{\varepsilon}_{c,T} = \min\{\tilde{\varepsilon}_T, \varepsilon_{c,T}\} > 0. \tag{71}$$

We consider $\varepsilon \in (0, \hat{\varepsilon}_{c,T})$. According to the strong maximum principle, we have $u_\varepsilon(n_\varepsilon T, \cdot) > 0$ on $\mathbb{R}$, where $n_\varepsilon \in \mathbb{N}$ is such that (46) holds for all $(t,u) \in [n_\varepsilon T, +\infty) \times \mathbb{R}^+$. Consequently, there exists a nonnegative and nontrivial compactly supported function $\tilde{u}_{\varepsilon,0} : \mathbb{R} \to \mathbb{R}$ such that

$$u_\varepsilon(n_\varepsilon T, x) \geq \tilde{u}_{\varepsilon,0}, \ \forall x \in \mathbb{R}. \tag{72}$$

Let $\tilde{u}_\varepsilon : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be the solution of the Cauchy problem

$$\begin{cases} (\tilde{u}_\varepsilon)_t - (\tilde{u}_\varepsilon)_{xx} = f_{-\varepsilon}^T(t, \tilde{u}_\varepsilon) \text{ on } (0,+\infty) \times \mathbb{R}, \\ \tilde{u}_\varepsilon(0,\cdot) = \tilde{u}_{0,\varepsilon} \text{ on } \mathbb{R}. \end{cases}$$

Owing to (46), (72) and the fact that $f_{-\varepsilon}^T$ is T-periodic, we can apply a comparison principle and get that

$$\tilde{u}_\varepsilon(t,x) \leq u_\varepsilon(t + n_\varepsilon T, x), \ \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}. \tag{73}$$

According to (49), we have $f_\varepsilon \leq 0$ on $\mathbb{R}^+ \times [2, +\infty)$. Hence, since u_0 is bounded, if we define

$$\tilde{C} = \max\{2, \sup_{\mathbb{R}} u_0\},$$

then according to the maximum principle, we have $u_\varepsilon \leq \tilde{C}$ on $\mathbb{R}^+ \times \mathbb{R}$. In particular

$$u_\varepsilon(n_\varepsilon T, x) \leq \tilde{C}, \ \forall x \in \mathbb{R}. \tag{74}$$

Let $v_\varepsilon : \mathbb{R}^+ \to \mathbb{R}$ be the solution of

$$\begin{cases} (v_\varepsilon)_t = f_\varepsilon^T(t, v_\varepsilon) \text{ on } \mathbb{R}^+, \\ v_\varepsilon(0) = \tilde{C}. \end{cases} \tag{75}$$

Owing to (46) and (74), we can still apply a comparison principle to get that

$$u_\varepsilon(t + n_\varepsilon T, x) \leq v_\varepsilon(t), \ \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}. \tag{76}$$

According to (49) and the fact that $\tilde{C} \geq 2$, it occurs that $v_\varepsilon(T) \leq v_\varepsilon(0)$. So the sequence $(v_\varepsilon(nT))_n$ is nonincreasing. Furthermore, this sequence is bounded below by 0. Consequently, it converges to a real number $l \geq 0$. For any $n \in \mathbb{N}$, we define $v_{\varepsilon,n} : \mathbb{R}^+ \to \mathbb{R}$ by $v_{\varepsilon,n}(t) = v_\varepsilon(t+nT)$. The sequence $(v_{\varepsilon,n})_n$ converges up to extraction of a subsequence to $v_\varepsilon^* \geq 0$ in $\mathcal{C}^1([0,T])$ satisfying

$$\begin{cases} (v_\varepsilon^*)' = f_\varepsilon^T(t, v_\varepsilon^*) \text{ on } [0,T], \\ v_\varepsilon^*(0) = v_\varepsilon^*(T) = l. \end{cases}$$

So v_ε^* is equal to 0 or w_ε^T. Yet, there exists $\kappa_\varepsilon > 0$ such that $0 < \kappa_\varepsilon \Phi_{0,f_\varepsilon^T}(0) \leq \tilde{C}$ and

$$\left| f_\varepsilon^T\big(t, \kappa_\varepsilon \Phi_{0,f_\varepsilon^T}(t)\big) - (f_\varepsilon^T)_u(t,0)\kappa_\varepsilon \Phi_{0,f_\varepsilon^T}(t) \right| \leq -\frac{\lambda_{0,f_\varepsilon^T}}{2}\kappa_\varepsilon \Phi_{0,f_\varepsilon^T}(t), \ \forall t \in [0,T].$$

Consequently, we have on $\mathbb{R}^+$

$$\begin{aligned} (\kappa_\varepsilon \Phi_{0,f_\varepsilon^T})' - f_\varepsilon^T(t, \kappa_\varepsilon \Phi_{0,f_\varepsilon^T}) &\leq \kappa_\varepsilon \Phi_{0,f_\varepsilon^T}\big(\lambda_{0,f_\varepsilon^T} + (f_\varepsilon^T)_u(t,0)\big) \\ &\quad - \Big(\kappa_\varepsilon \Phi_{0,f_\varepsilon^T}(f_\varepsilon^T)_u(t,0) + \frac{\lambda_{0,f_\varepsilon^T}}{2}\kappa_\varepsilon \Phi_{0,f_\varepsilon^T}\Big) \\ &\leq \frac{\lambda_{0,f_\varepsilon^T}}{2}\kappa_\varepsilon \Phi_{0,f_\varepsilon^T} \leq 0. \end{aligned}$$

Hence, the function $\kappa_\varepsilon \Phi_{0,f_\varepsilon^T}$ is a subsolution of the problem (75) on $\mathbb{R}^+$. Therefore

$$0 < \kappa_\varepsilon \Phi_{0,f_\varepsilon^T}(t) \leq v_\varepsilon(t), \ \forall t \in \mathbb{R}^+.$$

Using the T-periodicity of Φ_{0,f_ε^T} and passing to the limit as $n \to +\infty$, we obtain

$$0 < \kappa_\varepsilon \Phi_{0,f_\varepsilon^T}(t) \leq v_\varepsilon^*(t), \ \forall t \in \mathbb{R}^+.$$

Consequently, we have necessarily $v_\varepsilon^* \equiv w_\varepsilon^T$ on $[0,T]$. In particular, the uniqueness of accumulation point of the sequence $(v_{\varepsilon,n})_n$ implies that the convergence to w_ε^T holds for the whole sequence. Let $\eta > 0$. There exists $n_{\eta,\varepsilon} \in \mathbb{N}$ such that

$$n \geq n_{\eta,\varepsilon} \Rightarrow \sup_{t\in[0,T]} |v_\varepsilon(t+nT) - w_\varepsilon^T(t)| \leq \eta. \tag{77}$$

On the other hand, according to (70), the spreading properties in periodic case (Proposition 1.3) give the existence of $t_{c,\eta,\varepsilon} \geq 0$ such that

$$t \geq t_{c,\eta,\varepsilon} \Rightarrow \sup_{|x|<ct} |w_{-\varepsilon}^T(t) - \tilde{u}_\varepsilon(t,x)| \leq \eta. \tag{78}$$

Let $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$ such that $t \geq \max\{t_{c,\eta,\varepsilon}, n_{\eta,\varepsilon}T\}$ and $|x| < ct$. According to (73) and (76), it occurs that

$$\tilde{u}_\varepsilon(t,x) \leq u_\varepsilon(t+n_\varepsilon T, x) \leq v_\varepsilon(t)$$

The fact that $t \geq n_{\eta,\varepsilon}T$ implies that we can write $t = n_t T + \tilde{t}$, where $\tilde{t} \in [0,T)$ and $n_t \in \mathbb{N}$ such that $n_t \geq n_{\eta,\varepsilon}$. Consequently, as the function w^T is T-periodic, we have

$$\tilde{u}_\varepsilon(t,x) - w^T(t) \leq u_\varepsilon(t+n_\varepsilon T, x) - w^T(t+n_\varepsilon T) \leq v_\varepsilon(n_t T + \tilde{t}) - w^T(\tilde{t})$$

Hence, according to (77) and Lemma 3.1

$$u_\varepsilon(t+n_\varepsilon T, x) - w^T(t+n_\varepsilon T) \leq |v_\varepsilon(n_t T+\tilde{t}) - w_\varepsilon^T(\tilde{t})| + |w_\varepsilon^T(\tilde{t}) - w^T(\tilde{t})| \leq \eta + M_T\varepsilon,$$

and on the other hand, owing to (78) and Lemma 3.1, it occurs that

$$\begin{aligned} u_\varepsilon(t+n_\varepsilon T, x) - w^T(t+n_\varepsilon T) &\geq -\sup_{|y|<ct} |w_{-\varepsilon}^T(t) - \tilde{u}_\varepsilon(t,y)| - \sup_{[0,T]} |w_{-\varepsilon}^T - w^T| \\ &\geq -\eta - M_T\varepsilon. \end{aligned}$$

To conclude, for any $t \geq \max\{t_{c,\eta,\varepsilon}, n_{\eta,\varepsilon}T\} + n_\varepsilon T$, we have

$$\sup_{|x|<ct} |u_\varepsilon(t,x) - w^T(t)| \leq \eta + M_T\varepsilon,$$

which concludes the proof of the first assertion of Theorem 1.5.

We now show the second part of the theorem. We consider $c > c_T^*$ and c' such that $c_T^* < c' < c$. There exists $\varepsilon'_{c,T} > 0$ such that for all $\varepsilon \in (0, \varepsilon'_{c,T})$ we have

$$c' > \min\{c^*_{T,-\varepsilon}, c^*_{T,\varepsilon}\}. \tag{79}$$

Furthermore, according to (4), (23) and (25), there exists $D > 0$ such that for all $\varepsilon \in [0,1)$, we have

$$f_\varepsilon(t,u) \leq Du, \ \forall t \in \mathbb{R}^+, \ \forall u \in \mathbb{R}^+. \tag{80}$$

We define $\overline{\varepsilon}_{c,T} = \min\{1, \hat{\varepsilon}_{c,T}, \varepsilon'_{c,T}\} > 0$, where $\hat{\varepsilon}_{c,T}$ is defined in (71). Let $\varepsilon \in (0, \overline{\varepsilon}_{c,T})$. We consider $H : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ solving the heat equation

$$\begin{cases} H_t - H_{xx} = 0 \text{ on } (0,+\infty) \times \mathbb{R}, \\ H(0,\cdot) = u_0 \text{ on } \mathbb{R}. \end{cases}$$

The function H is given by

$$H(t,x) = \frac{1}{2\sqrt{\pi t}} \int_{\mathrm{Supp}(u_0)} e^{-\frac{(x-y)^2}{4t}} u_0(y)dy, \ \forall t \in (0,+\infty), \forall x \in \mathbb{R}, \tag{81}$$

where $\mathrm{Supp}(u_0)$ is the support of u_0, which is here assumed to be compact. We define the function $H_D : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ by $H_D(t,x) = H(t,x)e^{Dt}$. We have $(H_D)_t - (H_D)_{xx} = DH_D$ on $(0,+\infty) \times \mathbb{R}$. Furthermore, owing to (80), we have $(u_\varepsilon)_t - (u_\varepsilon)_{xx} = f_\varepsilon(t,u_\varepsilon) \leq Du_\varepsilon$ on $(0,+\infty) \times \mathbb{R}$. Consequently, since $H_D(0,\cdot) = u_\varepsilon(0,\cdot) = u_0$ on $\mathbb{R}$, the comparison principle yields

$$u_\varepsilon(t,x) \leq H(t,x)e^{Dt}, \ \forall t \in \mathbb{R}^+, \forall x \in \mathbb{R}.$$

In particular, owing to (81), it occurs that

$$u_\varepsilon(n_\varepsilon T, x) \leq \frac{e^{Dn_\varepsilon T}}{2\sqrt{\pi n_\varepsilon T}} \int_{\mathrm{Supp}(u_0)} e^{-\frac{(x-y)^2}{4n_\varepsilon T}} u_0(y)dy, \ \forall x \in \mathbb{R}. \tag{82}$$

We define the real number

$$\gamma_{c',\varepsilon} = \frac{c' + \sqrt{(c')^2 + 4\lambda_{0,f_\varepsilon^T}}}{2}.$$

Let us note that $(c')^2 + 4\lambda_{0,f_\varepsilon^T} > 0$ because $c' > c^*_{T,\varepsilon} = 2\sqrt{-\lambda_{0,f_\varepsilon^T}}$. According to (82), $u_\varepsilon(n_\varepsilon T, \cdot)$ has a Gaussian decay as $|x| \to +\infty$ and in particular, there exists a real number $M_{c',\varepsilon} > 0$ such that

$$u_\varepsilon(n_\varepsilon T, x) \leq M_{c',\varepsilon}\Phi_{0,f_\varepsilon^T}(0)e^{-\gamma_{c',\varepsilon}x}, \ \forall x \in \mathbb{R}. \tag{83}$$

We also define the function $v_{c',\varepsilon} : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ by

$$v_{c',\varepsilon}(t,x) = M_{c',\varepsilon}\Phi_{0,f_\varepsilon^T}(t)e^{-\gamma_{c',\varepsilon}(x-c't)},$$

We have on $\mathbb{R}^+ \times \mathbb{R}$

$$(v_{c',\varepsilon})_t - (v_{c',\varepsilon})_{xx} = (-\gamma^2_{c',\varepsilon} + \gamma_{c',\varepsilon}c' + \lambda_{0,f_\varepsilon^T})M_{c',\varepsilon}\Phi_{0,f_\varepsilon^T}e^{-\gamma_{c',\varepsilon}(x-c't)} + (f_\varepsilon^T)_u(t,0)v_{c',\varepsilon}.$$

Hence according to (5) and the fact that $-\gamma^2_{c',\varepsilon} + \gamma_{c',\varepsilon}c' + \lambda_{0,f_\varepsilon^T} = 0$, we obtain on $\mathbb{R}^+ \times \mathbb{R}$

$$(v_{c',\varepsilon})_t - (v_{c',\varepsilon}) \geq f_\varepsilon^T(t, v_{c',\varepsilon})$$

Furthermore, owing to (46), (47) and the T-periodicity of f_ε^T, it occurs that on $\mathbb{R}^+ \times \mathbb{R}$

$$(u_\varepsilon)_t - (u_\varepsilon)_{xx} = f_\varepsilon(t + n_\varepsilon T, u_\varepsilon) \leq f_\varepsilon^T(t + n_\varepsilon T, u_\varepsilon) = f_\varepsilon^T(t, u_\varepsilon)$$

Consequently, since (83) implies that $u_\varepsilon(n_\varepsilon T, \cdot) \leq v_{c',\varepsilon}(0, \cdot)$ on $\mathbb{R}$, the comparison principle implies that

$$0 \leq u_\varepsilon(t + n_\varepsilon T, x) \leq v_{c',\varepsilon}(t, x), \ \forall (t, x) \in \mathbb{R}^+ \times \mathbb{R}.$$

For all $t \geq 0$, since $v_{c',\varepsilon}(t, \cdot)$ is decreasing on $\mathbb{R}$, we have

$$0 \leq \sup_{x>ct} u_\varepsilon(t, x) \leq \sup_{x>ct} v_{c',\varepsilon}(t, x) \leq v_{c',\varepsilon}(t, ct)$$
$$= M_{c',\varepsilon} \Phi_{0,f_\varepsilon^T}(t) e^{-\gamma_{c',\varepsilon}(c-c')t} \xrightarrow{t\to+\infty} 0.$$

In the same way, we can show that

$$0 \leq \sup_{x<-ct} u_\varepsilon(t + n_\varepsilon T, x) \xrightarrow{t\to+\infty} 0.$$

To summarize

$$\lim_{t\to+\infty} \sup_{|x|>ct} u_\varepsilon(t, x) = 0,$$

which concludes the proof of the second assertion of Theorem 1.5. □

4. **Influence of the protocol of the treatment.** We begin by proving Proposition 6.

Proof. Owing to (28), the principal eigenvalue associated with 0 and f_τ^T is given by

$$\lambda_{0,f_\tau^T} = -g'(0) + \int_0^T m_\tau^T(t)\ dt = -g'(0) + \int_0^1 \varphi(t) dt = \lambda_{0,f^T}.$$

□

We now demonstrate Proposition 7.

Proof. Let $T > T^*$. We denote P_τ^T the Poincaré map associated with f_τ^T. We recall that P_τ^T is defined on $\mathbb{R}^+$ by

$$P_\tau^T(\alpha) = y_{\tau,\alpha}(T),$$

where $y_{\tau,\alpha}$ is the solution of the Cauchy problem

$$\begin{cases} (y_{\tau,\alpha})' = f_\tau^T(t, y_{\tau,\alpha}) \text{ on } \mathbb{R}^+, \\ y_{\tau,\alpha}(0) = \alpha. \end{cases} \tag{84}$$

In the same way as in the proof of Proposition 1, we show that the function P_τ^T has a unique positive fixed point α_τ^T. Furthermore $\alpha_\tau^T \in (0, 1]$. Consequently there is a unique equilibrium state $w_\tau^T : \mathbb{R} \to (0, 1]$ associated with f_τ^T. It is the solution of the Cauchy problem (84) with $\alpha = \alpha_\tau^T$.

We begin by showing the continuity property. Let $\tau^* \in (0, T)$ and $(\tau_n)_n$ be a sequence of $(0, T)$ such that $\tau_n \xrightarrow{n\to+\infty} \tau^*$. We will demonstrate that $w_{\tau_n}^T(0) \xrightarrow{n\to+\infty} w_{\tau^*}^T(0)$. The sequence $(w_{\tau_n}^T)_n$ converges up to extraction of a subsequence to a function w^* in $\mathcal{C}^{0,\delta}([0, T])$ for any $\delta \in (0, 1)$. The equilibrium state $w_{\tau_n}^T$ satisfies

$$\begin{cases} w_{\tau_n}^T(t) = w_{\tau_n}^T(0) + \int_0^t f_{\tau_n}^T(s, w_{\tau_n}^T(s)) ds, \ \forall t \in [0, T], \\ w_{\tau_n}^T(0) = w_{\tau_n}^T(T). \end{cases}$$

Passing to the limit as $n \to +\infty$, we obtain

$$\begin{cases} w^*(t) = w^*(0) + \int_0^t f_{\tau^*}^T(s, w^*(s))ds, \ \forall t \in [0,T], \\ w^*(0) = w^*(T). \end{cases}$$

The function $t \mapsto \int_0^t f_{\tau^*}^T(s, w^*(s))ds$ is of class $\mathcal{C}^1([0,T])$. Consequently w^* is of class $\mathcal{C}^1([0,T])$ and it satisfies

$$\begin{cases} (w^*)' = f_\tau^T(t, w^*) \text{ on } [0,T], \\ w^*(0) = w^*(T). \end{cases}$$

Owing to Proposition 1, it follows that $w^* \equiv 0$, or $w^* \equiv w_{\tau^*}^T$. If $w^* = 0$, then $w_{\tau_n}^T \to 0$ as $n \to +\infty$ uniformly on $[0,T]$. For any $n \in \mathbb{N}$, we have

$$\frac{(w_{\tau_n}^T)'(t)}{w_{\tau_n}^T(t)} = \frac{f^T(t, w_{\tau_n}^T(t))}{w_{\tau_n}^T(t)}, \ \forall t \in [0,T].$$

We integrate the previous equation over $[0,T]$, then we pass to the limit as $n \to +\infty$. We obtain $-T\lambda_{0,f_{\tau^*}^T} = 0$. It is a contradiction because since $T > T^*$, we have $\lambda_{0,f_{\tau^*}^T} = \lambda_{0,f^T} < 0$. Hence, we have necessarily $w^* \equiv w_{\tau^*}^T$. So the function $\tau \mapsto w_\tau^T(0)$ is continuous on $(0,T)$.

We now study the monotonicity of this function. We consider two real numbers τ_1 and τ_2 such that $0 < \tau_1 < \tau_2 < T$. The Poincaré map $P_{\tau_i}^T$ associated with $f_{\tau_i}^T$ is defined on $\mathbb{R}^+$ by

$$P_{\tau_i}^T(\alpha) = y_{\tau_i,\alpha}(T),$$

where $y_{\tau_i,\alpha}$ is the solution of (84), with $\tau = \tau_i$. We recall that the equilibrium state $w_{\tau_i}^T$ is the solution on $\mathbb{R}^+$ of (84) with $\alpha = \alpha_{\tau_i}^T$. Consequently, if we prove that $P_{\tau_1}^T > P_{\tau_2}^T$ on $(0,+\infty)$, then we will deduce that $\alpha_{\tau_1}^T > \alpha_{\tau_2}^T$, that is, $w_{\tau_1}^T(0) > w_{\tau_2}^T(0)$. Fix $\alpha > 0$. We define the function $z_{\tau_i,\alpha} : \mathbb{R}^+ \to \mathbb{R}$ by

$$z_{\tau_i,\alpha}(t) = y_{\tau_i,\alpha}(t) e^{\int_0^t m_{\tau_i}^T(s)ds}. \tag{85}$$

This function solves on $\mathbb{R}^+$ the equation

$$(z_{\tau_i,\alpha})' = \frac{g\left(z_{\tau_i,\alpha} e^{-\int_0^t m_{\tau_i}^T(s)ds}\right)}{e^{-\int_0^t m_{\tau_i}^T(s)ds}}$$

For any $t \in [0,T]$, we have

$$e^{-\int_0^t m_{\tau_1}^T(s)ds} \leq e^{-\int_0^t m_{\tau_2}^T(s)ds}. \tag{86}$$

According to (4) and the fact that $z_{\tau_1,\alpha} > 0$, it follows that for any $t \in [0,T]$

$$\frac{g\left(z_{\tau_1,\alpha} e^{-\int_0^t m_{\tau_1}^T(s)ds}\right)}{z_{\tau_1,\alpha} e^{-\int_0^t m_{\tau_1}^T(s)ds}} \geq \frac{g\left(z_{\tau_1,\alpha} e^{-\int_0^t m_{\tau_2}^T(s)ds}\right)}{z_{\tau_1,\alpha} e^{-\int_0^t m_{\tau_2}^T(s)ds}}. \tag{87}$$

In other terms, $z_{\tau_1,\alpha}$ is a subsolution of the equation satisfied by $z_{\tau_2,\alpha}$. Since $z_{\tau_1,\alpha}(0) = z_{\tau_2,\alpha}(0) = \alpha$, we can apply a comparison principle and we obtain

$$z_{\tau_1,\alpha}(t) \geq z_{\tau_2,\alpha}(t), \ \forall t \in [0,T].$$

Actually, the previous inequality is strict with $t = T$ because (86) and (87) are strict on $(0,\tau_2)$. Owing to (85), we have

$$y_{\tau_1,\alpha}(T) e^{\int_0^T m_{\tau_1}^T(s)ds} > y_{\tau_2,\alpha}(T) e^{\int_0^T m_{\tau_2}^T(s)ds}$$

According to (28), it occurs that

$$\int_0^T m_{\tau_1}^T(s)ds = \int_0^T m_{\tau_2}^T(s)ds = \int_0^1 \varphi(s)ds.$$

Consequently

$$y_{\tau_1,\alpha}(T) > y_{\tau_2,\alpha}(T).$$

In other words, $P_{\tau_1}^T(\alpha) > P_{\tau_2}^T(\alpha)$, which concludes the proof. □

REFERENCES

[1] D. G. Aronson and H. F. Weinberger, Multidimensional nonlinear diffusions arising in population genetics, *Adv. Math.*, **30** (1978), 33–76.

[2] H. Berestycki and F. Hamel, Front propagation in periodic excitable media, *Comm. Pure Appl. Math.*, **55** (2002), 949–1032.

[3] H. Berestycki, F. Hamel and N. Nadirashvili, The principal eigenvalue of elliptic operators with large drift and applications to nonlinear propagation phenomena, *Comm. Math. Phys.*, **253** (2005), 451–480.

[4] H. Berestycki, F. Hamel and N. Nadirashvili, The speed of propagation for KPP type problems. I - Periodic framework, *J. Europ. Math. Soc.,* **7** (2005), 173–213.

[5] H. Berestycki, F. Hamel and L. Roques, Analysis of the periodically fragmented environment model: I - Species persistence, *J. Math. Bio.*, **51** (2005), 75–113.

[6] H. Berestycki, F. Hamel and L. Roques, Analysis of the periodically fragmented environment model: II - Biological invasions and pulsating traveling fronts, *J. Math. Pures Appl.*, **84** (2005), 1101–1146.

[7] B. Contri, Pulsating fronts for bistable on average reaction-diffusion equations in a time periodic environment, *Journal of Mathematical Analysis and Applications*, **437** (2016), 90–132.

[8] W. Ding, F. Hamel and X. Zhao, Bistable pulsating fronts for reaction-diffusion equations in a periodic habitat, *Indiana Univ. Math. J.*, **66** (2017), 1189–1265.

[9] J. Fang, X. Yu and X.-Q. Zhao, Traveling waves and spreading speeds for time-space periodic monotone systems, *J. Func. Anal.*, **272** (2017), 4222–4262.

[10] J. Fang and X.-Q. Zhao, Bistable traveling waves for monotone semiflows with applications, *J. Europ. Math. Soc.*, **17** (2015), 2243–2288.

[11] P. C. Fife and J. B. McLeod, The approach of solutions of non-linear diffusion equations to traveling front solutions, *Arch. Ration. Mech. Anal.,* **65** (1977), 335–361.

[12] R. A. Fisher, The wave of advance of advantageous genes, *Ann. Eugenics*, **7** (1937), 335–369.

[13] M. Freidlin and J. Gartner, On the propagation of concentration waves in periodic and random media, *Sov. Math. Dokl.*, **20** (1979), 1282–1286.

[14] G. Frejacques, *Travelling Waves In Infinite Cylinders With Time-periodic Coefficients,* Ph. D thesis, Université d'Aix-Marseille, 2005.

[15] P. Hess, *Periodic-parabolic Boundary Value Problems and Positivity,* Longman Scientific and Technical, 1991.

[16] W. Hudson and B. Zinner, Existence of travelling waves for reaction-diffusion equations of fisher type in periodic media, *Boundary Value Problems for Functional-Differential Equations*, J. Henderson (ed.), World Scientific, 1995, 187–199.

[17] A. N. Kolmogorov, I. G. Petrovsky and N. S. Piskunov, Etude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique, *Bulletin Université D'Etat à Moscou (Bjul. Moskowskogo Gos. Univ.),* **1** (1937), 1–26.

[18] X. Liang and X.-Q. Zhao, Asymptotic speeds of spread and traveling waves for monotone semiflows with applications, *Comm. Pure Appl. Math.*, **60** (2007), 1–40.

[19] X. Liang and X.-Q Zhao, Spreading speeds and traveling waves for abstract monostable evolution systems, *J. Funct. Anal.*, **259** (2010), 857–903.

[20] G. Nadin, The principal eigenvalue of a space-time periodic parabolic operator, *Ann. Mat. Pura Appl.*, **188** (2009), 269–295.

[21] G. Nadin, Traveling fronts in space-time periodic media, *J. Math. Pures Appl.*, **92** (2009), 232–262.

[22] G. Nadin, Existence and uniqueness of the solutions of a space-time periodic reaction-diffusion equation, *J. Diff. Equations*, **249** (2010), 1288–1304.

[23] J. Nolen, M. Rudd and J. Xin, Existence of KPP fronts in spatially-temporally periodic advection and variational principle for propagation speeds, *Dynamics of PDE*, **2** (2005), 1–24.

[24] W. Shen, Travelling waves in time almost periodic structures governed by bistable nonlinearities. II. Existence, *J. Diff. Equations*, **159** (1999), 55–101.

[25] W. Shen, Travelling waves in time almost periodic structures governed by bistable nonlinearities. I. Stability and uniqueness, *J. Diff. Equations*, **159** (1999), 1–54.

[26] N. Shigesada, K. Kawasaki and E. Teramoto, Traveling periodic waves in heterogeneous environments, *Theor. Pop. Bio.*, **30** (1986), 143–160.

[27] H. F. Weinberger, On spreading speeds and traveling waves for growth and migration models in a periodic habitat, *J. Math. Bio*, **45** (2002), 511–548.

[28] J. X. Xin, Existence of planar flame fronts in convective-diffusive periodic media, *Arch. Ration. Mech. Anal.*, **121** (1992), 205–233.

[29] J. X. Xin, Analysis and modeling of front propagation in heterogeneous media, *SIAM Review*, **42** (2000), 161–230.

Received August 2016; revised January 2018.

E-mail address: benjamin.contri@univ-amu.fr