

DECAY RATES FOR ELASTIC-THERMOELASTIC STAR-SHAPED NETWORKS

ZHONG-JIE HAN

School of Mathematics
Tianjin University
300354 Tianjin, China

ENRIQUE ZUAZUA

(1) DeustoTech - Fundación Deusto
Avda. Universidades, 24
48007 Bilbao, Basque Country, Spain

(2) Departamento de Matemáticas
Universidad Autónoma de Madrid
28049 Madrid, Spain

(3) Facultad Ingeniería
Universidad de Deusto, Avda. Universidades, 24
48007 Bilbao, Basque Country, Spain

ABSTRACT. This work discusses the asymptotic behaviour of a transmission problem on star-shaped networks of interconnected elastic and thermoelastic rods. Elastic rods are undamped, of conservative nature, while the thermoelastic ones are damped by thermal effects. We analyse the overall decay rate depending of the number of purely elastic components entering on the system and the irrationality properties of its lengths.

First, a sufficient and necessary condition for the strong stability of the thermoelastic-elastic network is given. Then, the uniform exponential decay rate is proved by frequency domain analysis techniques when only one purely elastic undamped rod is present. When the network involves more than one purely elastic undamped rod the lack of exponential decay is proved and nearly sharp polynomial decay rates are deduced under suitable irrationality conditions on the lengths of the rods, based on Diophantine approximation arguments. More general slow decay rates are also derived. Finally, we present some numerical simulations supporting the analytical results.

1. Introduction. In recent years the problems of control and stabilisation for thermoelastic systems have been studied intensively. We refer for instance to [22], [42] in the context of controllability, and to [14], [18], [23], [24], [35] for stabilization, among others.

2010 *Mathematics Subject Classification.* Primary: 35B40, 93D20; Secondary: 35M10, 74F05.

Key words and phrases. Thermoelasticity, elasticity, rods, star-shaped network, exponential decay, polynomial decay, asymptotic behavior, resolvent estimates, irrationality conditions.

The first author was supported by the Natural Science Foundation of China grant NSFC-61573252 and China Scholarship Council. The second author was supported by the Advanced Grant DYCON of the European Research Council Executive Agency, ICON of the ANR-2016-ACHN-0014-01 (France), FA9550-14-1-0214 of the EOARD-AFOSR, FA9550-15-1-0027 of AFOSR, the MTM2014-52347 Grant of the MINECO and a Humboldt Research Award at the University of Erlangen-Nürnberg.

A closely related interesting issue is the asymptotic behaviour of a system consisting of two different materials joined together at the interface, one being purely elastic and the other thermoelastic (see Figure 1). In these systems, other than the intrinsic coupling effects of thermal and elastic components, typical in thermoelasticity, the thermoelastic and purely elastic rods are also coupled through the interface.

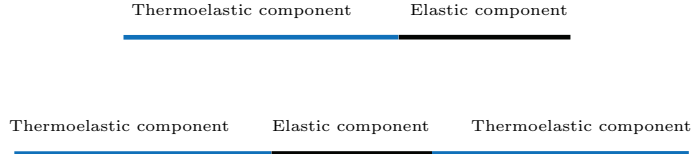


FIGURE 1. Transmission problem in 1-d elasticity-thermoelasticity

Exponential and polynomial decay properties for this kind of systems were proved by Rivera *et al.* in [12], [28], [31] and Messaoudi *et al.* in [30], using the energy multiplier method. Han and Xu in [15] got a sharp polynomial decay rate for a thermoelastic transmission problem with joint mass, based on a detailed spectral analysis and resolvent operator estimates. We also refer to [29] for the analysis of the large time behaviour of transmission problems of multi-dimensional thermoelasticity.

In this work, we consider similar transmission problems in multi-connected networks. More precisely, we are interested in the large time behaviour of star-shaped networks constituted by coupled thermoelastic and purely elastic rods (see Figure 2).

The aim of this work is to give a complete analysis on the large time behaviour of these systems proving exponential, polynomial and slow decay rates. The optimality of these results is also discussed.

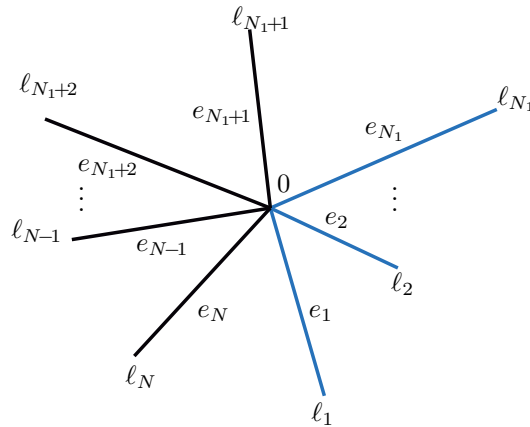


FIGURE 2. Star-shaped thermoelastic-elastic network

In order to present the problems under consideration more precisely some notations are needed. We denote by e_j , $j = 1, 2, \dots, n$ the segments occupying the intervals $(0, \ell_j)$, $\ell_j \geq 0$, respectively. Denote by $x = 0$ the common node of this

star-shaped network. Assume that the edges $j = 1, 2, \dots, N_1$ ($0 < N_1 < N$) in the network are constituted by thermoelastic rods, given by the following equations:

$$\begin{cases} u_{k,tt}(x, t) - u_{k,xx}(x, t) + \alpha_k \theta_{k,x}(x, t) = 0, & x \in (0, \ell_k), k = 1, 2, \dots, N_1, t > 0, \\ \theta_{k,t}(x, t) - \theta_{k,xx}(x, t) + \beta_k u_{k,tx}(x, t) = 0, & x \in (0, \ell_k), k = 1, 2, \dots, N_1, t > 0, \end{cases} \quad (1)$$

and the other edges in the network are all purely elastic ones given by

$$u_{j,tt}(x, t) - u_{j,xx}(x, t) = 0, \quad x \in (0, \ell_j), j = N_1 + 1, \dots, N, t > 0. \quad (2)$$

Here and in the sequel $u_j(x, t)$, $j = 1, 2, \dots, N$ denote the displacements of the rods at time t , and $\theta_k(x, t)$, $k = 1, 2, \dots, N_1$ the temperature difference with respect to a fixed reference temperature.

Assume that the exterior nodes of the network are all clamped, and the displacements and temperatures are all continuous at the common node. There is no heat exchange between thermoelastic components and purely elastic ones and the balance of forces at the common node is fulfilled. Thus, the boundary and transmission conditions read as follows:

$$\begin{cases} u_j(\ell_j, t) = 0, & j = 1, 2, \dots, N, t > 0, \\ u_j(0, t) = u_k(0, t), & \forall j, k = 1, 2, 3, \dots, N, t > 0, \\ \theta_k(\ell_k, t) = 0, & k = 1, 2, \dots, N_1, t > 0, \\ \theta_k(0, t) = \theta_j(0, t), & \forall j, k = 1, 2, \dots, N_1, t > 0, \\ \sum_{j=1}^N u_{j,x}(0, t) = \sum_{j=1}^{N_1} \alpha_j \theta_j(0, t), & \sum_{j=1}^{N_1} \frac{\alpha_j}{\beta_j} \theta_{j,x}(0, t) = 0, t > 0 \end{cases} \quad (3)$$

with initial conditions

$$u|_{t=0} = u^{(0)} := (u_j^{(0)})_{j=1}^N, \quad u_t|_{t=0} = u^{(1)} := (u_j^{(1)})_{j=1}^N, \quad \theta|_{t=0} = \theta^{(0)} := (\theta_j^{(0)})_{j=1}^{N_1}. \quad (4)$$

Overall the thermoelastic-elastic network system under consideration reads as follows:

$$\begin{cases} u_{j,tt}(x, t) - u_{j,xx}(x, t) + \alpha_j \theta_{j,x}(x, t) = 0, & x \in (0, \ell_j), j = 1, 2, \dots, N_1, t > 0, \\ \theta_{j,t}(x, t) - \theta_{j,xx}(x, t) + \beta_j u_{j,tx}(x, t) = 0, & x \in (0, \ell_j), j = 1, 2, \dots, N_1, t > 0, \\ u_{j,tt}(x, t) - u_{j,xx}(x, t) = 0, & x \in (0, \ell_j), j = N_1 + 1, \dots, N, t > 0, \\ u_j(\ell_j, t) = 0, & j = 1, 2, \dots, N, t > 0, \\ u_j(0, t) = u_k(0, t), & \forall j, k = 1, 2, 3, \dots, N, t > 0, \\ \theta_k(\ell_k, t) = 0, & k = 1, 2, \dots, N_1, t > 0, \\ \theta_k(0, t) = \theta_j(0, t), & \forall j, k = 1, 2, \dots, N_1, t > 0, \\ \sum_{j=1}^N u_{j,x}(0, t) = \sum_{j=1}^{N_1} \alpha_j \theta_j(0, t), & \sum_{j=1}^{N_1} \frac{\alpha_j}{\beta_j} \theta_{j,x}(0, t) = 0, t > 0, \\ u(t=0) = u^{(0)}, & u_t(t=0) = u^{(1)}, \quad \theta(t=0) = \theta^{(0)}. \end{cases} \quad (5)$$

Remark 1. The transmission problem in Figure 1 can be considered as a special case of the above network ($N = 2, N_1 = 1$).

The natural energy of this system is as follows

$$E(t) = \frac{1}{2} \sum_{j=1}^N \int_0^{\ell_j} [u_{j,t}^2 + u_{j,x}^2] dx + \frac{1}{2} \sum_{j=1}^{N_1} \frac{\alpha_j}{\beta_j} \int_0^{\ell_j} \theta_j^2 dx,$$

and a direct calculation yields the dissipation law

$$E'(t) = - \sum_{k=1}^{N_1} \frac{\alpha_k}{\beta_k} \int_0^{\ell_k} \theta_{k,x}^2 dx \leq 0. \quad (6)$$

Hence, the energy of system (5) is decreasing. Moreover, from (6), it is easy to see that the dissipation mechanism only acts in the thermoelastic rods. This motivates the problem of whether or not the dissipation is strong enough to make the total energy of the network decay to zero, and with which rate.

In this paper, the large time behaviour of system (5) is mainly discussed based on frequency domain analysis ([5], [6], [13], [19], [25], [26] and [34]). In [37], Shel showed the exponential stability of networks of thermoelastic and elastic materials for some special cases by similar methods. However, it was assumed that there was no heat exchange between the thermoelastic rods connected at the common nodes. In this paper, we allow for the heat exchange between thermoelastic rods at common nodes, which is a natural assumption.

By estimating the resolvent operator along the imaginary axis and employing multiplier techniques, we get a necessary and sufficient condition for system (5) to decay uniformly exponentially, namely that there is no more than one purely elastic rod entering in the network. If this condition fails, the system lacks exponential decay and we further show that the decay rate of the networks can not be faster than t^{-1} . Moreover, for a very special case that there are two purely elastic rods involved in the network, the optimal polynomial decay result is obtained and for the general case that there are more purely elastic rods involved, a nearly optimal polynomial decay result is also derived under certain Diophantine approximation conditions, that refer to the lengths of the purely elastic rods involved in the network.

To discuss the sharpness of slow decay rates it is useful to get explicit information on the spectrum of the system, and compare its real and imaginary parts (see [6] and [41]). However, spectra of PDE networks are often difficult to calculate. Thus, we prove the optimality by estimating the norm of the resolvent operator along the imaginary axis (see [1]). But resolvent estimates are hard to be achieved due to the thermoelastic coupling. Thus, we employ diagonalisation argument to deal with the resolvent problem. This allows building explicit approximations of solutions ensuring that the polynomial decay rates we get are nearly optimal.

The rest of the paper is organised as follows. In section 2, the main result of this paper is given. Section 3 is devoted to show the well-posedness and strong asymptotic stability of the system (5). In section 4, we prove the exponential and nearly optimal slow decay rates, under different conditions. Section 5 is devoted to discuss some more general slow decay rates for system (5). Especially, for the special case $N - N_1 = 2$, the condition to achieve optimal decay of the network is obtained. In section 6, the numerical simulations of the dynamical behaviour of system (5) are presented.

The results in this paper contribute to the understanding of the decay properties of wave and thermoelastic wave networks, a topic in which important issues are still to be understood.

There has been an extensive literature on other closely related issues such as the large time behaviour and controllability properties of elastic networks with node and boundary feedback controls. We refer, among others (the present list of references is by no means complete), to Lagnese *et al.* [21] for the modelling and control of elastic networks; Ammari *et al.* [2], [3] and [4] and Nicaise *et al.* [32]

for stabilisation problems on networks of wave and Euler-Bernoulli beams with star-shaped and tree-shaped configurations; Dáger and Zuazua [8], [9] and [10] for boundary controllability of wave networks; Xu *et al.* [16], [17] and [40] for the stabilisation and spectral properties of the wave networks.

2. Main results. This section is devoted to state the main result of this paper.

As we will see later, system (5) can be rewritten as an abstract Cauchy problem in an appropriate Hilbert space \mathcal{H} :

$$\frac{dU(t)}{dt} = \mathcal{A}U(t), \quad t > 0; \quad U(0) = U_0, \tag{7}$$

where $U(t) = (u, u_t, \theta)^T$ and $U(0) = (u^{(0)}, u^{(1)}, \theta^{(0)})^T \in \mathcal{H}$ are given.

The problem of whether the energy of solutions tends to zero as time goes to infinity or not has a simple answer:

Theorem 2.1. *Operator \mathcal{A} generates a C_0 semigroup of contractions on \mathcal{H} . Moreover, the energy of the system (5) decays to zero as $t \rightarrow \infty$ if and only if one of the following two conditions is fulfilled,*

- 1). $N - N_1 = 1$;
- 2). $N - N_1 \geq 2$ and $\ell_i/\ell_j \notin Q$, $i, j = N_1 + 1, N_1 + 2, \dots, N$, $i \neq j$.

We now obtain explicit decay rates for network (5).

It is well known that if all the components in the network are thermoelastic, that is $N = N_1$, the energy of the system decays exponentially to zero.

In fact, in Propositions 1 and 2 in section 4, we obtain the following necessary and sufficient condition for the exponential decay.

Theorem 2.2. *The energy of system (5) decays to zero exponentially if and only if $N - N_1 \leq 1$, that is, if no more than one purely elastic undamped rod is involved in the network.*

Accordingly, when $N - N_1 > 1$, one can only expect slow decay rate for network (5). In order to address this issue we need the following definition from [36] and [11].

Definition 2.3. ([36], [11]) Real numbers $\ell_1, \ell_2, \dots, \ell_m$ are said to verify the conditions (S), if $\ell_1, \ell_2, \dots, \ell_m$ are linearly independent over the field Q of rational numbers; and the ratios ℓ_i/ℓ_j are algebraic numbers for $i, j = 1, 2, \dots, m$.

The notation (S) for this condition was introduced in [11] to refer to the fundamental contribution by Schmidt [36], that defined this class of irrational numbers for the simultaneous approximation by rational ones. It should be noted that the condition (S) denotes a narrow class of irrationals, since the set of algebraic numbers is countable and has Lebesgue measure zero.

By a detailed frequency domain analysis, we have the following explicit polynomial decay rate for system (5).

Theorem 2.4. *When $N - N_1 > 1$, The following estimation always holds.*

$$\liminf_{t \rightarrow \infty} tE(t) > 0. \tag{8}$$

Thus, we can not expect a decay rate which is beyond first order polynomial. Furthermore, if $\ell_{N_1+1}, \ell_{N_1+2}, \dots, \ell_N$ ($N - N_1 > 1$) satisfy the conditions (S), then

for any $\epsilon > 0$, there always exists a constant $C_\epsilon > 0$ such that the energy of network (5) satisfies

$$E(t) \leq C_\epsilon t^{-\frac{1}{1+\epsilon}} \|(u^{(0)}, u^{(1)}, \theta^{(0)})\|_{\mathcal{D}(\mathcal{A})}^2, \quad \forall t \geq 0, \quad (9)$$

for all $(u^{(0)}, u^{(1)}, \theta^{(0)}) \in \mathcal{D}(\mathcal{A})$. Thus, this decay rate is nearly sharp in the sense of (8).

Remark 2. Using the method of proof of Theorem 2.4, we can obtain more general slow decay rates (polynomial, logarithmic or arbitrarily slow decay), which will be presented in section 5. Especially, for a very special case that there are two purely elastic rods involved in the network ($N - N_1 = 2$), we show that the network can achieve the optimal decay rate t^{-1} if the mutual ratio of the lengths of the purely elastic rods belongs to the set of irrational numbers having a continuous fraction expansion $[a_0, a_1, \dots, a_n, \dots]$ with bounded (a_n) .

Remark 3. The optimality result in (8) is well-known for wave-like equations with velocity damping in the case of one single string with damping on an internal point, which is equivalent to a simple star-like network constituted only by two strings ([20], [11]).

Here we show the same lower bound on the decay rate for the more general system involving thermoelastic rods. However, for general cases that more than two purely elastic rods entering in the networks, it is still an open problem to find the condition to guarantee achieving a sharp polynomial decay rate.

3. Well-posedness and strong stability. This section is devoted to show the well-posedness of network (5) by the semigroup theory and prove Theorem 2.1, with a necessary and sufficient condition for strong stability of this system.

Let us first introduce an appropriate Hilbert space setting for the well-posedness of the system.

For $n \geq 1$, define

$$L^2(\mathcal{R}^n) = \{u | u_j \in L^2(0, \ell_j), \forall j = 1, 2, \dots, n\},$$

$$V_n := \{\phi \in \prod_{j=1}^n H^1(0, \ell_j) | \phi_j(0) = \phi_k(0), \phi_j(\ell_j) = 0, \forall k, j = 1, 2, \dots, n\}.$$

Set the state space \mathcal{H} as follows:

$$\mathcal{H} = V_N \times L^2(\mathcal{R}^N) \times L^2(\mathcal{R}^{N_1}),$$

equipped with inner product:

$$(W, \widetilde{W})_{\mathcal{H}} = \sum_{j=1}^N \int_0^{\ell_j} u_{j,x} \overline{\widetilde{u}_{j,x}} dx + \sum_{j=1}^N \int_0^{\ell_j} w_j \overline{\widetilde{w}_j} dx + \sum_{k=1}^{N_1} \frac{\alpha_k}{\beta_k} \int_0^{\ell_k} \theta_k \overline{\widetilde{\theta}_k} dx,$$

for $W = (u, w, \theta)$, $\widetilde{W} = (\widetilde{u}, \widetilde{w}, \widetilde{\theta}) \in \mathcal{H}$.

$(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ is a Hilbert space.

Then, define the system operator \mathcal{A} in \mathcal{H} as follows:

$$\mathcal{A} \begin{pmatrix} u \\ w \\ \theta \end{pmatrix} = \begin{pmatrix} w \\ u_{xx} - \alpha I_{N \times N_1} \theta_x \\ \theta_{xx} - \beta w_{1,x} \end{pmatrix} = \begin{pmatrix} 0 & I & 0 \\ \partial_{xx} & 0 & -\alpha I_{N \times N_1} \partial_x \\ 0 & -\beta I_{N \times N_1}^T \partial_x & \partial_{xx} \end{pmatrix} \begin{bmatrix} u \\ w \\ \theta \end{bmatrix}$$

where $\alpha = \text{diag}(\alpha_1, \alpha_2, \dots, \alpha_{N_1})$, $\beta = \text{diag}(\beta_1, \beta_2, \dots, \beta_{N_1})$, $I_{N \times N_1} = [I_{N_1}, 0]^T$ and I_{N_1} is the N_1 -unit matrix with domain

$$\mathcal{D}(\mathcal{A}) = \left\{ (u, w, \theta) \in [V_N \cap \prod_{j=1}^N H^2(0, \ell_j)] \times V_N \times \prod_{j=1}^{N_1} H^2(0, \ell_j) \right. \\ \left. \begin{array}{l} \sum_{j=1}^N u_{j,x}(0) = \sum_{k=1}^{N_1} \alpha_k \theta_k(0) \\ \theta_j(\ell_j) = 0, \quad j = 1, 2, \dots, N_1 \\ \theta_j(0) = \theta_k(0), \quad j, k = 1, 2, \dots, N_1 \\ \sum_{j=1}^{N_1} \frac{\alpha_j}{\beta_j} \theta_{j,x}(0) = 0 \end{array} \right\}.$$

Thus, system (5) can be rewritten as the evolution equation (7) in \mathcal{H} .

It is easy to check that \mathcal{A} is dissipative in \mathcal{H} . Moreover, \mathcal{A} is injective and surjective and hence $0 \in \rho(\mathcal{A})$. Then, by Lummer-Phillips theorem (see [33]), \mathcal{A} generates a C_0 semigroup of contractions $S(t)$ on \mathcal{H} .

Now, we focus on proving the strong stability property of the system in Theorem 2.1. The proof by contradiction is mainly used here.

Proof of Theorem 2.1.

Sufficiency. If the strong stability of system (5) does not hold, then by the Lyubich-Phóng strong stability theorem (see [27]), there exists at least one $\tilde{\lambda} = i\tilde{\sigma} \in \sigma(\mathcal{A})$, $\tilde{\sigma} \in \mathbb{R}, \tilde{\sigma} \neq 0$ on the imaginary axis. Assume that

$$\tilde{W} = \left((u_j)_{j=1}^N, \tilde{\lambda}(u_j)_{j=1}^N, (\theta_k)_{k=1}^{N_1} \right)^T \in \mathcal{D}(\mathcal{A})$$

is an eigenvector of \mathcal{A} corresponding to such $\tilde{\lambda}$. We get

$$0 = \Re \tilde{\lambda} \|\tilde{W}\|_{\mathcal{H}}^2 = \Re(\mathcal{A}\tilde{W}, \tilde{W})_{\mathcal{H}} = - \sum_{k=1}^{N_1} \frac{\alpha_k}{\beta_k} \int_0^{\ell_k} \theta_{k,x}^2 dx,$$

which yields $\theta_k = \theta_{k,x} = 0, k = 1, 2, \dots, N_1$. Then by the boundary and transmission conditions in (5), $u_j(x), j = 1, 2, \dots, N$ satisfy the following equations:

$$\begin{cases} \tilde{\lambda}^2 u_j(x) - u_{j,xx}(x) = 0, \quad x \in (0, \ell_j), \quad j = 1, 2, \dots, N, \\ \beta_j \tilde{\lambda} u_{j,x}(x) = 0, \quad x \in (0, \ell_j), \quad j = 1, 2, \dots, N_1, \\ u_j(\ell_j) = 0, \quad j = 1, 2, \dots, N, \\ u_j(0) = u_k(0), \quad \forall j, k = 1, 2, 3, \dots, N, \\ \sum_{j=1}^N u_{j,x}(0) = 0. \end{cases} \tag{10}$$

If $N - N_1 = 1$, it is easy to get $(u, \tilde{\lambda}u, \theta) = 0$, which contradicts the fact that $(u, \tilde{\lambda}u, \theta) = 0$ is an eigenvector.

If $N - N_1 \geq 2$, we get by direct calculation,

$$\begin{cases} u_k = 0, \quad k = 1, 2, \dots, N_1, \\ u_j = c_j \sinh \tilde{\lambda}x, \quad j = N_1 + 1, N_1 + 2, \dots, N, \end{cases}$$

which satisfy

$$c_j \sinh \tilde{\lambda} \ell_j = 0, \quad j = N_1 + 1, N_1 + 2, \dots, N, \quad \text{and} \quad \sum_{j=N_1+1}^N c_j = 0.$$

Since $(u, \tilde{\lambda}u, \theta)$ is the eigenvector corresponding to $\tilde{\lambda}$, then there are at least c_{j_1}, c_{j_2} for some $j_1, j_2 \geq N_1 + 1$ such that

$$c_{j_1}, c_{j_2} \neq 0.$$

Hence, $\sinh \tilde{\lambda} \ell_{j_1} = \sinh \tilde{\lambda} \ell_{j_2} = 0$. Thus, $\ell_{j_1}/\ell_{j_2} \in Q$, which is in contradiction with condition (2) in Theorem 2.1.

Necessity. If there exist i_0, j_0 satisfying $N_1 + 1 \leq i_0, j_0 \leq N$, $\ell_{i_0}/\ell_{j_0} = p/q$, with p, q nonzero integers, we get easily that $\left((\widehat{u}_j(x))_{j=1}^N, \tilde{\lambda}(\widehat{u}_j(x))_{j=1}^N, (0)_{k=1}^{N_1} \right)$ is an eigenvector corresponding to $\tilde{\lambda} = iq\pi/\ell_{j_0} = ip\pi/\ell_{i_0}$, in which

$$\begin{aligned} \widehat{u}_{i_0}(x) &= \sin(p\pi x/\ell_{i_0}), \quad \widehat{u}_{j_0}(x) = -\sin(q_0\pi x/\ell_{j_0}), \\ \widehat{u}_j(x) &= 0, \quad j = 1, 2, \dots, N, j \neq i_0, j_0. \end{aligned}$$

This contradicts the strong stability property of system (5). Therefore, $\ell_i/\ell_j \notin Q$, $i, j = N_1 + 1, N_1 + 2, \dots, N$, $i \neq j$. The proof is complete. \square

Remark 4. For the proof of ‘‘Sufficiency’’, we also can use the unique continuation property for wave networks in Dager and Zuazua [11] (See Corollary 5.28, p.135). Indeed, (10), in the absence of thermal components, corresponds to the eigenproblem associated with the pure wave system and, according to the results in [11], its unique solution is the trivial one, which contradicts that $\left((u_j)_{j=1}^N, \tilde{\lambda}(u_j)_{j=1}^N, (\theta_k)_{k=1}^{N_1} \right)^T$ is an eigenvector. Then the desired result follows.

4. Decay rates. This section is devoted to achieve explicit decay rates of the total energy of solutions of system (5). The exponential decay and slow decay rates are deduced under different assumptions of the various components of the network.

4.1. Exponential decay rate: Case $N - N_1 = 1$. In this subsection, we analyse the decay rate of network (5) when $N - N_1 = 1$, namely, there is only one purely elastic rod involved in the network. To do this, let us introduce the following lemma as described in [13], [19], [25] and [34].

Lemma 4.1. *Let $(S(t))_{t \geq 0}$ be a C_0 semigroup on a Hilbert space \mathcal{H} generated by \mathcal{A} . Then the semigroup is exponentially stable if and only if*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \tag{11}$$

and

$$\|(i\sigma I - \mathcal{A})^{-1}\|_{\mathcal{H}} \leq C, \quad \forall \sigma \in \mathbb{R}. \tag{12}$$

We are then in conditions to prove the following proposition, which is one of the main statements in Theorem 2.1:

Proposition 1. *When $N - N_1 = 1$, the energy of system (5) decays exponentially to zero.*

Proof. It is sufficient to show that the conditions in Lemma 4.1 are fulfilled. It should be noted that although the idea of this proof is similar to the one in [37], some different multipliers are employed to get certain estimates so as to deal with the transmission conditions in the present paper.

By the argument of the proof of Theorem 2.1, it is easy to see that, in the present case, there is no eigenvalue of \mathcal{A} on the imaginary axis, that is, the condition (11) in Lemma 4.1 holds. Now we proceed to prove condition (12) and to deduce the exponential decay rate.

If condition (12) is not fulfilled, then there exists a sequence of real numbers σ_n such that the corresponding resolvents $T_n = (i\sigma_n I - \mathcal{A})^{-1}$ satisfy $\|T_n\|_{\mathcal{H}} \rightarrow \infty$, $n \rightarrow \infty$. By Banach-Steinhaus theorem, there exists $F \in \mathcal{H}$ such that

$$T_n F = (i\sigma_n I - \mathcal{A})^{-1} F = \tilde{\Psi}_n \rightarrow \infty, \text{ in } \mathcal{H}, \quad n \rightarrow \infty.$$

Thus,

$$(i\sigma_n I - \mathcal{A}) \frac{\widetilde{\Psi}_n}{\|\widetilde{\Psi}_n\|_{\mathcal{H}}} = \frac{F}{\|\widetilde{\Psi}_n\|_{\mathcal{H}}} \rightarrow 0, \text{ in } \mathcal{H}, n \rightarrow \infty.$$

So there exists a sequence $\Phi^n = (U^n, V^n, \Theta^n) \in \mathcal{D}(\mathcal{A})$, with $\|\Phi^n\|_{\mathcal{H}} = 1$, where $U^n = (u_j^n)_{j=1}^N$, $V^n = (v_j^n)_{j=1}^N$, $\Theta^n = (\theta_j^n)_{j=1}^{N-1}$, and a sequence $\sigma_n \in \mathbb{R}$ with $\sigma_n \rightarrow \infty$ such that

$$\lim_{n \rightarrow \infty} \|(i\sigma_n I - \mathcal{A})\Phi^n\|_{\mathcal{H}} = 0,$$

namely,

$$i\sigma_n u_j^n - v_j^n \rightarrow 0, \text{ in } H^1(0, \ell_j), j = 1, 2, \dots, N, \tag{13}$$

$$i\sigma_n v_j^n - u_{j,xx}^n + \alpha_j \theta_{j,x}^n \rightarrow 0, \text{ in } L^2(0, \ell_j), j = 1, 2, \dots, N - 1, \tag{14}$$

$$i\sigma_n \theta_j^n - \theta_{j,xx}^n + \beta_j v_{j,x}^n \rightarrow 0, \text{ in } L^2(0, \ell_j), j = 1, 2, \dots, N - 1, \tag{15}$$

$$i\sigma_n v_N^n - u_{N,xx}^n \rightarrow 0, \text{ in } L^2(0, \ell_N). \tag{16}$$

Note that $((i\sigma_n - \mathcal{A})\Phi^n, \Phi^n)_{\mathcal{H}} = - \sum_{j=1}^{N-1} \frac{\alpha_j}{\beta_j} \int_0^{\ell_j} |\theta_{j,x}^n|^2 dx \rightarrow 0$. Thus,

$$\theta_{j,x}^n \rightarrow 0, \text{ in } L^2(0, \ell_j), j = 1, 2, \dots, N - 1 \tag{17}$$

and hence by Poincaré inequality, we get $\theta_j^n \rightarrow 0$, in $L^2(0, \ell_j)$, $j = 1, 2, \dots, N - 1$. Moreover, by the Gagliardo-Nirenberg inequality (see [25]), we get $\|\theta_j^n\|_{L^\infty} \leq d_1 \|\theta_{j,x}^n\|^{\frac{1}{2}} \|\theta_j^n\|^{\frac{1}{2}} + d_2 \|\theta_j^n\| \rightarrow 0$, which implies that

$$\theta_j^n(0) \rightarrow 0, \quad j = 1, 2, \dots, N_1. \tag{18}$$

Removing $\theta_{j,x}^n$ in (14) and substituting (13) into (14), we get

$$i\sigma_n v_j^n - u_{j,xx}^n \rightarrow 0, \text{ in } L^2(0, \ell_j), j = 1, 2, \dots, N - 1. \tag{19}$$

and taking the inner product of (19) with $xu_{j,x}^n$ in $L^2(0, \ell_j)$, we have

$$(-\sigma_n^2 u_j^n, xu_{j,x}^n) - (u_{j,xx}^n, xu_{j,x}^n) \rightarrow 0, \quad j = 1, 2, \dots, N - 1.$$

Note that

$$2\Re(-\sigma_n^2 u_j^n, xu_{j,x}^n) = -\sigma_n^2 u_j^n(\ell_j) \overline{\ell_j u_{j,x}^n(\ell_j)} - (-\sigma_n^2 u_j^n, u_j^n) = -(-\sigma_n^2 u_j^n, u_j^n),$$

$$2\Re(u_{j,xx}^n, xu_{j,x}^n) = u_{j,x}^n(\ell_j) \overline{\ell_j u_{j,x}^n(\ell_j)} - (u_{j,x}^n, u_{j,x}^n).$$

Hence, $u_{j,x}^n(\ell_j)$, $j = 1, 2, \dots, N - 1$ are bounded. Similarly, taking the inner product of (19) with $(x - \ell_j)u_{j,x}^n$ in $L^2(0, \ell_j)$ yields

$$(-\sigma_n^2 u_j^n, (x - \ell_j)u_{j,x}^n) - (u_{j,xx}^n, (x - \ell_j)u_{j,x}^n) \rightarrow 0, \quad j = 1, 2, \dots, N - 1.$$

Hence,

$$\begin{aligned} & 2\Re(-\sigma_n^2 u_j^n, (x - \ell_j)u_{j,x}^n) - 2\Re(u_{j,xx}^n, (x - \ell_j)u_{j,x}^n) \\ & = -\sigma_n^2 u_j^n(0) \overline{\ell_j u_{j,x}^n(0)} - (-\sigma_n^2 u_j^n, u_j^n) - u_{j,x}^n(0) \overline{\ell_j u_{j,x}^n(0)} + (u_{j,x}^n, u_{j,x}^n). \end{aligned} \tag{20}$$

Thus, $u_{j,x}^n(0)$ and $\sigma_n u_j^n(0)$ are bounded.

Dividing (15) by $i\sigma_n$, together with (13), we get $-\theta_{j,xx}^n/(i\sigma_n) + \beta_j u_{j,x}^n \rightarrow 0$, $j = 1, 2, \dots, N - 1$. Hence, $\theta_{j,xx}^n/i\sigma_n$ is bounded in $L^2(0, \ell_j)$. Then taking the L^2 -product of the above with $u_{j,x}^n$ yields

$$\left(-\frac{\theta_{j,xx}^n}{i\sigma_n}, u_{j,x}^n\right) + (\beta_j u_{j,x}^n, u_{j,x}^n) \rightarrow 0, \quad j = 1, 2, \dots, N - 1. \tag{21}$$

Integrating the above by parts, we have

$$-\frac{\theta_{j,x}^n(\ell_j)}{i\sigma_n}u_{j,x}^n(\ell_j)+\frac{\theta_{j,x}^n(0)}{i\sigma_n}u_{j,x}^n(0)+\left(\frac{\theta_{j,x}^n}{i\sigma_n},u_{j,xx}^n\right)+(\beta_j u_{j,x}^n,u_{j,x}^n)\rightarrow 0, j=1,2,\dots,N-1. \quad (22)$$

Note that dividing (14) by $i\sigma_n$, we obtain the boundedness of $u_{j,xx}^n/(i\sigma_n)$ in $L^2(0,\ell_j)$. Thus,

$$\left(\frac{\theta_{j,x}^n}{i\sigma_n},u_{j,xx}^n\right)\rightarrow 0, \quad j=1,2,\dots,N-1.$$

By the Gagliardo-Nirenberg inequality,

$$\frac{\|\theta_{j,x}^n\|_{L^\infty}}{\sqrt{i\sigma_n}}\leq d_1\left\|\frac{\theta_{j,xx}^n}{i\sigma_n}\right\|^{\frac{1}{2}}\|\theta_{j,x}^n\|^{\frac{1}{2}}+d_2\frac{\|\theta_{j,x}^n\|}{\sqrt{i\sigma_n}}\rightarrow 0, \quad (23)$$

we have $\theta_{j,x}^n(\ell_j)/(\sqrt{i\sigma_n})$, $\theta_{j,x}^n(0)/(\sqrt{i\sigma_n})\rightarrow 0$, $j=1,2,\dots,N-1$. By the boundedness of $u_{j,x}^n(\ell_j)$ and $u_{j,x}^n(0)$, together with (22), we have

$$u_{j,x}^n\rightarrow 0, \text{ in } L^2(0,\ell_j), \quad j=1,2,\dots,N-1. \quad (24)$$

Thus, $v_j^n\rightarrow 0$, in $H^1(0,\ell_j)$, $j=1,2,\dots,N-1$. Dividing (14) by $i\sigma_n$ and taking the product of the obtained identity with v_j^n from 0 to ℓ_j yields

$$(v_j^n,v_j^n)-\left(\frac{u_{j,xx}^n}{i\sigma_n},v_j^n\right)\rightarrow 0, \quad j=1,2,\dots,N-1,$$

and hence $(v_j^n,v_j^n)-(u_{j,xx}^n/i\sigma_n,i\sigma_n v_j^n)\rightarrow 0$, $j=1,2,\dots,N-1$.

Integrating by parts, we have

$$(v_j^n,v_j^n)-u_{j,x}^n(0)u_j^n(0)-(u_{j,x}^n,u_{j,x}^n)\rightarrow 0, \quad j=1,2,\dots,N-1. \quad (25)$$

Note that $u_j^n(0)\rightarrow 0$, $j=1,2,\dots,N-1$, due to u_j^n , $u_{j,x}^n\rightarrow 0$, in $L^2(0,\ell_j)$, $j=1,2,\dots,N-1$ and Gagliardo-Nirenberg inequality. Hence, due to the boundedness of $u_{j,x}^n(0)$,

$$v_j^n\rightarrow 0, \text{ in } L^2(0,\ell_j), \quad j=1,2,\dots,N-1. \quad (26)$$

Then from (20), we have

$$u_{j,x}^n(0), \sigma_n u_j^n(0)\rightarrow 0, \quad j=1,2,\dots,N-1. \quad (27)$$

On the other hand, on the segment $(0,\ell_N)$, $u_N^n(0)$, $\sigma_n u_N^n(0)\rightarrow 0$ due to (18), (27) and the transmission condition at $x=0$.

Taking the inner product of (16) with $(x-\ell_N)u_{N,x}^n$, we have

$$(i\sigma_n u_N^n,(x-\ell_N)u_{N,x}^n)-(u_{N,xx}^n,(x-\ell_N)u_{N,x}^n)\rightarrow 0. \quad (28)$$

Note that

$$2\Re(i\sigma_n u_N^n,(x-\ell_N)u_{N,x}^n)=-i\sigma_n u_N^n(0)\overline{\ell_N u_N^n(0)}+(v_N^n,i\sigma_n u_N^n)\rightarrow (v_N^n,v_N^n),$$

$$2\Re(u_{N,xx}^n,(x-\ell_N)u_{N,x}^n)=-u_{N,x}^n(0)\overline{\ell_N u_{N,x}^n(0)}-(u_{N,x}^n,u_{N,x}^n)\rightarrow -(u_{N,x}^n,u_{N,x}^n).$$

Thus,

$$\Re(i\sigma_n v_N^n,(x-\ell_N)u_{N,x}^n)-\Re(w_{j,xx}^n,(x-\ell_N)u_{N,x}^n)=(v_N^n,v_N^n)+(u_{N,x}^n,u_{N,x}^n)\rightarrow 0. \quad (29)$$

Hence,

$$u_{N,x}^n,v_N^n\rightarrow 0, \text{ in } L^2(0,\ell_N). \quad (30)$$

Thus, by (17), (24) and (26), we get $\Phi^n\rightarrow 0$, in \mathcal{H} , which is in contradiction with $\|\Phi^n\|=1$. The desired result follows. \square

4.2. Lack of exponential decay: Case $N - N_1 > 1$. In this subsection, we shall show that if more than one purely elastic undamped rod is involved in the network, the exponential decay rate does not hold.

Proposition 2. *If $N - N_1 > 1$ the network (5) lacks the property of exponential decay.*

Proof. Note that from Lemma 4.1, it is sufficient to show that the norm of the resolvent operator of system (5) along the imaginary axis is necessarily unbounded when $N - N_1 > 1$.

To do it we consider the resolvent problem

$$(\lambda I - \mathcal{A})U = F, \quad \lambda = -i\sigma, \quad \sigma \in \mathbb{R}, \quad (31)$$

where $U = ((u_j)_{j=1}^N, (v_j)_{j=1}^N, (\theta_j)_{j=1}^{N_1})$, $F = ((f_j)_{j=1}^N, (g_j)_{j=1}^N, (\eta_j)_{j=1}^{N_1})$, which is equivalent to

$$\begin{cases} \lambda u_j - v_j = f_j, & j = 1, 2, \dots, N, \\ \lambda v_j - (u_{j,xx} - \alpha_j \theta_{j,x}) = g_j, & j = 1, 2, \dots, N_1, \\ \lambda \theta_j - (\theta_{j,xx} - \beta_j v_{j,x}) = \eta_j, & j = 1, 2, \dots, N_1, \\ \lambda v_j - u_{j,xx} = g_j, & j = N_1 + 1, N_1 + 2, \dots, N. \end{cases} \quad (32)$$

Choose $f_j = 0$, $j = 1, 2, \dots, N$, and $g_j = \eta_j = 0$, $j = 1, 2, \dots, N_1$. Then we have

$$\begin{cases} \lambda u_j - v_j = 0, & j = 1, 2, \dots, N, \\ \lambda v_j - (u_{j,xx} - \alpha_j \theta_{j,x}) = 0, & j = 1, 2, \dots, N_1, \\ \lambda \theta_j - (\theta_{j,xx} - \beta_j v_{j,x}) = 0, & j = 1, 2, \dots, N_1, \\ \lambda v_j - u_{j,xx} = g_j, & j = N_1 + 1, N_1 + 2, \dots, N. \end{cases} \quad (33)$$

Using explicit representation formulas we get

$$\begin{aligned} u_j(x) &= \frac{u(0)}{\sin \sigma \ell_j} \sin \sigma(\ell_j - x) + \frac{\sin \sigma(\ell_j - x)}{\sigma \sin \sigma \ell_j} \int_0^{\ell_j} g_j(\ell_j - s) \sin \sigma(\ell_j - s) ds \\ &\quad - \frac{1}{\sigma} \int_0^{\ell_j - x} g_j(\ell_j - s) \sin \sigma(\ell_j - x - s) ds, \quad j = N_1 + 1, N_1 + 2, \dots, N. \end{aligned} \quad (34)$$

In order to calculate u_j , θ_j , $j = 1, 2, \dots, N_1$, let us consider the following coupled system.

$$\begin{cases} \lambda u_j - v_j = 0, & j = 1, 2, \dots, N_1, \\ \lambda v_j - (u_{j,xx} - \alpha_j \theta_{j,x}) = 0, & j = 1, 2, \dots, N_1, \\ \lambda \theta_j - (\theta_{j,xx} - \beta_j v_{j,x}) = 0, & j = 1, 2, \dots, N_1, \end{cases}$$

that is,

$$\begin{cases} \lambda^2 u_j - (u_{j,xx} - \alpha_j \theta_{j,x}) = 0, & j = 1, 2, \dots, N_1, \\ \lambda \theta_j - (\theta_{j,xx} - \beta_j \lambda u_{j,x}) = 0, & j = 1, 2, \dots, N_1. \end{cases} \quad (35)$$

We rewrite (35) in the vector form

$$\frac{dY_j(x)}{dx} = A_j Y_j(x), \quad j = 1, 2, \dots, N_1,$$

$$\text{where } Y_j(x) = (u_j, \frac{u_{j,x}}{\lambda}, \theta_j, \frac{\theta_{j,x}}{\lambda})^T, \quad A_j = \begin{bmatrix} 0 & \lambda & 0 & 0 \\ \lambda & 0 & 0 & \alpha_j \\ 0 & 0 & 0 & \lambda \\ 0 & \beta_j \lambda & 1 & 0 \end{bmatrix}.$$

Set $d_{j,1} = \frac{\lambda[\lambda+\alpha_j\beta_j+1+\sqrt{(\lambda+\alpha_j\beta_j+1)^2-4\lambda}]}{2}$, $d_{j,2} = \frac{\lambda[\lambda+\alpha_j\beta_j+1-\sqrt{(\lambda+\alpha_j\beta_j+1)^2-4\lambda}]}{2}$. In order to diagonalize the matrix A_j , we employ the transformation $Y_j := P_j Z_j, j = 1, 2, \dots, N_1$, where

$$P_j = \begin{bmatrix} 1 & 1 & \tilde{b}_j & -\tilde{b}_j \\ \frac{\sqrt{d_{j,1}}}{\lambda} & -\frac{\sqrt{d_{j,1}}}{\lambda} & \tilde{b}_j \frac{\sqrt{d_{j,2}}}{\lambda} & \tilde{b}_j \frac{\sqrt{d_{j,2}}}{\lambda} \\ \tilde{a}_j & -\tilde{a}_j & 1 & 1 \\ \tilde{a}_j \frac{\sqrt{d_{j,1}}}{\lambda} & \tilde{a}_j \frac{\sqrt{d_{j,1}}}{\lambda} & \frac{\sqrt{d_{j,2}}}{\lambda} & -\frac{\sqrt{d_{j,2}}}{\lambda} \end{bmatrix}$$

and

$$\tilde{a}_j = \frac{1}{\alpha_j \sqrt{d_{j,1}}}(-\lambda^2 + d_{j,1}), \quad \tilde{b}_j = \frac{1}{\beta_j \sqrt{d_{j,2}}}(-1 + \frac{d_{j,2}}{\lambda}).$$

We then have $\frac{dZ_j(x)}{dx} = P_j^{-1} A_j P_j Z_j = \begin{bmatrix} \sqrt{d_{j,1}} & 0 & 0 & 0 \\ 0 & -\sqrt{d_{j,1}} & 0 & 0 \\ 0 & 0 & \sqrt{d_{j,2}} & 0 \\ 0 & 0 & 0 & -\sqrt{d_{j,2}} \end{bmatrix} Z_j.$

In this way we obtain the following system

$$Z_j(x) = \begin{bmatrix} e^{\sqrt{d_{j,1}}x} & 0 & 0 & 0 \\ 0 & e^{-\sqrt{d_{j,1}}x} & 0 & 0 \\ 0 & 0 & e^{\sqrt{d_{j,2}}x} & 0 \\ 0 & 0 & 0 & e^{-\sqrt{d_{j,2}}x} \end{bmatrix} Z(0),$$

and

$$Y_j(x) = P_j Z_j(x) = P_j \begin{bmatrix} e^{\sqrt{d_{j,1}}x} & 0 & 0 & 0 \\ 0 & e^{-\sqrt{d_{j,1}}x} & 0 & 0 \\ 0 & 0 & e^{\sqrt{d_{j,2}}x} & 0 \\ 0 & 0 & 0 & e^{-\sqrt{d_{j,2}}x} \end{bmatrix} P_j^{-1} Y_j(0). \tag{36}$$

By the boundary and transmission conditions in (5), together with (34), we get the following estimate (the technical details are given as in Appendix):

$$\begin{aligned} & \sigma u(0) \\ &= \frac{\sum_{j=N_1+1}^N \prod_{\substack{k \neq j \\ k=N_1+1}}^N \sin \sigma \ell_k \cos \sigma \ell_j \int_0^{\ell_j} g_j(\ell_j - s) \sin \sigma(\ell_j - s) ds}{\prod_{k=N_1+1}^N \sin \sigma \ell_k \sum_{j=1}^{N_1} \left(i \frac{\cosh(i\sigma + \frac{\alpha_j \beta_j}{2}) \ell_j}{\sinh(i\sigma + \frac{\alpha_j \beta_j}{2}) \ell_j} + O(\frac{1}{\sqrt{\sigma}}) \right) - \sum_{j=N_1+1}^N \prod_{\substack{k \neq j \\ k=N_1+1}}^N \sin \sigma \ell_k \cos \sigma \ell_j} \\ & - \frac{\sum_{j=N_1+1}^N \prod_{k=N_1+1}^N \sin \sigma \ell_k \int_0^{\ell_j} g_j(\ell_j - s) \cos \sigma(\ell_j - s) ds}{\prod_{k=N_1+1}^N \sin \sigma \ell_k \sum_{j=1}^{N_1} \left(i \frac{\cosh(i\sigma + \frac{\alpha_j \beta_j}{2}) \ell_j}{\sinh(i\sigma + \frac{\alpha_j \beta_j}{2}) \ell_j} + O(\frac{1}{\sqrt{\sigma}}) \right) - \sum_{j=N_1+1}^N \prod_{\substack{k \neq j \\ k=N_1+1}}^N \sin \sigma \ell_k \cos \sigma \ell_j}. \tag{37} \end{aligned}$$

Choosing $\sigma_n \ell_{N_1+1} = n\pi, n \rightarrow +\infty$, then $\sin \sigma_n \ell_{N_1+1} = 0$. Since the irrational numbers always can be approximated by rational ones, we can find a subsequence σ_{n_k} , such that

$$\sin \sigma_{n_k} \ell_j \rightarrow 0, \quad j = N_1 + 2, N_1 + 3, \dots, N.$$

In fact, note that there is always a sequence of rational numbers $\frac{p_m^n}{q_m^n}, p_m^n, q_m^n \in \mathbb{Z}^+, n = 1, 2, \dots$ such that $|\frac{\ell_m}{\ell_{N_1+1}} - \frac{p_m^n}{q_m^n}| \rightarrow 0, m = N_1 + 2, N_1 + 3, \dots, N, n \rightarrow \infty$.

Thus choose $n_k = \prod_{m=N_1+2}^N q_m^k$. Then we have

$$\begin{aligned} \sin \sigma_{n_k} \ell_j &= \sin(\sigma_{n_k} \ell_{N_1+1} \frac{\ell_j}{\ell_{N_1+1}}) = \sin(\pi(\prod_{m=N_1+2}^N q_m^k) \frac{\ell_j}{\ell_{N_1+1}}) \\ &\rightarrow \sin(\pi(\prod_{m=N_1+2}^N q_m^k) \frac{p_j^k}{q_j^k}) = 0, k \rightarrow \infty, j = N_1 + 2, N_1 + 3, \dots, N. \end{aligned}$$

Set $g_{N_1+1}^{n_k}(s) := \sin \sigma_{n_k}(\ell_{N_1+1} - s), g_j^{n_k}(s) = 0, j = N_1 + 2, N_1 + 3, \dots, N$. Then we have

$$\begin{aligned} \int_0^{\ell_{N_1+1}} g_{N_1+1}^{n_k}(\ell_j - s) \sin \sigma_{n_k}(\ell_j - s) ds &\rightarrow -\frac{\ell_{N_1+1}}{2}, n_k \rightarrow \infty, \\ \int_0^{\ell_{N_1+1}} g_{N_1+1}^{n_k}(\ell_j - s) \cos \sigma_{n_k}(\ell_j - s) ds &\rightarrow 0, n_k \rightarrow \infty. \end{aligned}$$

Thus, from (37), we get

$$\sigma_{n_k} u^{n_k}(0) \rightarrow \frac{\ell_{N_1+1}}{2}, n_k \rightarrow \infty.$$

Hence, by (34), we have

$$i\sigma_{n_k} u_j^{n_k}(x) \rightarrow \infty, \text{ in } L^2(0, \ell_j), j = N_1 + 2, N_1 + 3, \dots, N,$$

due to the fact that $\sin \sigma_{n_k} \ell_j \rightarrow 0, j = N_1 + 2, N_1 + 3, \dots, N$. Note that $v_j^{n_k} = i\sigma_{n_k} u_j^{n_k}(x)$. Therefore,

$$v_j^{n_k} \rightarrow \infty, \text{ in } L^2(0, \ell_j), j = N_1 + 2, N_1 + 3, \dots, N.$$

Summarising the developments above we find a sequence (σ_n, F^n) , where $F^n \in \mathcal{H}$ and $\|F^n\|_{\mathcal{H}}$ is bounded, satisfying

$$\|(i\sigma_n I - \mathcal{A})^{-1} F^n\|_{\mathcal{H}} \rightarrow \infty, \sigma_n \rightarrow +\infty,$$

which implies the lack of exponential decay rate of system (5). The proof is complete. \square

4.3. Lower bounds on the polynomial decay rate. This subsection is devoted to get the lower bounds on the polynomial decay rate as stated in Theorem 2.4.

We need the following result from [6] (see also [26]).

Lemma 4.2. *A C_0 semigroup $e^{t\mathcal{A}}$ of contractions on a Hilbert space satisfies*

$$\|e^{t\mathcal{A}} U_0\| \leq C t^{-\frac{1}{2}} \|U_0\|_{\mathcal{D}(\mathcal{A})}, \forall U_0 \in \mathcal{D}(\mathcal{A}), t \rightarrow \infty$$

for some constant $C > 0$, if and only if the following conditions hold:

- 1). $\{i\beta | \beta \in \mathbb{R}\} \subset \rho(\mathcal{A})$;
- 2). $\limsup_{|\beta| \rightarrow \infty} \frac{1}{\beta \varepsilon} \|(i\beta - \mathcal{A})^{-1}\| < \infty$.

Proposition 3. *The decay rate of the energy of system (5) can be, at most, polynomial of order t^{-1} .*

Proof. From Lemma 4.2, it is sufficient to show that there exists at least one sequence (σ_n, F^n) such that $\sigma_n \rightarrow +\infty$, $n \rightarrow \infty$ and

$$\|(i\sigma_n I - \mathcal{A})^{-1} F^n\| > \tilde{C}\sigma_n^2, \tag{38}$$

where $F^n \in \mathcal{H}$ and $\|F^n\|_{\mathcal{H}}$ is bounded, \tilde{C} being some positive constant.

For simplicity, we consider the case $N - N_1 = 2$, that is, there are two purely elastic rods involved in the network. Based on the proof of Proposition 2, we choose the sequence (σ_{n_k}, F_{n_k}) satisfying (38) in the following three steps.

Step 1). Choose $\sigma_n = \frac{1}{\ell_{N_1+1} + \ell_{N_1+2}} n\pi$. Thus we get $\sin \sigma_n(\ell_{N_1+1} + \ell_{N_1+2}) = 0$.

Since

$$\begin{aligned} \sin \sigma \ell_{N_1+1} &= \sin \sigma(\ell_{N_1+1} + \ell_{N_1+2} - \ell_{N_1+2}) \\ &= \sin \sigma(\ell_{N_1+1} + \ell_{N_1+2}) \cos \sigma \ell_{N_1+2} - \cos \sigma(\ell_{N_1+1} + \ell_{N_1+2}) \sin \sigma \ell_{N_1+2}, \end{aligned}$$

we have

$$\sin \sigma_n \ell_{N_1+1} = \pm \sin \sigma_n \ell_{N_1+2}, \quad \cos \sigma_n \ell_{N_1+1} = \pm \cos \sigma_n \ell_{N_1+2}.$$

Step 2). Choose $f_j = 0$, $j = 1, 2, \dots, N$, $g_j = \eta_j = 0$, $j = 1, 2, \dots, N_1$ and $g_{N_1+1}^n(s) = \sin \sigma_n(\ell_{N_1+1} - s)$, $g_{N_1+2}^n(s) = 0$ in (32).

Substituting the above into (37), we get

$$\begin{aligned} &\sigma_n u(0) \\ &= \pm \frac{\sin \sigma_n \ell_{N_1+2} \cos \sigma_n \ell_{N_1+1} \int_0^{\ell_{N_1+1}} \sin \sigma_n s \sin \sigma_n(\ell_{N_1+1} - s) ds}{(\sin \sigma_n \ell_{N_1+2})^2 \sum_{j=1}^{N_1} \left(i \frac{\cosh(i\sigma_n + \frac{\alpha_j \beta_j}{2}) \ell_j}{\sinh(i\sigma_n + \frac{\alpha_j \beta_j}{2}) \ell_j} + O\left(\frac{1}{\sqrt{\sigma_n}}\right) \right)} \\ &\pm \frac{(\sin \sigma_n \ell_{N_1+2})^2 \int_0^{\ell_{N_1+1}} \sin \sigma_n s \cos \sigma_n(\ell_{N_1+1} - s) ds}{(\sin \sigma_n \ell_{N_1+2})^2 \sum_{j=1}^{N_1} \left(i \frac{\cosh(i\sigma_n + \frac{\alpha_j \beta_j}{2}) \ell_j}{\sinh(i\sigma_n + \frac{\alpha_j \beta_j}{2}) \ell_j} + O\left(\frac{1}{\sqrt{\sigma_n}}\right) \right)}. \end{aligned} \tag{39}$$

Note that

$$\begin{aligned} \int_0^{\ell_{N_1+1}} \sin \sigma_n s \sin \sigma_n(\ell_{N_1+1} - s) ds &\rightarrow -\frac{\ell_{N_1+1} \cos \sigma_n \ell_{N_1+1}}{2}, \quad n \rightarrow \infty, \\ \int_0^{\ell_{N_1+1}} \sin \sigma_n s \cos \sigma_n(\ell_j - s) ds &\rightarrow \frac{\ell_{N_1+1} \sin \sigma_n \ell_{N_1+1}}{2}, \quad n \rightarrow \infty, \end{aligned}$$

and

$$\sum_{j=1}^{N_1} \left(i \frac{\cosh(i\sigma_n + \frac{\alpha_j \beta_j}{2}) \ell_j}{\sinh(i\sigma_n + \frac{\alpha_j \beta_j}{2}) \ell_j} + O\left(\frac{1}{\sqrt{\sigma_n}}\right) \right) = O(1) \notin \mathbb{R}.$$

Hence,

$$|\sigma_n u(0)| = \left| \frac{\ell_{N_1+1} \cos^2 \sigma_n \ell_{N_1+2}}{2 \sin \sigma_n \ell_{N_1+2}} \pm \frac{\ell_{N_1+1} \sin \sigma_n \ell_{N_1+2}}{2} \right| O(1). \tag{40}$$

Step 3). Note that we have chosen in Step 1) that $\sigma_n = \frac{1}{\ell_{N_1+1} + \ell_{N_1+2}} n\pi$ and thus

$$\sin \sigma_n \ell_{N_1+2} = \sin\left(\frac{\ell_{N_1+2}}{\ell_{N_1+1} + \ell_{N_1+2}} (\ell_{N_1+1} + \ell_{N_1+2}) \sigma_n\right) = \sin\left(\frac{\ell_{N_1+2}}{\ell_{N_1+1} + \ell_{N_1+2}} n\pi\right).$$

By Dirichlet theorem, we can always find an infinite subsequence $n_k \in \mathbb{Z}$ such that

$$\left| \left| \frac{\ell_{N_1+2}}{\ell_{N_1+1} + \ell_{N_1+2}} n_k \right| \right| < \frac{1}{n_k},$$

and hence $|\sin \sigma_{n_k} \ell_{N_1+2}| = |\sin(\frac{\ell_{N_1+2}}{\ell_{N_1+1} + \ell_{N_1+2}} n_k \pi)| < \frac{\pi}{n_k}$.

Thus, there exists a positive constant C

$$|\sigma_{n_k} u(0)| > C n_k. \tag{41}$$

Note that from (33) and (34), we get $v_{N_1+2}^{n_k}(x) = i \sigma_{n_k} u_{N_1+2}^{n_k}(x)$ and

$$\begin{aligned} u_{N_1+2}(x) &= \frac{u(0)}{\sin \sigma \ell_{N_1+2}} \sin \sigma(\ell_{N_1+2} - x) \\ &+ \frac{\sin \sigma(\ell_{N_1+2} - x)}{\sigma \sin \sigma \ell_{N_1+2}} \int_0^{\ell_{N_1+2}} g_{N_1+2}(\ell_{N_1+2} - s) \sin \sigma(\ell_{N_1+2} - s) ds \\ &- \frac{1}{\sigma} \int_0^{\ell_{N_1+2} - x} g_{N_1+2}(\ell_{N_1+2} - s) \sin \sigma(\ell_{N_1+2} - x - s) ds. \end{aligned}$$

Hence, by (41), there exists a positive constant \widehat{C} satisfying

$$\|v_{N_1+2}^{n_k}\| > \widehat{C} n_k^2.$$

Summarizing above, we have found a sequence (σ_{n_k}, F_{n_k}) , where

$$\sigma_{n_k} = \frac{1}{\ell_{N_1+1} + \ell_{N_1+2}} n_k \pi, \quad F_{n_k} = F = ((f_j)_{j=1}^N, (g_j)_{j=1}^N, (\eta_j)_{j=1}^{N_1}),$$

in which $f_j = 0$, $j = 1, 2, \dots, N$, $g_j = \eta_j = 0$, $j = 1, 2, \dots, N_1$ and $g_{N_1+1}^n(s) = \sin \sigma_n(\ell_{N_1+1} - s)$, $g_{N_1+2}^n(s) = 0$, such that

$$\|(i \sigma_{n_k} I - \mathcal{A})^{-1} F_{n_k}\| > \widetilde{C} \sigma_{n_k}^2, \quad \sigma_{n_k} \rightarrow \infty,$$

where $\widetilde{C} > 0$ is some constant. The proof is complete. \square

Remark 5. For general case $N - N_1 > 2$, We can prove that the networks also satisfy (38) by modifying the above construction slightly. Indeed, choose $g_{N_1+1}^n(s) = \sin \sigma_n(\ell_{N_1+1} - s)$, $g_j^n(s) = 0$, $N \geq j > N_1 + 1$ in step 2), and choose $n_k \in \mathbb{Z}$ such that $|\frac{\ell_{N_1+2}}{\ell_{N_1+1} + \ell_{N_1+2}} n_k| < \frac{1}{n_k}$, and $\sin \sigma_{n_k} \ell_j \rightarrow 0$, $N \geq j > N_1 + 2$, $k \rightarrow \infty$ in step 3). Thus, by the same discussion as above, the desired result follows.

4.4. Explicit polynomial decay rate: Case $N - N_1 > 1$ (Proof of Theorem 2.4). In subsection 4.2 we proved that if there are more than one purely elastic rods involved in the network (5), the system can not achieve the exponential decay. We have also shown that the decay rate can be, at most, polynomial of order t^{-1} .

The mutual-irrationality of the radii of ℓ_j , $j = N_1 + 1, N_1 + 2, \dots, N$ in Theorem 2.1 is a sufficient and necessary condition for strong stability. So, for the rest of this section, we always assume this condition to be fulfilled.

Now, we prove effective and explicit polynomial decay results when $N - N_1 > 1$, and ℓ_j , $j = N_1 + 1, N_1 + 2, \dots, N$ satisfy the conditions (S) in Definition 2.3.

Proof of Theorem 2.4. Similarly to the proof of Theorem 2.2, we argue by contradiction, on the basis of Lemma 4.2.

Let Φ_n be a sequence such that $\Phi^n = (U^n, V^n, \Theta^n) \in \mathcal{D}(\mathcal{A})$, with $\|\Phi^n\|_{\mathcal{H}} = 1$, where $U^n = (u_j^n)_{j=1}^N$, $V^n = (v_j^n)_{j=1}^N$, $\Theta^n = (\theta_j^n)_{j=1}^{N_1}$, and a sequence $\sigma_n \in \mathbb{R}$ with $\sigma_n \rightarrow \infty$ such that

$$\lim_{n \rightarrow \infty} \sigma_n^{2(1+\epsilon)} \|(i \sigma_n I - \mathcal{A}) \Phi^n\|_{\mathcal{H}} = 0,$$

where ϵ is given as in Theorem 2.4. Hence,

$$\sigma_n^{2(1+\epsilon)}(i\sigma_n u_j^n - v_j^n) = f_j^n \rightarrow 0, \text{ in } H^1(0, \ell_j), j = 1, 2, \dots, N, \quad (42)$$

$$\sigma_n^{2(1+\epsilon)}(i\sigma_n v_j^n - u_{j,xx}^n + \alpha_j \theta_{j,x}^n) = g_j^n \rightarrow 0, \text{ in } L^2(0, \ell_j), j = 1, 2, \dots, N_1, \quad (43)$$

$$\sigma_n^{2(1+\epsilon)}(i\sigma_n \theta_j^n - \theta_{j,xx}^n + \beta_j v_{j,x}^n) = y_j^n \rightarrow 0, \text{ in } L^2(0, \ell_j), j = 1, 2, \dots, N_1, \quad (44)$$

and

$$\sigma_n^{2(1+\epsilon)}(i\sigma_n v_j^n - u_{j,xx}^n) = g_j \rightarrow 0, \text{ in } L^2(0, \ell_j), j = N_1 + 1, N_1 + 2, \dots, N. \quad (45)$$

Note that $(\sigma_n^{2(1+\epsilon)}(i\sigma_n - \mathcal{A})\Phi^n, \Phi^n)_{\mathcal{H}} = -\sum_{j=1}^{N_1} \alpha_j \beta_j \int_0^{\ell_j} |\sigma_n^{(1+\epsilon)} \theta_{j,x}^n|^2 dx \rightarrow 0$.

Hence,

$$\sigma_n^{(1+\epsilon)} \theta_{j,x}^n \rightarrow 0, \text{ in } L^2(0, \ell_j), j = 1, 2, \dots, N_1. \quad (46)$$

Thus, by Poincaré inequality,

$$\sigma_n^{(1+\epsilon)} \theta_j^n \rightarrow 0, \text{ in } L^2(0, \ell_j), j = 1, 2, \dots, N_1.$$

Similar to the proof of Theorem 2.2, we can get that

$$\sigma_n^{(1+\epsilon)} u_{j,x}^n(x), \sigma_n^{(1+\epsilon)} v_j^n(x), \sigma_n^{(1+\epsilon)} \theta_j^n(x) \rightarrow 0, \text{ in } L^2(0, \ell_j), j = 1, 2, \dots, N_1, \quad (47)$$

$$\sigma_n^{(1+\epsilon)} u_{j,x}^n(0), \sigma_n^{(1+\epsilon)} \sigma_n u_j^n(0), \sigma_n^{(1+\epsilon)} \theta_j^n(0) \rightarrow 0, j = 1, 2, \dots, N_1. \quad (48)$$

Thus, by the transmission conditions at the common node, we get

$$\sigma_n^{(1+\epsilon)} \sigma_n u_j^n(0) \rightarrow 0, j = N_1 + 1, N_1 + 2, \dots, N, \quad (49)$$

$$\sigma_n^{(1+\epsilon)} \sum_{j=N_1+1}^N u_{j,x}^n(0) \rightarrow 0. \quad (50)$$

Then we will prove that $\Phi^n \rightarrow 0$ in \mathcal{H} , which leads to a contradiction. To do this, we discuss it by the following two cases:

Case 1). Assume that $\sin \sigma_n \ell_j \neq 0$, $j = N_1 + 1, N_1 + 2, \dots, N$.

By (42), (45), we can easily get that $u_{j,xx}^n + \sigma_n^2 u_j^n = -\frac{g_j^n + i\sigma_n f_j^n}{\sigma_n^{2(1+\epsilon)}}$.

Thus a direct calculation yields

$$\begin{aligned} u_j^n(x) &= \frac{u^n(0)}{\sin \sigma_n \ell_j} \sin \sigma_n(\ell_j - x) \\ &+ \frac{\sin \sigma_n(\ell_j - x)}{\sigma_n \sin \sigma_n \ell_j} \int_0^{\ell_j} \frac{g_j^n(\ell_j - s) + i\sigma_n f_j^n(\ell_j - s)}{\sigma_n^{2(1+\epsilon)}} \sin(\sigma_n(\ell_j - s)) ds \\ &- \frac{1}{\sigma_n} \int_0^{\ell_j - x} \frac{g_j^n(\ell_j - s) + i\sigma_n f_j^n(\ell_j - s)}{\sigma_n^{2(1+\epsilon)}} \sin \sigma_n(\ell_j - x - s) ds, \\ &j = N_1 + 1, N_1 + 2, \dots, N. \end{aligned}$$

Hence,

$$\begin{aligned} u_{j,x}^n(0) &= -\frac{\sigma_n u^n(0) \cos \sigma_n \ell_j}{\sin \sigma_n \ell_j} \\ &- \frac{\cos \sigma_n \ell_j}{\sin \sigma_n \ell_j} \int_0^{\ell_j} \frac{g_j^n(\ell_j - s) + i\sigma_n f_j^n(\ell_j - s)}{\sigma_n^{2(1+\epsilon)}} \sin(\sigma_n(\ell_j - s)) ds \\ &+ \int_0^{\ell_j} \frac{g_j^n(\ell_j - s) + i\sigma_n f_j^n(\ell_j - s)}{\sigma_n^{2(1+\epsilon)}} \cos \sigma_n(\ell_j - s) ds, \\ &j = N_1 + 1, N_1 + 2, \dots, N. \end{aligned} \quad (51)$$

It is easy to show that

$$\begin{aligned} \int_0^{\ell_j} [g_j^n(\ell_j - s) + i\sigma_n f_j^n(\ell_j - s)] \sin \sigma_n(\ell_j - s) ds &\rightarrow 0, \\ \int_0^{\ell_j} [g_j^n(\ell_j - s) + i\sigma_n f_j^n(\ell_j - s)] \cos \sigma_n(\ell_j - s) ds &\rightarrow 0. \end{aligned} \quad (52)$$

Set

$$\gamma_n = \max_{j=N_1+1, N_1+2, \dots, N} \prod_{i=N_1+1, i \neq j}^N |\sin(\sigma_n \ell_i)|, \quad \forall n \geq 1. \quad (53)$$

When ℓ_j , $j = N_1 + 1, N_1 + 2, \dots, N$ satisfy the conditions (S) as given in Definition 2.3, by Corollary A.10 in [11], we get

$$\gamma_n \geq \frac{c_\epsilon}{\sigma_n^{1+\epsilon}}, \quad \forall \epsilon > 0. \quad (54)$$

For any given sequence σ_n , assume that there exists j_0^n ($N_1 + 1 \leq j_0^n \leq N$) such that

$$\gamma_n = \prod_{i=N_1+1, i \neq j_0^n}^N |\sin(\sigma_n \ell_i)|.$$

Hence, for any $j \neq j_0^n$, $N_1 + 1 \leq j \leq N$, we have

$$|\sin \sigma_n \ell_j| \geq \gamma_n,$$

which implies that

$$\frac{1}{|\sin \sigma_n \ell_j|} \leq \frac{1}{\gamma_n} \leq \frac{1}{c_\epsilon} \sigma_n^{1+\epsilon}.$$

Therefore, by (49) and (52) together with (51), we get that

$$u_{j,x}^n(0) \rightarrow 0, \quad N_1 + 1 \leq j \leq N, \quad j \neq j_0^n, \quad n \rightarrow \infty. \quad (55)$$

Then by the transmission condition (50), we obtain that

$$u_{j,x}^n(0) \rightarrow 0, \quad N_1 + 1 \leq j \leq N, \quad n \rightarrow \infty. \quad (56)$$

Taking the inner product of (45) with $(x - \ell_j)u_{j,x}^n$, $j = N_1 + 1, N_1 + 2, \dots, N$, respectively,

$$(i\sigma_n v_j^n, (x - \ell_j)u_{j,x}^n) - (u_{j,xx}^n, (x - \ell_j)u_{j,x}^n) \rightarrow 0, \quad j = N_1 + 1, N_1 + 2, \dots, N. \quad (57)$$

We have

$$2\Re(i\sigma_n v_j^n, (x - \ell_j)u_{j,x}^n) = i\sigma_n v_j^n(0) \overline{\ell_j u_j^n(0)} + (v_j^n, i\sigma_n u_j^n) \rightarrow (v_j^n, v_j^n),$$

$$2\Re(u_{j,xx}^n, (x - \ell_j)u_{j,x}^n) = u_{j,x}^n(0) \overline{\ell_j u_{j,x}^n(0)} - (u_{j,x}^n, u_{j,x}^n) \rightarrow -(u_{j,x}^n, u_{j,x}^n).$$

Hence,

$$\Re(i\sigma_n v_j^n, (x - \ell_j)u_{j,x}^n) - \Re(u_{j,xx}^n, (x - \ell_j)u_{j,x}^n) = (v_j^n, v_j^n) + (u_{j,x}^n, u_{j,x}^n) \rightarrow 0. \quad (58)$$

Therefore,

$$u_{j,x}^n, v_j^n \rightarrow 0, \quad \text{in } L^2(0, \ell_j), \quad j = N_1 + 1, N_1 + 2, \dots, N. \quad (59)$$

Thus, by (47) and (59), we have obtained

$$\Phi^n = ((u_j^n)_{j=1}^N, (v_j^n)_{j=1}^N, (\theta_j^n)_{j=1}^{N_1}) \rightarrow 0, \quad \text{in } \mathcal{H}, \quad n \rightarrow \infty,$$

which contradicts $\|\Phi^n\|_{\mathcal{H}} = 1$.

Now let us continue to consider the second step:

Case 2). Assume that there exists j_0 , $N_1 + 1 \leq j_0 \leq N$ such that $\sin \sigma_n \ell_{j_0} = 0$.

Note that there exists at most one j_0^n , $N_1 + 1 \leq j_0 \leq N$ satisfying $\sin \sigma_n \ell_{j_0} = 0$ due to the irrationality of the mutual ratios between ℓ_j , $N_1 + 1 \leq j \leq N$. In this case,

$$\gamma_n = \prod_{i=N_1+1, i \neq j_0^n}^N |\sin(\sigma_n \ell_i)|,$$

where γ_n is given as in (53). Thus, by the same arguments as in *Step 1*), we can get

$$u_{j,x}^n(0) \rightarrow 0, \quad N_1 + 1 \leq j \leq N, \quad j \neq j_0^n, \quad n \rightarrow \infty,$$

which together with the transmission condition (50), implies

$$u_{j,x}^n(0) \rightarrow 0, \quad N_1 + 1 \leq j \leq N, \quad n \rightarrow \infty.$$

Then by (57)–(59) in *Step 1*), we get

$$u_{j,x}^n, v_j^n \rightarrow 0, \quad \text{in } L^2(0, \ell_j), \quad j = N_1 + 1, N_1 + 2, \dots, N.$$

Hence, the same contradiction holds as in *Step 1*).

Therefore, by Lemma 4.2, we get the polynomial decay rate of system (5), that is

$$E(t) \leq C_\epsilon t^{-\frac{1}{1+\epsilon}} \|(u^{(0)}, u^{(1)}, \theta^{(0)})\|_{\mathcal{D}(\mathcal{A})}^2, \quad \forall t \geq 0.$$

Note that in Proposition 3, we have proved that the polynomial decay order of the energy of system (5) is at most t^{-1} . Hence, $t^{-\frac{1}{1+\epsilon}}$ is the nearly sharp. The proof of Theorem 2.4 is complete. \square

5. More general slow decay rates. In the last section, we have shown that, when $N - N_1 > 1$, namely, when the network (5) involves more than one purely elastic rod, the system can not achieve the exponential decay rate. Then we further derived the nearly optimal polynomial decay rate, if the conditions (S) in Definition 2.3 are fulfilled.

In fact, the discussion of the proof of Theorem 2.4 (see subsection 4.3), shows that the slow decay rate of the system is determined by γ_n given as (53), which depends on the property of the lengths of the purely elastic rods entering in the network. Hence, other more general slow decay rates can be obtained when different conditions on ℓ_j , $j = N_1 + 1, N_1 + 2, \dots, N$ are imposed.

To do this, let us introduce the following definition on the irrational sets (see p. 209 in [11]), which is deduced from [7] (see Theorem I, p. 120).

Definition 5.1. (Theorem [7], [11]) 1. *Set B_ϵ :* for all $\epsilon > 0$ there exists a set $B_\epsilon \subset \mathbb{R}$, such that the Lebesgue measure of $\mathbb{R} \setminus B_\epsilon$ vanishes, and a constant $C_\epsilon > 0$ for which, if $\xi \in B_\epsilon$, then $|\xi m| \geq \frac{C_\epsilon}{m^{1+\epsilon}}$ for all integer number m . All the algebraic irrational numbers belong to B_ϵ .

2. *Set \mathcal{F} :* the set of all real numbers ρ such that $\rho \notin \mathbb{Q}$ and so that its expansion as a continued fraction $[0, a_1, a_2, \dots, a_n, \dots]$ is such that (a_n) is bounded. In particular \mathcal{F} is contained in the sets B_ϵ for every $\epsilon > 0$. (see p. 209 in [11] for more details). It contains all quadratic algebraic irrational numbers and also some transcendental numbers. This set \mathcal{F} has Lebesgue measure zero and is not denumerable.

We have the following result:

Corollary 1. *For any $(u^{(0)}, u^{(1)}, \theta^{(0)}) \in \mathcal{D}(\mathcal{A})$, there always exists a constant $C > 0$ such that the energy of network (5) satisfies*

$$E(t) \leq C_\delta t^{-\frac{1}{\delta}} \|(u^{(0)}, u^{(1)}, \theta^{(0)})\|_{\mathcal{D}(\mathcal{A})}^2, \quad \forall t \geq 0, \quad (60)$$

where \tilde{s} is given as follows:

- if $\frac{\ell_i}{\ell_j} \in B_\epsilon$, $i, j = N_1 + 1, N_1 + 2, \dots, N$, $i \neq j$, then $\tilde{s} = N - N_1 - 1 + \epsilon$;
- if $\frac{\ell_i}{\ell_j} \in \mathcal{F}$, $i, j = N_1 + 1, N_1 + 2, \dots, N$, $i \neq j$, then $\tilde{s} = N - N_1 - 1$.

Proof. Since the proof is similar to the one of Theorem 2.4, we only give a sketch of it.

If the decay rate is not fulfilled, there exists a sequence $\Phi^n = (U^n, V^n, \Theta^n) \in \mathcal{D}(\mathcal{A})$, with $\|\Phi^n\|_{\mathcal{H}} = 1$, where $U^n = (u_j^n)_{j=1}^N$, $V^n = (v_j^n)_{j=1}^N$, $\Theta^n = (\theta_j^n)_{j=1}^{N_1}$, and a sequence $\sigma_n \in \mathbb{R}$ with $\sigma_n \rightarrow \infty$ such that

$$\lim_{n \rightarrow \infty} \sigma_n^{2\tilde{s}} \|(i\sigma_n I - \mathcal{A})\Phi^n\|_{\mathcal{H}} = 0,$$

where \tilde{s} is given as in Corollary 1.

By Diophantine approximation (see [11]), different estimates can be gotten for γ_n , when the irrational numbers ℓ_i/ℓ_j belong to the sets B_ϵ and \mathcal{F} (as in as in Definition (5.1), respectively. By Corollary A. 10 in [11] (see also [39]), we get that

- (1) if $\frac{\ell_i}{\ell_j} \in B_\epsilon$, $i, j = N_1 + 1, N_1 + 2, \dots, N$, $i \neq j$, then $\gamma_n \geq c_\epsilon/\sigma_n^{N-N_1-1+\epsilon}$, $n \geq 1$, $\epsilon > 0$;
- (2) if $\frac{\ell_i}{\ell_j} \in \mathcal{F}$, $i, j = N_1 + 1, N_1 + 2, \dots, N$, $i \neq j$, then $\gamma_n \geq c/\sigma_n^{N-N_1-1}$, $n \geq 1$.

Thus, proceeding as in the discussion of the proof of Theorem 2.4, finally we can get

$$\Phi^n \rightarrow 0, \quad \text{in } \mathcal{H}, \quad n \rightarrow \infty.$$

This contradicts $\|\Phi^n\|_{\mathcal{H}} = 1$. Hence, the desired result follows. \square

Remark 6. By Corollary 1, it is easy to see that when $N - N_1 = 2$, that is there are two purely elastic rods in the network, the system can achieve optimal polynomial decay rate t^{-1} , if $\frac{\ell_{N_1+1}}{\ell_N} \in \mathcal{F}$.

More generally, by the similar proof as the one for Theorem 2.4, together with the so called M_{\log} -Theorem in [5], we obtain the general slow decay rate of system (5) as follows.

Corollary 2. Set $M_{\log}(s) = M(s)(\log(1 + M(s)) + \log(1 + s))$, where $M(s) : \mathbb{R}^+ \rightarrow (0, \infty)$ is continuous and increasing.

If

$$\gamma_n \geq \frac{1}{\sqrt{M(\sigma_n)}}, \quad n > 0,$$

where

$$\gamma_n = \max_{j=N_1+1, N_1+2, \dots, N} \prod_{i=N_1+1, i \neq j}^N |\sin(\sigma_n \ell_i)|, \quad \forall \sigma_n \rightarrow \infty,$$

then

$$E(t) \leq C \frac{1}{(M_{\log}^{-1}(ct))^2} \|(u^{(0)}, u^{(1)}, \theta^{(0)})\|_{\mathcal{D}(\mathcal{A})}^2, \quad \forall t \geq 0, \tag{61}$$

where $M_{\log}^{-1}(\cdot)$ is the inverse function of M_{\log} .

Especially, if $M(s) \leq ce^{as}$, $a, c > 0$, then the network (5) achieves logarithmic decay rate.

Proof. We can still use the proof of Theorem 2.4 to derive

$$\limsup_{|\sigma| \rightarrow \infty} \frac{1}{(M(\sigma))} \|(i\sigma - \mathcal{A})^{-1}\| < \infty,$$

which together with M_{\log} -Theorem in Batty and Duyckaerts [5] yields (61). \square

Remark 7. From Corollary 2, we see that in order to obtain an explicit decay rate, it is very important to estimate the lower bound of γ_n . Some techniques such as Diophantine approximations can be used to estimate it. However, it is still open to get a sharp estimate for it.

6. Numerical simulations. This section is devoted to present some numerical simulations on the dynamical behaviour of system (5) to support the results obtained above.

The backward Euler method in time (time step: $dt = 0.01$) and the Chebyshev spectral method in space (spatial grid size $S = 40$) were employed, in a MatLab environment (see [38]).

For simplicity, we assume that the star-shaped network consists of three edges, the lengths of which are given as

$$\ell_1 = 1, \ell_2 = 2, \ell_3 = 1.$$

The following cases are considered:

Case A. $N_1 = 3$ (Three thermoelastic rods)

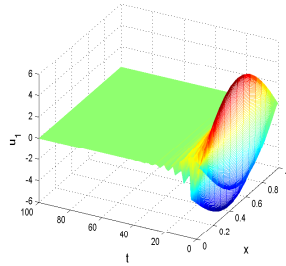


Figure A-1: $u_1(x, t)$

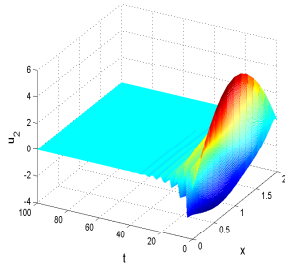


Figure A-2: $u_2(x, t)$

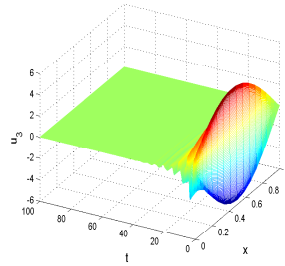


Figure A-3: $u_3(x, t)$

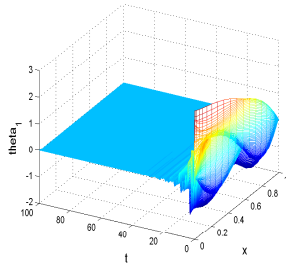


Figure A-4: $\theta_1(x, t)$

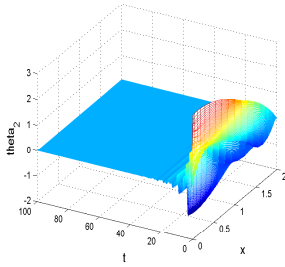


Figure A-5: $\theta_2(x, t)$

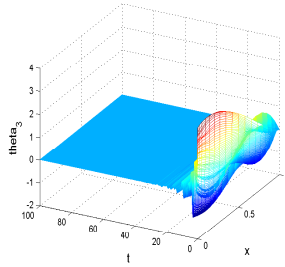


Figure A-6: $\theta_3(x, t)$

In this case, all the three edges of the network are constituted by thermoelastic ones. First, choose the initial conditions as follows:

$$\begin{cases} u_1(x, 0) = 5 \sin(\pi x), & u_2(x, 0) = 5 \sin(\frac{\pi x}{\ell_2}), & u_3(x, 0) = -5 \sin(\pi x), \\ u_{1,t}(x, 0) = 9 \sin(\pi x), & u_{2,t}(x, 0) = 4 \sin(\frac{\pi x}{\ell_2}), & u_{3,t}(x, 0) = -9 \sin(\pi x), \\ \theta_1(x, 0) = 3 \sin(\frac{1}{2}\pi x + \frac{\pi}{2}), & \theta_2(x, 0) = 3 \sin(\frac{\pi}{2\ell_2}x + \frac{\pi}{2}), & \theta_3(x, 0) = 3 \sin(\frac{1}{2}\pi x + \frac{\pi}{2}). \end{cases} \tag{62}$$

and the parameters in system (5):

$$\alpha_1 = 1, \beta_1 = 1, \alpha_2 = 2, \beta_2 = 1, \alpha_3 = 3, \beta_3 = 1. \tag{63}$$

The dynamical behaviour of $u_j, \theta_j, j = 1, 2, 3$ in the time interval $[0, 100]$ is given as in Figure A-1, 2, 3, 4, 5, 6. We observe the exponential decay rate of system (5).

Case B. $N_1 = 0$ (Three purely elastic rods)

In this case, the network consists of purely elastic rods, that is, no thermal-damping dissipates energy. The initial conditions are the same as in Case A. Solutions are plotted in Figure B-1, 2, 3, which confirms the conservative character of the system.

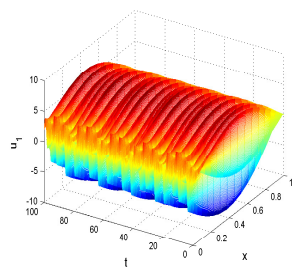


Figure B-1: $u_1(x, t)$

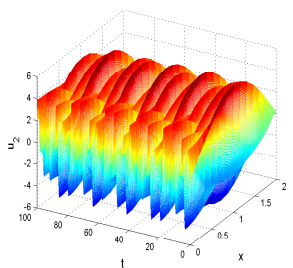


Figure B-2: $u_2(x, t)$

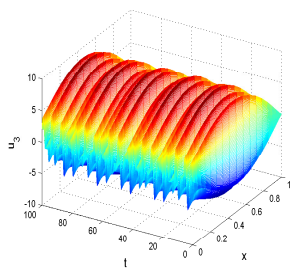


Figure B-3: $u_3(x, t)$

Case C. $N_1 = 2$ (Two thermoelastic rods and one purely elastic one)

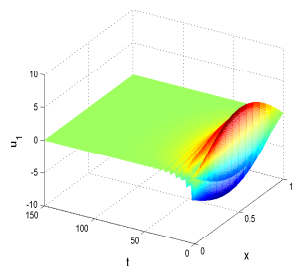


Figure C-1: $u_1(x, t)$

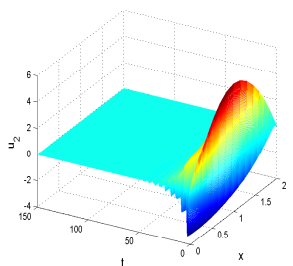


Figure C-2: $u_2(x, t)$

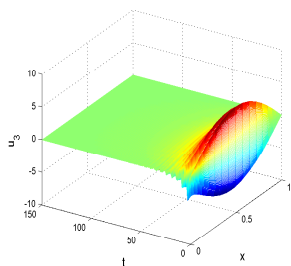


Figure C-3: $u_3(x, t)$

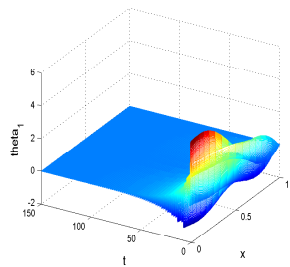


Figure C-4: $\theta_1(x, t)$

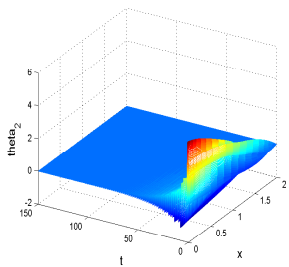


Figure C-5: $\theta_2(x, t)$

In this case, the network is constituted by two thermoelastic rods and one purely elastic one. The initial conditions and parameters in system (5) are chosen as (62) and (63). We get the dynamical behaviour in Figure C-1, 2, 3, 4, 5.

In these figures, we can see that the behaviour of each $u_j, j = 1, 2, 3$ and $\theta_j, j = 1, 2$ are convergent to zero very fast. This shows numerically that the energy of

system (5) decays to zero exponentially, which is consistent with the result on exponential decay rate in previous sections.

Case D. $N_1 = 1$ (One thermoelastic rod and two purely elastic rods)

In this case, the network consists of one thermoelastic rod and two purely elastic ones. The lengths of rods are still given as $\ell_1 = 1$, $\ell_2 = 2$, $\ell_3 = 1$. Simulations are plotted in Figure D-1, 2, 3, 4. We observe the lack of decay as predicted by the theory.

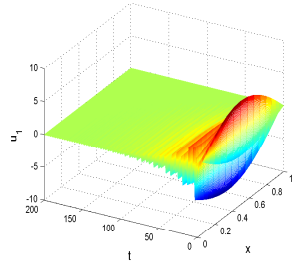


Figure D-1: $u_1(x, t)$

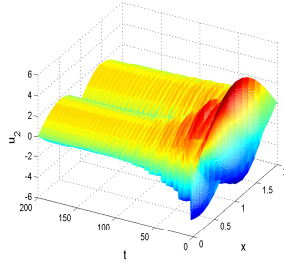


Figure D-2: $u_2(x, t)$

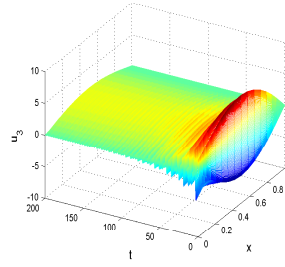


Figure D-3: $u_3(x, t)$

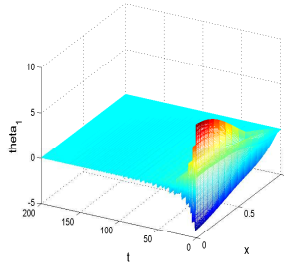


Figure D-4: $\theta_1(x, t)$

Case E. $N_1 = 1$, $\ell_2 = \sqrt{2}$ (One thermoelastic rod and two purely elastic ones)

In this case, the network is still constituted by one thermoelastic rod and two purely elastic ones. This time the length $\ell_2 = \sqrt{2}$, which leads to $\frac{\ell_2}{\ell_3} = \sqrt{2} \notin Q$.

In Figure E-1, 2, 3, 4, we observe a very slow decay rate. It implies that the energy of this system decays to zero but lacks exponential growth rate.

Moreover, we presented Figure F-1, 2 to compare the decay rate of the energy for each case. Figure F-1 shows the dynamical behaviours of the logarithmic scale of the energy for Case A, B and C. Figure F-2 shows the dynamical behaviours of the energies for Case D and E. From these two figures, we can clearly see the behaviours of energies for each case respect time, which are consistent with our theoretical results obtained in this paper.

Appendix A. Proof of (37). This appendix is devoted to show how to get (37).

Note that $\theta_j(0) = \theta_k(0)$, $u_j(0) = u_k(0)$ due to the continuity at the common node in system (5).

Thus, by (36), for $j = 1, 2, \dots, N_1$,

$$u_j(x) = \frac{u(0)}{-\sqrt{d_{j,2}} + \tilde{a}_j \tilde{b}_j \sqrt{d_{j,1}}} [-\sqrt{d_{j,2}} \cosh \sqrt{d_{j,1}} x + \tilde{a}_j \tilde{b}_j \sqrt{d_{j,1}} \cosh \sqrt{d_{j,2}} x]$$

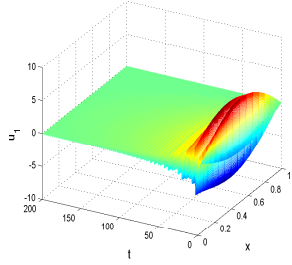
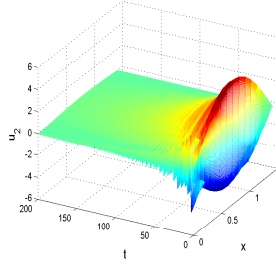
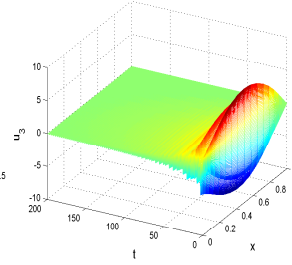
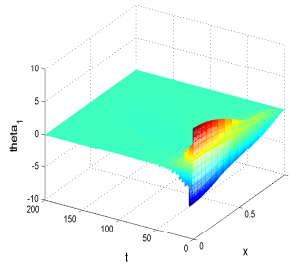
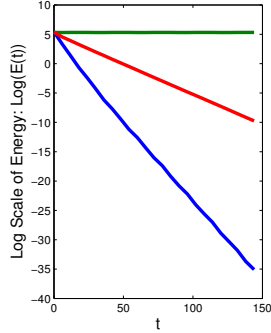

 Figure E-1: $u_1(x, t)$

 Figure E-2: $u_2(x, t)$

 Figure E-3: $u_3(x, t)$

 Figure E-4: $\theta_1(x, t)$


Figure F-1: Logarithmic scale of energy for Case A, B, C

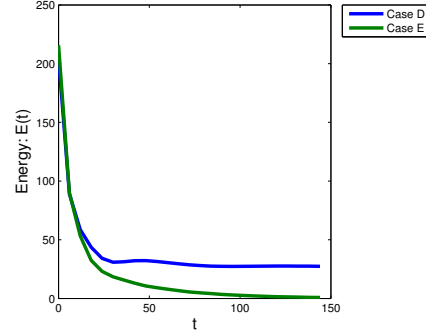


Figure F-2: Energy for Case D, E

$$\begin{aligned}
 & + \frac{u_{j,x}(0)}{\sqrt{d_{j,1}} - \tilde{a}_j \tilde{b}_j \sqrt{d_{j,2}}} [\sinh \sqrt{d_{j,1}} x - \tilde{a}_j \tilde{b}_j \sinh \sqrt{d_{j,2}} x] \\
 & + \frac{\theta(0)}{\sqrt{d_{j,1}} - \tilde{a}_j \tilde{b}_j \sqrt{d_{j,2}}} [-\tilde{b}_j \sqrt{d_{j,2}} \sinh \sqrt{d_{j,1}} x + \tilde{b}_j \sqrt{d_{j,1}} \sinh \sqrt{d_{j,2}} x] \\
 & + \frac{\theta_{j,x}(0)}{-\sqrt{d_{j,2}} + \tilde{a}_j \tilde{b}_j \sqrt{d_{j,1}}} [\tilde{b}_j \cosh \sqrt{d_{j,1}} x - \tilde{b}_j \cosh \sqrt{d_{j,2}} x], \\
 \theta_j(x) = & \frac{u(0)}{-\sqrt{d_{j,2}} + \tilde{a}_j \tilde{b}_j \sqrt{d_{j,1}}} [-\sqrt{d_{j,2}} \tilde{a}_j \sinh \sqrt{d_{j,1}} x + \tilde{a}_j \sqrt{d_{j,1}} \sinh \sqrt{d_{j,2}} x] \\
 & + \frac{u_{j,x}(0)}{\sqrt{d_{j,1}} - \tilde{a}_j \tilde{b}_j \sqrt{d_{j,2}}} [\tilde{a}_j \cosh \sqrt{d_{j,1}} x - \tilde{a}_j \cosh \sqrt{d_{j,2}} x]
 \end{aligned}$$

$$\begin{aligned}
& + \frac{\theta(0)}{\sqrt{d_{j,1}} - \tilde{a}_j \tilde{b}_j \sqrt{d_{j,2}}} [-\tilde{a}_j \tilde{b}_j \sqrt{d_{j,2}} \cosh \sqrt{d_{j,1}} x + \sqrt{d_{j,1}} \cosh \sqrt{d_{j,2}} x] \\
& + \frac{\theta_{j,x}(0)}{-\sqrt{d_{j,2}} + \tilde{a}_j \tilde{b}_j \sqrt{d_{j,1}}} [\tilde{a}_j \tilde{b}_j \sinh \sqrt{d_{j,1}} x - \sinh \sqrt{d_{j,2}} x].
\end{aligned}$$

Thus, by the boundary condition $u_j(\ell_j) = \theta_j(\ell_j) = 0$, $j = 1, 2, \dots, N_1$, we get

$$\begin{aligned}
0 &= \frac{u(0)}{-\sqrt{d_{j,2}} + \tilde{a}_j \tilde{b}_j \sqrt{d_{j,1}}} [-\sqrt{d_{j,2}} \cosh \sqrt{d_{j,1}} \ell_j + \tilde{a}_j \tilde{b}_j \sqrt{d_{j,1}} \cosh \sqrt{d_{j,2}} \ell_j] \\
&+ \frac{u_{j,x}(0)}{\sqrt{d_{j,1}} - \tilde{a}_j \tilde{b}_j \sqrt{d_{j,2}}} [\sinh \sqrt{d_{j,1}} \ell_j - \tilde{a}_j \tilde{b}_j \sinh \sqrt{d_{j,2}} \ell_j] \\
&+ \frac{\theta(0)}{\sqrt{d_{j,1}} - \tilde{a}_j \tilde{b}_j \sqrt{d_{j,2}}} [-\tilde{b}_j \sqrt{d_{j,2}} \sinh \sqrt{d_{j,1}} \ell_j + \tilde{b}_j \sqrt{d_{j,1}} \sinh \sqrt{d_{j,2}} \ell_j] \\
&+ \frac{\theta_{j,x}(0)}{-\sqrt{d_{j,2}} + \tilde{a}_j \tilde{b}_j \sqrt{d_{j,1}}} [\tilde{b}_j \cosh \sqrt{d_{j,1}} \ell_j - \tilde{b}_j \cosh \sqrt{d_{j,2}} \ell_j], \\
0 &= \frac{u(0)}{-\sqrt{d_{j,2}} + \tilde{a}_j \tilde{b}_j \sqrt{d_{j,1}}} [-\sqrt{d_{j,2}} \tilde{a}_j \sinh \sqrt{d_{j,1}} \ell_j + \tilde{a}_j \sqrt{d_{j,1}} \sinh \sqrt{d_{j,2}} \ell_j] \\
&+ \frac{u_{j,x}(0)}{\sqrt{d_{j,1}} - \tilde{a}_j \tilde{b}_j \sqrt{d_{j,2}}} [\tilde{a}_j \cosh \sqrt{d_{j,1}} \ell_j - \tilde{a}_j \cosh \sqrt{d_{j,2}} \ell_j] \\
&+ \frac{\theta(0)}{\sqrt{d_{j,1}} - \tilde{a}_j \tilde{b}_j \sqrt{d_{j,2}}} [-\tilde{a}_j \tilde{b}_j \sqrt{d_{j,2}} \cosh \sqrt{d_{j,1}} \ell_j + \sqrt{d_{j,1}} \cosh \sqrt{d_{j,2}} \ell_j] \\
&+ \frac{\theta_{j,x}(0)}{-\sqrt{d_{j,2}} + \tilde{a}_j \tilde{b}_j \sqrt{d_{j,1}}} [\tilde{a}_j \tilde{b}_j \sinh \sqrt{d_{j,1}} \ell_j - \sinh \sqrt{d_{j,2}} \ell_j].
\end{aligned}$$

Then a direct calculation yields, for $j = 1, 2, \dots, N_1$,

$$\begin{pmatrix} u_{j,x}(0) \\ \theta_{j,x}(0) \end{pmatrix} = \begin{pmatrix} C_{11}^j & C_{12}^j \\ C_{21}^j & C_{22}^j \end{pmatrix} \begin{pmatrix} u(0) \\ \theta(0) \end{pmatrix}, \quad (\text{A.1})$$

where

$$\begin{aligned}
C_{11}^j &= [(\sqrt{d_{j,1}} - \tilde{a}_j \tilde{b}_j \sqrt{d_{j,2}})(-\tilde{a}_j \tilde{b}_j \sinh \sqrt{d_{j,1}} \ell_j \cosh \sqrt{d_{j,2}} \ell_j + \cosh \sqrt{d_{j,1}} \ell_j \sinh \sqrt{d_{j,2}} \ell_j)] / D_j, \\
C_{12}^j &= [-\tilde{a}_j \tilde{b}_j^2 \sqrt{d_{j,2}} - \tilde{b}_j \sqrt{d_{j,1}} + \tilde{b}_j \sqrt{d_{j,1}} (-\tilde{a}_j \tilde{b}_j \sinh \sqrt{d_{j,1}} \ell_j \sinh \sqrt{d_{j,2}} \ell_j + \cosh \sqrt{d_{j,1}} \ell_j \cosh \sqrt{d_{j,2}} \ell_j) \\
&+ \tilde{b}_j \sqrt{d_{j,2}} (-\sinh \sqrt{d_{j,1}} \ell_j \sinh \sqrt{d_{j,2}} \ell_j + \tilde{a}_j \tilde{b}_j \cosh \sqrt{d_{j,1}} \ell_j \cosh \sqrt{d_{j,2}} \ell_j)] / D_j, \\
C_{21}^j &= [-\tilde{a}_j^2 \tilde{b}_j \sqrt{d_{j,1}} - \tilde{a}_j \sqrt{d_{j,2}} + \tilde{a}_j \sqrt{d_{j,1}} (\tilde{a}_j \tilde{b}_j \cosh \sqrt{d_{j,1}} \ell_j \cosh \sqrt{d_{j,2}} \ell_j - \sinh \sqrt{d_{j,1}} \ell_j \sinh \sqrt{d_{j,2}} \ell_j) \\
&+ \tilde{a}_j \sqrt{d_{j,2}} (\cosh \sqrt{d_{j,1}} \ell_j \cosh \sqrt{d_{j,2}} \ell_j - \tilde{a}_j \tilde{b}_j \sinh \sqrt{d_{j,1}} \ell_j \sinh \sqrt{d_{j,2}} \ell_j)] / D_j, \\
C_{22}^j &= [(-\sqrt{d_{j,2}} + \tilde{a}_j \tilde{b}_j \sqrt{d_{j,1}})(\tilde{a}_j \tilde{b}_j \cosh \sqrt{d_{j,1}} \ell_j \sinh \sqrt{d_{j,2}} \ell_j - \sinh \sqrt{d_{j,1}} \ell_j \cosh \sqrt{d_{j,2}} \ell_j)] / D_j,
\end{aligned}$$

and

$$D_j = -2\tilde{a}_j \tilde{b}_j - (1 + \tilde{a}_j^2 \tilde{b}_j^2) \sinh \sqrt{d_{j,1}} \ell_j \sinh \sqrt{d_{j,2}} \ell_j + 2\tilde{a}_j \tilde{b}_j \cosh \sqrt{d_{j,1}} \ell_j \cosh \sqrt{d_{j,2}} \ell_j.$$

Then by the transmission condition $\sum_{j=1}^{N_1} \frac{\alpha_j}{\beta_j} \theta_{j,x}(0) = 0$, we get

$$\sum_{j=1}^{N_1} \frac{\alpha_j}{\beta_j} (C_{21}^j u(0) + C_{22}^j \theta(0)) = 0, \quad (\text{A.2})$$

which implies that

$$\theta(0) = -u(0) \frac{\sum_{j=1}^{N_1} \frac{\alpha_j}{\beta_j} C_{21}^j}{\sum_{j=1}^{N_1} \frac{\alpha_j}{\beta_j} C_{22}^j}.$$

Hence by (A.1),

$$u_{j,x}(0) = C_{11}^j u(0) + C_{12}^j \theta(0) = u(0) \left(C_{11}^j - C_{12}^j \frac{\sum_{j=1}^{N_1} \frac{\alpha_j}{\beta_j} C_{21}^j}{\sum_{j=1}^{N_1} \frac{\alpha_j}{\beta_j} C_{22}^j} \right), \quad j = 1, 2, \dots, N_1. \tag{A.3}$$

On the other hand, by (34),

$$\begin{aligned} u_{j,x}(0) &= -\frac{\sigma u(0) \cos \sigma \ell_j}{\sin \sigma \ell_j} - \frac{\cos \sigma \ell_j}{\sin \sigma \ell_j} \int_0^{\ell_j} g_j(\ell_j - s) \sin \sigma(\ell_j - s) ds \\ &\quad + \int_0^{\ell_j} g_j(\ell_j - s) \cos \sigma(\ell_j - s) ds, \quad j = N_1 + 1, N_1 + 2, \dots, N. \end{aligned} \tag{A.4}$$

Then by the transmission condition $\sum_{j=1}^N u_{j,x}(0) = \sum_{j=1}^{N_1} \alpha_j \theta_j(0)$, together with (A.3) and (A.4), we have

$$\begin{aligned} &u(0) \left(\sum_{j=1}^{N_1} \left(C_{11}^j - C_{12}^j \frac{\sum_{j=1}^{N_1} \frac{\alpha_j}{\beta_j} C_{21}^j}{\sum_{j=1}^{N_1} \frac{\alpha_j}{\beta_j} C_{22}^j} + \alpha_j \frac{\sum_{j=1}^{N_1} \frac{\alpha_j}{\beta_j} C_{21}^j}{\sum_{j=1}^{N_1} \frac{\alpha_j}{\beta_j} C_{22}^j} \right) - \sum_{j=N_1+1}^N \frac{\sigma \cos \sigma \ell_j}{\sin \sigma \ell_j} \right) \\ &= \sum_{j=N_1+1}^N \frac{\cos \sigma \ell_j}{\sin \sigma \ell_j} \int_0^{\ell_j} g_j(\ell_j - s) \sin \sigma(\ell_j - s) ds \\ &\quad - \sum_{j=N_1+1}^N \int_0^{\ell_j} g_j(\ell_j - s) \cos \sigma(\ell_j - s) ds. \end{aligned} \tag{A.5}$$

Hence,

$$\begin{aligned} \sigma u(0) &= \frac{\sum_{j=N_1+1}^N \left(\frac{\cos \sigma \ell_j}{\sin \sigma \ell_j} \int_0^{\ell_j} g_j(\ell_j - s) \sin \sigma(\ell_j - s) ds - \int_0^{\ell_j} g_j(\ell_j - s) \cos \sigma(\ell_j - s) ds \right)}{\frac{1}{\sigma} \sum_{j=1}^{N_1} \left(C_{11}^j - C_{12}^j \frac{\sum_{j=1}^{N_1} \frac{\alpha_j}{\beta_j} C_{21}^j}{\sum_{j=1}^{N_1} \frac{\alpha_j}{\beta_j} C_{22}^j} + \alpha_j \frac{\sum_{j=1}^{N_1} \frac{\alpha_j}{\beta_j} C_{21}^j}{\sum_{j=1}^{N_1} \frac{\alpha_j}{\beta_j} C_{22}^j} \right) - \sum_{j=N_1+1}^N \frac{\cos \sigma \ell_j}{\sin \sigma \ell_j}}. \end{aligned} \tag{A.6}$$

As $\sigma \rightarrow \infty$, we have the following estimation:

$$d_{j,1} = (i\sigma)^2 + \alpha_j \beta_j (i\sigma) + \alpha_j \beta_j + O\left(\frac{1}{\sigma}\right), \quad d_{j,2} = i\sigma - \alpha_j \beta_j + O\left(\frac{1}{\sigma}\right).$$

Hence,

$$\sqrt{d_{j,1}} = i\sigma + \frac{\alpha_j \beta_j}{2} + O\left(\frac{1}{\sigma}\right), \quad \sqrt{d_{j,2}} = \sqrt{i\sigma} - \frac{\alpha_j \beta_j}{2\sqrt{i\sigma}} + O\left(\frac{1}{\sigma^{3/2}}\right),$$

$$\tilde{a}_j = \beta_j + O\left(\frac{1}{\sigma}\right), \quad \tilde{b}_j = -\alpha_j(i\sigma)^{-\frac{3}{2}} + O\left(\frac{1}{\sigma^{5/2}}\right).$$

and

$$-\sqrt{d_{j,2}} + \tilde{a}_j \tilde{b}_j \sqrt{d_{j,1}} = -\sqrt{i\sigma} - \frac{\alpha_j \beta_j}{\sqrt{i\sigma}} + O\left(\frac{1}{\sigma^{3/2}}\right),$$

$$\sqrt{d_{j,1}} - \tilde{a}_j \tilde{b}_j \sqrt{d_{j,2}} = i\sigma + \frac{\alpha_j \beta_j}{2} + \frac{\alpha_j \beta_j}{\sqrt{i\sigma}} + O\left(\frac{1}{\sigma}\right).$$

Substituting the above into (A.1), we get that

$$C_{11}^j = \frac{(\sqrt{d_{j,1}} - \tilde{a}_j \tilde{b}_j \sqrt{d_{j,2}}) \cosh \sqrt{d_{j,1}} \ell_j}{(1 + \tilde{a}_j^2 \tilde{b}_j^2) \sinh \sqrt{d_{j,1}} \ell_j} + O\left(\frac{1}{\sigma^{3/2}}\right) = \left(i\sigma + \frac{\alpha_j \beta_j}{2} + \frac{\alpha_j \beta_j}{\sqrt{i\sigma}}\right) \frac{\cosh \sqrt{d_{j,1}} \ell_j}{\sinh \sqrt{d_{j,1}} \ell_j} + O\left(\frac{1}{\sigma}\right), \quad (\text{A.7})$$

$$C_{12}^j = \frac{(\tilde{b}_j \sqrt{d_{j,1}}) \cosh \sqrt{d_{j,1}} \ell_j}{(1 + \tilde{a}_j^2 \tilde{b}_j^2) \sinh \sqrt{d_{j,1}} \ell_j} + O\left(\frac{1}{\sigma^{3/2}}\right) = -\frac{\alpha_j}{\sqrt{i\sigma}} \frac{\cosh \sqrt{d_{j,1}} \ell_j}{\sinh \sqrt{d_{j,1}} \ell_j} + O\left(\frac{1}{\sigma}\right), \quad (\text{A.8})$$

$$C_{21}^j = -\frac{(\tilde{a}_j \sqrt{d_{j,1}})}{(1 + \tilde{a}_j^2 \tilde{b}_j^2)} + O\left(\frac{1}{\sigma^{3/2}}\right) = -i\beta_j \sigma + \frac{\alpha_j \beta_j^2}{2} + O\left(\frac{1}{\sigma}\right), \quad (\text{A.9})$$

$$C_{22}^j = -\frac{(-\sqrt{d_{j,2}} + \tilde{a}_j \tilde{b}_j \sqrt{d_{j,1}})}{(1 + \tilde{a}_j^2 \tilde{b}_j^2)} + O\left(\frac{1}{\sigma^{3/2}}\right) = \sqrt{i\sigma} + \frac{\alpha_j \beta_j}{\sqrt{i\sigma}} + O\left(\frac{1}{\sigma}\right). \quad (\text{A.10})$$

Then by (A.6),

$$\begin{aligned} \sigma u(0) &= \frac{\sum_{j=N_1+1}^N \left(\frac{\cos \sigma \ell_j}{\sin \sigma \ell_j} \int_0^{\ell_j} g_j(\ell_j - s) \sin \sigma(\ell_j - s) ds - \int_0^{\ell_j} g_j(\ell_j - s) \cos \sigma(\ell_j - s) ds \right)}{\sum_{j=1}^{N_1} \left(i \frac{\cosh(i\sigma + \frac{\alpha_j \beta_j}{2}) \ell_j}{\sinh(i\sigma + \frac{\alpha_j \beta_j}{2}) \ell_j} + O\left(\frac{1}{\sqrt{\sigma}}\right) \right) - \sum_{j=N_1+1}^N \frac{\cos \sigma \ell_j}{\sin \sigma \ell_j}} \\ &= \frac{\sum_{j=N_1+1}^N \left(\frac{\cos \sigma \ell_j}{\sin \sigma \ell_j} \int_0^{\ell_j} g_j(\ell_j - s) \sin \sigma(\ell_j - s) ds - \int_0^{\ell_j} g_j(\ell_j - s) \cos \sigma(\ell_j - s) ds \right)}{\sum_{j=1}^{N_1} \left(i \frac{\cosh(i\sigma + \frac{\alpha_j \beta_j}{2}) \ell_j}{\sinh(i\sigma + \frac{\alpha_j \beta_j}{2}) \ell_j} + O\left(\frac{1}{\sqrt{\sigma}}\right) \right) - \sum_{j=N_1+1}^N \frac{\cos \sigma \ell_j}{\sin \sigma \ell_j}} \\ &= \frac{\sum_{j=N_1+1}^N \prod_{\substack{k=N_1+1 \\ k \neq j}}^N \sin \sigma \ell_k \cos \sigma \ell_j \int_0^{\ell_j} g_j(\ell_j - s) \sin \sigma(\ell_j - s) ds}{\prod_{k=N_1+1}^N \sin \sigma \ell_k \sum_{j=1}^{N_1} \left(i \frac{\cosh(i\sigma + \frac{\alpha_j \beta_j}{2}) \ell_j}{\sinh(i\sigma + \frac{\alpha_j \beta_j}{2}) \ell_j} + O\left(\frac{1}{\sqrt{\sigma}}\right) \right) - \sum_{j=N_1+1}^N \prod_{\substack{k=N_1+1 \\ k \neq j}}^N \sin \sigma \ell_k \cos \sigma \ell_j} \\ &= \frac{\sum_{j=N_1+1}^N \prod_{k=N_1+1}^N \sin \sigma \ell_k \int_0^{\ell_j} g_j(\ell_j - s) \cos \sigma(\ell_j - s) ds}{\prod_{k=N_1+1}^N \sin \sigma \ell_k \sum_{j=1}^{N_1} \left(i \frac{\cosh(i\sigma + \frac{\alpha_j \beta_j}{2}) \ell_j}{\sinh(i\sigma + \frac{\alpha_j \beta_j}{2}) \ell_j} + O\left(\frac{1}{\sqrt{\sigma}}\right) \right) - \sum_{j=N_1+1}^N \prod_{\substack{k=N_1+1 \\ k \neq j}}^N \sin \sigma \ell_k \cos \sigma \ell_j}. \end{aligned}$$

Thus, (37) has been obtained.

REFERENCES

[1] M. Alves, J. Muñoz Rivera, M. Sepúlveda and O. V. Villagrán, [The lack of exponential stability in certain transmission problems with localized Kelvin–Voigt dissipation](#), *SIAM J. Appl. Math.*, **74** (2014), 345–365.

[2] K. Ammari and M. Jellouli, [Stabilization of star-shaped networks of strings](#), *Differential and Integral Equations*, **17** (2004), 1395–1410.

[3] K. Ammari, M. Jellouli and M. Khenissi, [Stabilization of generic trees of strings](#), *Journal of Dynamical and Control Systems*, **11** (2005), 177–193.

[4] K. Ammari, [Asymptotic behaviour of some elastic planar networks of Bernoulli-Euler beams](#), *Applicable Analysis*, **86** (2007), 1529–1548.

[5] C. J. K. Batty and T. Duyckaerts, [Non-uniform stability for bounded semi-groups on Banach spaces](#), *J. Evol. Equ.*, **8** (2008), 765–780.

- [6] A. Borichev and Y. Tomilov, [Optimal polynomial decay of functions and operator semigroups](#), *Math. Ann.*, **347** (2010), 455–478.
- [7] J. W. S. Cassals, *An Introduction to Diophantine Approximation*, Cambridge University Press, Cambridge, UK, 1957.
- [8] R. Dáger and E. Zuazua, [Controllability of star-shaped networks of strings](#), *C. R. Acad. Sci. Paris Ser. I Math.*, **332** (2001), 621–626.
- [9] R. Dáger and E. Zuazua, [Controllability of tree-shaped networks of vibrating strings](#), *C. R. Acad. Sci. Paris Ser. I Math.*, **332** (2001), 1087–1092.
- [10] R. Dáger and E. Zuazua, [Spectral boundary controllability of networks of strings](#), *C. R. Math. Acad. Sci. Paris*, **334** (2002), 545–550.
- [11] R. Dáger and E. Zuazua, *Wave Propagation, Observation and Control in 1-d Flexible Multi-structures*, Mathématiques et Applications, 50, Springer-Verlag, Berlin, 2006.
- [12] H. D. Fernández-Sare, J. E. Muñoz-Rivera and R. Racke, Stability for a transmission problem in thermoelasticity with second sound, *Journal of Thermal Stresses*, **31** (2008), 1170–1189.
- [13] L. Gearhart, [Spectral theory for contraction semigroups on Hilbert space](#), *Trans. Amer. Math. Soc.*, **236** (1978), 385–394.
- [14] Z. J. Han and G. Q. Xu, [Spectral analysis and stability of thermoelastic Bresse system with second sound and boundary viscoelastic damping](#), *Math. Meth. Appl. Sci.*, **38** (2015), 94–112.
- [15] Z. J. Han and G. Q. Xu, [Spectrum and stability analysis for a transmission problem in thermoelasticity with a concentrated mass](#), *Z. Angew. Math. Phys.*, **66** (2015), 1717–1736.
- [16] Z. J. Han and G. Q. Xu, [Spectrum and dynamical behavior of a kind of planar network of non-uniform strings with non-located feedbacks](#), *Networks and Heterogeneous Media*, **5** (2010), 315–334.
- [17] Z. J. Han and G. Q. Xu, [Output feedback stabilisation of a tree-shaped network of vibrating strings with non-located observation](#), *International Journal of Control*, **84** (2011), 458–475.
- [18] D. B. Henry, O. Lopes and A. Perissinotto Jr., [On the essential spectrum of a semigroup of thermoelasticity](#), *Nonlinear Analysis: Theory, Methods & Applications*, **21** (1993), 65–75.
- [19] F. L. Huang, Characteristic condition for exponential stability of linear dynamical systems in Hilbert spaces, *Ann. Diff. Eqs.*, **1** (1985), 43–56.
- [20] S. Jaffard, M. Tucsnak and E. Zuazua, [Singular internal stabilization of the wave equation](#), *Journal of Differential Equations*, **145** (1998), 184–215.
- [21] J. Lagnese, G. Leugering and E. J. P. G. Schmidt, *Modeling, Analysis and Control of Dynamic Elastic Multi-link Structures*, Systems & Control: Foundations & Applications, Birkhäuser, Boston, 1994.
- [22] G. Lebeau and E. Zuazua, [Null controllability of a system of linear thermoelasticity](#), *Archive Rat. Mech. Anal.*, **141** (1998), 297–329.
- [23] G. Lebeau and E. Zuazua, [Decay rates for the linear system of three-dimensional system of thermoelasticity](#), *Archive Rat. Mech. Anal.*, **148** (1999), 179–231.
- [24] Z. Liu and B. Rao, [Energy decay rate of the thermoelastic Bresse system](#), *Z. Angew. Math. Phys.*, **60** (2009), 54–69.
- [25] Z. Liu and S. Zheng, *Semigroups Associated with Dissipative Systems*, CRC Research Notes in Mathematics, vol. 398, Chapman and Hall/CRC, Boca Raton, 1999.
- [26] Z. Liu and R. Rao, [Characterization of polynomial decay rate for the solution of linear evolution equation](#), *Z. Angew. Math. Phys.*, **56** (2005), 630–644.
- [27] Yu. I. Lyubich and V. Q. Phóng, Asymptotic stability of linear differential equations in Banach spaces, *Studia Math.*, **88** (1988), 37–42.
- [28] A. Marzocchi, J. E. Muñoz Rivera and M. G. Naso, [Asymptotic behaviour and exponential stability for a transmission problem in thermoelasticity](#), *Math. Meth. Appl. Sci.*, **25** (2002), 955–980.
- [29] A. Marzocchi, J. E. Muñoz Rivera and M. G. Naso, [Transmission problem in thermoelasticity with symmetry](#), *IMA J. Appl. Math.*, **68** (2003), 23–46.
- [30] S. A. Messaoudi and B. Said-Houari, [Energy decay in a transmission problem in thermoelasticity of type III](#), *IMA J. Appl. Math.*, **74** (2009), 344–360.
- [31] J. E. Muñoz Rivera and H. Portillo Oquendo, [The transmission problem for thermoelastic beams](#), *Journal of Thermal Stresses*, **24** (2001), 1137–1158.
- [32] S. Nicaise and J. Valein, [Stabilization of the wave equation on 1-D networks with a delay term in the nodal feedbacks](#), *Networks and Heterogeneous Media*, **2** (2007), 425–479.

- [33] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, Berlin, 1983.
- [34] J. Priüss, On the spectrum of C_0 -semigroups, *Trans. Amer. Math. Soc.*, **284** (1984), 847–857.
- [35] R. Racke, Thermoelasticity with second sound-Exponential stability in linear and non-linear 1-d, *Math. Meth. Appl. Sci.*, **25** (2002), 409–441.
- [36] W. M. Schmidt, Simultaneous approximation to algebraic numbers by rationals, *Acta Math.*, **125** (1970), 189–201.
- [37] F. Shel, Exponential stability of a network of elastic and thermoelastic materials, *Math. Meth. Appl. Sci.*, **36** (2013), 869–879.
- [38] L. N. Trefethen, *Spectral Methods in Matlab*, PA: SIAM, Philadelphia, 2000.
- [39] J. Valein and E. Zuazua, Stabilization of the wave equation on 1-d networks, *SIAM J. Contr. Optim.*, **48** (2009), 2771–2797.
- [40] G. Q. Xu, D. Y. Liu and Y. Q. Liu, Abstract second order hyperbolic system and applications to controlled networks of strings, *SIAM J. Control Optim.*, **47** (2008), 1762–1784.
- [41] X. Zhang and E. Zuazua, Polynomial decay and control of a 1-d hyperbolic-parabolic coupled system, *J. Differ. Equations*, **204** (2004), 380–438.
- [42] E. Zuazua, Controllability of the linear system of thermoelasticity, *Journal de Mathématiques Pures et Appliquées*, **74** (1995), 291–315.

Received September 2016; revised July 2017.

E-mail address: zjhan@tju.edu.cn

E-mail address: enrique.zuazua@uam.es