# THE WIGNER-LOHE MODEL FOR QUANTUM SYNCHRONIZATION AND ITS EMERGENT DYNAMICS

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Abstract. We present the Wigner-Lohe model for quantum synchronization which can be derived from the Schrödinger-Lohe model using the Wigner formalism. For identical one-body potentials, we provide a priori sufficient framework leading the complete synchronization, in which $L^2$-distances between all wave functions tend to zero asymptotically.

1. **Introduction.** Synchronization represents a phenomenon in which rhythms of weakly coupled oscillators are adjusted to the common frequency due to their weak interactions. It is often observed in many complex systems, e.g., the flashing of fireflies, clapping of hands in a concert hall, and heartbeat regulation by pacemaker cells, etc., [1, 6, 7, 33, 35]. However, rigorous mathematical treatment of such collective phenomena were begun only several decades ago by two scientists Winfree [37] and Kuramoto [26, 27]. For the mathematical modeling of synchronization, they adopted a continuous dynamical system approach based on their heuristic and intuitive arguments. In this paper, we are mainly interested in quantum Lohe oscillators with all-to-all couplings under one-body potential. To fix the idea, consider a classical complete network consisting of $N$ nodes, where each pair of nodes is connected with an equal capacity which is assumed to be unity. We also assume that quantum Lohe oscillators with the same unit mass are positioned on the nodes of the underlying complete network. To avoid unnecessary physical complexity, we ignore entanglement and decoherence effects inherent to the quantum many-body

systems. For a better physical modeling, such genuine quantum effects need to be taken into account.

Let $\psi_i = \psi_i(t,x)$ be the wave function of the $i$-th Lohe oscillator on the spatial domain $\mathbb{R}^d$. Then, the dynamics of Lohe oscillators with unit mass is governed by the Schrödinger-Lohe (S-L) model: for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$.

$$
\mathrm{i}\partial_t \psi_i = -\frac{1}{2}\Delta\psi_i + V_i\psi_i + \frac{\mathrm{i}K}{2N}\sum_{k=1}^{N}\Big(\frac{\|\psi_i\|\psi_k}{\|\psi_k\|} - \frac{\langle\psi_k,\psi_i\rangle\psi_i}{\|\psi_i\|\|\psi_k\|}\Big), \quad 1 \le i \le N, \qquad (1)
$$

where $\|\cdot\| := \|\cdot\|_{L^2}$ and $\langle\cdot,\cdot\rangle$ are the standard $L^2$ norm and an inner product on $\mathbb{R}^d$, and $V_i = V_i(x)$ and $K$ correspond to the one-body potential and nonnegative coupling strength, respectively. The S–L model (1) was first introduced by Australian physicist Max Lohe [28] several years ago as an infinite state generalization of the Lohe matrix model [29]. As discussed in [28, 29], quantum synchronization has received much attention from the quantum optics community because of its possible applications in quantum computing and quantum information [14, 23, 24, 25, 30, 36, 39, 40]. The emergent dynamics of the S-L system (1) has been partially treated in [11, 12] for some restricted class of initial data and a large coupling strength. Recently, a new approach based on the finite-dimensional reduction has been proposed in [5, 15] which significantly improve the previous results [11, 12] by the Lyapunov functional approach. However, a complete resolution of the synchronization problem for (1) is still far from complete.

Our main purpose of this paper is to present a quantum kinetic analogue of the S-L model (1) and study its emergent dynamics. The study on the quantum kinetic model for the Schrödinger equation dates back to Wigner's paper [38], in which Wigner considered the quantum mechanical motion of a large ensemble of electrons in a vacuum under the action of the Coulomb force generated by the charge of the electrons. For the modeling of large ensemble, he introduced a quasi one-particle distribution function, so called the Wigner function and showed that it satisfies the quantum Liouville equation [8, 9, 19, 20, 32, 41].

Before we briefly describe our main results, we first recall the Wigner transform of wave function on $\mathbb{R}^d$. For more basic facts on Wigner transforms we refer the reader to [22, 21].

**Definition 1.1.** For any two wave functions $\psi, \phi \in L^2$, we define the Wigner transform

$$
w[\psi,\phi](x,p) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{iy\cdot p}\bar{\psi}\Big(x+\frac{y}{2}\Big)\phi\Big(x-\frac{y}{2}\Big)\,dy.
$$

If we choose $\psi = \phi$, then we write $w[\psi] := w[\psi,\psi]$.

In order to shorten the formulas, we are going to introduce the following notation: if $\psi_j$, $j = 1,\dots,N$ is the solution to the S-L system (1), then we write

$$
w_j := w[\psi_j], \quad w_{jk} := w[\psi_j,\psi_k], \quad w_{jk}^+ := \operatorname{Re} w_{jk} \quad \text{and} \quad w_{jk}^- := \operatorname{Im} w_{jk}.
$$

Our main results of this paper are as follows. First, we show that the Wigner transforms $w_i$ and $w_{ij}^{\pm}$ satisfies a coupled non-local system:

$$
\begin{cases}
\partial_t w_j + p \cdot \nabla_x w_j + \Theta[V](w_j) = \dfrac{K}{N} \sum_{k=1}^{N} \Big[ w_{jk}^+ - \Big( \int w_{jk}^+ dp dx \Big) w_j \Big], \\[2mm]
\partial_t w_{jk}^+ + p \cdot \nabla_x w_{jk}^+ + \Theta[V](w_{jk}^+) \\[1mm]
= \dfrac{K}{2N} \sum_{\ell=1}^{N} \Big[ w_{j\ell}^+ + w_{\ell k}^+ - \Big( \int (w_{j\ell}^+ + w_{\ell k}^+) dp dx \Big) w_{jk}^+ + \Big( \int (w_{j\ell}^- + w_{\ell k}^-) dp dx \Big) w_{jk}^- \Big], \\[2mm]
\partial_t w_{jk}^- + p \cdot \nabla_x w_{jk}^- + \Theta[V](w_{jk}^-) \\[1mm]
= \dfrac{K}{2N} \sum_{\ell=1}^{N} \Big[ w_{j\ell}^- + w_{\ell k}^- - \Big( \int (w_{j\ell}^+ + w_{\ell k}^+) dp dx \Big) w_{jk}^- + \Big( \int (w_{j\ell}^- + w_{\ell k}^-) dp dx \Big) w_{jk}^+ \Big].
\end{cases}
\tag{2}
$$

Second, we derive a sufficient condition for the complete synchronization of the coupled system (2). Finally, we also investigate the hydrodynamic formulation for the Schrödinger-Lohe system (1) and derive synchronization estimates in some special cases.

The rest of this paper is organized as follows. In Section 2, we present the Schrödinger-Lohe model for quantum synchronization and discuss previous works on the complete synchronization of the S-L model. In Section 3, we derive our augmented Wigner-Lohe model from the S-L model using the Wigner transform. In Section 4, we present a priori complete synchronization estimates for some restricted class of initial data. In Section 5, we also discuss a hydrodynamic model which can be obtained from the S-L model for two-oscillator case.

2. **Preliminaries.** In this section, we briefly present the Schrödinger-Lohe (S-L) model for Lohe synchronization, and review earlier results on the synchronization problem for the S-L model.

2.1. **The Schrödinger-Lohe model.** As a phenomenological model for the quantum synchronization generalizing classical Kuramoto synchronization, Lohe proposed a coupled Schrödinger-type model in [28]. For $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ and $1 \leq i \leq N$,

$$
\mathrm{i}\partial_t \psi_i = -\frac{1}{2}\Delta \psi_i + V_i \psi_i + \frac{\mathrm{i}K}{2N} \sum_{k=1}^{N} \Big( \frac{\|\psi_i\|\psi_k}{\|\psi_k\|} - \frac{\langle \psi_k, \psi_i \rangle \psi_i}{\|\psi_i\|\|\psi_k\|} \Big),
\tag{3}
$$

where we normalized $\hbar = 1$ and $m = 1$.

**Lemma 2.1.** [28] *Let $\Psi = (\psi_1, \cdots \psi_N)$ be a smooth solution to (3) with initial data $\Psi_0 = (\psi_1^0, \cdots, \psi_N^0)$. Then, the $L^2$ norm of $\psi_i$ is constant along the flow (3):*

$$
\|\psi_i(t)\| = \|\psi_i^0\| \ \ for \ \ t \geq 0, \ 1 \leq i \leq N.
$$

In view of the previous lemma, from now on we will assume that $\|\psi_i^0\| = 1$, $1 \leq i \leq N$, so that system (3) becomes

$$
\mathrm{i}\partial_t \psi_j = -\frac{1}{2}\Delta \psi_j + V_j \psi_j + \frac{\mathrm{i}K}{2N} \sum_{k=1}^{N} \big( \psi_k - \langle \psi_k, \psi_j \rangle \psi_j \big).
\tag{4}
$$

For the space-homogeneous case, we set the spatial domain to be a periodic domain $\mathbb{T}^d$ and choose a special choice of $V_i$:

$$
V_i(x) = \Omega_i : \text{constant}, \qquad \psi_i(t, x) = \psi_i(t), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{T}^d.
$$

system (3) can be reduced to the Kuramoto model which is a prototype model for classical synchronization. In this special setting, the S-L model becomes

$$\mathrm{i}\frac{d\psi_i}{dt} = \Omega_i\psi_i + \frac{\mathrm{i}K}{2N}\sum_{k=1}^{N}\Big(\frac{|\psi_i|\psi_k}{|\psi_k|} - \frac{\langle\psi_i,\psi_k\rangle\psi_i}{|\psi_i||\psi_k|}\Big). \tag{5}$$

We next simply take the ansatz for $\psi_i$:

$$\psi_i := e^{-\mathrm{i}\theta_i}, \quad 1 \le i \le N \tag{6}$$

and substitute this ansatz into (5) to obtain

$$\dot{\theta}_i\psi_i = \Omega_i\psi_i + \frac{\mathrm{i}K}{2N}\sum_{k=1}^{N}\Big(\psi_k - e^{-\mathrm{i}(\theta_i-\theta_k)}\psi_i\Big).$$

Then, we take the inner product of the above relation with $\psi_i$ and compare the real part of the resulting relation to get the Kuramoto model for classical synchronization [1, 6, 13, 16, 17, 18]:

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{k=1}^{N}\sin(\theta_k - \theta_i). \tag{7}$$

Thus, the S-L model can be viewed as a quantum generalization of the Kuramoto model.

2.2. **Previous results.** In this subsection, we briefly review the previous results [11, 12, 15, 5] on the complete synchronization of the S-L model. For this, we first recall the definition of the complete synchronization as follows.

**Definition 2.2.** Let $\Psi = (\psi_1, \cdots \psi_N)$ be a smooth solution to (3) with initial data $\Psi^0 = (\psi_1^0, \cdots, \psi_N^0)$. Then, the S-L model exhibits an asymptotic phase-locking if the following relations holds:

$$\exists \lim_{t\to\infty}\langle\psi_i,\psi_j\rangle = \alpha_{ij} \in \mathbb{C}. \tag{8}$$

**Remark 1.** For the classical phase models such as (7), asymptotic phase-locking is defined as the following condition:

$$\exists \lim_{t\to\infty}|\theta_i(t) - \theta_j(t)| = \theta_{ij}^\infty. \tag{9}$$

Via the relation (6), we can see that (8) and (9) are closely related. In fact, in [12] for identical potentials $V_i = V_j$, the complete synchronization is defined as

$$\lim_{t\to\infty}\|\psi_i(t) - \psi_j(t)\| = 0, \quad 1 \le i,j \le N. \tag{10}$$

Note that the condition (10) and normalization condition $\|\psi_i\| = 1$ yield

$$\lim_{t\to\infty}\langle\psi_i(t),\psi_j(t)\rangle = 1.$$

Thus, the condition (10) satisfies the condition (8). Recently, in [5, 15] the case with different one-body potentials was treated, at least for $N = 2$. In this framework it is shown that, in some regimes, the limit in (8) is not 1 but depends on the difference between the potentials. Hence the limit in (10) gives a non-zero constant. This is indeed the more general case, when the system (4) exhibits complete frequency synchronization but not phase synchronization. For more details we address the reader to [5, 15].

As mentioned in the Introduction, the S-L model was first considered in Lohe's work [28] for the non-Abelian generalization of the Kuramoto model. However, the first rigorous mathematical studies of the S-L model were treated by the second author and his collaborators in [12, 15] in two different methodologies. The first methodology is to use $L^2$-diameters for $\{\psi_i\}$ as a Lyapunov functional and derive a Gronwall type differential inequality to conclude the complete synchronization with $\alpha_{ij} = 1$. More precisely, we set

$$D(\Psi) := \max_{i,j} ||\psi_i - \psi_j||.$$

In [12], authors derived a differential inequality for the diameter $D(\Psi)$:

$$\frac{d}{dt} D(\Psi) \leq K(D(\Psi))\Big(D(\Psi) - \frac{1}{2}\Big), \quad t > 0.$$

This leads to an exponential synchronization of the (1).

**Theorem 2.3.** [12] *Suppose that the coupling strength and initial data satisfy*

$$K > 0, \quad V_i = V, \quad \|\psi_i^0\|_{L^2} = 1, \quad 1 \leq i \leq N, \quad D(\Psi^0) < \frac{1}{2}.$$

*Then, for any solution $\Psi = (\psi_1, \dots, \psi_N)$ to (1), the diameter $D(\Psi)$ satisfies*

$$D(\Psi(t)) \leq \frac{D(\Psi^0)}{D(\Psi^0) + (1 - 2D(\Psi^0))e^{Kt}}, \quad t \geq 0.$$

**Remark 2.** For distinct one-body potentials, we do not have an asymptotic phase-locking estimate for the S-L model yet, however in [11], for some restricted class of initial data and large coupling strength, a weaker concept of synchronization, namely practical synchronization estimates have been obtained:

$$\lim_{K \to \infty} \limsup_{t \to \infty} \max_{i,j} ||\psi_i - \psi_j|| = 0.$$

On the other hand, at least in the two oscillator case, it is possible to improve considerably the practical synchronization result: indeed in [5, 15] a complete picture of different regimes is shown, where the system (4) exhibits complete synchronization or dephasing, i.e. time periodic orbits for the correlation function.

Recently, an alternative approach to prove synchronization for the S-L model was proposed both in [5] and [15], by using a finite dimensional reduction. More precisely, in both papers the authors consider the correlations between the wave functions,

$$z_{jk}(t) := \langle \psi_j, \psi_k \rangle(t) = r_{jk}(t) + is_{jk}(t), \tag{11}$$

and they study their asymptotic behavior. Moreover, in [5] the introduction of the order parameter, defined in analogy with the classical Kuramoto model, allows to give a more general result.

**Theorem 2.4.** [5] *Let $(\psi_1, \dots, \psi_N) \in \mathcal{C}(\mathbb{R}_+; L^2(\mathbb{R}^d))^N$ be the solution to (4) with initial data $(\psi_1(0), \dots, \psi_N(0)) = (\psi_1^0, \dots, \psi_N^0) \in L^2(\mathbb{R}^d)^N$, and we assume that*

$$\sum_{k=1}^{N} \operatorname{Re} z_{jk}(0) > 0, \quad for\ any\ j = 1, \dots, N. \tag{12}$$

*Then we have*

$$|1 - z_{jk}(t)| \lesssim e^{-Kt}, \quad as\ t \to \infty.$$

As we will see in the next sections, the same approach used in [5, 15] will also be exploited to infer the synchronization results for the Wigner-Lohe model (13) and the hydrodynamical system (19). More precisely, for the quantum hydrodynamic system (19) we are going to need also some synchronization estimates proved at the $H^1$ regularity level. Such estimates are proved in [5].

3. **From Schrödinger-Lohe to Wigner-Lohe.** In this section we present a kinetic quantum analogue "the Wigner-Lohe (W-L) model" for the quantum synchronization, which can be derived from the Schrödinger-Lohe (S-L) model [28, 29] via the Wigner transform. In this and following sections, we assume that all one-body potentials are identical

$$V_j(x) = V(x), \quad 1 \le j \le N.$$

Recall that for a given a solution $\psi$ to the free Schrödinger equation:

$$i\partial_t \psi = -\frac{1}{2}\Delta\psi + V\psi,$$

then its Wigner transform $w = w[\psi]$ satisfies

$$\partial_t w + p \cdot \nabla_x w + \Theta[V]w = 0,$$

where the operator $\Theta[V]$ is defined by

$$\Theta[V](w)(x,p) := -\frac{i}{(2\pi)^d} \int e^{i(p-p')\cdot y} \left( V\left(x+\frac{y}{2}\right) - V\left(x-\frac{y}{2}\right) \right) w(x,p')\, dp'dy.$$

Hence, to derive the Wigner-Lohe system (2) we just need to see how the nonlocal coupling in (4) translates at the Wigner level. More precisely, let $\psi_j$ be a solution to (4), then by defining $w_j = w[\psi_j]$, we see that it satisfies

$$\partial_t w_j + p \cdot \nabla w_j + \Theta[V]w_j = R_j,$$

where the remainder term $R_j$ is given by

$$
\begin{aligned}
R_j =& \frac{1}{(2\pi)^d}\frac{K}{2N}\sum_{k=1}^N \int e^{ip\cdot y}\Big[\bar{\psi}_k\left(t,x+\frac{y}{2}\right)\psi_j\left(t,x-\frac{y}{2}\right) \\
&+ \bar{\psi}_j\left(t,x+\frac{y}{2}\right)\psi_k\left(t,x-\frac{y}{2}\right)\Big]\, dy - \frac{1}{(2\pi)^d}\frac{K}{2N}\sum_{k=1}^N 2r_{jk}w_j \\
=& \frac{K}{N}\sum_{k=1}^N \left(w_{jk}^+ - r_{jk}w_j\right),
\end{aligned}
$$

where $r_{jk}(t) := \operatorname{Re}\langle\psi_j,\psi_k\rangle(t) = \int w_{jk}^+(t,x,p)\, dxdp$. Let us recall that this last equality comes from one of the basic properties of Wigner transforms, namely

$$\int w[f,g](x,p)\, dxdp = \langle f,g\rangle,$$

for any $f,g \in L^2$. Resuming, the equation for $w_j$ is given by

$$\partial_t w_j + p \cdot \nabla_x w_j + \Theta[V]w_j = \frac{K}{N}\sum_{k=1}^N \left(w_{jk}^+ - r_{jk}w_j\right).$$

We now need to derive the equation for $w_{jk} = w[\psi_j,\psi_k]$. Since the linear part in the S-L model (4) is common for every wave functions (remember we chose identical

potentials, $V_j \equiv V$), then the linear part in the Wigner equation for $w_{jk}$ will be exactly the same as for $w_j$. Consequently we also have

$$\partial_t w_{jk} + p \cdot \nabla_x w_{jk} + \Theta[V]w_{jk} = R_{jk},$$

where

$$R_{jk} = \frac{1}{(2\pi)^d} \frac{K}{2N} \sum_{\ell=1}^{N} \int e^{iy \cdot p} \left( \bar{\psi}_\ell \left( t, x + \frac{y}{2} \right) \psi_k \left( t, x + \frac{y}{2} \right) + \bar{\psi}_j \left( t, x + \frac{y}{2} \right) \psi_\ell \right.$$

$$\left. \left( t, x + \frac{y}{2} \right) \right) dy - \frac{1}{(2\pi)^d} \frac{K}{2N} \sum_{\ell=1}^{N} (z_{j\ell} w_{jk} + z_{\ell k} w_{jk}).$$

Let us recall that $z_{jk}$ is defined in (11) and we notice that $\overline{z_{jk}} = z_{kj}$. Hence we obtain

$$R_{jk} = \frac{K}{2N} \sum_{\ell=1}^{N} (w_{j\ell} + w_{\ell k} - (z_{j\ell} + z_{\ell k})w_{jk})$$

and the equation for $w_{jk}$ becomes

$$\partial_t w_{jk} + p \cdot \nabla w_{jk} + \Theta[V]w_{jk} = \frac{K}{2N} \sum_{\ell=1}^{N} (w_{j\ell} + w_{\ell k} - (z_{j\ell} + z_{\ell k})w_{jk}).$$

By using definitions for $w_{jk}^{\pm}$ and the linearity of operator $\Theta[V]$, we then obtain the Wigner-Lohe system

$$\begin{cases} \partial_t w_j + p \cdot \nabla_x w_j \Theta[V]w_j = \dfrac{K}{N} \sum_{k=1}^{N} \left( w_{jk}^+ - r_{jk}w_j \right), \\ \partial_t w_{jk}^+ + p \cdot \nabla_x w_{jk}^+ \Theta[V](w_{jk}^+) \\ \qquad = \dfrac{K}{2N} \sum_{\ell=1}^{N} \left[ w_{j\ell}^+ + w_{\ell k}^+ - (r_{j\ell} + r_{\ell k})w_{jk}^+ - i(s_{j\ell} + s_{\ell k})w_{jk}^- \right], \\ \partial_t w_{jk}^- + p \cdot \nabla_x w_{jk}^- \Theta[V](w_{jk}^-) \\ \qquad = \dfrac{K}{2N} \sum_{\ell=1}^{N} \left[ w_{j\ell}^- + w_{\ell k}^- - (r_{j\ell} + R_{\ell k})w_{jk}^- - i(s_{j\ell} + s_{\ell k})w_{jk}^+ \right], \end{cases}$$

$$\tag{13}$$

4. **Emergent dynamics of the W-L model for identical potentials.** In this section, we focus on the Wigner-Lohe model with $N = 2$. In this case, system (13) becomes

$$\begin{cases} \partial_t w_1 + p \cdot \nabla_x w_1 + \Theta[V]w_1 = \dfrac{K}{2}(w_{12}^+ - r_{12}w_1), \\ \partial_t w_2 + p \cdot \nabla_x w_2 + \Theta[V]w_2 = \dfrac{K}{2}(w_{12}^+ - r_{12}w_2), \\ \partial_t w_{12} + p \cdot \nabla_x w_{12} + \Theta[V]w_{12} = \dfrac{K}{4}(w_1 + w_2 - 2z_{12}w_{12}), \end{cases} \tag{14}$$

where we have

$$w_{12}^+ = \text{Re}\, w_{12}, \quad z_{12} = z_{12}(t) = \int w_{12}\, dxdp, \quad r_{12} = \text{Re}\, z_{12}. \tag{15}$$

Let us remark that the system (14), complemented with the definitions (15) above, can be considered independently on the S-L system (4). For such a system we will prescribe initial data $w_1^0, w_2^0, w_{12}^0$ such that $w_1^0$ and $w_2^0$ are real valued, $\int w_1^0\, dxdp =$

$\int w_2^0 \, dxdp = 1$, $w_{12}^0$ is complex valued, $|\int w_{12}^0 \, dxdp| \leq 1$. Let us also notice that the last equation is complex valued, so that we don't split it into two coupled equations for $w_{12}^+$ and $w_{12}^-$ as in (13).

Let us now prove the synchronization for (14). First of all we remark that, by integrating the last equation over the whole phase space, we find the following ODE

$$\dot{z}_{12} = \frac{K}{2}(1 - z_{12}^2), \tag{16}$$

for which it is straightforward to give its asymptotic behavior.

**Lemma 4.1.** *Let $z_{12}(0) \in \mathbb{C}$ be such that $|z_{12}(0)| \leq 1$ and $z_{12}(0) \neq -1$, then the solution $z_{12}(t)$ to (16) satisfies*

$$|1 - z_{12}(t)| \lesssim e^{-Kt}.$$

*Proof.* By integrating (16) we obtain

$$z_{12}(t) = \frac{(1 + z_{12}(0))e^{Kt} - (1 - z_{12}(0))}{(1 + z_{12}(0))e^{Kt} + (1 - z_{12}(0))}.$$

$\square$

By using the Lemma above it is then possible to show the complete synchronization for the W-L model (14).

**Theorem 4.2.** *Let $(w_1, w_2, w_{12})$ be a solution to (14) with initial data $(w_1(0), w_2(0), w_{12}(0)) = (w_1^0, w_2^0, w_{12}^0)$ such that*

$$\int w_1^0(x, p) \, dxdp = \int w_2^0(x, p) \, dxdp = 1,$$

*and*

$$\left| \int w_{12}^0(x, p) \, dxdp \right| \leq 1, \quad \int w_{12}^0(x, p) \, dxdp \neq -1.$$

*Then we have*

$$\|w_1(t) - w_2(t)\|_{L^2}^2 \leq e^{-Kt}, \quad as \ t \to \infty.$$

*Proof.* It follows from (14) that it is possible to write down the equation for the difference $w_1 - w_2$,

$$\partial_t(w_1 - w_2) + p \cdot \nabla_x(w_1 - w_2) + \Theta[V](w_1 - w_2) = -\frac{Kr_{12}}{2}(w_1 - w_2).$$

By multiplying it by $2(w_1 - w_2)$ and by integrating over the whole phase space we obtain

$$\frac{d}{dt}\|w_1(t) - w_2(t)\|_{L^2}^2 = -Kr_{12}(t)\|w_1(t) - w_2(t)\|_{L^2}^2.$$

By Lemma 4.1 we know that $|1 - r_{12}(t)| \lesssim e^{-Kt}$, hence by Gronwall's inequality we obtain the synchronization result. $\square$

**Remark 3.** In Theorem 4.2 the case $\int w_{12}^0 \, dxdp = -1$ has been excluded. This case can be treated under the additional hypothesis $w_1^0 = w_2^0 = -w_{12}^0$, which can be interpreted as a natural consistency assumption with the analogue Schrödinger-Lohe model. Indeed, let us consider initial wave functions $\psi_1^0, \psi_2^0 \in L^2$, such that $\|\psi_1^0\|_{L^2} = \|\psi_2^0\|_{L^2} = 1$ and $\langle \psi_1^0, \psi_2^0 \rangle = \int w[\psi_1^0, \psi_2^0] \, dxdp = \int w_{12}^0 \, dxdp = -1$, then $\psi_1^0 = -\psi_2^0$ and hence $w[\psi_1^0] = w[\psi_2^0] = -w[\psi_1^0, \psi_2^0]$. In this particular case, $w_1, w_2, w_{12}$ in (14) evolve independently, according to the free Wigner equation

$$\partial_t w + p \cdot \nabla_x w + \Theta[V]w = 0. \tag{17}$$

To see that, first of all we notice that $w_1, w_2$ both satisfy the same equation with the same initial data, so they coincide. Moreover, under the above assumptions on the initial data, we have $z_{12}(0) = -1$ and hence $z_{12}(t) = -1$ for all $t > 0$. Consequently we have

$$\partial_t w_{12} + p \cdot \nabla_x w_{12} + \Theta[V] w_{12} = \frac{K}{2}(w_1 - w_{12})$$

and hence

$$\|w_1(t) + w_{12}(t)\|_{L^2} = \|w_1(0) + w_{12}(0)\|_{L^2} = 0.$$

Concluding, we see that the right hand sides in (14) are all zero and the dynamics is determined by (17). This case corresponds to complete decorrelation between the quantum nodes.

5. **Quantum hydrodynamics.** In this Section, we derive the hydrodynamic equations associated to the Schrödinger-Lohe model (4). Here we follow the approach developed in [2, 3, 4] where a polar factorisation method is exploited in order to define the hydrodynamical quantities also in the vacuum region. In order to simplify the exposition we mainly focus on the case of two identical oscillators. In this case the Schrödinger-Lohe model reads

$$\begin{cases} \mathrm{i}\partial_t \psi_1 = -\dfrac{1}{2}\Delta\psi_1 + V\psi_1 + \dfrac{\mathrm{i}K}{4}(\psi_2 - \langle\psi_2, \psi_1\rangle\psi_1) \\ \mathrm{i}\partial_t \psi_2 = -\dfrac{1}{2}\Delta\psi_2 + V\psi_2 + \dfrac{\mathrm{i}K}{4}(\psi_1 - \langle\psi_1, \psi_2,\rangle\psi_2). \end{cases} \tag{18}$$

The case with $N$ non-identical oscillators can be treated similarly with obvious modifications, but the study of this special case will simplify substantially the exposition.

In order to derive the hydrodynamics associated to system (18), we first need to ensure that it is globally well-posed in $H^1(\mathbb{R}^d)$. This is indeed a straightforward application of the standard theory for nonlinear Schrödinger equations [10], see for example Proposition 2.1 in [5]. Furthermore, let us notice that by defining $z_{12}(t) = \langle\psi_1, \psi_2\rangle(t)$, then this function satisfies the ODE (16). This is not surprising because the W-L model was indeed derived from (18) and because of the property $\int w[\psi_1, \psi_2] \, dx dp = \langle\psi_1, \psi_2\rangle$. This implies that, under the same assumptions of Lemma 4.1, in this case we also have

$$|1 - z_{12}(t)| \lesssim e^{-Kt}.$$

However, this synchronization result is too weak to be exploited for quantum hydrodynamic system derived from (18). Indeed, as we already remarked above, the natural space for the hydrodynamics is the finite energy space, namely $H^1$ for the wave functions. Hence we need to improve the result in the space of energy. Here we will make use of Theorem 4.5 in [5], where we address the reader for more general results in this direction.

Let us now consider the solution $(\psi_1, \psi_2) \in \mathcal{C}(\mathbb{R}_+; H^1)$ to system (18), given by Proposition 2.1 in [5]. To derive the hydrodynamic system associated with (18), we first define the mass densities, namely $\rho_1 = |\psi_1|^2$ and $\rho_2 = |\psi_2|^2$. By differentiating those quantities with respect to time and by using the equations above, we obtain

$$\begin{cases} \partial_t \rho_1 + \operatorname{div} J_1 = \dfrac{K}{2}(\rho_{12} - r_{12}\rho_1), \\ \partial_t \rho_2 + \operatorname{div} J_2 = \dfrac{K}{2}(\rho_{12} - r_{12}\rho_2), \end{cases}$$

where the associated current densities are respectively given by

$$J_1 := \mathrm{Im}(\bar{\psi}_1 \nabla \psi_1), \quad J_2 := \mathrm{Im}(\bar{\psi}_2 \nabla \psi_2).$$

Furthermore, in the equation for the mass density we also find the interaction term $\rho_{12} = \mathrm{Re}(\bar{\psi}_1 \psi_2)$, so that $r_{12} = \mathrm{Re}\langle \psi_1, \psi_2 \rangle = \int \rho_{12} \, dx$.

Let us notice that $\rho_{12}$ is not a mass density, since in general it can also be negative. By using those definitions we can derive the evolution equations for the current densities $J_1$ and $J_2$. For instance, by differentiating $J_1$ with respect to time we find that

$$\partial_t J_1 + \mathrm{div}(\mathrm{Re}(\nabla \bar{\psi}_1 \otimes \nabla \psi_1)) + \rho_1 \nabla V = \frac{1}{4} \nabla \Delta \rho_1 + \frac{K}{2}(J_{12} - r_{12}J_1),$$

where the new interaction term here is given by

$$J_{12} = \frac{1}{2} \mathrm{Im}(\bar{\psi}_1 \nabla \psi_2 + \bar{\psi}_2 \nabla \psi_1).$$

Next, we use the polar factorisation Lemma in [2, 3] to infer that, for $\psi_1 \in H^1(\mathbb{R}^d)$, we have

$$\mathrm{Re}(\nabla \bar{\psi}_1 \otimes \nabla \psi_1) = \nabla \sqrt{\rho_1} \otimes \nabla \sqrt{\rho_1} + \Lambda_1 \otimes \Lambda_1, \qquad \text{a.e. in } \mathbb{R}^d,$$

where $\sqrt{\rho_1} = |\psi_1|, \Lambda_1 = \mathrm{Im}(\bar{\phi}_1 \nabla \psi_1)$, $\phi_1$ is the polar factor for the wave function $\psi_1$ and we have $\sqrt{\rho_1}\Lambda_1 = J_1$, see [2, 3, 4] for more details on the polar factorisation. In this way we can write down the following equation for the current density $J_1$:

$$\partial_t J_1 + \mathrm{div}\left(\frac{J_1 \otimes J_1}{\rho_1}\right) + \rho_1 \nabla V = \frac{1}{2}\rho_1 \nabla\left(\frac{\Delta \sqrt{\rho_1}}{\sqrt{\rho_1}}\right) + \frac{K}{2}(J_{12} - r_{12}J_1).$$

By using the equation for $\psi_2$ we obtain an analogous equation for $J_2$:

$$\partial_t J_2 + \mathrm{div}\left(\frac{J_2 \otimes J_2}{\rho_2}\right) + \rho_2 \nabla V = \frac{1}{2}\rho_2 \nabla\left(\frac{\Delta \sqrt{\rho_2}}{\sqrt{\rho_2}}\right) + \frac{K}{2}(J_{12} - r_{12}J_2).$$

Resuming, by defining the hydrodynamical quantities $\rho_1, J_1, \rho_2, J_2$ associated to $\psi_1, \psi_2$, respectively, we can derive the following system:

$$\begin{cases} \partial_t \rho_1 + \mathrm{div}\, J_1 = \dfrac{K}{2}(\rho_{12} - r_{12}\rho_1), \\[2mm] \partial_t \rho_2 + \mathrm{div}\, J_2 = \dfrac{K}{2}(\rho_{12} - r_{12}\rho_2), \\[2mm] \partial_t J_1 + \mathrm{div}\left(\dfrac{J_1 \otimes J_1}{\rho_1}\right) + \rho_1 \nabla V = \dfrac{1}{2}\rho_1 \nabla\left(\dfrac{\Delta \sqrt{\rho_1}}{\sqrt{\rho_1}}\right) + \dfrac{K}{2}(J_{12} - r_{12}J_1), \\[2mm] \partial_t J_2 + \mathrm{div}\left(\dfrac{J_2 \otimes J_2}{\rho_2}\right) + \rho_2 \nabla V = \dfrac{1}{2}\rho_2 \nabla\left(\dfrac{\Delta \sqrt{\rho_2}}{\sqrt{\rho_2}}\right) + \dfrac{K}{2}(J_{12} - r_{12}J_2). \end{cases}$$

Note that the above hydrodynamical system is not closed, as we need to derive also the evolution equations for the quantities $\rho_{12}, J_{12}$. However, it is quite troublesome to derive a hydrodynamical equation for the quantity $J_{12}$. For this reason we consider the following auxiliary variables

$$\rho_d := |\psi_1 - \psi_2|^2, \quad J_d := \mathrm{Im}((\overline{\psi_1 - \psi_2})\nabla(\psi_1 - \psi_2)).$$

By using those variables, it is straightforward to derive their dynamical equations,

$$\partial_t \rho_d = 2\,\mathrm{Re}\left\{ \overline{(\psi_1 - \psi_2)} \left( \frac{i}{2}\Delta(\psi_1 - \psi_2) - iV(\psi_1 - \psi_2) \right.\right.$$
$$\left.\left. + \frac{K}{4}(\psi_2 - \psi_1 - \langle\psi_2,\psi_1\rangle\psi_1 + \langle\psi_1,\psi_2\rangle\psi_2) \right) \right\}$$
$$= -\,\mathrm{div}\,J_d + \frac{K}{2}\,\mathrm{Re}\left\{ \overline{(\psi_1 - \psi_2)}\left((1 + \langle\psi_1,\psi_2\rangle)(\psi_2 - \psi_1) + 2i\,\mathrm{Im}\langle\psi_1,\psi_2\rangle\psi_1\right) \right\}$$
$$= -\,\mathrm{div}\,J_d - \frac{K}{2}(1 + r_{12})\rho_d + Ks_{12}\sigma_{12},$$

where we denoted $\sigma_{12} = \mathrm{Im}(\bar\psi_1\psi_2)$, so that

$$s_{12} = \mathrm{Im}\langle\psi_1,\psi_2\rangle = \int_{\mathbb{R}^d} \sigma_{12}\,dx.$$

Define $\psi_d := \psi_1 - \psi_2$, then by following some similar calculations as before we find out

$$\partial_t J_d = -\frac{1}{2}\,\mathrm{Re}(\Delta\bar\psi_d\nabla\psi_d - \bar\psi_d\nabla\Delta\psi_d) - \rho_d\nabla V$$
$$+ \frac{K}{4}\left[-2J_d + \mathrm{Im}\left(\langle\psi_2,\psi_1\rangle(-\bar\psi_1\nabla\psi_d + \bar\psi_d\nabla\psi_2)\right.\right.$$
$$\left.\left. + \langle\psi_1,\psi_2\rangle(\bar\psi_2\nabla\psi_d - \bar\psi_d\nabla\psi_1)\right)\right].$$

After some simple algebra, we obtain that

$$\mathrm{Im}\left(\langle\psi_2,\psi_1\rangle(-\bar\psi_1\nabla\psi_d + \bar\psi_d\nabla\psi_2) + \langle\psi_1,\psi_2\rangle(\bar\psi_2\nabla\psi_d - \bar\psi_d\nabla\psi_1)\right)$$
$$= -2r_{12}J_d - 4s_{12}G_{12},$$

where $G_{12} := \frac{1}{2}\,\mathrm{Re}(\bar\psi_2\nabla\psi_1 - \bar\psi_1\nabla\psi_2)$. Hence the equation for $J_d$ is given by

$$\partial_t J_d + \mathrm{div}\left(\frac{J_d \otimes J_d}{\rho_d}\right) + \rho_d\nabla V = \frac{1}{2}\rho_d\nabla\left(\frac{\Delta\sqrt{\rho_d}}{\sqrt{\rho_d}}\right) - \frac{K}{2}\left((1 + r_{12})J_d + 2s_{12}G_{12}\right).$$

Once again, to close the hydrodynamic equations, we still need to determine the evolution for $\sigma_{12}, G_{12}$. As before, the equation derived for $G_{12}$ would be too involved, for this reason we alternatively define $\psi_a = \psi_1 - i\psi_2$ and its hydrodynamical quantities $\rho_a = \frac{1}{2}|\psi_a|^2$, $J_a = \frac{1}{2}\,\mathrm{Im}(\bar\psi_a\nabla\psi_a)$. If we write down the equation for $\psi_a$,

$$i\partial_t\psi_a = -\frac{1}{2}\Delta\psi_a + V\psi_a + \frac{K}{4}\left(i\psi_2 + \psi_1 - i\langle\psi_2,\psi_1\rangle\psi_1 - \langle\psi_1,\psi_2\rangle\psi_2\right),$$

we can then infer the equations for $\rho_a$ and $J_a$. By proceeding as before with some straightforward but long calculations, we find out

$$\partial_t\rho_a + \mathrm{div}\,J_a = \frac{K}{2}\left((1 - s_{12})\rho_{12} - r_{12}\rho_a\right)$$
$$\partial_t J_a + \mathrm{div}\left(\frac{J_a \otimes J_a}{\rho_a}\right) + \rho_a\nabla V = \frac{1}{2}\rho_a\nabla\left(\frac{\Delta\sqrt{\rho_a}}{\sqrt{\rho_a}}\right) + \frac{K}{2}\left((1 - s_{12})J_{12} - r_{12}J_a\right).$$

We can now resume and write down the whole set of hydrodynamic equations associated to the Schrödinger-Lohe system (18):

$$\partial_t \rho_1 + \operatorname{div} J_1 = \frac{K}{2}(\rho_{12} - r_{12}\rho_1)$$

$$\partial_t \rho_2 + \operatorname{div} J_2 = \frac{K}{2}(\rho_{12} - r_{12}\rho_2),$$

$$\partial_t J_1 + \operatorname{div}\left(\frac{J_1 \otimes J_1}{\rho_1}\right) + \rho_1 \nabla V = \frac{1}{2}\rho_1 \nabla \left(\frac{\Delta\sqrt{\rho_1}}{\sqrt{\rho_1}}\right) + \frac{K}{2}(J_{12} - r_{12}J_1),$$

$$\partial_t J_2 + \operatorname{div}\left(\frac{J_2 \otimes J_2}{\rho_2}\right) + \rho_2 \nabla V = \frac{1}{2}\rho_2 \nabla \left(\frac{\Delta\sqrt{\rho_2}}{\sqrt{\rho_2}}\right) + \frac{K}{2}(J_{12} - r_{12}J_2),$$

$$\partial_t \rho_d + \operatorname{div} J_d = -\frac{K}{2}(1 + r_{12})\rho_d + Ks_{12}\sigma_{12},$$

$$\partial_t J_d + \operatorname{div}\left(\frac{J_d \otimes J_d}{\rho_d}\right) + \rho_d \nabla V = \frac{1}{2}\rho_d \nabla \left(\frac{\Delta\sqrt{\rho_d}}{\sqrt{\rho_d}}\right) - \frac{K}{2}\left((1 + r_{12})J_d + 2s_{12}G_{12}\right),$$

$$\partial_t \rho_a + \operatorname{div} J_a = \frac{K}{2}\left((1 - s_{12})\rho_{12} - r_{12}\rho_a\right)$$

$$\partial_t J_a + \operatorname{div}\left(\frac{J_a \otimes J_a}{\rho_a}\right) + \rho_a \nabla V = \frac{1}{2}\rho_a \nabla \left(\frac{\Delta\sqrt{\rho_a}}{\sqrt{\rho_a}}\right) + \frac{K}{2}\left((1 - s_{12})J_{12} - r_{12}J_a\right),$$

$$\tag{19}$$

where

$$\rho_{12} := \frac{1}{2}(\rho_1 + \rho_2 - \rho_d), \quad J_{12} := \frac{1}{2}(J_1 + J_2 - J_d),$$

$$\sigma_{12} := \rho_a - \frac{1}{2}(\rho_1 + \rho_2), \quad G_{12} := J_a - \frac{1}{2}(J_1 + J_2).$$

By considering the system (19) above we can now prove the synchronization property. In view of the previous synchronization results we expect that

$$\lim_{t \to \infty} \left(\|\nabla\sqrt{\rho_1} - \nabla\sqrt{\rho_2}\|_{L^2} + \|\Lambda_1 - \Lambda_2\|_{L^2}\right) = 0$$

and furthermore

$$\lim_{t \to \infty} \left(\|\nabla\sqrt{\rho_d}\|_{L^2} + \|\Lambda_d\|_{L^2}\right) = 0.$$

To show the above properties we are going to use a synchronization result in $H^1$ for system (18) given in [5], which will be stated in the following Theorem. The result below actually holds in a more general case, see [5] for more details, however here we will only state the synchronization property we are going to use for our system (19).

**Theorem 5.1.** [5] *Let $(\psi_1^0, \psi_2^0) \in H^1$ be such that*

$$\langle \psi_1^0, \psi_2^0 \rangle \neq -1.$$

*Then, for the solution $(\psi_1, \psi_2) \in \mathcal{C}(\mathbb{R}_+; H^1)$ emanated from such initial data, we have*

$$\|\psi_1(t) - \psi_2(t)\|_{H^1} \lesssim e^{-Kt}, \quad as \ t \to \infty. \tag{20}$$

We apply now this result for the synchronization of system (19), First of all, from (20) we then infer that

$$\lim_{t \to \infty} \|\psi_d(t)\|_{H^1} = 0.$$

This and the polar factorisation Lemma 3 in [3] then readily implies that

$$\lim_{t \to \infty} (\|\nabla \sqrt{\rho_d}(t)\|_{L^2} + \|\Lambda_d(t)\|_{L^2}) = 0.$$

Furthermore, from the fact that $\psi_d \to 0$ in $H^1$ as $t \to \infty$ and the polar factorisation Lemma, again, we can also show that

$$\lim_{t \to \infty} (\|\nabla \sqrt{\rho_1}(t) - \nabla \sqrt{\rho_2}(t)\|_{L^2} + \|\Lambda_1(t) - \Lambda_2(t)\|_{L^2}) = 0.$$

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