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HOMOGENIZATION OF PINNING CONDITIONS ON PERIODIC NETWORKS

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Abstract. This paper deals with the description of the overall effect of pinning conditions in discrete systems. We study a variational problem on the discrete in which pinning sites are modeled as network subsets on which concentrated forces are imposed. We want to determine the asymptotic effect of pinning conditions on a periodic lattice as its size vanishes. Our analysis is performed in the framework of Γ-convergence and highlights the analogies and differences with the corresponding continuous problem, i.e. periodically perforated domains. We derive a functional form for the limit energies which depends on the relationship between the space dimension and the growth rate of the interaction functions.

1. **Introduction.** This paper deals with the description of the overall effect of pinning conditions in discrete systems, highlighting the analogies and differences with the corresponding continuous case. In variational problems on the continuum, pinning sites are usually modeled as small zones where concentrated forces or Dirichlet conditions are imposed. Their effect can be described by exhibiting suitable effective problems. In the simplest (but already presenting most of the general features) case of *periodically-perforated domains* one imposes homogeneous Dirichlet conditions on a periodic array $U_{\delta,R}$ of small balls of radius R and centers on a δ-periodic lattice, and considers, e.g., minimum problems of the form

$$\min\Big\{\int_\Omega \big(|Du|^p - fu\big)\,dx : u = 0 \text{ on } U_{\delta,R}\Big\}. \tag{1}$$

As $\delta, R \to 0$ these problems can be approximated by

$$\min\Big\{\int_\Omega \big(|Du|^p + C|u|^p - fu\big)\,dx\Big\},$$

where the middle term replaces the constraint; the constant C depends on the mutual asymptotic behavior of the two parameters. It is suggestive to think of u as a temperature field of a mixture of water and ice, with $U_{\delta,R}$ representing the ice distribution, and the second problem as an effective approximation when the ice particles are small. Note that there is a critical ratio between R and δ below which the constant C is 0 (if the percentage of "ice" is too small then it does not influence

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the limit) and above which C is $+\infty$ (i.e., the percentage of ice is so high that in the limit it forces $u = 0$).

The study of problems of the form above dates back to an early work by Khrushlov and Marchenko [17]. It has been subsequently popularized by a well-known paper of Cioranescu and Murat [8] and comprises a number of generalizations which cover also non-periodic geometries and give rise to the so-called *Relaxed Dirichlet Problems* (see e.g [11],[12],[13],[14],[15],[19] and [10] for an overview on the subject). In the framework of Γ-convergence recent papers as [3] and [18] deal with general vector energies (for a general introduction to Γ-convergence see e.g. [4], [5], [9]). At the critical scale the basis of the asymptotic description of problems (1) is a separation-of-scales argument: the contribution of the energy that "concentrates" near each of the small balls can be decoupled from the others and from the energy that is "diffused" elsewhere (this is formalized in the procedure highlighted in the paper by Ansini and Braides [3]), and can be then computed by means of suitable "capacitary formulas" that give C. It must be noted that in the subcritical case $p < n$ the contribution of each ball is of the form

$$CR^{n-p}|u|^p,$$

which gives the scaling $R \sim \delta^{n/(n-p)}$, while in the critical scale $p = n$ that contribution is

$$C|\log R|^{n-1}|u|^n,$$

which gives the scaling $|\log R| \sim \delta^{n/(n-1)}$. If $p > n$ there is no critical scaling leading to a non-trivial limit energy, so this case does not allow any interesting analysis.

In the simplest discrete case, the integrals $\int_\Omega |Du|^p$ are replaced by finite-difference energies on a cubic lattice $\varepsilon\mathbb{Z}^n$ of the form

$$\sum_{NN} \varepsilon^n \Big| \frac{u(i) - u(j)}{\varepsilon} \Big|^p, \tag{2}$$

where the sum ranges over all *nearest-neighbors* in $\varepsilon\mathbb{Z}^n \cap \Omega$. The continuous approximation of (2) is indeed

$$\int_\Omega \|Du\|^p \, dx, \tag{3}$$

where

$$\|Du\|_p^p = \sum_{j=1}^n \Big| \frac{\partial u}{\partial x_j} \Big|^p.$$

The pinning condition which replicates the perforated domain constraint is then expressed as

$$u = 0 \quad \text{on } \delta\mathbb{Z}^n,$$

where of course in addition one requires $\delta/\varepsilon \in \mathbb{N}$. A classical interpretation of this problem is the case of a network of thermal conductors, where u represents a temperature field and the pinning sites correspond to points where the temperature is forced to be zero. In the discrete setting the constrained minimum problem can be also thought as giving equilibrium configurations for an atomistic model, e.g., with hardening conditions due to the presence of transverse dislocations as in the paper by Garroni and Müller [16].

We can observe right away that the small parameter ε plays at the same time the role of both the discrete lattice scale and of the perforation size R, thus giving the critical scalings

$$\varepsilon \sim \delta^{n/(n-p)} \quad \text{and} \quad |\log \varepsilon| \sim \delta^{n/(n-1)}.$$

If suitable discretizations of a forcing term are added, the choice of the critical scaling leads to limit problems of the form

$$\min\Big\{\int_\Omega \Big(\|Du\|_p^p + C|u|^p - fu\Big)\,dx\Big\},$$

analogous to the ones we get in the continuous setting. The computation of the constant C presents some differences from the computation in the continuous case, even though a separation-of-scales procedure can be followed by proving a decoupling lemma (Lemma 6.1), which allows to analyze the single effect of each pinning site. In the critical case $p = n$ the energy "concentrating close to the pinning sites" indeed concentrates at a scale much larger than ε. In this way the capacitary computation reduces to the continuous one with a perforation of size $R = \varepsilon$ and with the anisotropic energy (3). In dimension $n = 2$ the constant is exactly the "classical" one since $\|Du\|_2$ equals the Euclidean gradient norm $|Du|$. In the subcritical case $p < n$, instead, the energy concentrates at scale ε, so that the constant C is expressed by the "discrete p-capacity" of a point in the lattice $\mathbb{Z}^n$.

In this paper we prove the convergence result outlined above in a general setting where u can be vector-valued and the discrete energies take the form

$$E_\varepsilon(u) = \sum_{i,j} \varepsilon^n f^{(i-j)/\varepsilon}\Big(\frac{u(i) - u(j)}{\varepsilon}\Big),$$

where the interactions range over all pairs in $\Omega \cap \varepsilon\mathbb{Z}^n$, and are governed by general pair potentials depending also on the mutual distance of i and j in the reference lattice $\varepsilon\mathbb{Z}^n$. The energy densities $f^\xi(z)$, with $\xi \in \mathbb{Z}^n$, satisfy polynomial growth conditions in z of order p, and decay conditions in ξ that allow to restrict to (long-range but) finite-range interactions in $\Omega \cap \varepsilon\mathbb{Z}^n$ (following the general convergence result for unconstrained functionals by Alicandro and Cicalese [1]). The main result of the paper is Theorem 3.1, where we show that, given an infinitesimal sequence (ε_j) and a family of functionals (E_{ε_j}) defined as outlined above, (E_{ε_j}) admits (a subsequence converging to) a Γ-limit of the form

$$F(u) = \int_\Omega \Big(f_0(Du) + \Phi(u)\Big) dx,$$

where f_0 is given by the unconstrained *homogenization formula* proved in [1], and Φ is described by suitable asymptotic formulas that generalize the capacitary argument outlined above. Again, the form of Φ differs if $p = n$ or $p < n$. Note that in general the limit function Φ may depend on the sequence ε_j, as a consequence of the possible lack of homogeneity of degree p of the energy densities f^ξ. This non-uniqueness of the limit for the non-homogeneous case has already been observed for the continuous case (see e.g. [3]). The main technical point is the adaptation of the separation-of-scales arguments to the general long-range case. While for nearest neighbors the approach of Ansini and Braides can be easily repeated, upon adapting it to the geometry of the lattice (e.g., considering squares in the place of balls, etc.), for long-range interactions the discrete functionals are non-local and some extra care must be taken to make that procedure work.

This paper is organized as follows. In Sections 2 and 3 we introduce some notation and state the main result of the paper, Theorem 3.1. In Section 4 we point out some analogies and differences between the problem we are dealing with and the corresponding continuous case, by looking at the asymptotic behavior of a family of relevant minimum problems. In Section 5 we study two families of auxiliary functions; by determining their properties we highlight the differences between the critical case ($p = n$) and the subcritical one ($p < n$). In Section 6 we prove two technical lemmas. In Sections 7 and 8 we prove the Γ-liminf inequality and the Γ-limsup inequality. Finally, Section 9 is devoted to the description of two special cases, which show some interesting features despite requiring restrictive assumptions.

2. **Notation.** Let $m, n \in \mathbb{N}$ with $n \geq 2$ and $m \geq 1$. For any measurable $B \subset \mathbb{R}^n$ we denote by $|B|$ the n-dimensional Lebesgue measure of B. Let $\{e_1, \dots, e_n\}$ be the set of unit vectors along the coordinate directions. For fixed $\varepsilon > 0$ we denote by B_ε the lattice $B_\varepsilon := \varepsilon\mathbb{Z}^n \cap B$. We denote by $\mathcal{A}_\varepsilon(B)$ the set of functions

$$\mathcal{A}_\varepsilon(B) = \{u : B_\varepsilon \to \mathbb{R}\}.$$

Let I be the set of vectors

$$I = \{\xi \in \mathbb{Z}^n : -\xi <^l \xi\},$$

where $<^l$ denotes the lexicographical order: given two vectors $\xi = (\xi_1, \dots, \xi_n)$ and $\zeta = (\zeta_1, \dots, \zeta_n)$, we say that $\xi <^l \zeta$ if and only if there exists $m \in \{1, \dots, n\}$ such that $\xi_i = \zeta_i$ for all $i < m$ and $\xi_m < \zeta_m$. We introduce this notion since we decided not to count the interactions twice. Equivalently, we could have chosen to pick both ξ and $-\xi$ and add some symmetry requirement on the interaction densities. For any vector $\xi \in I$ and $B \subseteq \mathbb{R}^n$, we define

$$R^\xi_\varepsilon(B) = \{a \in B_\varepsilon : \; a + \varepsilon\xi \in B_\varepsilon\}.$$

Given a function $v \in \mathcal{A}_\varepsilon(B)$, we indicate by $D^\xi_\varepsilon v$ its difference quotient along ξ; i.e.,

$$D^\xi_\varepsilon v(a) = \frac{v(a) - v(a + \varepsilon\xi)}{\varepsilon|\xi|} \qquad \text{for } a \in R^\xi_\varepsilon(B).$$

Having fixed a constant $M > 0$, we denote by I_M the subset of I given by

$$I_M = \{\xi \in \mathbb{Z}^n : |\xi| \leq M \text{ and } -\xi <^l \xi\}.$$

Sometimes it will be convenient to use a specific notation for the set of all *nearest neighbors*, defined as

$$M_\varepsilon(B) = \{\{a, b\} \; : \; a, b \in B_\varepsilon \text{ and } |a - b| = \varepsilon\}.$$

Since nearest neighbors are defined as sets containing two points, and not as pairs in $B_\varepsilon \times B_\varepsilon$, we will count each interaction along the coordinate directions only once.

Let Ω be a bounded open subset of $\mathbb{R}^n$ with $|\partial\Omega| = 0$. For fixed $\varepsilon > 0$ we consider the lattice $\varepsilon\mathbb{Z}^n \cap \Omega =: \Omega_\varepsilon$; we will often write Ω_j in place of Ω_{ε_j}. A function $u \in \mathcal{A}_\varepsilon(\Omega)$ is identified with the piecewise-constant measurable function given by $u(x) = u(z^\varepsilon_x)$, where z^ε_x is the closest point to x in $\varepsilon\mathbb{Z}^n$ (which is uniquely defined up to a set of zero measure). In this definition, we set $u(z) = 0$ if $z \in \varepsilon\mathbb{Z}^n \setminus \Omega$. $\mathcal{A}_\varepsilon(\Omega)$ is then regarded as a subset of $L^1(\Omega)$. We will often use the notations

$$\mathcal{A}_\varepsilon(\Omega) = \{u : \Omega_\varepsilon \to \mathbb{R}\} \quad \text{and} \quad R^\xi_\varepsilon(\Omega) = \{a \in \Omega_\varepsilon : \; a + \varepsilon\xi \in \Omega_\varepsilon\}, \text{ with } \xi \in I.$$

Given $l > 0$, let $[l]$ be its integer part. For all $l > 0$ and $x \in \mathbb{R}^n$ we denote by $Q(l,x)$ the closed hypercube $x+[-l,l]^n$. In particular $Q_\varepsilon(l,x) = \varepsilon\mathbb{Z}^n \cap (x+[-l,l]^n)$. Moreover, for fixed $h \geq l > 0$ and $x \in \mathbb{R}^n$, we define $\mathcal{S}_\varepsilon(l,h;x) = \Omega_\varepsilon \cap (x+([-h,h]^n \setminus (-l,l)^n))$. If $l = h$, we write $\mathcal{S}_\varepsilon(l;x) = \mathcal{S}_\varepsilon(l,h;x) = \Omega_\varepsilon \cap \partial(x+[-l,l]^n)$. If $x = 0$ we write $Q_\varepsilon(l)$, $\mathcal{S}_\varepsilon(l,h)$, $\mathcal{S}_\varepsilon(l)$ instead of $Q_\varepsilon(l,0)$, $\mathcal{S}_\varepsilon(l,h;0)$, $\mathcal{S}_\varepsilon(l;0)$ respectively.

Given a set of points $A \subseteq \Omega_\varepsilon$, we denote by $\mathbf{A}$ the union of all the ε-cells centered in elements of A:

$$\mathbf{A} = \cup_{a\in A}\mathcal{C}(a), \quad \text{where } \mathcal{C}(a) = a + [-\varepsilon/2, \varepsilon/2]^n.$$

3. **Main result.** In this section we state the main result of the paper.

Theorem 3.1. *Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geq 2$. Let $m \in \mathbb{N}$, $m \geq 1$ and $1 < p \leq n$. Let I be the set of vectors $I = \{\xi \in \mathbb{Z}^n : -\xi <^l \xi\}$. For all $\xi \in I$, we consider a function $f^\xi : \mathbb{R}^m \to [0,+\infty)$ such that $f^\xi(0) = 0$. We assume that the functions f^ξ satisfy the following conditions:*

1. *there exists a constant $c_1 > 0$ such that for all $i \in \{1,\dots,n\}$*

$$\begin{array}{ll} f^{e_i}(z) \geq c_1|z|^n \text{ for all } z \in \mathbb{R}^m & \text{if } p = n \\ f^{e_i}(z) \geq c_1(|z|^p - 1) \text{ for all } z \in \mathbb{R}^m & \text{if } p < n, \end{array} \tag{4}$$

2. *there exists a family of constants $c_2^\xi > 0$ such that for all $\xi \in I$*

$$\begin{array}{ll} f^{\xi}(z) \leq c_2^\xi|z|^n \text{ for all } z \in \mathbb{R}^m & \text{if } p = n \\ f^{\xi}(z) \leq c_2^\xi(|z|^p + 1) \text{ for all } z \in \mathbb{R}^m & \text{if } p < n \end{array} \tag{5}$$

   and

$$\sum_{\xi\in I} c_2^\xi < +\infty,$$

3. *there exists a constant $c_3 > 0$ such that for all $\xi \in I$*

$$\begin{array}{ll} |f^\xi(z) - f^\xi(w)| \leq c_3|z-w|(|z|^{n-1} + |w|^{n-1}) \text{ for all } z,w \in \mathbb{R}^m & \text{if } p = n \\ |f^\xi(z) - f^\xi(w)| \leq c_3|z-w|(1 + |z|^{p-1} + |w|^{p-1}) \text{ for all } z,w \in \mathbb{R}^m & \text{if } p < n. \end{array} \tag{6}$$

Let (ε_j) be a sequence of positive numbers converging to zero. Let (δ_j) be a positive infinitesimal sequence such that $\delta_j/\varepsilon_j \in \mathbb{N}$ and $\lim_j \delta_j/\varepsilon_j = +\infty$. We assume that (ε_j) and (δ_j) satisfy

$$\varepsilon_j = \begin{cases} e^{-r(1+o(1))\delta_j^{n/(1-n)}} \text{ as } j \to +\infty & \text{if } p = n \\ r^{(1-n)/(n-p)}\delta_j^{n/(n-p)}(1+o(1)) \text{ as } j \to +\infty & \text{if } p < n, \end{cases} \tag{7}$$

where r is a positive constant.

- *In the case $p = n$, for all $j \in \mathbb{N}$, $\alpha > 0$ and $M > 0$ we define the function $g^\alpha_{j,M} : \mathbb{R}^m \to [0,+\infty)$ as*

$$\begin{aligned} g^\alpha_{j,M}(z) \quad = \quad \inf\Big\{ & \sum_{\xi\in I_M} \sum_{A\in R_1^\xi(Q(\alpha S_j))} f^\xi\big(\varepsilon_j^{-1} D_1^\xi v(A)\big)\varepsilon_j^n : \\ & v(0) = 0,\ v = z \text{ on } \mathcal{S}_1([\alpha S_j - M], [\alpha S_j])\Big\}, \end{aligned} \tag{8}$$

  where $S_j = \varepsilon_j^{-1}|\log\varepsilon_j|^{(1-n)/n}$. Then, upon possibly passing to subsequences, there exists a function $\varphi : \mathbb{R}^m \to [0,+\infty)$ such that

$$\varphi(z) = \lim_{M\to+\infty}\lim_{\alpha\to0^+}\lim_{j\to+\infty} |\log\varepsilon_j|^{n-1} g^\alpha_{j,M}(z) \text{ for all } z \in \mathbb{R}^m.$$

- *In the case $p < n$, for all $j \in \mathbb{N}$, $N > 0$ and $M > 0$ we define the function $\phi^N_{j,M} : \mathbb{R}^m \to [0, +\infty)$ as*

$$\begin{aligned} \phi^N_{j,M}(z) \;\; = \;\; \inf\Big\{ \sum_{\xi \in I_M} \sum_{A \in R^\xi_1(Q(N))} f^\xi(\varepsilon_j^{-1} D^\xi_1 v(A))\varepsilon_j^p : \\ v(0) = 0,\; v = z \text{ on } \mathcal{S}_1([N-M],[N]) \Big\}. \end{aligned} \tag{9}$$

Then, upon possibly passing to subsequences, there exists a function $\phi : \mathbb{R}^m \to [0, +\infty)$ such that

$$\phi(z) = \lim_{M\to+\infty} \lim_{N\to+\infty} \lim_{j\to+\infty} \phi^N_{j,M}(z) \text{ for all } z \in \mathbb{R}^m.$$

- *Moreover, for all $j \in \mathbb{N}$ we consider the functional $F_{\varepsilon_j} : \mathcal{A}_{\varepsilon_j}(\Omega) \to [0, +\infty]$ defined by*

$$F_{\varepsilon_j}(u) = \begin{cases} \displaystyle\sum_{\xi \in I} \sum_{a \in R^\xi_{\varepsilon_j}(\Omega)} f^\xi(D^\xi_{\varepsilon_j} u(a))\varepsilon_j^n & \text{if } u = 0 \text{ on } \Omega_{\delta_j} \\ +\infty & \text{otherwise.} \end{cases}$$

Upon extracting a subsequence such that the function

$$\Phi : \mathbb{R}^m \to [0, +\infty), \qquad \Phi(z) = \begin{cases} \varphi(z) & \text{if } p = n \\ \phi(z) & \text{if } p < n \end{cases}$$

is well defined, the family (F_{ε_j}) Γ-converges in the $L^1(\Omega; \mathbb{R}^m)$-topology to the functional $F : L^1(\Omega; \mathbb{R}^m) \to [0, +\infty]$ given by

$$F(u) = \begin{cases} \displaystyle\int_\Omega f_0(Du)\,dx + r^{1-n} \int_\Omega \Phi(u)\,dx & \text{if } u \in W^{1,p}(\Omega; \mathbb{R}^m) \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$f_0(A) = \lim_{h\to+\infty} \frac{1}{h^n} \min\Big\{ \sum_{\xi \in I} \sum_{a \in R^\xi_1(Q(h))} f(D^\xi_1 u(a)),\; u = Ax \text{ on } \mathcal{S}_1(h) \Big\}$$

for all $A \in \mathbb{M}^{m\times n}$.

In the following corollary we formulate Theorem 3.1 in the simplest case (scalar functions, nearest neighbors interactions and Dirichlet integrals) for the sake of clarity.

Corollary 1. *Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geq 2$, and p be such that $1 < p \leq n$. Let (ε_j) and (δ_j) be positive infinitesimal sequences such that $\delta_j/\varepsilon_j \in \mathbb{N}$ and $\lim_j \delta_j/\varepsilon_j = +\infty$. We assume that (ε_j) and (δ_j) satisfy*

$$\varepsilon_j = \begin{cases} r^{(1-n)/(n-p)} \delta_j^{n/(n-p)}(1 + o(1)) \text{ as } j \to +\infty & \text{if } p < n \\ e^{-r(1+o(1))\delta_j^{n/(1-n)}} \quad \text{as } j \to +\infty & \text{if } p = n, \end{cases}$$

where r is a positive constant. For all $j \in \mathbb{N}$ we consider the functional $F_{\varepsilon_j} : \mathcal{A}_{\varepsilon_j}(\Omega) \to [0, +\infty]$ defined by

$$F_{\varepsilon_j}(u) = \begin{cases} \displaystyle\sum_{\{a,b\} \in M_{\varepsilon_j}(\Omega)} \varepsilon_j^{n-p} |u(a) - u(b)|^p & \text{if } u = 0 \text{ on } \Omega_{\delta_j} \\ +\infty & \text{otherwise.} \end{cases}$$

Then for all $1 < p \leq n$ *there exists a positive constant* C_p *given by*

$$C_p = \lim_{T\to+\infty} \left\{ \sum_{\{A,B\}\in M_1(Q(T))} |u(A)-u(B)|^p : \begin{array}{c} u \in \mathcal{A}_1(Q(T)), \\ u(0)=0,\ u=1 \text{ on } \mathcal{S}_1([T]) \end{array} \right\}$$

if $p < n$ *and*

$$C_n = \lim_{T\to+\infty} (\log T)^{n-1} \inf \Big\{ \sum_{\{A,B\}\in M_1(Q(T))} |v(A)-v(B)|^n : v \in \mathcal{A}_1(Q(T)),\ v(0)=0,\ v=1 \text{ on } \mathcal{S}_1([T]) \Big\}$$

if $p = n$*. In particular, for* $p = n$ *the constant* C_n *equals* ω_{n-1}*; i.e., the surface area of the unit-sphere* $\mathcal{S}^{n-1}$*. Moreover, for* $1 < p \leq n$ *the family* (F_{ε_j}) Γ*-converges in the* $L^1(\Omega)$*-topology to the functional* $F : L^1(\Omega) \to [0,+\infty]$ *given by*

$$F(u) = \begin{cases} \displaystyle\int_\Omega |Du|^p\,dx + r^{1-n} C_p \int_\Omega |u|^p\,dx & \text{if } u \in W^{1,p}(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

3.1. **More notation and preliminaries.** Assume that all the conditions of Theorem 3.1 are satisfied. For all $j \in \mathbb{N}$ we set

$$\mathcal{F}_{\varepsilon_j}(u) = \sum_{\xi\in I} \sum_{a\in R^\xi_{\varepsilon_j}(\Omega)} f^\xi(D^\xi_{\varepsilon_j} u(a))\varepsilon_j^n.$$

Note that $\mathcal{F}_{\varepsilon_j}$ differs from F_{ε_j} since in the latter we add the constraint $u = 0$ on Ω_{δ_j}. Namely,

$$F_{\varepsilon_j}(u) = \begin{cases} \mathcal{F}_{\varepsilon_j}(u) & \text{if } u = 0 \text{ on } \Omega_{\delta_j} \\ +\infty & \text{otherwise.} \end{cases}$$

For all $D \subseteq \Omega$ we denote by $F_{\varepsilon_j}(u; D)$ and $\mathcal{F}_{\varepsilon_j}(u; D)$ the localized functionals

$$\mathcal{F}_{\varepsilon_j}(u; D) = \sum_{\xi\in I} \sum_{a\in R^\xi_{\varepsilon_j}(D)} f^\xi(D^\xi_{\varepsilon_j} u(a))\varepsilon_j^n$$

and

$$F_{\varepsilon_j}(u; D) = \begin{cases} \mathcal{F}_{\varepsilon_j}(u; D) & \text{if } u = 0 \text{ on } \Omega_{\delta_j} \cap D \\ +\infty & \text{otherwise.} \end{cases}$$

Throughout the paper we will use a homogenization result proved by Alicandro and Cicalese in [1, Theorem 4.1]. We recall it in the form we need for our purposes.

Proposition 1. *Let* f^ξ*,* $\xi \in I$*, satisfy the assumptions of Theorem 3.1. For all* $\varepsilon > 0$ *we define* $\mathcal{F}_\varepsilon : \mathcal{A}_\varepsilon(\Omega) \to [0,+\infty)$ *as*

$$\mathcal{F}_\varepsilon(u) = \sum_{\xi\in I} \sum_{a\in R^\xi_\varepsilon(\Omega)} f^\xi(D^\xi_\varepsilon u(a))\varepsilon^n.$$

Then, $(\mathcal{F}_\varepsilon)$ Γ*-converges with respect to the* $L^p(\Omega;\mathbb{R}^m)$*-topology to the functional* $\mathcal{F}_0 : L^p(\Omega;\mathbb{R}^m) \to [0,+\infty]$ *defined as*

$$\mathcal{F}_0(u) = \begin{cases} \displaystyle\int_\Omega f_0(Du)dx & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^m) \\ +\infty & \text{otherwise,} \end{cases}$$

where $f_0 : \mathcal{M}^{m\times n} \to [0, +\infty)$ *is given by the homogenization formula*

$$f_0(A) = \lim_{h\to+\infty} \frac{1}{h^n} \min\Big\{ \sum_{\xi\in I} \sum_{a\in R_1^\xi(Q(h))} f(D_1^\xi u(a)),\ u = Ax\ \textit{on}\ \mathcal{S}_1(h)\Big\}. \tag{10}$$

Remark 1. (Finite range interactions) In order not to overburden the notation, in what follows we will focus on long but finite-range interactions: we will limit out attention to a set of functions f^ξ with $\xi \in I_M = \{\xi \in \mathbb{Z}^n : |\xi| \le M \text{ and } -\xi <^l \xi\}$, for some fixed $M \ge 1$. This is not restrictive thanks to the general convergence result for unconstrained functionals by Alicandro and Cicalese, recalled in Proposition 1. When no confusion can arise, we will simply write I, g_j^α and ϕ_j^N instead of $I_M, g_{j,M}^\alpha$ and $\phi_{j,M}^N$. Note that, under this simplifying assumption, condition (5) can be rewritten as follows: there exists a constant $c_2 > 0$ such that for all $\xi \in I$

$$\begin{array}{ll} f^\xi(z) \le c_2|z|^n \text{ for all } z \in \mathbb{R}^m & \text{if } p = n \\ f^\xi(z) \le c_2(|z|^p + 1) \text{ for all } z \in \mathbb{R}^m & \text{if } p < n. \end{array} \tag{11}$$

4. **Comparison with the continuous case.** In this paragraph we point out the basic difference between the critical case and the subcritical one, by analyzing the asymptotic behavior of the family of minimum problems $\{m_T^d :\ T \in \mathbb{N}\}$, defined as

$$m_T^d = \inf\Big\{ \sum_{\{a,b\}\in M_1(Q(T))} |u(a) - u(b)|^p :\ \begin{array}{c} u \in \mathcal{A}_1(Q(T)), \\ u(0) = 0,\ u = 1 \text{ on } \mathcal{S}_1(T) \end{array} \Big\}. \tag{12}$$

Note that m_T^d is the simplest version of the minimum problems which appear in (8) and (9): we deal with nearest-neighbors interactions only, the test functions are scalar ($m = 1$) and $f^\xi(z) = |z|^p$ for all ξ. These simplifying assumptions correspond to the ones of Corollary 1. In the proof of Theorem 3.1 we will use a separation-of-scales procedure: a decoupling lemma (Lemma 6.1) will allow to analyze the single effect of each pinning site independently. In the simplest case, the energy "concentrating close to the pinning sites" is exactly the one we minimize in (12).

In what follows, we will determine the asymptotic behavior of m_T^d in the critical-exponent case and in the subcritical-exponent one (step **1** and **2** respectively).

1. Critical case $p = n$. We will show that in the case of the critical exponent the sequence m_T^d has the same asymptotic behavior as its continuous analogue. In the continuous setting, we consider the minimum $m_{t,T}^c$

$$m_{t,T}^c = \min\Big\{ \int_{Q(T)} \|Du\|_n^n :\ u - 1 \in W_0^{1,n}(Q_T),\ u = 0 \text{ on } Q(t)\Big\}, \tag{13}$$

where $0 < t < T$ and $\|Du\|_n = (\sum_{i=1}^n |\partial u/\partial x_i|^n)^{1/n}$. This case has been studied in the framework of Γ-convergence in [18]. In particular, we know that the sequence $(m_{t,T}^c)$ has a logarithmic behavior as T goes to $+\infty$:

$$\lim_{T\to+\infty} (\log T)^{n-1} m_{t,T}^c = \omega_{n-1}. \tag{14}$$

We recall that this convergence can be proved by an argument based on a *telescopic construction*, as in [18, Section 5]. If in particular $p = n = 2$, then the $\|Du\|_2$ norm is the same as the Euclidean norm $|Du|$ and the constant l_2 equals 2π. We notice that the minimum in (13) is scale-invariant: if we rescale our sets by a constant $\alpha > 0$, we get $m_{\alpha t,\alpha T}^c = m_{t,T}^c$. In this paragraph we will prove the following lemma:

Lemma 4.1. *If $p = n$ the family of discrete infima (m^d_T) defined in (12) satisfies*

$$\lim_{T\to+\infty}(\log T)^{n-1}m^d_T = \lim_{T\to+\infty}(\log T)^{n-1}m^c_{1,T} = \omega_{n-1}, \tag{15}$$

where $m^c_{1,T}$ is defined as in (13).

Proof. For $t \geq 1$ we introduce the discrete infima

$$m^d_{t,T} = \inf\left\{ \sum_{\{A,B\}\in M_1(Q(T))} |u(A)-u(B)|^n : \begin{array}{c} u \in \mathcal{A}_1(Q(T)), \\ u = 0 \text{ on } Q_1(t),\ u = 1 \text{ on } \mathcal{S}_1(T) \end{array} \right\}. \tag{16}$$

Note that $m^d_{t,T}$ differs from m^d_T since in the former the test functions vanish on $Q_1(t)$, while in the latter they satisfy the (less restrictive) condition $u(0) = 0$. By a two-step argument we will prove that

$$\lim_{T\to+\infty}(\log T)^{n-1}m^d_{1,T} = \lim_{T\to+\infty}(\log T)^{n-1}m^c_{1,T} = \omega_{n-1}$$

and then we will show that

$$\lim_{T\to+\infty}(\log T)^{n-1}m^d_{1,T} = \lim_{T\to+\infty}(\log T)^{n-1}m^d_T,$$

thus obtaining (15).

1.1. In this step we show that

$$\lim_{T\to+\infty}(\log T)^{n-1}m^d_{1,T} = \lim_{T\to+\infty}(\log T)^{n-1}m^c_{1,T}.$$

First of all, we can identify each test function $u \in \mathcal{A}_1(Q(T))$ in the definition of $m^d_{1,T}$ with a function $\tilde{u}$ obtained as the piecewise affine interpolation of u on the lattice $Q_1(T)$. The function $\tilde{u}$ can be defined using the construction developed by Alicandro and Cicalese in [2, Section 4.1]; following this procedure we get an interpolating function $\tilde{u}$ which satisfies

$$\int_{Q(T)} \|D\tilde{u}\|^n_n\,dx = \sum_{\{A,B\}\in M_1(Q(T))} |u(A)-u(B)|^n.$$

Since $u = 0$ on $Q_1(1)$ and $u = 1$ on $\mathcal{S}_1(T)$, the piecewise affine interpolation $\tilde{u}$ vanishes on the cube $Q(1)$ and belongs to the space $1 + W^{1,n}_0(Q(T))$. Then $\tilde{u}$ is a test function for $m^c_{1,T}$. There follows that

$$m^c_{1,T} \leq m^d_{1,T}.$$

We want to show that the converse inequality holds, up to an infinitesimal error. Let $T \in \mathbb{N}$. Due to scale-invariance, we have $m^c_{1,T-1} = m^c_{2,2T-2}$. Let $v \in \operatorname{argmin}\{m^c_{2,2T-2}\}$; i.e., $v \in 1 + W^{1,n}_0(Q(2T-2))$, $v = 0$ on $Q(2)$ and

$$E(v) := \int_{Q(2T-2)} \|Dv\|^n_n\,dx = m^c_{2,2T-2}.$$

By (14) we deduce that $m^c_{2,2T-2} = m^c_{2,2T} + o((\log T)^{1-n})$ as $T \to +\infty$, hence $E(v) = m^c_{2,2T} + o((\log T)^{1-n})$. For all fixed $x \in [0,1)^n$ we denote by L^x the lattice $L^x = (x+\mathbb{Z}^n)\cap Q(2T+2)$ and by v^x the discretization of v over L^x, i.e. $v^x(y) = v(y)$ for $y \in L^x$. By construction we have $v^x = 0$ on $Q_1(1;x)$ and $v^x = 1$ on $\mathcal{S}_1(2T;x)$. Moreover, we indicate by $E^x(v)$ the sum of the one-dimensional integrals of the restriction of v over the set of lines parallel to the coordinate axes and passing through the points of the lattice L^x:

$$E^x(v) = \sum_{i=1}^{n} \sum_{z \in P_i} \sum_{j=-2T}^{2T-1} \int_{\sigma(i,z,j)} \Big| \frac{\partial v}{\partial y_i} \Big|^n,$$

where $P_i = (\{y \in \mathbb{Z}^n : \ y_i = 0\} + x) \cap Q(2T+2)$ and $\sigma(i,z,j)$ is the unit segment $\sigma(i,z,j) = \{(j+s)e_i + z, \ s \in [0,1]\}$. Now, by Jensen's inequality and the definition of v^x we have

$$E^x(v) \geq \sum_{i=1}^{n} \sum_{z \in P_i} \sum_{j=-2T}^{2T-1} \Big(\int_{\sigma(i,z,j)} \Big| \frac{\partial v}{\partial y_i} \Big| \Big)^n \geq \sum_{\{a,b\} \in M_1(Q(2T))} |v^x(a+x) - v^x(b+x)|^n.$$

By Fubini's Theorem we have $E(v) = \int_{[0,1)^n} E^x(v)dx$. Then, there exists $\overline{x} \in [0,1)^n$ such that

$$E(v) \geq E^{\overline{x}}(v) \geq \sum_{\{a,b\} \in M_1(Q(2T))} |v^{\overline{x}}(a+\overline{x}) - v^{\overline{x}}(b+\overline{x})|^n \geq m^d_{1,2T}.$$

To sum up, we got

$$m^c_{2,2T-2} = m^c_{2,2T} + o((\log T)^{1-n}) = m^c_{1,T} + o((\log T)^{1-n}) \geq m^d_{1,2T} + o((\log T)^{1-n}). \tag{17}$$

Since the limit in (14) is independent of t, we have $m^c_{1,2T} = m^c_{1,T} + o((\log T)^{1-n})$. Plugging this equation into (17), we conclude that

$$m^d_{1,2T} \leq m^c_{1,2T} + o((\log T)^{1-n}).$$

we finally obtain

$$\lim_{T \to +\infty} (\log T)^{n-1} m^d_{1,T} = \lim_{T \to +\infty} (\log T)^{n-1} m^c_{1,T} = \omega_{n-1},$$

as desired.

1.2. In this step we complete the proof of the lemma by showing that

$$\lim_{T \to +\infty} (\log T)^{n-1} m^d_{1,T} = \lim_{T \to +\infty} (\log T)^{n-1} m^d_T. \tag{18}$$

We will first consider an intermediate step, which will be generalized to derive (18). We introduce an additional discrete minimum problem:

$$m^{d,B(1)}_T = \inf \Big\{ \sum_{\{A,B\} \in M_1(Q(T))} |u(A) - u(B)|^n : \begin{array}{c} u \in \mathcal{A}_1(Q(T)), \\ u = 0 \text{ on } B_1(1), \ u = 1 \text{ on } \mathcal{S}_1(T) \end{array} \Big\},$$

where $B_1(1) = \mathbb{Z}^n \cap \{x \in \mathbb{R}^n : |x| \leq 1\} = \{\pm e_h, h = 1, \dots, n\}$. Let $T \in \mathbb{N}$ be fixed and let $u \in \mathcal{A}_1(Q(T))$ be a test function for $m^{d,B(1)}_T$; i.e., $u = 0$ on $B_1(1)$ and $u = 1$ on $\mathcal{S}_1(T)$. Since in particular $u(0) = 0$, then u is also a test function for m^d_T, hence

$$m^d_T \leq m^{d,B(1)}_T.$$

Analogously, we get $m^{d,B(1)}_T \leq m^d_{1,T}$. Now, since

$$0 \leq \lim_{T \to +\infty} m^d_T \leq \lim_{T \to +\infty} m^d_{1,T} = 0,$$

for fixed $\nu > 0$ there exists T_0 such that for $T > T_0$ we have $m^d_T < \nu$. For $T > T_0$ and a fixed $\eta > 0$, let $\tilde{u}_T \in \mathcal{A}_1(Q(T))$ be such that $u(0) = 0$, $u = 1$ on $\mathcal{S}_1(T)$ and

$$\sum_{\{A,B\} \in M_1(Q(T))} |\tilde{u}_T(A) - \tilde{u}_T(B)|^n < m^d_T + \eta < \nu + \eta.$$

In particular, $|\tilde{u}_T(A) - \tilde{u}_T(B)|^n < \nu + \eta$ for all $\{A, B\} \in M_1(Q(1))$. Having fixed $A = 0$, we get $|\tilde{u}_T(A) - \tilde{u}_T(B)|^n = |\tilde{u}_T(B)|^n < \eta + \nu$ for all $B \in \{\pm e_h, h = 1, \dots, n\} = B_1(1)$. This implies that $\tilde{u}_T$ is an appropriate test function for the minimum problem

$$\bar{m}_T^d = \inf\Big\{ \sum_{\{A,B\} \in M_1(Q(T))} |u(A) - u(B)|^n : \\ u \in \mathcal{A}_1(Q(T)),\ u \le (\eta + \nu)^{1/n} \text{ on } B_1(1),\ u = 1 \text{ on } \mathcal{S}_1(T) \Big\},$$

hence

$$\bar{m}_T^d < m_T^d + \eta.$$

By a truncation and a scaling argument it is easy to see that

$$\begin{aligned} \bar{m}_T^d &= \inf\Big\{ \sum_{\{A,B\} \in M_1(Q(T))} |u(A) - u(B)|^n : \\ &\qquad u \in \mathcal{A}_1(Q(T)),\ u = (\eta + \nu)^{1/n} \text{ on } B_1(1),\ u = 1 \text{ on } \mathcal{S}_1(T) \Big\} \\ &= (1 - (\eta + \nu)^{1/n})^n \inf\Big\{ \sum_{\{A,B\} \in M_1(Q(T))} |u(A) - u(B)|^n : \\ &\qquad u \in \mathcal{A}_1(Q(T)),\ u = 0 \text{ on } B_1(1),\ u = 1 \text{ on } \mathcal{S}_1(T) \Big\} \\ &= (1 - (\eta + \nu)^{1/n})^n m_T^{d,B(1)}. \end{aligned}$$

To sum up, for all $\nu > 0$ there exists T_0 such that for $T > T_0$ and $\eta > 0$ we have

$$(1 - (\eta + \nu)^{1/n})^n m_T^{d,B(1)} < m_T^d + \eta.$$

By the arbitrariness of η we get

$$(1 - \nu^{1/n})^n m_T^{d,B(1)} < m_T^d.$$

Then for all $\nu > 0$ there exists T_0 such that for $T > T_0$

$$0 \le \frac{m_T^{d,B(1)}}{m_T^d} - 1 < \frac{1 - (1 - \nu^{1/n})^n}{(1 - \nu^{1/n})^n}.$$

At this point it is easy to repeat the argument we have seen so far (replacing $B_1(1)$ by $Q_1(1)$ and $\{0\}$ by $B_1(1)$) to prove that

$$\lim_{T \to +\infty} \frac{m_T^{d,B(1)}}{m_{1,T}^d} = 1.$$

Then we get (18). □

The following generalization is straightforward.

Lemma 4.2. *Let $n \ge 2$ and $m \ge 1$. Let $z \in \mathbb{R}^m$ be fixed. For all $T \in \mathbb{N}$ we define*

$$m_T^d(z) = \inf\Big\{ \sum_{\{a,b\} \in M_1(Q(T))} |u(a) - u(b)|^n : \begin{array}{c} u \in \mathcal{A}_1(Q(T); \mathbb{R}^m), \\ u(0) = 0,\ u = z \text{ on } \mathcal{S}_1(T) \end{array} \Big\}.$$

Then

$$\lim_{T \to +\infty} (\log T)^{n-1} m_T^d(z) = \omega_{n-1} |z|^n.$$

2. Subcritical case $p < n$. In the subcritical case, $p < n$, we do not have the same correspondence with the continuous setting. In this scenario, the infima m^d_T converge to a positive constant C_p which can be interpreted as the discrete p-capacity of a point in $\mathbb{Z}^n$: with an abuse of notation we write

$$C_p = \inf\Big\{ \sum_{\{a,b\}\in M_1(\mathbb{Z}^n)} |u(a)-u(b)|^p : \begin{array}{c} u\in\mathcal{A}_1(\mathbb{Z}^n), \\ u(0)=0,\ u=1 \text{ on } \mathcal{S}_1(+\infty) \end{array} \Big\}.$$

We will prove the following lemma

Lemma 4.3. *If $p < n$ there exists a positive constant C_p such that*

$$\lim_{T\to+\infty} m^d_T = C_p,$$

where m^d_T is defined as in (12).

Proof. By definition m^d_T is a decreasing sequence of positive numbers, hence it admits a limit $C_p \geq 0$. Let us show that $C_p > 0$. For $N \in \mathbb{N}$ sufficiently large, we consider a function $u \in \mathcal{A}_1(Q(N))$ such that $u(0) = 0$, $u = 1$ on $\mathcal{S}_1(N)$ and

$$\sum_{\{a,b\}\in M_1(Q(N))} |u(a)-u(b)|^p < C_p + \frac{1}{N}.$$

Now, two events can occur: either $u \neq 0$ in at least one point of $Q_1(1)$, or $u = 0$ on all the points of $Q_1(1)$. In the first case, the energy of u over $Q_1(N)$ must be greater than a positive constant α, given by the non-zero interaction we certainly have in $Q_1(1)$, and then $C_p + 1/N > \alpha$. By letting $N \to +\infty$ we get $C_p \geq \alpha > 0$. In the second case, since $u = 0$ on $Q_1(1)$, we can identify it with a piecewise affine function $\tilde{u}$ such that $\tilde{u} = 0$ on $Q(1)$, $\tilde{u} = 1$ on $\partial Q(N)$ and

$$\int_{Q(N)} \|D\tilde{u}\|_p^p = \sum_{\{a,b\}\in M_1(Q(N))} |u(a)-u(b)|^p < C_p + \frac{1}{N}.$$

Now,

$$\int_{Q(N)} \|D\tilde{u}\|_p^p \geq c \inf\Big\{ \int_{Q(N)} |Dv|^p\,dx :\ v = 0 \text{ on } Q(1)\Big\} \geq c\ \mathrm{Cap}_p(Q(1);\mathbb{R}^n),$$

where $\mathrm{Cap}_p(Q(1);\mathbb{R}^n) > 0$ is the p-capacity of the cube $Q(1)$ in $\mathbb{R}^n$. By letting $N \to +\infty$, we conclude that C_p is strictly positive. □

Lemma 4.3 can be easily generalized to the case of vector valued functions.

Lemma 4.4. *Let $n \geq 2$, $m \geq 1$ and $1 < p < n$. Let $z \in \mathbb{R}^m$ be fixed. For all $T \in \mathbb{N}$ we define*

$$m^d_T(z) = \inf\Big\{ \sum_{\{a,b\}\in M_1(Q(T))} |u(a)-u(b)|^p : \begin{array}{c} u\in\mathcal{A}_1(Q(T);\mathbb{R}^m), \\ u(0)=0,\ u=z \text{ on } \mathcal{S}_1(T) \end{array} \Big\}.$$

Then

$$\lim_{T\to+\infty} m^d_T(z) = C_p|z|^p.$$

5. **Building blocks of the Γ-limit.** In this section we list some properties of the auxiliary functions $(\varphi^\alpha_{j,M})$ and $(\phi^N_{j,M})$ we introduced in the statement of Theorem 3.1. We show that these families converge to some functions φ and ϕ respectively, upon possibly passing to subsequences. The limit densities φ and ϕ will account for the contribution of the pinning sites in the Γ-limit.

5.1. **Critical case.** In this paragraph we list some properties of the auxiliary functions g_j^α we introduced in (8) for the critical case (the notation is simplified according to Remark 1). Let all the assumptions of Theorem 3.1 be satisfied. It is convenient to set $T_j = \varepsilon_j^{-1}$ and $S_j = T_j(\log T_j)^{(1-n)/n}$; by construction T_j, S_j tend to $+\infty$ as $j \to +\infty$. For fixed $\alpha > 0$, $j \in \mathbb{N}$ we define $g_j^\alpha : \mathbb{R}^m \to [0, +\infty)$ as

$$g_j^\alpha(z) = \inf\Big\{ \sum_{\xi\in I} \sum_{A\in R_1^\xi(Q(\alpha S_j))} T_j^{-n} f^\xi\big(T_j D_1^\xi v(A)\big) : \begin{array}{l} v(0)=0 \\ v = z \text{ on } \mathcal{S}_1([\alpha S_j - M], [\alpha S_j]) \end{array} \Big\}.$$

Now, $\varphi_j^\alpha : \mathbb{R}^m \to [0, +\infty)$ is given by:

$$\varphi_j^\alpha(z) = (\log T_j)^{n-1} g_j^\alpha(z).$$

We will apply Ascoli-Arzel's Theorem to the family (φ_j^α) in order to prove the following result:

Proposition 2. *For all $\alpha > 0$ there exists a function $\varphi^\alpha : \mathbb{R}^m \to [0, +\infty)$ such that φ_j^α tends to φ^α as $j \to +\infty$ upon passing to subsequences, uniformly on the compact sets of $\mathbb{R}^m$.*

1 Equi-boundedness. Taking into account (4), (11) and Lemma 4.2, we can show that there exist two constants $C_1, C_2 > 0$ such that for all $j \in \mathbb{N}$ and $\alpha > 0$ the functions φ_j^α satisfy a growth condition of the form

$$C_1|z|^n \le \varphi_j^\alpha(z) \le C_2|z|^n \quad \text{for all } z \in \mathbb{R}^m. \tag{19}$$

2 Equi-Lipschitz continuity. Firstly we fix a compact set $K \subset \mathbb{R}^m \setminus \{0\}$ and we denote by L a positive constant such that $K \subseteq (-L, L)^m$.

(i) Let $z, z' \in K$ be such that $z' = kz$ for some $k \neq 0$. Having fixed $\eta > 0$, we consider a function $v \in \mathcal{A}_1(Q(\alpha S_j); \mathbb{R}^m)$ such that $v(0) = 0$, $v = z$ on $\mathcal{S}_1([\alpha S_j - M], [\alpha S_j])$ and

$$(\log T_j)^{n-1} \sum_{\xi\in I} \sum_{A\in R_1^\xi(Q(\alpha S_j))} T_j^{-n} f^\xi(T_j D_1^\xi v(A)) < \varphi_j^\alpha(z) + \eta|z|^n.$$

We define $w \in \mathcal{A}_1(Q(\alpha S_j); \mathbb{R}^m)$ as $w = kv$. By construction w is a test function for the infimum in $\phi_j^\alpha(z')$. By (6), (4) and (19) we can deduce that

$$\varphi_j^\alpha(z') \le \varphi_j^\alpha(z) + c|z' - z|(|z'|^{n-1} + |z|^{n-1}) + c\eta L^n,$$

where the constant c is independent of j and α. By the arbitrariness of η we get

$$\varphi_j^\alpha(z') \le \varphi_j^\alpha(z) + c|z' - z|(|z'|^{n-1} + |z|^{n-1}).$$

By symmetry reasons we can conclude that

$$|\varphi_j^\alpha(z') - \varphi_j^\alpha(z)| \le c|z - z'|(|z|^{n-1} + |z'|^{n-1}). \tag{20}$$

(ii) Let $z, z' \in K$ be such that $z' = \mathcal{R}z$ for some $\mathcal{R} \in SO(m)$. Arguing similarly to (i), we get

$$\varphi_j^\alpha(z') \le \varphi_j^\alpha(z) + c|z' - z|(|z'|^{n-1} + |z|^{n-1}).$$

By symmetry reasons, we conclude that

$$|\varphi_j^\alpha(z') - \varphi_j^\alpha(z)| \le c|z - z'|(|z|^{n-1} + |z'|^{n-1}), \tag{21}$$

for some positive constant c independent of j and α.

(iii) By combining (20) and (21) we deduce that there exists a constant c, independent of j and α, such that

$$|\varphi_j^\alpha(z') - \varphi_j^\alpha(z)| \leq c|z-z'|(|z|^{n-1} + |z'|^{n-1}) \quad \text{for all } z, z' \in K. \tag{22}$$

By (22) we can infere that the sequence (φ_j^α) satisfies an equi-Lipschitz condition on all compact subsets of $\mathbb{R}^m$.

In conclusion, by Ascoli-Arzel's Theorem, for all $\alpha > 0$ there exist a subsequence $\varphi_{j_k}^\alpha$ and a function $\varphi^\alpha : \mathbb{R}^m \to [0, +\infty)$ such that

$$\varphi^\alpha(z) = \lim_{k \to +\infty} \varphi_{j_k}^\alpha(z),$$

uniformly on the compact subsets of $\mathbb{R}^m$. ∎

Remark 2. By construction $\varphi^\alpha(0) = \varphi_j^\alpha(0) = 0$ for all α, j. Furthermore, by passing to the limit as $j \to +\infty$ in (22) we deduce that φ^α satisfies

$$|\varphi^\alpha(z) - \varphi^\alpha(z')| \leq c|z-z'|(|z|^{n-1} + |z'|^{n-1}) \text{ for all } z, z' \in \mathbb{R}^m,$$

for some constant $c > 0$.

5.2. **Subcritical case.** In this paragraph we analyze some properties of the functions ϕ_j^N we introduced in Theorem 3.1 for the subcritical case. For all $N > 0$, $j \in \mathbb{N}$ and $\xi \in I$ we define $h_j^\xi : \mathbb{R}^m \to [0, +\infty)$ as

$$h_j^\xi(z) = T_j^{-p} f^\xi(T_j z), \qquad \text{for all } z \in \mathbb{R}^m.$$

By assumptions (4)-(6) we deduce that h_j^ξ is locally Lipschitz-continuous and satisfies the following condition:

$$|h_j^\xi(z) - h_j^\xi(w)| \leq c(T_j^{-p+1} + |z|^{p-1} + |w|^{p-1})|z-w| \quad \text{for all } z, w \in \mathbb{R}^m, \tag{23}$$

where the positive constant c is independent of j. Therefore, for all $\xi \in I$ there exists a (locally Lipschitz-continuous) function $h^\xi : \mathbb{R}^m \to [0, +\infty)$ such that h_j^ξ converges pointwise to h^ξ, upon possibly passing to subsequences. We recall that for $N, j \in \mathbb{N}$ the function $\phi_j^N : \mathbb{R}^m \to [0, +\infty)$ is defined as

$$\phi_j^N(z) = \inf\Big\{ \sum_{\xi \in I} \sum_{A \in R_1^\xi(Q(N))} h_j^\xi(D_1^\xi v(A)) : \begin{array}{l} v(0) = 0 \\ v = z \text{ on } \mathcal{S}_1([N-M],[N]) \end{array} \Big\}.$$

Moreover, for all $N \in \mathbb{N}$ we can define $\phi_N : \mathbb{R}^m \to [0, +\infty)$ as

$$\phi^N(z) = \inf\Big\{ \sum_{\xi \in I} \sum_{A \in R_1^\xi(Q(N))} h^\xi(D_1^\xi v(A)) : \begin{array}{l} v(0) = 0 \\ v = z \text{ on } \mathcal{S}_1([N-M],[N]) \end{array} \Big\}.$$

Finally, we set

$$\phi(z) = \lim_{N \to +\infty} \phi^N(z).$$

Note that the limit over N in the definition of ϕ coincides with the infimum over $N \in \mathbb{N}$. Let us deduce some convergence properties of the functions above.

1. By the pointwise convergence of h_j^ξ to h^ξ as $j \to +\infty$, we deduce that, for fixed $N \in \mathbb{N}$, ϕ_j^N converges pointwise to ϕ^N up to subsequences.

2. For all $N \in \mathbb{N}$ and $\eta > 0$ there exists a positive constant $c_{N,\eta} = c(N)\eta^p$ such that

$$\begin{aligned}|\phi_j^N(z) - \phi_j^N(w)| \quad &\leq \quad c_{N,\eta}\delta_j^{n(p-1)/(n-p)}|z-w|(1+|w|^{p-1}+|z|^{p-1}) \\ &\qquad +c|z-w|(|w|^{p-1}+|z|^{p-1}) \end{aligned} \tag{24}$$

for $|z|, |w| > \eta$, for all $j \in \mathbb{N}$. Taking into account (23) and the growth conditions (4)-(11), we can prove this inequality by slightly modifying the argument we followed in the critical case. For fixed N, (24) corresponds to a Lipschitz condition on the compact subsets of $\mathbb{R}^m \setminus \{0\}$, uniformly on the index j.
3. For all $N \in \mathbb{N}$ there exists a positive constant c_N such that

$$\phi_j^N(z) \leq c_N T_j^{-p} + c|z|^p \tag{25}$$

for all $z \in \mathbb{R}^m$, $j \in \mathbb{N}$. This property follows from the growth condition (11) and a comparison with the case $f^\xi(z) = |z|^p$. Note that for fixed N (25) is an equi-boundedness condition on $(\phi_j^N)_j$.
4. By (24) and (25) we can apply Ascoli-Arzel's Theorem to the family of functions (ϕ_j^N), where N is fixed. We deduce that the convergence of ϕ_j^N to ϕ^N is uniform on the compact subsets of $\mathbb{R}^m \setminus \{0\}$, upon possibly passing to subsequences.
5. Letting $j \to +\infty$ in (25) we obtain $\phi^N(z) \leq c|z|^p$. By the growth condition from below (4), we deduce that ϕ^N satisfies the following inequality:

$$c_1 c|z|^p \leq \phi^N(z) \leq c_2 c|z|^p \text{ for all } z \in \mathbb{R}^m.$$

6. Arguing as in **1**, for fixed $\eta > 0$ we get a Lipschitz condition for ϕ^N in the form

$$|\phi^N(z) - \phi^N(w)| \leq c(\eta^p + |z-w|(|w|^{p-1} + |z|^{p-1})) \text{ for all } z, w \in \mathbb{R}^m.$$

7. By applying Ascoli-Arzel's Theorem to (ϕ^N), we deduce that the convergence of ϕ^N to ϕ is not only pointwise but also uniform on the compact subsets of $\mathbb{R}^m$, upon passing to subsequences.

6. **Two technical lemmas.** In this section we will prove two technical lemmas which will be used in the proof of Theorem 3.1. The first one is a "decoupling lemma", in the spirit of [3, Lemma 3.1], which relies on the standard De Giorgi's averaging method. Unlike the case of periodically perforated domains, we are dealing with non-local functionals, due to presence of long-range interactions. As a consequence, the "separation of scales" procedure requires some extra care. The second lemma describes how to recombine the decoupled energies to obtain the extra term of the Γ-limit. We will prove it in a general form, which comprises both the critical and the subcritical case.

Lemma 6.1. *Let (u_j) be a sequence such that $u_j \in \mathcal{A}_{\varepsilon_j}(\Omega;\mathbb{R}^m)$ and $u_j \to u$ in $L^1(\Omega;\mathbb{R}^m)$ for some $u \in W^{1,p}(\Omega;\mathbb{R}^m)$. We assume that*

$$\sup_j \sum_{\xi \in I} \sum_{a \in R^\xi_{\varepsilon_j}(\Omega)} \varepsilon_j^n f^\xi(D^\xi_{\varepsilon_j} u_j(a)) < +\infty. \tag{26}$$

Let (ρ_j) be a sequence of the form $\rho_j = \beta\delta_j$, with $\beta < 1/2$. We denote by Z_j the set of indices $Z_j = \{i \in \mathbb{Z}^n : \ dist(i\delta_j, \partial\Omega) > \delta_j\}$. Let $k \in \mathbb{N}$ be fixed. Then for all

$i \in Z_j$ there exists $k_i \in \{0, \dots, k-1\}$ such that, having set

$$C_j^i = Q_{\varepsilon_j}\Big(\Big[\frac{\rho_j}{\varepsilon_j}2^{-k_i}\Big]\varepsilon_j; i\delta_j\Big) \setminus Q_{\varepsilon_j}\Big(\Big[\frac{\rho_j}{\varepsilon_j}2^{-k_i-1}\Big]\varepsilon_j; i\delta_j\Big),$$
$$\rho_j^i = \Big[\frac{3}{4}\frac{\rho_j}{\varepsilon_j}2^{-k_i}\Big]\varepsilon_j,$$
$$u_j^i = \frac{1}{\sharp C_j^i}\sum_{a\in C_j^i} u_j(a),$$

there exists a sequence $w_j \in \mathcal{A}_{\varepsilon_j}(\Omega;\mathbb{R}^m)$ such that $w_j \to u$ in $L^1(\Omega;\mathbb{R}^m)$ and

$$w_j(a) = u_j^i \text{ for all } a \in \mathcal{S}_{\varepsilon_j}(\rho_j^i; i\delta_j), \tag{27}$$
$$w_j(a) = u_j(a) \text{ for all } a \in \Omega_j \setminus \bigcup_{i\in Z_j} C_j^i, \tag{28}$$
$$\Big|\sum_{\xi\in I}\sum_{a\in R^\xi_{\varepsilon_j}(\Omega)} \varepsilon_j^n\big(f^\xi\big(D^\xi_{\varepsilon_j}u_j(a)\big) - f^\xi\big(D^\xi_{\varepsilon_j}w_j(a)\big)\big)\Big| \le \frac{c}{k}. \tag{29}$$

Proof. We fix $i \in Z_j$ and $h \in \{0, \dots, k-1\}$. We set:

$$C_j^{h,i} = Q_{\varepsilon_j}\Big(\Big[\frac{\rho_j}{\varepsilon_j}2^{-h}\Big]\varepsilon_j; i\delta_j\Big) \setminus Q_{\varepsilon_j}\Big(\Big[\frac{\rho_j}{\varepsilon_j}2^{-h-1}\Big]\varepsilon_j; i\delta_j\Big),$$
$$\rho_j^{h,i} = \Big[\frac{3}{4}\frac{\rho_j}{\varepsilon_j}2^{-h}\Big]\varepsilon_j,$$
$$u_j^{h,i} = \frac{1}{\sharp C_j^{h,i}}\sum_{a\in C_j^{h,i}} u_j(a).$$

We denote by $C_{j,M}^{h,i}$ the following subset of $C_j^{h,i}$:

$$C_{j,M}^{h,i} = Q_{\varepsilon_j}\Big(\Big[\frac{\rho_j}{\varepsilon_j}2^{-h}\Big]\varepsilon_j - M\varepsilon_j; i\delta_j\Big) \setminus Q_{\varepsilon_j}\Big(\Big[\frac{\rho_j}{\varepsilon_j}2^{-h-1}\Big]\varepsilon_j + M\varepsilon_j; i\delta_j\Big).$$

Let $\phi_j^{h,i} \in C_0^\infty(\mathbf{C}_{j,M}^{h,i})$ be such that $\phi_j^{h,i} = 1$ on $\partial Q(\rho_j^{h,i}, i\delta_j)$ and $\big|\nabla\phi_j^{h,i}\big| \le c(\rho_j^{h,i})^{-1}$. In particular, the support of $\phi_j^{h,i}$ satisfies $\mathrm{supp}(\phi_j^{h,i}) \subseteq \mathbf{C}_{j,M}^{h,i} = \cup_{a\in C_{j,M}^{h,i}} Q(\varepsilon_j/2, a)$. For all $a \in C_j^{h,i}$ we set

$$w_j^{h,i}(a) := \phi_j^{h,i}(a)u_j^{h,i} + \big(1 - \phi_j^{h,i}(a)\big)u_j(a).$$

Note that for all $a \in C_j^{h,i} \setminus C_{j,M}^{h,i}$ we have $w_j^{h,i}(a) = u_j(a)$. Now, by the growth condition (5) on f^ξ we have

$$\begin{aligned} F_{\varepsilon_j}(w_j^{h,i}; C_j^{h,i}) &= \sum_{\xi\in I}\sum_{a\in R^\xi_{\varepsilon_j}(C_j^{h,i})} \varepsilon_j^n f^\xi(D^\xi_{\varepsilon_j}w_j^{h,i}(a)) \\ &\le c\sum_{\xi\in I}\sum_{a\in R^\xi_{\varepsilon_j}(C_j^{h,i})} \varepsilon_j^n|D^\xi_{\varepsilon_j}w_j^{h,i}(a)|^p + c|\mathbf{C}_j^{h,i}|. \end{aligned} \tag{30}$$

Let $a \in R^\xi_{\varepsilon_j}(C_j^{h,i})$ be fixed and $b = a + \varepsilon_j\xi$. Then by construction

$$D^\xi_{\varepsilon_j}w_j^{h,i}(a) = \big(u_j^{h,i} - u_j(b)\big)\frac{\phi_j^{h,i}(a) - \phi_j^{h,i}(b)}{\varepsilon_j|\xi|} + \big(1 - \phi_{\varepsilon_j}^{h,i}(a)\big)\frac{u_j(a) - u_j(b)}{\varepsilon_j|\xi|}.$$

There follows that

$$\begin{aligned}\left|D^{\xi}_{\varepsilon_j} w^{h,i}_j(a)\right|^p &\le c|u^{h,i}_j - u_j(b)|^p\Big|\frac{\phi^{h,i}_j(a)-\phi^{h,i}_j(b)}{\varepsilon_j|\xi|}\Big|^p\\ &\quad + c|1-\phi^{h,i}_j(a)|^p\Big|\frac{u_j(a)-u_j(b)}{\varepsilon_j|\xi|}\Big|^p\\ &\le c|u^{h,i}_j - u_j(b)|^p\Big|\frac{\phi^{h,i}_j(a)-\phi^{h,i}_j(b)}{\varepsilon_j|\xi|}\Big|^p + c\Big|\frac{u_j(a)-u_j(b)}{\varepsilon_j|\xi|}\Big|^p.\end{aligned}$$

We want to estimate the term $\left|\phi^{h,i}_j(a)-\phi^{h,i}_j(b)\right|^p(\varepsilon_j|\xi|)^{-p}$. Since $b = a+\varepsilon_j\xi$, we have

$$\begin{aligned}\Big|\frac{\phi^{h,i}_j(a)-\phi^{h,i}_j(b)}{\varepsilon_j|\xi|}\Big|^p &= \Big|\int_0^1 \frac{\partial}{\partial\xi}\big(\phi^{h,i}_j(a+(1-s)\varepsilon_j\xi)\big)\frac{1}{|\xi|}\,ds\Big|^p\\ &\le \int_0^1\Big|\frac{\partial}{\partial\xi}\big(\phi^{h,i}_j(a+(1-s)\varepsilon_j\xi)\big)\frac{1}{|\xi|}\Big|^p\,ds\\ &\le c\left|\nabla\phi^{h,i}_j\right|^p_\infty \le c(\rho^{h,i}_j)^{-p}.\end{aligned}$$

Summing up over $a\in R^{\xi}_{\varepsilon_j}(C^{h,i}_j)$ and $\xi\in I$, we get

$$\begin{aligned}&\sum_{\xi\in I}\sum_{a\in R^{\xi}_{\varepsilon_j}(C^{h,i}_j)}\varepsilon^n_j\left|D^{\xi}_{\varepsilon_j}w^{h,i}_j(a)\right|^p\\ &\le c\sum_{\xi\in I}\sum_{a\in R^{\xi}_{\varepsilon_j}(C^{h,i}_j)}|u^{h,i}_j-u_j(a+\varepsilon_j\xi)|^p\Big|\frac{\phi^{h,i}_j(a)-\phi^{h,i}_j(b)}{\varepsilon_j|\xi|}\Big|^p\varepsilon^n_j\\ &\quad + c\sum_{\xi\in I}\sum_{a\in R^{\xi}_{\varepsilon_j}(C^{h,i}_j)}\left|D^{\xi}_{\varepsilon_j}u_j(a)\right|^p\varepsilon^n_j\\ &\le \frac{c}{(\rho^{h,i}_j)^p}\sum_{\xi\in I}\sum_{a\in R^{\xi}_{\varepsilon_j}(C^{h,i}_j)}|u^{h,i}_j-u_j(a+\varepsilon_j\xi)|^p\varepsilon^n_j\\ &\quad + c\sum_{\xi\in I}\sum_{a\in R^{\xi}_{\varepsilon_j}(C^{h,i}_j)}\left|D^{\xi}_{\varepsilon_j}u_j(a)\right|^p\varepsilon^n_j. \qquad (31)\end{aligned}$$

Now,

$$\sum_{\xi\in I}\sum_{a\in R^{\xi}_{\varepsilon_j}(C^{h,i}_j)}|u^{h,i}_j-u_j(a+\varepsilon_j\xi)|^p\varepsilon^n_j \le c\sum_{a\in C^{h,i}_j}|u^{h,i}_j-u_j(a)|^p\varepsilon^n_j.$$

By [7, Lemma 5.2] (a discrete version of Poincar's inequality), we get

$$\sum_{a\in C^{h,i}_j}|u^{h,i}_j-u_j(a)|^p\varepsilon^n_j \le C(\rho^{h,i}_j)^p\sum_{\{a,b\}\in M_{\varepsilon_j}(C^{h,i}_j)}\Big|\frac{u_j(a)-u_j(b)}{\varepsilon_j}\Big|^p\varepsilon^n_j, \qquad (32)$$

and by construction

$$\sum_{\{a,b\}\in M_{\varepsilon_j}(C^{h,i}_j)}\Big|\frac{u_j(a)-u_j(b)}{\varepsilon_j}\Big|^p\varepsilon^n_j \le \sum_{\xi\in I}\sum_{a\in R^{\xi}_{\varepsilon_j}(C^{h,i}_j)}\left|D^{\xi}_{\varepsilon_j}u_j(a)\right|^p\varepsilon^n_j. \qquad (33)$$

Taking into account (31)-(33), we deduce that

$$\sum_{\xi\in I}\sum_{a\in R^{\xi}_{\varepsilon_j}(C^{h,i}_j)}\varepsilon_j^n\big|D^{\xi}_{\varepsilon_j}w^{h,i}_j(a)\big|^p\leq c\sum_{\xi\in I}\sum_{a\in R^{\xi}_{\varepsilon_j}(C^{h,i}_j)}\big|D^{\xi}_{\varepsilon_j}u_j(a)\big|^p\varepsilon_j^n. \tag{34}$$

By (30) and (34) we get

$$\sum_{\xi\in I}\sum_{a\in R^{\xi}_{\varepsilon_j}(C^{h,i}_j)}\varepsilon_j^n f^{\xi}(D^{\xi}_{\varepsilon_j}w^{h,i}_j(a))\leq c\sum_{\xi\in I}\sum_{a\in R^{\xi}_{\varepsilon_j}(C^{h,i}_j)}\big|D^{\xi}_{\varepsilon_j}u_j(a)\big|^p\varepsilon_j^n+c|\mathbf{C}^{h,i}_j|. \tag{35}$$

Now, by (35) and growth condition (5) we get

$$\begin{aligned}\Big|\sum_{\xi\in I}\sum_{a\in R^{\xi}_{\varepsilon_j}(C^{h,i}_j)}&\varepsilon_j^n\big(f^{\xi}(D^{\xi}_{\varepsilon_j}w^{h,i}_j(a))-f^{\xi}(D^{\xi}_{\varepsilon_j}u_j(a))\big)\Big|\\ &\leq\sum_{\xi\in I}\sum_{a\in R^{\xi}_{\varepsilon_j}(C^{h,i}_j)}\varepsilon_j^n f^{\xi}(D^{\xi}_{\varepsilon_j}w^{h,i}_j(a))+\sum_{\xi\in I}\sum_{a\in R^{\xi}_{\varepsilon_j}(C^{h,i}_j)}\varepsilon_j^n f^{\xi}(D^{\xi}_{\varepsilon_j}u_j(a))\\ &\leq c\sum_{\xi\in I}\sum_{a\in R^{\xi}_{\varepsilon_j}(C^{h,i}_j)}\big|D^{\xi}_{\varepsilon_j}u_j(a)\big|^p\varepsilon_j^n+c|\mathbf{C}^{h,i}_j|.\end{aligned}$$

Summing up over $h\in\{0,1,\dots,k-1\}$ we obtain

$$\begin{aligned}\sum_{h=0}^{k-1}\Big|\sum_{\xi\in I}\sum_{a\in R^{\xi}_{\varepsilon_j}(C^{h,i}_j)}&\varepsilon_j^n\big(f^{\xi}(D^{\xi}_{\varepsilon_j}w^{h,i}_j(a))-f^{\xi}(D^{\xi}_{\varepsilon_j}u_j(a))\big)\Big|\\ &\leq c\sum_{h=0}^{k-1}\sum_{\xi\in I}\sum_{a\in R^{\xi}_{\varepsilon_j}(C^{h,i}_j)}\big|D^{\xi}_{\varepsilon_j}u_j(a)\big|^p\varepsilon_j^n+c\sum_{h=0}^{k-1}|\mathbf{C}^{h,i}_j|\\ &\leq c\sum_{\xi\in I}\sum_{a\in R^{\xi}_{\varepsilon_j}(Q(\rho_j;i\delta_j))}\big|D^{\xi}_{\varepsilon_j}u_j(a)\big|^p\varepsilon_j^n+c|\mathbf{Q}(\rho_j;i\delta_j)|.\end{aligned}$$

Therefore there exists $k_i\in\{0,1,\dots,k-1\}$ such that

$$\begin{aligned}\Big|\sum_{\xi\in I}\sum_{a\in R^{\xi}_{\varepsilon_j}(C^{k_i,i}_j)}&\varepsilon_j^n\big(f^{\xi}(D^{\xi}_{\varepsilon_j}w^{h,i}_j(a))-f^{\xi}(D^{\xi}_{\varepsilon_j}u_j(a))\big)\Big|\\ &\leq\frac{c}{k}\sum_{\xi\in I}\sum_{a\in R^{\xi}_{\varepsilon_j}(Q(\rho_j;i\delta_j))}\big|D^{\xi}_{\varepsilon_j}u_j(a)\big|^p\varepsilon_j^n+\frac{c}{k}|\mathbf{Q}(\rho_j;i\delta_j)|\\ &\leq\frac{c}{k}F_{\varepsilon_j}(u_j;Q(\rho_j;i\delta_j))+\frac{c}{k}|\mathbf{Q}(\rho_j;i\delta_j)|,\end{aligned} \tag{36}$$

where the latter inequality follows from (4). With this choice of k_i for all $i\in Z_j$, conditions (27)-(29) are satisfied by picking $h=k_i$ in the definitions above; i.e.,

$$C^i_j=C^{k_i,i}_j,\qquad u^i_j=u^{k_i,i}_j,\qquad \rho^i_j=\rho^{k_i,i}_j,$$

$$\text{and}\quad w_j(a)=\begin{cases}u^i_j\phi^{i,k_i}_j(a)+\big(1-\phi^{i,k_i}_j(a)\big)u_j(a) & \text{for } a\in C^i_j,\ i\in Z_j,\\ u_j(a) & \text{otherwise.}\end{cases} \tag{37}$$

In fact by (36), (37) and the fact that $u_j = w_j$ on $\Omega_j \setminus \bigcup_{i\in Z_j} C^{k_i,i}_{j,M}$ we get:

$$\begin{aligned}
&\Big|\sum_{\xi\in I}\sum_{a\in R^{\xi}_{\varepsilon_j}(\Omega)} \varepsilon_j^n\big(f^{\xi}(D^{\xi}_{\varepsilon_j}w_j(a)) - f^{\xi}(D^{\xi}_{\varepsilon_j}u_j(a))\big)\Big| \\
&\quad\leq \sum_{i\in Z_j}\Big|\sum_{\xi\in I}\sum_{a\in R^{\xi}_{\varepsilon_j}(C^i_j)} \varepsilon_j^n\big(f^{\xi}(D^{\xi}_{\varepsilon_j}w_j(a)) - f^{\xi}(D^{\xi}_{\varepsilon_j}u_j(a))\big)\Big| \\
&\quad\leq \frac{c}{k}\sum_{i\in Z_j}\Big(F_{\varepsilon_j}(u_j;Q(\rho_j;i\delta_j)) + |\mathbf{Q}(\rho_j;i\delta_j)|\Big) \\
&\quad\leq \frac{c}{k}\Big(F_{\varepsilon_j}(u_j;\Omega) + |\Omega|\Big) \leq \frac{c}{k},
\end{aligned}$$

where the latter inequality follows from (26). Finally, we prove that $w_j \to u$ in $L^1(\Omega;\mathbb{R}^m)$. By construction

$$\int_{\Omega}|w_j - u|\,dx = \int_{\Omega\setminus\bigcup_{i\in Z_j}\mathbf{C}^i_j}|u_j - u|\,dx + \sum_{i\in Z_j}\int_{\mathbf{C}^i_j}|w_j - u|\,dx. \tag{38}$$

Now, the first term in (38) is infinitesimal:

$$\int_{\Omega\setminus\bigcup_{i\in Z_j}\mathbf{C}^i_j}|u_j - u|\,dx \leq \int_{\Omega}|u_j - u|\,dx \to 0 \qquad \text{as } j\to+\infty.$$

By (37) the second term in (38) can be estimated as follows:

$$\begin{aligned}
\sum_{i\in Z_j}\int_{\mathbf{C}^i_j}|w_j - u|\,dx &\leq c\sum_{i\in Z_j}\Big(\int_{\mathbf{C}^i_j}|u^i_j - u_j|\,dx + \int_{\mathbf{C}^i_j}|u_j - u|\,dx\Big) \\
&\leq c\sum_{i\in Z_j}\sum_{a\in C^i_j}|u_j(a) - u^i_j|\varepsilon_j^n + \int_{\Omega}|u_j - u|\,dx
\end{aligned}$$

Now, by discrete Hölder's inequality, [7, Lemma 5.2] and the concavity of $y\mapsto y^{\frac{1}{p}}$, we get

$$\begin{aligned}
&\sum_{i\in Z_j}\sum_{a\in C^i_j}|u_j(a) - u^i_j|\varepsilon_j^n \\
&\leq \sum_{i\in Z_j}\varepsilon_j^n\Big(\sum_{a\in C^i_j}|u_j(a) - u^i_j|^p\Big)^{\frac{1}{p}}\Big(\sharp C^i_j\Big)^{1-\frac{1}{p}} \\
&\leq c\varepsilon_j^n\varepsilon_j^{-n/p}\Big(\sum_{a\in C^i_j}|u_j(a) - u^i_j|^p\varepsilon_j^n\Big)^{\frac{1}{p}}\Big(\frac{\delta_j^n}{\varepsilon_j^n}\Big)^{1-\frac{1}{p}} \\
&\leq c\delta_j^{n-\frac{n}{p}}\Big(\delta_j^p\sum_{\{a,b\}\in M_{\varepsilon_j}(C^i_j)}\Big|\frac{u_j(a) - u_j(b)}{\varepsilon_j}\Big|^p\varepsilon_j^n\Big)^{\frac{1}{p}} \\
&\leq c\delta_j^{n-\frac{n}{p}}\delta_j(\sharp Z_j)^{1-\frac{1}{p}}\Big(\sum_{i\in Z_j}\sum_{\{a,b\}\in M_{\varepsilon_j}(C^i_j)}\Big|\frac{u_j(a) - u_j(b)}{\varepsilon_j}\Big|^p\varepsilon_j^n\Big)^{\frac{1}{p}} \\
&\leq c\delta_j.
\end{aligned}$$

In conclusion,

$$\sum_{i\in Z_j}\int_{\mathbf{C}_j^i}|w_j-u|\,dx\le c\delta_j+\int_\Omega|u-u_j|\,dx\to 0\qquad\text{as } j\to+\infty.$$

□

Lemma 6.2. *Let $1<p\le n$. Let (ε_j) and (δ_j) be as in (7). Let (u_j) be a sequence such that $u_j\in\mathcal{A}_{\varepsilon_j}(\Omega;\mathbb{R}^m)$. Assume that $u_j\to u$ in $L^1(\Omega;\mathbb{R}^m)$ for some $u\in W^{1,p}(\Omega;\mathbb{R}^m)$ and that (u_j) is bounded in $L^\infty(\Omega;\mathbb{R}^m)$. Let $k\in\mathbb{N}$ be fixed. Let (ρ_j) be a sequence of the form $\rho_j=\beta\delta_j$, with $\beta<1/2$. For all $i\in Z_j$ we define the set*

$$C_j^i=Q_{\varepsilon_j}\Big(\Big[\frac{\rho_j}{\varepsilon_j}2^{-k_i}\Big]\varepsilon_j;i\delta_j\Big)\setminus Q_{\varepsilon_j}\Big(\Big[\frac{\rho_j}{\varepsilon_j}2^{-k_i-1}\Big]\varepsilon_j;i\delta_j\Big),$$

where k_i is arbitrarily chosen in $\{0,1,\dots,k-1\}$. Let

$$u_j^i=\frac{1}{\sharp C_j^i}\sum_{a\in C_j^i}u_j(a)\qquad\text{and}\qquad Q_j^i=Q_{\varepsilon_j}(\delta_j;i\delta_j).$$

For all $N,j\in\mathbb{N}$ we consider two families of functions $r_{N,j},r_N:\mathbb{M}^{m\times n}\to[0,+\infty)$ such that the following assumptions hold:

1. *$r_{N,j}\to r_N$ as $j\to+\infty$, uniformly on the compact sets of $\mathbb{R}^m\setminus\{0\}$, for all $N\in\mathbb{N}$;*
2. *there exist a positive infinitesimal sequence ν_j and a constant $c>0$ such that*
$$r_{N,j}(z)\le\nu_j+c|z|^p\qquad\text{for all } z\in\mathbb{R}^m;\tag{39}$$
3. *for fixed $\eta>0$ there exists a constant $c>0$ such that for all $w,z\in\mathbb{R}^m$ we have*
$$|r_N(z)-r_N(w)|\le c(\eta^p+|z-w|(|w|^{p-1}+|z|^{p-1}));\tag{40}$$
4. *for $z=0$ we have*
$$r_N(0)=r_{N,j}(0)=0.\tag{41}$$

We define $\psi_j^N\in\mathcal{A}_{\varepsilon_j}(\Omega;\mathbb{R}^m)$ as

$$\psi_j^N(a)=\sum_{i\in Z_j}r_{N,j}(u_j^i)\chi_{Q_j^i}(a),\quad a\in\Omega_j,$$

where χ indicates the characteristic function. Then,

$$\lim_{j\to+\infty}\sum_{a\in\Omega_j}\psi_j^N(a)\varepsilon_j^n=\lim_{j\to+\infty}\sum_{i\in Z_j}r_{N,j}(u_j^i)\delta_j^n=\int_\Omega r_N(u)\,dx.$$

Proof. Let $\eta>0$ be fixed. For $\eta\le|z|\le\sup_j\|u_j\|_\infty$ we have $|r_{N,j}(z)-r_N(z)|\to 0$ as $j\to+\infty$ by assumption 1. For all $|z|<\eta$ conditions (39)-(41) imply that

$$|r_{N,j}(z)-r_N(z)|\le\nu_j+c\eta^p.$$

Since $u_j \to u$ in $L^1(\Omega;\mathbb{R}^m)$, we get

$$\begin{aligned}
&\limsup_{j\to+\infty}\Big|\sum_{i\in Z_j} r_{N,j}(u_j^i)\delta_j^n - \int_\Omega r_N(u)\,dx\Big| \\
&\le \limsup_{j\to+\infty}\Big|\sum_{i\in Z_j} r_N(u_j^i)\delta_j^n - \int_\Omega r_N(u)\,dx\Big| + c\eta^p \\
&\le \limsup_{j\to+\infty}\sum_{i\in Z_j}\int_{\mathbf{Q}_j^i}\big|r_N(u_j^i) - r_N(u)\big|\,dx + c\eta^p \\
&= \limsup_{j\to+\infty}\sum_{i\in Z_j}\sum_{a\in Q_j^i}\big|r_N(u_j^i) - r_N(u_j(a))\big|\varepsilon_j^n + c\eta^p.
\end{aligned}$$

By (40) and the boundedness of (u_j), we obtain

$$\big|r_N(u_j^i)-r_N(u_j(a))\big| \le c(|u_j^i-u_j(a)|(|u_j^i|^{p-1}+|u_j(a)|^{p-1})\big|+\eta^p) \le c(|u_j^i-u_j(a)|+\eta^p),$$

where the constant c is independent of j. There follows that

$$\limsup_{j\to+\infty}\Big|\sum_{i\in Z_j} r_{N,j}(u_j^i)\delta_j^n - \int_\Omega r_N(u)\,dx\Big| \le c\limsup_{j\to+\infty}\sum_{i\in Z_j}\sum_{a\in Q_j^i}|u_j^i - u_j(a)|\varepsilon_j^n + c\eta^p. \tag{42}$$

By the discrete version of Hölder's inequality we get

$$\begin{aligned}
\sum_{i\in Z_j}\sum_{a\in Q_j^i}|u_j^i - u_j(a)|\varepsilon_j^n &\le \varepsilon_j^n\sum_{i\in Z_j}\Big(\sum_{a\in Q_j^i}|u_j^i - u_j(a)|^p\Big)^{\frac1p}\big(\sharp Q_j^i\big)^{1-\frac1p} \\
&\le c\delta_j^n\delta_j^{-n/p}\sum_{i\in Z_j}\Big(\sum_{a\in Q_j^i}|u_j^i - u_j(a)|^p\Big)^{\frac1p}.
\end{aligned}$$

By [7, Lemma 5.2], we deduce that

$$\sum_{a\in Q_j^i}|u_j^i - u_j(a)|^p \le c\delta_j^p\sum_{\{a,b\}\in M_{\varepsilon_j}(Q_j^i)}\Big|\frac{u_j(a)-u_j(b)}{\varepsilon_j}\Big|^p\varepsilon_j^n.$$

Note that in the inequality above the constant c can be chosen to be independent of i, since for fixed j the family $\{C_j^i, i\in Z_j\}$ is s finite collection of homothetic sets. Therefore,

$$\sum_{i\in Z_j}\sum_{a\in Q_j^i}|u_j^i - u_j(a)|\varepsilon_j^n \le c\delta_j^n\delta_j^{-n/p}\sum_{i\in Z_j}\delta_j\Big(\sum_{\{a,b\}\in M_{\varepsilon_j}(Q_j^i)}\Big|\frac{u_j(a)-u_j(b)}{\varepsilon_j}\Big|^p\varepsilon_j^n\Big)^{\frac1p}.$$

Taking into account the concavity of the real function $x\mapsto x^{\frac1p}$, we get

$$\begin{aligned}
&\sum_{i\in Z_j}\sum_{a\in Q_j^i}|u_j^i - u_j(a)|\varepsilon_j^n \\
&\le c\delta_j^n\delta_j^{-n/p}\delta_j\big(\sharp Z_j^i\big)^{1-\frac1p}\Big(\sum_{i\in Z_j}\sum_{\{a,b\}\in M_{\varepsilon_j}(Q_j^i)}\Big|\frac{u_j(a)-u_j(b)}{\varepsilon_j}\Big|^p\varepsilon_j^n\Big)^{\frac1p} \\
&\le c\delta_j\Big(\sum_{\{a,b\}\in M_{\varepsilon_j}(\Omega)}\Big|\frac{u_j(a)-u_j(b)}{\varepsilon_j}\Big|^p\varepsilon_j^n\Big)^{\frac1p} \le c\delta_j.
\end{aligned} \tag{43}$$

By (42), (43) and the arbitrariness of η we conclude that

$$\limsup_{j\to+\infty}\Big|\sum_{i\in Z_j} r_{N,j}(u_j^i)\delta_j^n - \int_\Omega r_N(u)\,dx\Big| \leq \limsup_{j\to+\infty} c\delta_j = 0.$$

□

Remark 3. In the subcritical case $p < n$, we will apply Lemma 6.2 with $r_{N,j} = \phi_j^N$ and $r_N = \phi^N$. Then

$$\psi_j^N(a) = \sum_{i\in Z_j} \phi_j^N(u_j^i)\chi_{Q_j^i}(a), \quad a \in \Omega_j.$$

Remark 4. In the critical case $p = n$, we will apply Lemma 6.2 with $r_{N,j} = \varphi_j^{1/N}$ and $r_N = \varphi^{1/N}$. Setting $\alpha = N^{-1}$ and writing ψ_j^α in place of $\psi_j^{\alpha^{-1}}$, we will have

$$\psi_j^\alpha(a) = \sum_{i\in Z_j} \varphi_j^\alpha(u_j^i)\chi_{Q_j^i}(a), \quad a \in \Omega_j.$$

7. Γ-$\liminf$ **inequality.**

Proposition 3 (Γ-$\liminf$ inequality)**.** *Let $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ and $u_j \in \mathcal{A}_{\varepsilon_j}(\Omega;\mathbb{R}^m)$ be such that $u_j \to u$ in $L^1(\Omega;\mathbb{R}^m)$. Then*

$$\liminf_{j\to+\infty} F_{\varepsilon_j}(u_j) \geq F(u). \tag{44}$$

In the proof we will use the following truncation Lemma, which is a discrete version of [6, Lemma 3.5], and can be proved by adjusting to the discrete setting the arguments used in [6].

Lemma 7.1. *Let (u_j) be a sequence such that $u_j \in \mathcal{A}_{\varepsilon_j}(\Omega;\mathbb{R}^m)$, (u_j) is bounded in $L^1(\Omega;\mathbb{R}^m)$ and $\sup_j \mathcal{F}_{\varepsilon_j}(u_j) < +\infty$. Then, for all $L \in \mathbb{N}$ and $\eta > 0$ there exist a subsequence ε_j (not relabeled), a constant $R_L > L$ and a Lipschitz function $t_L : \mathbb{R}^m \to \mathbb{R}^m$ of Lipschitz constant 1 such that*

$$t_L(z) = z \text{ if } |z| < R_L$$
$$t_L(z) = 0 \text{ if } |z| > 2R_L$$
$$\text{and } \lim_j \mathcal{F}_{\varepsilon_j}(u_j) \geq \liminf_j \mathcal{F}_{\varepsilon_j}(t_L(u_j)) - \eta.$$

Proof of Proposition 3. With no loss of generality we assume that $\liminf_j F_{\varepsilon_j}(u_j) < +\infty$. We will first derive the $\liminf$ inequality under a boundedness assumption, and then we will deal with the general case (step **A** and **B** respectively).
A. We assume that (u_j) is bounded in $L^\infty(\Omega;\mathbb{R}^m)$ (we will remove this assumption through a truncation argument). We fix $k \in \mathbb{N}$ and we consider a sequence (ρ_j) of the form $\rho_j = \beta\delta_j$, with $\beta < 1/2$. We apply Lemma 6.1 to (u_j) in order to get a new sequence $w_j \to u$ in $L^1(\Omega;\mathbb{R}^m)$ satisfying (27)-(29). We denote by E_j the discrete set

$$E_j = \bigcup_{i\in Z_j} Q_{\varepsilon_j}(\rho_j^i; i\delta_j).$$

By construction

$$\liminf_j F_{\varepsilon_j}(u_j) \geq \liminf_j F_{\varepsilon_j}(u_j; E_j) + \liminf_j F_{\varepsilon_j}(u_j; \Omega_j \setminus E_j).$$

First of all, we want to find a lower bound for the contribution of (u_j) on $\Omega_j \setminus E_j$ and then we will estimate the energy on E_j (steps **A.1** and **A.2** respectively).

A.1. In this step we will find a lower bound for the contribution of the energy far from the pinning sites; i.e., the term $\liminf_j F_{\varepsilon_j}(u_j;\Omega_j \setminus E_j)$. The proof of this estimate is formally the same for the critical case $p = n$ and the subcritical one, $p < n$; note that the formula defining the bulk term of the Γ-limit has the same structure for any order of growth. However, the critical scaling for δ_j (and hence ρ_j) as a function of ε_j is obviously different, so the set E_j has a different "size" in the two cases.

We define a new sequence $v_j \in \mathcal{A}_{\varepsilon_j}(\Omega;\mathbb{R}^m)$ by modifying w_j as follows:

$$v_j(a) = \begin{cases} u^i_j & \text{for } \ a \in Q^i_j := Q_{\varepsilon_j}(\rho^i_j; i\delta_j), \quad i \in Z_j \\ w_j(a) & \text{otherwise.} \end{cases}$$

Note that $v_j(a) = u_j(a)$ for all $a \in \Omega_j \setminus \bigcup_{i\in Z_j} Q([2^{-k_i}\rho_j/\varepsilon_j]\varepsilon_j; i\delta_j)$, since w_j is such that $u_j = w_j$ on $\Omega_j \setminus \bigcup_{i\in Z_j} C^i_j$. Note moreover that $v_j \to u$ in $L^1(\Omega;\mathbb{R}^m)$. In fact

$$\begin{aligned}
\lim_j \int_\Omega |v_j - u|\,dx &\le \lim_j \int_\Omega |u_j - v_j|\,dx + \lim_j \int_\Omega |u_j - u|\,dx \\
&= \lim_j \sum_{a\in\Omega_j} |u_j(a) - v_j(a)|\varepsilon^n_j \\
&\le \lim_j \sum_{a\in\Omega_j\setminus E_j} |u_j(a) - v_j(a)|\varepsilon^n_j + \lim_j \sum_{a\in E_j} |u_j(a) - v_j(a)|\varepsilon^n_j \\
&\le \lim_j \sum_{a\in\Omega_j\setminus E_j} |u_j(a) - w_j(a)|\varepsilon^n_j + \lim_j \sum_{a\in E_j} |u_j(a) - v_j(a)|\varepsilon^n_j \\
&\le \lim_j \Big(\int_\Omega |u_j - u|\,dx + \int_\Omega |w_j - u|\,dx \Big) \\
&\quad + \lim_j \sum_{i\in Z_j} \sum_{i\in Q^i_j} |u^i_j - u_j(a)|\varepsilon^n_j \\
&= \lim_j \sum_{i\in Z_j} \sum_{i\in Q^i_j} |u^i_j - u_j(a)|\varepsilon^n_j.
\end{aligned}$$

Arguing as in Lemma 6.1 we get

$$\begin{aligned}
\lim_j \sum_{i\in Z_j} \sum_{i\in Q^i_j} |u^i_j - u_j(a)|\varepsilon^n_j &\le \lim_j c\delta_j \Big(\sum_{\{a,b\}\in M_{\varepsilon_j}(\Omega)} \varepsilon^{n-p}_j |u_j(a) - u_j(b)|^p \Big)^{1/p} \\
&\le \lim_j c\delta_j = 0.
\end{aligned}$$

Now, Lemma 6.1 implies that

$$\liminf_j F_{\varepsilon_j}(u_j;\Omega_j \setminus E_j) + \frac{c}{k} \ge \liminf_j F_{\varepsilon_j}(w_j;\Omega_j \setminus E_j).$$

We can write

$$F_{\varepsilon_j}(w_j;\Omega_j \setminus E_j) = \mathcal{F}_{\varepsilon_j}(v_j) - R_j, \tag{45}$$

where

$$R_j = \sum_{i\in Z_j} \sum_{\xi\in I} \sum_{a\in\mathcal{Y}^\xi_{\varepsilon_j}(\rho^i_j;i\delta_j)} \varepsilon^n_j f^\xi(D^\xi_{\varepsilon_j} v_j(a))$$

and $\mathcal{Y}^\xi_{\varepsilon_j}(l;c) = \{a \in \Omega_{\varepsilon_j} :\ a \in Q_{\varepsilon_j}(l;c),\ a + \varepsilon_j\xi \in \Omega_j \setminus Q_{\varepsilon_j}(l;c)\}$ accounts for the interactions across $\partial(c + [-l,l]^n)$ for all $c \in \mathbb{R}^n$ and $l > 0$.

We want to show that R_j is negligible. Note that for each $a \in \mathcal{Y}^{\xi}_{\varepsilon_j}(\rho^i_j; i\delta_j)$ we have $a, a+\varepsilon_j\xi \in C^i_j$, since $\mathrm{dist}(a; \mathcal{S}_{\varepsilon_j}(\rho^i_j; i\delta_j)) \le M\varepsilon_j < ([2^{-k_i}\rho_j/\varepsilon_j]\varepsilon_j - [2^{-k_i-1}\rho_j/\varepsilon_j]\varepsilon_j)/2$ (and the same holds for $a+\varepsilon_j\xi$). Hence

$$\begin{aligned} R_j &\le \sum_{i\in Z_j}\sum_{\xi\in I}\sum_{a\in R^{\xi}_{\varepsilon_j}(C^i_j)} \varepsilon_j^n f^{\xi}(D^{\xi}_{\varepsilon_j} v_j(a)) \\ &\le c\sum_{i\in Z_j}\sum_{\{a,b\}\in M_{\varepsilon_j}(C^i_j)} \varepsilon_j^n \Big(1 + \Big|\frac{v_j(a)-v_j(b)}{\varepsilon_j}\Big|^p\Big) \\ &\le c\sum_{i\in Z_j}\sum_{\{a,b\}\in M_{\varepsilon_j}(C^i_j)} \varepsilon_j^n + c\sum_{i\in Z_j}\sum_{\{a,b\}\in M_{\varepsilon_j}(C^i_j\setminus Q(\rho^i_j; i\delta_j))} \varepsilon_j^n \Big|\frac{w_j(a)-w_j(b)}{\varepsilon_j}\Big|^p \\ &\le c\varepsilon_j^n(\sharp Z_j)(\sharp C^i_j) + c\sum_{i\in Z_j}\sum_{\{a,b\}\in M_{\varepsilon_j}(C^i_j)} \varepsilon_j^n \Big|\frac{w_j(a)-w_j(b)}{\varepsilon_j}\Big|^p \\ &\le c\beta^n + c\sum_{i\in Z_j} F_{\varepsilon_j}(w_j; C^i_j). \end{aligned}$$

By Lemma 6.1 we deduce that

$$R_j \le c\beta^n + \frac{c}{k} \quad \text{for } j \text{ large enough.} \tag{46}$$

By (45) and (46) we get

$$\liminf_j F_{\varepsilon_j}(w_j; \Omega_j \setminus E_j) \ge \liminf_j \mathcal{F}_{\varepsilon_j}(v_j) - c\beta^n - \frac{c}{k}.$$

Since $v_j \to u$ in $L^1(\Omega;\mathbb{R}^m)$, by Proposition 1 we have

$$\liminf_j F_{\varepsilon_j}(w_j; \Omega_j \setminus E_j) \ge \liminf_j \mathcal{F}_{\varepsilon_j}(v_j) - c\beta^n - \frac{c}{k} \ge \int_{\Omega} f_0(Du)\,dx - c\beta^n - \frac{c}{k}, \tag{47}$$

where $f_0 : \mathbb{M}^{m\times n} \to [0,+\infty)$ is given by the homogenization formula in (10).
A.2. In this paragraph we focus our attention on the contribution of u_j on E_j; i.e., close to the pinning sites. By Lemma 6.1 we have

$$\liminf_j F_{\varepsilon_j}(u_j; E_j) + \frac{c}{k} \ge \liminf_j F_{\varepsilon_j}(w_j; E_j) \ge \liminf_j \sum_{i\in Z_j} F_{\varepsilon_j}(w_j; Q(\rho^i_j; i\delta_j)).$$

For fixed $j \in \mathbb{N}$ and $i \in Z_j$ we define the function $w_{i,j} \in \mathcal{A}_{\varepsilon_j}(\mathbb{R}^n;\mathbb{R}^m)$ as

$$w_{i,j}(a) = \begin{cases} w_j(a+i\delta_j) & \text{if } a \in Q_{\varepsilon_j}(\rho^i_j) \\ u^i_j & \text{if } a \in \varepsilon_j\mathbb{Z}^n \setminus Q_{\varepsilon_j}(\rho^i_j). \end{cases}$$

We will deal separately with the case $p = n$ and the case $p < n$ (steps **2.1** and **2.2** respectively), since the asymptotic behavior of the energies close to the pinning sites is determined by the growth exponent p.
A.2.1 Critical exponent $p = n$. Let $j \in \mathbb{N}$ and $i \in Z_j$ be fixed. By a rescaling argument on the space variable we define $\zeta^i_j \in \mathcal{A}_1(\mathbb{Z}^n;\mathbb{R}^m)$ as $\zeta^i_j(A) = w_{i,j}(A\varepsilon_j)$. By construction $\zeta^i_j(0) = 0$ and $\zeta^i_j = u^i_j$ on $\mathbb{Z}^n \setminus Q_1(\rho^i_j T_j - 1)$. In particular, we notice that $\zeta^i_j = u^i_j$ on $\mathcal{S}_1([\beta\delta_j T_j - M], [\beta\delta_j T_j])$ (provided that j is large enough).

Now,

$$\begin{aligned} F_{\varepsilon_j}(w_j; Q(\rho^i_j, i\delta_j)) &= F_{\varepsilon_j}(w_{i,j}; Q(\rho^i_j)) \\ &= \sum_{\xi\in I} \sum_{A\in R^\xi_1(Q_1(\rho^i_j T_j))} T_j^{-n} f^\xi(D^\xi_1 \zeta^i_j(A) T_j) \\ &= \sum_{\xi\in I} \sum_{A\in R^\xi_1(Q_1(\beta\delta_j T_j))} T_j^{-n} f^\xi(D^\xi_1 \zeta^i_j(A) T_j) - R^i_j, \end{aligned}$$

where

$$R^i_j = \sum_{\xi\in I} \sum_{A\in \mathcal{Y}^\xi_1(\rho^i_j T_j)} \varepsilon^n_j f^\xi(T_j D^\xi_1 \zeta^i_j(A)).$$

Summing up over $i \in Z_j$ we have

$$\sum_{i\in Z_j} F_{\varepsilon_j}(w_j; Q(\rho^i_j; i\delta_j)) \geq \sum_{i\in Z_j}\sum_{\xi\in I} \sum_{A\in R^\xi_1(Q_1(\beta\delta_j T_j))} \varepsilon^n_j f^\xi(D^\xi_1 \zeta^i_j(A) T_j) - \sum_{i\in Z_j} R^i_j.$$

Taking into account Lemma 6.1 we can show that $\sum_{i\in Z_j} R^i_j$ is negligible. In fact by a change of variables we get:

$$\begin{aligned} \sum_{i\in Z_j} R^i_j &\leq \sum_{i\in Z_j}\sum_{\xi\in I}\sum_{a\in R^\xi_{\varepsilon_j}(C^i_j)} \varepsilon^n_j f^\xi(D^\xi_{\varepsilon_j} w_{i,j}(a - i\delta_j)) \\ &\leq c\sum_{i\in Z_j}\sum_{\{a,b\}\in M_{\varepsilon_j}(C^i_j)} |w_{i,j}(a - i\delta_j) - w_{i,j}(b - i\delta_j)|^n \\ &\leq c\sum_{i\in Z_j}\sum_{\{a,b\}\in M_{\varepsilon_j}(C^i_j\cap Q(\rho^i_j, i\delta_j))} |w_j(a) - w_j(b)|^n \\ &\leq c\sum_{i\in Z_j}\sum_{\{a,b\}\in M_{\varepsilon_j}(C^i_j)} |w_j(a) - w_j(b)|^n \leq c\sum_{i\in Z_j} F_{\varepsilon_j}(w_j; C^i_j) \leq \frac{c}{k}. \end{aligned}$$

There follows that

$$\begin{aligned} &\liminf_j \sum_{i\in Z_j} F_{\varepsilon_j}(w_j; Q(\rho^i_j, i\delta_j)) \\ &\quad\geq \liminf_j \sum_{i\in Z_j}\sum_{\xi\in I}\sum_{A\in R^\xi_1(Q_1(\beta\delta_j T_j))} \varepsilon^n_j f^\xi(T_j D^\xi_1 \zeta^i_j(A)) - \frac{c}{k} \\ &\quad\geq \liminf_j \sum_{i\in Z_j} \inf\Big\{\sum_{\xi\in I}\sum_{A\in R^\xi_1(Q_1(\beta\delta_j T_j))} \varepsilon^n_j f^\xi(T_j D^\xi_1 \zeta(A)) :\ \zeta(0) = 0, \\ &\qquad\qquad \zeta = u^i_j \text{ on } \mathcal{S}_1([\beta\delta_j T_j - M], [\beta\delta_j T_j])\Big\} - \frac{c}{k}. \end{aligned}$$

Recalling that we set $S_j = T_j(\log T_j)^{(1-n)/n}$, we can write $\beta\delta_j T_j = \beta r^{(n-1)/n} S_j$. Letting $\alpha = \beta r^{(n-1)/n}$, we can re-write the inequality above as follows:

$$\begin{aligned} \liminf_j \sum_{i\in Z_j} F_{\varepsilon_j}(w_j; Q(\rho^i_j, i\delta_j)) &\geq \liminf_j \sum_{i\in Z_j} \frac{1}{(\log T_j)^{n-1}} \varphi^\alpha_j(u^i_j) - \frac{c}{k} \\ &= r^{1-n} \liminf_j \sum_{i\in Z_j} \delta^n_j \varphi^\alpha_j(u^i_j) - \frac{c}{k}. \end{aligned}$$

By Lemma 6.2 and Remark 3 we know that there exists the limit

$$\lim_j \sum_{i\in Z_j} \delta_j^n \varphi_j^\alpha(u_j^i) = \int_\Omega \varphi^\alpha(u)\,dx,$$

provided that we extract a suitable subsequence (not relabeled). Hence

$$\liminf_j F_{\varepsilon_j}(u_j;E_j) \geq \liminf_j F_{\varepsilon_j}(w_j;E_j) - \frac{c}{k} \geq r^{1-n}\int_\Omega \varphi^\alpha(u)\,dx - \frac{c}{k}, \tag{48}$$

with $\alpha = \beta r^{(n-1)/n}$. By (47) and (48) we can conclude that in the case $n = p$

$$\liminf_j F_{\varepsilon_j}(u_j) \geq \int_\Omega f_0(Du)\,dx + r^{1-n}\int_\Omega \varphi^\alpha(u)\,dx - \frac{c}{k} - c\beta^n.$$

By letting first $\beta \to 0^+$ and then $k \to +\infty$ we finally obtain the desired inequality:

$$\liminf_j F_{\varepsilon_j}(u_j) \geq \int_\Omega f_0(Du)\,dx + r^{1-n}\int_\Omega \varphi(u)\,dx = F(u).$$

A.2.2 Subcritical exponent $p < n$. Let $j \in \mathbb{N}$ and $i \in Z_j$ be fixed. By rescaling $w_{i,j}$ we define the function $\zeta_j^i \in \mathcal{A}_1(\mathbb{Z}^n, \mathbb{R}^m)$ as

$$\zeta_j^i(A) = \begin{cases} w_{i,j}(\varepsilon_j A) & \text{for } A \in Q_1(\rho_j^i T_j) \\ u_j^i & \text{for } A \in \mathbb{Z}^n \setminus Q_1(\rho_j^i T_j). \end{cases}$$

Note that $\zeta_j^i(0) = 0$ and $\zeta_j^i = u_j^i$ on $\mathcal{S}_1([\delta_j T_j - M], [\delta_j T_j])$. By a change of variables we have

$$F_{\varepsilon_j}(w_j; Q(\rho_j^i, i\delta_j)) = F_{\varepsilon_j}(w_{i,j}, Q(\rho_j^i)) = \varepsilon_j^{n-p} \sum_{\xi\in I}\sum_{A\in R_1^\xi(Q(\delta_j T_j))} h_j^\xi(D_1^\xi \zeta_j^i(A)) - R_j^i, \tag{49}$$

where $h_j^\xi(x) = T_j^{-p} f^\xi(T_j x)$ and the term R_j^i corresponds to the interactions across $\partial([-[\rho_j^i T_j], [\rho_j^i T_j]]^n)$:

$$R_j^i = \varepsilon_j^{n-p} \sum_{\xi\in I}\sum_{a\in\mathcal{Y}_1^\xi([\rho_j^i T_j])} h_j^\xi(D_1^\xi \zeta_j^i(A)).$$

By construction the function ζ_j^i satisfies

$$\begin{aligned} &\sum_{\xi\in I}\sum_{A\in R_1^\xi(Q(\delta_j T_j))} h_j^\xi(D_1^\xi \zeta_j^i(A)) \\ &\quad\geq \inf\Big\{ \sum_{\xi\in I}\sum_{A\in R_1^\xi(Q(\delta_j T_j))} h_j^\xi(D_1^\xi v(A)) : \begin{array}{l} v(0)=0 \\ v = u_j^i \text{ on } \mathcal{S}_1([\delta_j T_j - M], [\delta_j T_j]) \end{array} \Big\} \\ &\quad\geq \inf_N \inf\Big\{ \sum_{\xi\in I}\sum_{A\in R_1^\xi(Q(N))} h_j^\xi(D_1^\xi v(A)) : \begin{array}{l} v(0)=0 \\ v = u_j^i \text{ on } \mathcal{S}_1([N-M], [N]) \end{array} \Big\} \\ &\quad= \inf_N \phi_j^N(u_j^i). \end{aligned} \tag{50}$$

Summing up over the pinning sites $i \in Z_j$ and taking into account (49) and (50), we get

$$F_{\varepsilon_j}(w_j; E_j) \geq \sum_{i\in Z_j} F_{\varepsilon_j}(w_j; Q(\rho_j^i; i\delta_j)) \geq \inf_N \varepsilon_j^{n-p} \sum_{i\in Z_j} \phi_j^N(u_j^i) - \sum_{i\in Z_j} R_j^i.$$

The term $\sum_{i\in Z_j} R^i_j$ is negligible; in fact

$$\begin{aligned}\sum_{i\in Z_j} R^i_j &= \varepsilon_j^{n-p}\sum_{i\in Z_j}\sum_{\xi\in I}\sum_{a\in\mathcal{Y}_1^\xi([\rho^i_j T_j])} h^\xi_j(D^\xi_1\zeta^i_j(A))\\ &\le c\sum_{i\in Z_j}\sum_{\{a,b\}\in M_{\varepsilon_j}(C^i_j)}\varepsilon_j^n\Big(\Big|\frac{w_{i,j}(a-i\delta_j)-w_{i,j}(b-i\delta_j)}{\varepsilon_j}\Big|^p+1\Big)\\ &\le c\sum_{i\in Z_j}\sum_{\{a,b\}\in M_{\varepsilon_j}(C^i_j\cap Q(\rho^i_j,i\delta_j))}\varepsilon_j^{n-p}|w_j(a)-w_j(b)|^p+c\beta\\ &\le c\sum_{i\in Z_j}F_{\varepsilon_j}(w_j;C^i_j)+c\beta\le\frac{c}{k}+c\beta.\end{aligned}$$

Moreover, by Lemma 6.2 and Remark 4 we get that for fixed N there exists the limit

$$\lim_j\sum_{i\in Z_j}\varepsilon_j^{n-p}\phi^N_j(u^i_j)=\lim_j r^{1-n}\sum_{i\in Z_j}\delta^n_j\phi^N_j(u^i_j)=r^{1-n}\int_\Omega\phi^N(u)\,dx,$$

upon extracting a suitable subsequence. There follows that

$$\begin{aligned}\liminf_j F_{\varepsilon_j}(u_j;E_j)&\ge\liminf_j F_{\varepsilon_j}(w_j;E_j)-\frac{c}{k}\\ &\ge r^{1-n}\inf_N\int_\Omega\phi^N(u)\,dx-\frac{c}{k}-c\beta=r^{1-n}\int_\Omega\phi(u)\,dx-\frac{c}{k}-c\beta.\end{aligned}\tag{51}$$

By (47) and (51) we have

$$\liminf_j F_{\varepsilon_j}(u_j)\ge\int_\Omega f_0(Du)\,dx+r^{1-n}\int_\Omega\phi(u)\,dx-\frac{c}{k}-c\beta.$$

By letting $\beta\to 0^+$ and $k\to+\infty$ we conclude that

$$\liminf_j F_{\varepsilon_j}(u_j)\ge\int_\Omega f_0(Du)\,dx+r^{1-n}\int_\Omega\phi(u)\,dx.$$

B. It remains to show that the Γ-liminf inequality holds even if we remove the boundedness assumption on the sequence (u_j). For all $L\in\mathbb{N}$ and $\eta>0$ we apply the previous arguments to the truncated sequence $t_L(u_j)$, where t_L is as in the statement of Lemma 7.1; i.e.,

$$\begin{gathered}t_L(u_j)=z\text{ if }|u_j|<R_L\\ t_l(u_j)=0\text{ if }|u_j|>2R_L\\ \text{and }\liminf_j F_{\varepsilon_j}(u_j)\ge\liminf_j F_{\varepsilon_j}(t_L(u_j))-\eta.\end{gathered}\tag{52}$$

By step **A** we get

$$\liminf_j F_{\varepsilon_j}(t_L(u_j))\ge\int_\Omega f_0(Dt_L(u))\,dx+r^{1-n}\int_\Omega\varphi(t_L(u))\,dx$$

if $n=p$, and

$$\liminf_j F_{\varepsilon_j}(t_L(u_j))\ge\int_\Omega f_0(Dt_L(u))\,dx+r^{1-n}\int_\Omega\phi(t_L(u))\,dx$$

if $n>p$. Note that $t_L(u)\to u$ as $L\to+\infty$, with respect to the weak convergence of $W^{1,p}(\Omega;\mathbb{R}^m)$. By (52) and the arbitrariness of η, we can pass to the limit as $L\to+\infty$ and finally deduce that

$$\liminf_j F_{\varepsilon_j}(u_j)\ge F(u).$$

□

8. Γ-lim sup inequality.

Proposition 4 (Limsup inequality)**.** *For all $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ there exists a sequence (v_j) such that $v_j \in \mathcal{A}_{\varepsilon_j}(\Omega;\mathbb{R}^m)$, $v_j \to u$ in $L^1(\Omega;\mathbb{R}^m)$ and*

$$\limsup_j F_{\varepsilon_j}(v_j) \leq F(u).$$

Proof. First of all we will prove that the Γ-lim sup inequality holds for all piecewise affine functions and then we will obtain the general case through a density argument (step **A** and **B** respectively).

A. Let $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ be a piecewise affine function. Let $\eta > 0$ be fixed. By carefully applying the construction of [1, Theorem 4.1] to the function u, we can prove that there exists a sequence $u_j \in \mathcal{A}_{\varepsilon_j}(\Omega;\mathbb{R}^m)$ such that $u_j \to u$ in $L^1(\Omega;\mathbb{R}^m)$,

$$\limsup_j \mathcal{F}_{\varepsilon_j}(u_j) = \limsup_j \sum_{\xi\in I} \sum_{a\in R^{\xi}_{\varepsilon_j}(\Omega)} \varepsilon_j^n f^{\xi}(D^{\xi}_{\varepsilon_j} u_j(a)) \leq \int_{\Omega} f_0(Du)\,dx + \eta \tag{53}$$

and

$$\frac{|u_j(a) - u_j(b)|}{\varepsilon_j} \leq C \text{ for some } C > 0, \quad \text{for all } \{a,b\} \in M_{\varepsilon_j}(\Omega). \tag{54}$$

Moreover, the sequence (u_j) can be chosen to be bounded in $L^{\infty}(\Omega;\mathbb{R}^m)$.

In order to construct an approximate recovery sequence for u (for any value of the parameter η), we will deal separately with the case $p = n$ and the case $p < n$ (steps **A.1** and **A.2** respectively).

A.1 Critical exponent $p = n$. We want to modify (u_j) in order to get an approximate recovery sequence for u. We fix $k \in \mathbb{N}$ and $\beta > 0$ such that $2^{k+1}\beta < 1/2$. Let $\rho_j = 2^{k+1}\beta\delta_j$. Given this choice of ρ_j, we apply Lemma 6.1 to (u_j) and we get a sequence $w_j \to u$ in $L^1(\Omega;\mathbb{R}^m)$ satisfying (27)-(29). We denote by Z'_j the set of indices $Z'_j = \{i \in \mathbb{Z}^n \setminus Z_j : \ i\delta_j \in \Omega\}$, corresponding to the pinning sites close to the boundary of Ω. We define the sets

$$E_j = \bigcup_{i\in Z_j} Q_{\varepsilon_j}(\rho^i_j; i\delta_j), \qquad E'_j = \bigcup_{i\in Z'_j} Q_{\varepsilon_j}(\rho_j; i\delta_j) \cap \Omega.$$

By suitably modifying w_j on $E_j \cup E'_j$ we will get an approximate recovery sequence for u.

A.1.1. Firstly we deal with E_j. By construction we have $\rho^i_j \geq \beta\delta_j$ for all $i \in Z_j$. We set $T_j = \varepsilon_j^{-1}$ and $S_j = T_j(\log T_j)^{(1-n)/n}$. For fixed $j \in \mathbb{N}$ and $i \in Z_j$ we consider a function $\zeta^i_j \in \mathcal{A}_1(Q(r^{(n-1)/n}\beta S_j);\mathbb{R}^m)$ such that

$$\zeta^i_j(0) = 0, \qquad \zeta^i_j = u^i_j \text{ on } \mathcal{S}_1([r^{(n-1)/n}\beta S_j - M], [r^{(n-1)/n}\beta S_j])$$

and

$$(\log T_j)^{n-1} \sum_{\xi\in I} \sum_{A\in R^{\xi}_1(Q(\beta r^{(n-1)/n} S_j))} f^{\xi}(T_j D^{\xi}_1 \zeta^i_j(A)) T_j^{-n} < \varphi_j^{\beta r^{(n-1)/n}}(u^i_j) + \eta.$$

We define $v_j : E_j \to \mathbb{R}^m$ as follows:

$$v_j(a) = \begin{cases} \zeta^i_j\Big(\dfrac{a - i\delta_j}{\varepsilon_j}\Big) & \text{if } a \in Q_{\varepsilon_j}(\beta\delta_j; i\delta_j), \ i \in Z_j \\ u^i_j & \text{if } a \in Q_{\varepsilon_j}(\rho^i_j; i\delta_j) \setminus Q_{\varepsilon_j}(\beta\delta_j; i\delta_j), \ i \in Z_j \end{cases}$$

A.1.2. Now we focus our attention on the set E'_j. Let $\gamma_j \in \mathcal{A}_1(Q(\rho_j/\varepsilon_j);\mathbb{R})$ be a function such that $\gamma_j(0)=0$, $\gamma_j=1$ on $\mathcal{S}_1([\rho_j/\varepsilon_j])$ and

$$\sum_{\{A,B\}\in M_1(Q(\rho_j/\varepsilon_j))} |\gamma_j(A)-\gamma_j(B)|^n < \inf\Big\{ \sum_{\{A,B\}\in M_1(Q(\rho_j/\varepsilon_j))} |v(A)-v(B)|^n : \\ v\in\mathcal{A}_1(Q(\rho_j/\varepsilon_j)),\ v(0)=0,\ v=1 \text{ on } \mathcal{S}_1([\rho_j/\varepsilon_j])\Big\} + \eta.$$

By Lemma 4.1 we know that the infimum above satisfies

$$\lim_{j\to\infty} \Big|\log\Big(\frac{\rho_j}{\varepsilon_j}\Big)\Big|^{n-1} \inf\Big\{ \sum_{\{A,B\}\in M_1(Q(\rho_j/\varepsilon_j))} |v(A)-v(B)|^n : \\ v\in\mathcal{A}_1(Q(\rho_j/\varepsilon_j)),\ v(0)=0,\ v=1 \text{ on } \mathcal{S}_1([\rho_j/\varepsilon_j])\Big\} = \omega_{n-1}.$$

We define $v_j : E'_j \to \mathbb{R}^m$ as

$$v_j(a) = \gamma_j\Big(\frac{a-i\delta_j}{\varepsilon_j}\Big)u_j(a) \qquad \text{for } a\in Q_{\varepsilon_j}\Big(\frac{\rho_j}{\varepsilon_j}; i\delta_j\Big)\cap\Omega,\ i\in Z'_j.$$

A.1.3. Finally, we define $v_j(a)=w_j(a)$ for all $a\in\Omega_j\setminus(E_j\cup E'_j)$. To sum up, we set

$$v_j(a) = \begin{cases} \zeta^i_j\Big(\dfrac{a-i\delta_j}{\varepsilon_j}\Big) & \text{if } a\in Q_{\varepsilon_j}(\beta\delta_j; i\delta_j),\ i\in Z_j \\ u^i_j & \text{if } a\in Q_{\varepsilon_j}(\rho^i_j; i\delta_j)\setminus Q_{\varepsilon_j}(\beta\delta_j; i\delta_j),\ i\in Z_j \\ \gamma_j\Big(\dfrac{a-i\delta_j}{\varepsilon_j}\Big)u_j(a) & \text{for } a\in Q_{\varepsilon_j}(\rho_j/\varepsilon_j; i\delta_j)\cap\Omega,\ i\in Z'_j \\ w_j(a) & \text{if } a\in\Omega_j\setminus(E_j\cup E'_j). \end{cases}$$

Now we can prove that (v_j) is an approximate recovery sequence for u. By construction we have

$$\limsup_j F_{\varepsilon_j}(v_j) \le \limsup_j \sum_{i\in Z_j} F_{\varepsilon_j}(v_j; Q(\beta\delta_j, i\delta_j)) \tag{55}$$

$$+\limsup_j \sum_{i\in Z_j} F_{\varepsilon_j}(v_j; Q_{\varepsilon_j}(\rho^i_j, i\delta_j)\setminus Q_{\varepsilon_j}(\beta\delta_j, i\delta_j)) \tag{56}$$

$$+\limsup_j F_{\varepsilon_j}(v_j; \Omega_j\setminus(E_j\cup E'_j)) \tag{57}$$

$$+\limsup_j \sum_{i\in Z_j}\sum_{\xi\in I}\sum_{a\in\mathcal{Y}^\xi_{\varepsilon_j}(\beta\delta_j; i\delta_j)} f^\xi(D^\xi_{\varepsilon_j} v_j(a))\varepsilon_j^n \tag{58}$$

$$+\limsup_j \sum_{i\in Z_j}\sum_{\xi\in I}\sum_{a\in\mathcal{Y}^\xi_{\varepsilon_j}(\rho^i_j; i\delta_j)} f^\xi(D^\xi_{\varepsilon_j} v_j(a))\varepsilon_j^n \tag{59}$$

$$+\limsup_j \sum_{i\in Z'_j} F_{\varepsilon_j}(v_j; Q_{\varepsilon_j}(\rho_j, i\delta_j)\cap\Omega) \tag{60}$$

$$+\limsup_j \sum_{i\in Z'_j}\sum_{\xi\in I}\sum_{a\in\mathcal{Y}^\xi_{\varepsilon_j}(\rho_j; i\delta_j)} f^\xi(D^\xi_{\varepsilon_j} v_j(a))\varepsilon_j^n. \tag{61}$$

The terms above can be estimated separately. First of all we focus our attention on (55) and we notice that by a change of variables

$$\limsup_j \sum_{i\in Z_j} F_{\varepsilon_j}(v_j; Q(\beta\delta_j, i\delta_j)) = \limsup_j \sum_{i\in Z_j}\sum_{\xi\in I}\sum_{a\in R^\xi_{\varepsilon_j}(Q(\beta\delta_j,i\delta_j))} \varepsilon_j^n f^\xi(D^\xi_{\varepsilon_j} v_j(a))$$

$$= \limsup_j \sum_{i\in Z_j}\sum_{\xi\in I}\sum_{A\in R^\xi_1(Q(\beta r^{(n-1)/n}S_j))} T_j^{-n} f^\xi(T_j D^\xi_1 \zeta^i_j(A))$$

$$\leq \limsup_j \sum_{i\in Z_j} r^{1-n}\delta_j^n \left(\varphi_j^{\beta r^{(n-1)/n}}(u^i_j) + \eta\right)$$

$$\leq \limsup_j \sum_{i\in Z_j} r^{1-n}\delta_j^n \varphi_j^{\beta r^{(n-1)/n}}(u^i_j) + c\eta.$$

Taking into account Lemma 6.2 and Remark 4 we get

$$\limsup_j \sum_{i\in Z_j} F_{\varepsilon_j}(v_j; Q(\beta\delta_j, i\delta_j)) \leq r^{1-n}\int_\Omega \varphi^{\beta r^{(n-1)/n}}(u)\,dx + c\eta.$$

As far as (56) is concerned, by construction for all $i\in Z_j$ we have $v_j\equiv u^i_j$ on $Q_{\varepsilon_j}(\rho^i_j; i\delta_j)\setminus Q_{\varepsilon_j}(\beta\delta_j; i\delta_j)$. Since $f^\xi(0)=0$ for all $\xi\in I$, we get

$$\limsup_j \sum_{i\in Z_j} F_{\varepsilon_j}(v_j; Q(\rho^i_j; i\delta_j)\setminus Q(\beta\delta_j; i\delta_j)) = 0.$$

Now we focus our attention on (57); i.e.,

$$\limsup_j F_{\varepsilon_j}(v_j; \Omega_j\setminus(E_j\cup E'_j)) = \limsup_j F_{\varepsilon_j}(w_j; \Omega_j\setminus(E_j\cup E'_j)).$$

By Lemma 6.1 and (53) we get

$$\limsup_j F_{\varepsilon_j}(w_j; \Omega_j\setminus(E_j\cup E'_j)) \leq \limsup_j F_{\varepsilon_j}(u_j; \Omega_j\setminus(E_j\cup E'_j)) + \frac{c}{k}$$

$$\leq \limsup_j \mathcal{F}_{\varepsilon_j}(u_j) + \frac{c}{k} \leq \int_\Omega f_0(Du)\,dx + \frac{c}{k}.$$

Now we consider (58). By construction $\zeta^i_j = u^i_j$ on $\mathcal{S}_1([r^{(n-1)/n}\beta S_j - M], [r^{(n-1)/n}\beta S_j])$, hence $v_j = u^i_j$ on $Q_{\varepsilon_j}(\beta\delta_j, i\delta_j)\setminus Q_{\varepsilon_j}(\beta\delta_j - M\varepsilon_j - \varepsilon_j, i\delta_j)$. There follows that

$$\limsup_j \sum_{i\in Z_j}\sum_{\xi\in I}\sum_{a\in\mathcal{Y}^\xi_{\varepsilon_j}(\beta\delta_j,i\delta_j)} \varepsilon_j^n f^\xi(D^\xi_{\varepsilon_j} v_j(a)) = 0.$$

Moreover, we show that (59) is negligible. We have:

$$\limsup_j \sum_{i\in Z_j}\sum_{\xi\in I}\sum_{a\in\mathcal{Y}^\xi_{\varepsilon_j}(\rho^i_j,i\delta_j)} \varepsilon_j^n f^\xi(D^\xi_{\varepsilon_j} v_j(a))$$

$$\leq c\limsup_j \sum_{i\in Z_j}\sum_{\{a,b\}\in M_{\varepsilon_j}(Q(\rho^i_j+\varepsilon_j M, i\delta_j)\setminus Q(\rho^i_j-\varepsilon_j M, i\delta_j))} |v_j(a)-v_j(b)|^n$$

$$\leq c\limsup_j \sum_{i\in Z_j}\sum_{\{a,b\}\in M_{\varepsilon_j}(Q(\rho^i_j+\varepsilon_j M, i\delta_j)\setminus Q(\rho^i_j, i\delta_j))} |w_j(a)-w_j(b)|^n$$

$$\leq c\limsup_j \sum_{i\in Z_j} F_{\varepsilon_j}(w_j; C^j_i).$$

We recall that the computations in the proof of Lemma 6.1 imply that

$$\limsup_j \sum_{i\in Z_j} F_{\varepsilon_j}(w_j; C_i^j) \le \frac{c}{k},$$

hence

$$\limsup_j \sum_{i\in Z_j} \sum_{\xi\in I} \sum_{a\in\mathcal{Y}^{\xi}_{\varepsilon_j}(\rho^i_j, i\delta_j)} \varepsilon_j^n f^{\xi}(D^{\xi}_{\varepsilon_j} v_j(a)) \le \frac{c}{k}.$$

Finally, we deal with (60). By construction

$$\begin{aligned}
&\limsup_j \sum_{i\in Z'_j} F_{\varepsilon_j}(v_j; Q_{\varepsilon_j}(\rho_j, i\delta_j)\cap\Omega) \\
&\le c\limsup_j \sum_{i\in Z'_j} \sum_{\{a,b\}\in M_{\varepsilon_j}(Q(\rho_j,i\delta_j)\cap\Omega)} |v_j(a)-v_j(b)|^n \\
&\le c\limsup_j \sum_{i\in Z'_j} \sum_{\{a,b\}\in M_{\varepsilon_j}(Q(\rho_j,i\delta_j)\cap\Omega)} (|u_j(a)-u_j(b)|^n|\gamma_j(a-i\delta_j)|^n \\
&+ |u_j(b)|^n|\gamma_j(a-i\delta_j)-\gamma_j(b-i\delta_j)|^n).
\end{aligned}$$

Since (u_j) is bounded in $L^\infty(\Omega;\mathbb{R}^m)$ and (54) holds, we get

$$\begin{aligned}
&\limsup_j \sum_{i\in Z'_j} F_{\varepsilon_j}(v_j; Q(\rho_j, i\delta_j)\cap\Omega) \\
&\le c\limsup_j \sum_{i\in Z'_j} \sum_{\{a,b\}\in M_{\varepsilon_j}(Q(\rho_j,i\delta_j)\cap\Omega)} (\varepsilon_j^n|\gamma_j(a-i\delta_j)|^n + |\gamma_j(a-i\delta_j)-\gamma_j(b-i\delta_j)|^n).
\end{aligned}$$

By construction (γ_j) is bounded in $L^\infty(\Omega)$ and satisfies

$$(\log(\rho_jT_j))^{n-1} \sum_{\{a,b\}\in M_{\varepsilon_j}(Q(\rho_j,i\delta_j)\cap\Omega)} |\gamma_j(a-i\delta_j)-\gamma_j(b-i\delta_j)|^n \le c+\eta\log(\rho_jT_j))^{n-1}.$$

Since $(\log(\rho_jT_j))^{n-1}/(\log(T_j))^{n-1}\to 1$ as $j\to+\infty$ and $(\log T_j)^{n-1} = r^{n-1}\delta_j^{-n} + o(1)$, we get

$$\begin{aligned}
\limsup_j \sum_{i\in Z'_j} F_{\varepsilon_j}(v_j; Q(\rho_j, i\delta_j)\cap\Omega) &\le \limsup_j c\sum_{i\in Z'_j}\delta_j^n + \eta \\
&\le \limsup_j |\Omega'_j| + \eta|\Omega| = \eta|\Omega|,
\end{aligned}$$

where $\Omega'_j = \cup_{i\in Z'_j}\mathbf{Q}_{\varepsilon_j}(\rho_j; i\delta_j)\cap\Omega$.

To sum up the estimates we got so far, we have

$$\limsup_j F_{\varepsilon_j}(v_j) \le \int_\Omega f_0(Du)\,dx + r^{1-n}\int_\Omega \varphi^{\beta r^{(n-1)/n}}(u)\,dx + \frac{c}{k} + c\eta. \tag{62}$$

It remains to show that $v_j\to u$ in $L^1(\Omega)$. By construction $|\{u_j\neq v_j\}|\to 0$ and $u_j\to u$ in $L^1(\Omega)$. Since (Du_j) and (Dv_j) are bounded in $L^1(\Omega)$, by a compactness argument we deduce that $u_j - v_j\to 0$ in $L^1(\Omega)$ and then $v_j\to u$ in $L^1(\Omega)$.

Finally, we let $\beta\to 0^+$ and $k\to+\infty$ in (62) and we obtain

$$\limsup_j F_{\varepsilon_j}(v_j) \le \int_\Omega f_0(Du)\,dx + r^{1-n}\int_\Omega \varphi(u)\,dx,$$

as desired.

A.2 Subcritical exponent $p < n$. We want to modify (u_j) in order to get an approximate recovery sequence for u. Let $k \in \mathbb{N}$ be equal to $[1/\eta]$. Let $\rho_j = \beta\delta_j$, with $\beta < 1/2$. By applying Lemma 6.1 to the sequence (u_j), we get a modified sequence $w_j \to u$ in $L^1(\Omega;\mathbb{R}^m)$ such that conditions (27)-(29) are satisfied. We build an approximate recovery sequence v_j by carefully modifying w_j close to the pinning sites. To this purpose we define the sets

$$E_j = \bigcup_{i\in Z_j} Q_{\varepsilon_j}(\rho_j^i, i\delta_j) \qquad \text{and } E'_j = \bigcup_{i\in Z'_j} Q_{\varepsilon_j}(\rho_j, i\delta_j)\cap\Omega,$$

where $Z'_j = \{i \in \mathbb{Z}^n \setminus Z_j : i\delta_j \in \Omega\}$ indexes the pinning sites which are close to the boundary of Ω. We will deal separately with E_j, E'_j and $\Omega_j \setminus (E_j \cup E'_j)$ (steps **A.2.1**, **A.2.2** and **A.2.3** respectively). Let $N > 0$ be fixed.

A.2.1. Firstly, we deal with E_j. For all $i \in Z_j$ we consider a function $\mu^N_{i,j} \in \mathcal{A}_1(Q(N);\mathbb{R}^m)$ such that $\mu^N_{i,j}(0) = 0$, $\mu^N_{i,j} = u^i_j$ on $\mathcal{S}_1([N-M],[N])$ and

$$\sum_{\xi\in I}\sum_{A\in R_1^\xi(Q(N))} T_j^{-p} f^\xi(T_j D_1^\xi \mu^N_{i,j}(A)) < \phi_j^N(u_j^i) + \eta.$$

We define $v_j : E_j \to \mathbb{R}^m$ as

$$v_j(a) = \begin{cases} \mu^N_{i,j}\Big(\dfrac{a - i\delta_j}{\varepsilon_j}\Big) & \text{for } a \in Q_{\varepsilon_j}(N\varepsilon_j, i\delta_j),\ i \in Z_j \\ u_j^i & \text{for } a \in Q_{\varepsilon_j}(\rho_j^i, i\delta_j)\setminus Q_{\varepsilon_j}(N\varepsilon_j, i\delta_j),\ i \in Z_j. \end{cases}$$

A.2.2. In this step we focus on E'_j and the pinning sites which are close to the boundary of Ω. For N as in the previous step, we consider a scalar function $\mu^N \in \mathcal{A}_1(Q(N))$ such that $\mu^N(0) = 0$, $\mu^N = 1$ on $\mathcal{S}_1([N-M],[N])$ and $0 \le \mu^N \le 1$. We define $v_j : E'_j \to \mathbb{R}^m$ as

$$v_j(a) = u_j(a)\mu^N(a), \text{ for } a \in E'_j.$$

A.2.3. Finally we set $v_j(a) = w_j(a)$ for all $a \in \Omega_j \setminus (E_j \cup E'_j)$.

We then have:

$$\limsup_j F_{\varepsilon_j}(v_j) \le \limsup_j \sum_{i\in Z_j} F_{\varepsilon_j}(v_j; Q(N\varepsilon_j, i\delta_j)) \tag{63}$$

$$+\limsup_j \sum_{i\in Z_j} F_{\varepsilon_j}(v_j; Q_{\varepsilon_j}(\rho_j^i, i\delta_j)\setminus Q_{\varepsilon_j}(N\varepsilon_j, i\delta_j)) \tag{64}$$

$$+\limsup_j F_{\varepsilon_j}(v_j; \Omega_j \setminus (E_j \cup E'_j)) \tag{65}$$

$$+\limsup_j \sum_{i\in Z_j}\sum_{\xi\in I}\sum_{a\in \mathcal{Y}^\xi_{\varepsilon_j}(N\varepsilon_j; i\delta_j)} f^\xi(D^\xi_{\varepsilon_j} v_j(a))\varepsilon_j^n \tag{66}$$

$$+\limsup_j \sum_{i\in Z_j}\sum_{\xi\in I}\sum_{a\in \mathcal{Y}^\xi_{\varepsilon_j}(\rho_j^i; i\delta_j)} f^\xi(D^\xi_{\varepsilon_j} v_j(a))\varepsilon_j^n \tag{67}$$

$$+\limsup_j \sum_{i\in Z'_j} F_{\varepsilon_j}(v_j; Q_{\varepsilon_j}(\rho_j, i\delta_j)\cap\Omega) \tag{68}$$

$$+\limsup_j \sum_{i\in Z'_j}\sum_{\xi\in I}\sum_{a\in \mathcal{Y}^\xi_{\varepsilon_j}(\rho_j; i\delta_j)\cap\Omega} f^\xi(D^\xi_{\varepsilon_j} v_j(a))\varepsilon_j^n. \tag{69}$$

Arguing similarly to paragraph **A.1**, we deduce that (64), (66) and (67) are infinitesimal. As far as (63) is concerned, by construction we have

$$\begin{aligned}
&\limsup_j \sum_{i\in Z_j} F_{\varepsilon_j}(v_j; Q(N\varepsilon_j, i\delta_j)) \\
&= \limsup_j \sum_{i\in Z_j} \sum_{\xi\in I} \sum_{a\in R^{\xi}_{\varepsilon_j}(Q(N\varepsilon_j, i\delta_j))} f^{\xi}(D^{\xi}_{\varepsilon_j} v_j(a))\varepsilon_j^n \\
&= \limsup_j \sum_{i\in Z_j} \sum_{\xi\in I} \sum_{A\in R^{\xi}_1(Q(N))} f^{\xi}(T_j D^{\xi}_1 \mu^N_{i,j}(A)) T_j^{-n} \\
&\le \limsup_j \sum_{i\in Z_j} r^{1-n}\delta_j^n(\phi^N_j(u^i_j)+\eta) \le \limsup_j \sum_{i\in Z_j} r^{1-n}\delta_j^n \phi^N_j(u^i_j) + c\eta|\Omega|.
\end{aligned}$$

By Lemma 6.2 and Remark 3 we get

$$\limsup_j \sum_{i\in Z_j} F_{\varepsilon_j}(v_j; Q(N\varepsilon_j, i\delta_j)) \le r^{1-n}\int_\Omega \phi^N(u)\,dx + c\eta.$$

In order to estimate (65) we note that Lemma 6.1 implies

$$\begin{aligned}
\limsup_j F_{\varepsilon_j}(v_j; \Omega_j\setminus(E_j\cup E'_j)) &= \limsup_j F_{\varepsilon_j}(w_j; \Omega_j\setminus(E_j\cup E'_j)) \\
&\le \limsup_j F_{\varepsilon_j}(u_j; \Omega_j\setminus(E_j\cup E'_j)) + \frac{c}{k} \\
&\le \limsup_j \mathcal{F}_{\varepsilon_j}(u_j) + \frac{c}{k} \le \int_\Omega f_0(Du)\,dx + \frac{c}{k}.
\end{aligned}$$

It remains to show that (68) and (69) are negligible. By the definition of v_j on E'_j and the equiboundedness of (u_j) we get

$$\begin{aligned}
&\limsup_j \sum_{i\in Z'_j} F_{\varepsilon_j}(v_j; Q_{\varepsilon_j}(\rho_j, i\delta_j)\cap\Omega) \\
&\le c\limsup_j \sum_{i\in Z_j} \sum_{\{a,b\}\in M_{\varepsilon_j}(Q(\rho_j, i\delta_j)\cap\Omega)} \varepsilon_j^{n-p}(|u_j(a)\mu^N(a) - u_j(b)\mu^N(b)|^p + \varepsilon_j^p) \\
&\le c\limsup_j \sum_{i\in Z'_j} \sum_{\{a,b\}\in M_{\varepsilon_j}(Q(\rho_j, i\delta_j)\cap\Omega)} \varepsilon_j^{n-p}(|u_j(a) - u_j(b)|^p + |\mu^N(a)-\mu^N(b)|^p + \varepsilon_j^p).
\end{aligned}$$

By (54) we deduce that

$$\limsup_j \sum_{i\in Z'_j} F_{\varepsilon_j}(v_j; Q_{\varepsilon_j}(\rho_j, i\delta_j)\cap\Omega) \le c\limsup_j \sum_{i\in Z'_j} \delta_j^n = c\limsup_j |\mathbf{E}'_j| = |\partial\Omega| = 0.$$

Finally, we can prove that (69) is infinitesimal in a similar way, using the equiboundedness of (u_j) and the fact that $|\mathbf{E}'_j|$ tends to zero.

To sum up, we proved that

$$\limsup_j F_j(v_j) \le \int_\Omega f_0(Du)\,dx + r^{1-n}\int_\Omega \phi^N(u)\,dx + \frac{c}{k} + c\eta.$$

Note that the sequence v_j we built converges to u strongly in $L^1(\Omega;\mathbb{R}^m)$. This follows from $|\{u_j \neq v_j\}| \to 0$ and a compactness argument. Passing to the limit as

$N \to +\infty$ we have

$$\limsup_j F_j(v_j) \le \int_\Omega f_0(Du)\,dx + r^{1-n}\int_\Omega \phi(u)\,dx + \frac{c}{k} + c\eta,$$

which proves the existence of an approximate recovery sequence for u for each value of the parameter η. Hence, for all piecewise affine functions in $W^{1,p}(\Omega;\mathbb{R}^m)$ there exists a recovery sequence.

B. We can finally prove the Γ-lim sup inequality by using a density argument. For any $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ there exists a sequence (u_k) of piecewise affine functions such that $u_k \to u$ strongly in $W^{1,p}(\Omega;\mathbb{R}^m)$. In step **A** we proved that for all $k \in \mathbb{N}$ the Γ-lim sup $F''(u_k)$ satisfies

$$F''(u_k) \le F(u_k).$$

By the lower semicontinuity of F'' with respect to the strong $L^p(\Omega;\mathbb{R}^m)$-convergence and the continuity of F with respect to $W^{1,p}(\Omega;\mathbb{R}^m)$-convergence, we get

$$F''(u) \le \liminf_k F''(u_k) \le \liminf_k F(u_k) = F(u),$$

as desired. □

9. **Special cases.** In this section we focus on two cases in which the densities of the Γ-limit are given by explicit formulas.

9.1. **Convex energy densities.** If for all $\xi \in I$ $f^\xi : \mathbb{R}^m \to [0,+\infty)$ is a convex function, then the density function in the bulk term of the Γ-limit equals

$$f_0(A) = \sum_{\xi\in I} f^\xi\Big(A\cdot\frac{\xi}{|\xi|}\Big) \quad \text{for all } A \in \mathbb{M}^{m\times n}. \tag{70}$$

In fact, under the convexity condition we can use [1, Remark 5.3], which states that in this case Proposition 1 holds with f_0 as in (70). Then the Γ-limit is

$$F(u) = \sum_{\xi\in I}\int_\Omega f^\xi\Big(Du\cdot\frac{\xi}{|\xi|}\Big)\,dx + r^{1-n}\int_\Omega \Phi(u)\,dx.$$

9.2. **Nearest neighbors interactions and homogeneous density functions in the critical case.** In this paragraph we consider a special case which is of some interest on its own, despite being very specific. We are in the critical case $p = n$ and we consider nearest neighbors interactions only. Moreover, we assume that the functions f^ξ, $\xi \in I = \{e_1, \dots, e_n\}$, are all equal to the same function f, which is positively homogeneous of degree n and convex. In particular, these assumptions encompass the case $f(z) = \|z\|_n^n$, which has been analyzed in Section 4.

In this case the Γ-convergence result holds for the whole sequence F_{ε_j} and the limit functional F is given by

$$F(u) = \sum_{i=1}^n \int_\Omega f\Big(\frac{\partial u}{\partial x_i}\Big)\,dx + \int_\Omega d(u)\,dx,$$

where $d : \mathbb{R}^m \to [0,+\infty)$ equals

$$d(z) = \lim_{T\to+\infty}(\log T)^{n-1}\inf\Big\{\sum_{\{A,B\}\in M_1(Q(T))} f(v(A)-v(B)) :$$
$$v \in \mathcal{A}_1(Q(T);\mathbb{R}^m),\ v(0)=0,\ v=z \text{ on } \mathcal{S}_1([T])\Big\}.$$

Let us prove that the function d is well defined.

Lemma 9.1. *Let $f : \mathbb{R}^m \to [0, +\infty)$ be a convex function which is positively homogeneous of degree n and such that $f(0) = 0$. We assume that there exist two constants $c_1, c_2 > 0$ such that $c_1|z|^n \le f(z) \le c_2|z|^n$ for all $z \in \mathbb{R}^m$. Then for all $z \in \mathbb{R}^m$ there exists the limit*

$$d(z) = \lim_{T\to+\infty} (\log T)^{n-1} \inf\Big\{ \sum_{\{A,B\}\in M_1(Q(T))} f(v(A) - v(B)) : \\ v \in \mathcal{A}_1(Q(T);\mathbb{R}^m),\ v(0) = 0,\ v = z \text{ on } \mathcal{S}_1([T])\Big\}.$$

Proof. By the homogeneity of f, it suffices to prove the existence of $d(\nu)$, with $\nu \in \mathbb{R}^m$ and $|\nu| = 1$. We denote by μ_T the infimum which appears in the definition of $d(\nu)$:

$$\mu_T = \inf\Big\{ \sum_{\{A,B\}\in M_1(Q(T))} f(v(A) - v(B)) : \begin{array}{l} v \in \mathcal{A}_1(Q(T);\mathbb{R}^m) \\ v(0) = 0,\ v = \nu \text{ on } \mathcal{S}_1([T]) \end{array} \Big\}.$$

It is convenient to introduce a new family of infima $\tilde\mu_T$, defined as

$$\tilde\mu_T = \min\Big\{ \sum_{\{A,B\}\in M_1(Q(T))} f(v(A) - v(B)) : \begin{array}{l} v \in \mathcal{A}_1(Q(T);\mathbb{R}^m) \\ v = 0 \text{ on } Q_1(1),\ v = \nu \text{ on } \mathcal{S}_1([T]) \end{array} \Big\}.$$

The test functions for $\tilde\mu_T$ vanish on the whole set $Q_1(1)$ (not only on 0 as for μ_T). The proof is made of two steps: firstly we show that there exists the limit

$$\lim_{T\to+\infty} (\log T)^{n-1} \tilde\mu_T \in [0, +\infty);$$

and then we prove that the limit above equals

$$\lim_{T\to+\infty} (\log T)^{n-1} \mu_T = d(\nu).$$

1. Let $S \gg T$. Let $u_T \in \mathcal{A}_1(Q(T);\mathbb{R}^m)$ be such that $u_T = 0$ on $Q_1(1)$, $u_T(A) = \nu$ for all $A \in \mathcal{S}_1([T])$ and

$$\sum_{\{A,B\}\in M_1(Q(T))} f(u_T(A) - u_T(B)) \le \tilde\mu_T + \frac{1}{T}.$$

We will define a convenient test function for m_S by suitably modifying u_T and we will deduce an inequality of the form

$$(\log S)^{n-1}\tilde\mu_S \le (\log T)^{n-1}\tilde\mu_T + r(S,T) \qquad \text{with } \liminf_{T\to+\infty}\limsup_{S\to+\infty} r(S,T) = 0.$$

Let $k \in \mathbb{N}$ be such that $[T]^k \le [S] < [T]^{k+1}$; i.e., $k = [\log([S])/\log([T])]$. We consider the set $Q_1(S)$ and we denote by C_h its subsets

$$C_h = Q_1([T]^{h+1}) \setminus Q_1([T]^h - 1) \qquad h = 0, \dots, k-1.$$

In each C_h we consider an additional *meso-lattice* $C_h \cap [T]^h\mathbb{Z}^n$ and we use it to define a convenient test function u_S for $\tilde\mu_S$. For all $A \in C_h \cap [T]^h\mathbb{Z}^n$ we set

$$u_{S,h}(A) = \frac{\log([S])}{\log([T])}\Big(u_T\Big(\frac{A}{[T]^h}\Big) + h\nu\Big).$$

We denote by $\tilde u_{S,h}$ an interpolating function for $u_{S,h}$ which is piecewise affine on a triangulation defined by the lattice $C_h \cap [T]^h\mathbb{Z}^n$ and satisfies

$$\sum_{\{A,B\}\in M_{[T]^h}(C_h)} f(\tilde u_S(A) - \tilde u_S(B)) = \sum_{l=1}^n \int_{\mathbf{C}_h} f\Big(\frac{\partial \tilde u_S}{\partial x_l}(x)\Big)dx.$$

The function $\tilde{u}_{S,h}$ can be built by replicating the procedure developed in [2, Section 4.1]. In particular, we can choose our triangulations of $C_h \cap [T]^h\mathbb{Z}^n$ to be homothetic to each other (since the sets are obtained by rescaling). The test function $u_S \in \mathcal{A}_1(Q(S);\mathbb{R}^m)$ is defined as follows:

$$u_S(A) = \begin{cases} \tilde{u}_{S,h}(A) & \text{for } A \in C_h,\ h = 0,\dots,k-1 \\ \nu \dfrac{\log([T]^k + q)}{\log([S])} & \text{for } A \in \mathcal{S}_1([T]^k + q),\ q = 1,\dots,[S]-[T]^k. \end{cases}$$

Then u_S is an admissible test function for $\tilde{\mu}_S$; in fact $u_S = 0$ on $Q_1(1)$ and $u_S = \nu$ on $\mathcal{S}_1([S])$. Now we want to estimate the energy of u_S on $Q(S)$:

$$\begin{aligned} &\sum_{\{A,B\}\in M_1(Q(S))} f(u_S(A)-u_S(B)) \\ &\quad\le \sum_{h=0}^{k-1} \sum_{\{A,B\}\in M_1(C_h)} f(u_S(A)-u_S(B)) + \sum_{\{A,B\}\in M_1(Q(S)\setminus Q([T]^k))} f(u_S(A)-u_S(B)) \\ &\quad\le \sum_{h=0}^{k-1}\sum_{l=1}^{n} \int_{\mathbf{C}_h} f\Big(\frac{\partial \tilde{u}_{S,h}}{\partial x_l}(x)\Big)dx + \sum_{\{A,B\}\in M_1(Q(S)\setminus Q([T]^k))} f(u_S(A)-u_S(B)). \end{aligned}$$

If we set $y = [T]^{-h}x$ and we denote by $\tilde{u}_T$ the piecewise affine interpolation of u_T on the lattice $Q_1(T)$ (built on a triangulation that is homothetic to the one on which we constructed $\tilde{u}_S$), we obtain

$$\begin{aligned} \int_{\mathbf{C}_h} f\Big(\frac{\partial \tilde{u}_{S,h}}{\partial x_l}(x)\Big)dx &= \Big(\frac{\log([T])}{\log([S])}\Big)^n \int_{\mathbf{C}_1} f\Big(\frac{\partial \tilde{u}_T}{\partial y_l}(y)[T]^{-h}\Big)[T]^{hn}dy \\ &= \Big(\frac{\log([T])}{\log([S])}\Big)^n \sum_{A\in R_1^{e_l}(Q(T))} f(u_T(A+e_l)-u_T(A)) \end{aligned}$$

for all $l \in \{1,\dots,n\}$. There follows that

$$\begin{aligned} &\sum_{h=0}^{k-1} \sum_{\{A,B\}\in M_1(C_h)} f(u_S(A)-u_S(B)) \\ &\quad= \Big(\frac{\log([T])}{\log([S])}\Big)^n \sum_{h=0}^{k-1}\sum_{l=1}^{n} \sum_{A\in R_1^{e_l}(\overline{Q}(T))} f(u_T(A+e_l)-u_T(A)) \\ &\quad= k\Big(\frac{\log([T])}{\log([S])}\Big)^n \sum_{\{A,B\}\in M_1(Q(T))} f(u_T(A)-u_T(B)) \\ &\quad\le \Big[\frac{\log([S])}{\log([T])}\Big]\Big(\frac{\log([T])}{\log([S])}\Big)^n \Big(m_T + \frac{1}{T}\Big). \end{aligned} \tag{71}$$

Finally we consider the contribution of u_S on the set $Q_1(S) \setminus Q_1([T]^k)$. By construction

$$
\begin{aligned}
&\sum_{\{A,B\}\in M_1(Q(S)\setminus Q([T]^k))} f(u_S(A)-u_S(B)) \\
&\qquad \le cn^2 \sum_{q=0}^{[S]-[T]^k-1} f\Big(\nu \frac{\log([T]^k+q+1)-\log([T]^k+q)}{\log([S])}\Big) \\
&\qquad \le \frac{c}{(\log([S]))^n}([S]-[T]^k)\Big|\log\Big(1+\frac{1}{[T]^k}\Big)\Big|^n \\
&\qquad \le \frac{c}{(\log([S]))^n}([S]-[T]^k)\frac{1}{[T]^{kn}}. \qquad (72)
\end{aligned}
$$

By combining (71) and (72) we get

$$
\begin{aligned}
(\log S)^{n-1}\tilde{\mu}_S &\le (\log S)^{n-1} \sum_{\{A,B\}\in M_1(Q(S))} f(u_S(A)-u_S(B)) \\
&\le \Big[\frac{\log([S])}{\log([T])}\Big]\frac{\log([T])}{\log([S])}\Big((\log([T]))^{n-1}\tilde{\mu}_T + \frac{(\log([T]))^{n-1}}{T}\Big) \\
&\quad + \frac{c}{\log([S])}([S]-[T]^k)\frac{1}{[T]^{kn}}.
\end{aligned}
$$

Passing to the lim sup as $S \to +\infty$ we obtain

$$
\limsup_{S\to+\infty}(\log S)^{n-1}\tilde{\mu}_S \le (\log([T]))^{n-1}\tilde{\mu}_T + \frac{(\log([T]))^{n-1}}{T},
$$

since $k = [\log([S])/\log([T])]$. Finally, we take the lim inf as $T \to +\infty$ and we get

$$
\begin{aligned}
\limsup_{S\to+\infty}(\log S)^{n-1}\tilde{\mu}_S &\le \liminf_{T\to+\infty}(\log([T]))^{n-1}\tilde{\mu}_T + \lim_{T\to+\infty}\frac{(\log([T]))^{n-1}}{T} \\
&= \liminf_{T\to+\infty}(\log([T]))^{n-1}\tilde{\mu}_T.
\end{aligned}
$$

Hence, there exists the limit

$$
\lim_{T\to+\infty}(\log([T]))^{n-1}\tilde{\mu}_T. \qquad (73)
$$

Note that for all $\nu \in \mathbb{R}^m$, $|\nu| = 1$, the limit above is in $(0, +\infty)$. In fact, by the growth conditions on f there exist two constants $\tilde{c}_1, \tilde{c}_2 > 0$ such that

$$
\tilde{c}_1 m^d_{1,T} \le \tilde{\mu}_T \le \tilde{c}_2 m^d_{1,T},
$$

where $m^d_{1,T}$ is as in (16). In Section 4, Lemma 4.1, we proved that

$$
\lim_{T\to+\infty}(\log T)^{n-1}m^d_{1,T} = \omega_{n-1} \in (0, +\infty).
$$

By comparison, $\lim_T(\log T)^{n-1}\tilde{\mu}_T \in (0, +\infty)$.

2. It remains to show that the limit in (73) equals $d(\nu)$. First of all, we note that $\mu_T \le \tilde{\mu}_T$ by construction. Let $v_T \in \mathcal{A}_1(Q(T); \mathbb{R}^m)$ be such that $v_T(0) = 0$, $v_T = \nu$ on $\mathcal{S}_1([T])$ and

$$
\sum_{\{A,B\}\in M_1(Q(T))} f(v_T(A)-v_T(B)) \le \mu_T + \frac{1}{T}.
$$

Let $\eta > 0$ be a fixed constant. Then, for all T large enough we have $|v_T| \leq \eta$ on $Q_1(1)$. In fact: if $|v_T(a)| > \eta$ for some $a \in Q_1(1) \setminus \{0\}$, then we have

$$\tilde{\mu}_T + \frac{1}{T} \geq \mu_T + \frac{1}{T} \geq \sum_{\{A,B\} \in M_1(Q(T))} f(v_T(A) - v_T(B)) > c\eta^n.$$

By (73) we know that

$$\lim_{T \to +\infty} \tilde{\mu}_T + \frac{1}{T} = 0,$$

which leads to a contradiction. Therefore we have

$$\begin{aligned}
\mu_T + \frac{1}{T} &\geq \sum_{\{A,B\} \in M_1(Q(T))} f(v_T(A) - v_T(B)) \\
&\geq \inf\Big\{ \sum_{\{A,B\} \in M_1(Q(T))} f(v(A) - v(B)) : \\
&\qquad v \in \mathcal{A}_1(Q(T)),\ |v| \leq \eta \text{ on } Q_1(1),\ v = 1 \text{ on } \mathcal{S}_1([T]) \Big\} \\
&= \inf\Big\{ \sum_{\{A,B\} \in M_1(Q(T))} f(v(A) - v(B)) : \\
&\qquad v \in \mathcal{A}_1(Q(T)),\ v = \eta \text{ on } Q_1(1),\ v = 1 \text{ on } \mathcal{S}_1([T]) \Big\} \\
&= |1-\eta|^n \inf\Big\{ \sum_{\{A,B\} \in M_1(Q(T))} f(w(A) - w(B)) : \\
&\qquad w \in \mathcal{A}_1(Q(T)),\ w = 0 \text{ on } Q_1(1),\ w = 1 \text{ on } \mathcal{S}_1([T]) \Big\} \\
&= |1-\eta|^n \tilde{\mu}_T.
\end{aligned}$$

To sum up, we got

$$\tilde{\mu}_T + \frac{1}{T} \geq \mu_T + \frac{1}{T} \geq |1-\eta|^n \tilde{\mu}_T.$$

If we multiply by $(\log T)^{n-1}$, pass to the limit as $T \to +\infty$ and take into consideration the arbitrariness of η, we deduce that the limit in (73) equals $d(\nu)$.

Finally, we notice that d can by extended to any vector in $\mathbb{R}^m$ by n-homogeneity:

$$d(z) = \begin{cases} 0 & \text{if } z = 0 \\ |z|^n d\Big(\dfrac{z}{|z|}\Big) & \text{otherwise.} \end{cases}$$

□

In conclusion, we can state and prove the Γ-convergence result in this particular case.

Proposition 5. *Let $m, n \in \mathbb{N}$ with $m \geq 1$ and $n \geq 2$. Let Ω be a bounded open subset of $\mathbb{R}^n$ with $|\partial\Omega| = 0$. Let $f : \mathbb{R}^m \to [0, +\infty)$ be a convex function which is positively homogeneous of degree n and such that $f(0) = 0$. We assume that there exist two constants $c_1, c_2 > 0$ such that $c_1|z|^n \leq f(z) \leq c_2|z|^n$ for all $z \in \mathbb{R}^m$. Let (ε_j) be a positive infinitesimal sequence. We consider an additional sequence (δ_j) such that $\delta_j/\varepsilon_j \in \mathbb{N}$, $\delta_j \gg \varepsilon_j$, $\delta_j \to 0$ and*

$$\varepsilon_j = e^{-r(1+o(1))\delta_j^{n/(1-n)}}, \qquad \textit{for some constant } r > 0.$$

For all $j \in \mathbb{N}$ we define the functional $F_{\varepsilon_j} : \mathcal{A}_{\varepsilon_j}(\Omega) \to [0, +\infty]$ as

$$F_{\varepsilon_j}(u) = \begin{cases} \displaystyle\sum_{\{a,b\} \in M_{\varepsilon_j}(\Omega)} f(u(a) - u(b)) & \text{if } u = 0 \text{ on } \Omega_{\delta_j} \\ +\infty & \text{otherwise.} \end{cases}$$

Then F_{ε_j} Γ-converges, with respect to $L^1(\Omega; \mathbb{R}^m)$-convergence, to the limit functional $F : W^{1,n}(\Omega; \mathbb{R}^m) \to [0, +\infty)$ given by

$$F(u) = \sum_{i=1}^{n} \int_\Omega f\Big(\frac{\partial u}{\partial x_i}\Big)\, dx + \int_\Omega d(u)\, dx,$$

where $d : \mathbb{R}^m \to [0, +\infty)$ is obtained as

$$d(z) = \lim_{T \to +\infty} (\log T)^{n-1} \inf \Big\{ \sum_{\{A,B\} \in M_1(Q(T))} f(v(A) - v(B)) : \\ v \in \mathcal{A}_1(Q(T); \mathbb{R}^m),\ v(0) = 0,\ v = z \text{ on } \mathcal{S}_1([T]) \Big\}.$$

Proof. The proof follows immediately from Theorem 3.1, Lemma 9.1 and (70). By Theorem 3.1 and (70) we deduce that there exists a subsequence (ε_{j_k}) such that $F_{\varepsilon_{j_k}}$ Γ-converges to

$$F(u) = \sum_{i=1}^{n} \int_\Omega f\Big(\frac{\partial u}{\partial x_i}\Big)\, dx + \int_\Omega \varphi(u)\, dx,$$

where

$$\varphi(z) = \lim_{\alpha \to 0^+} \lim_{k \to +\infty} |\log \varepsilon_{j_k}|^{n-1} \\ \inf \Big\{ \sum_{\{A,B\} \in M_1(Q(\alpha S_{j_k}))} f(v(A) - v(B)) : \begin{array}{l} v \in \mathcal{A}_1(\overline{Q}(\alpha S)) \\ v(0) = 0 \\ v = z \text{ on } \mathcal{S}_1([\alpha S_{j_k}]) \end{array} \Big\}$$

and $S_{j_k} = \varepsilon_{j_k}^{-1} |\log \varepsilon_{j_k}|^{(1-n)/n}$. Note that $|\log \varepsilon_{j_k}| / \log(\alpha S_{j_k}) \to 1$ for any value of $\alpha > 0$. Then

$$\varphi(z) = \lim_{\alpha, k} |\log \alpha S_{j_k}|^{n-1} \\ \inf \left\{ \sum_{\{A,B\} \in M_1(Q(\alpha S_{j_k}))} f(v(A) - v(B)) : \begin{array}{l} v \in \mathcal{A}_1(\overline{Q}(\alpha S)) \\ v(0) = 0 \\ v = z \text{ on } \mathcal{S}_1([\alpha S_{j_k}]) \end{array} \right\}.$$

By Lemma 5 we can deduce that $\varphi(z) = d(z)$ for all $z \in \mathbb{R}^m$, and d is independent of the subsequence ε_{j_k}, as desired. □

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