NETWORKS AND HETEROGENEOUS MEDIA doi:10.3934/nhm.2012.7.525

Volume **7**, Number **3**, September **2012** pp. **525–541**

ON THE SIGNED POROUS MEDIUM FLOW

Edoardo Mainini

Département de Mathématiques, UMR 8628 Université Paris-Sud 11-CNRS,
Bâtiment 425, Faculté des Sciences d'Orsay, Université Paris-Sud 11,
F-91405 Orsay Cedex, France

(Communicated by Juan Luis Vazquez)

Abstract. We prove that the signed porous medium equation can be regarded as limit of an optimal transport variational scheme, therefore extending the classical result for positive solutions of [13] and showing that an optimal transport approach is suited even for treating signed densities.

1. Introduction.

Let us consider the signed porous medium equation

$$\partial_t \rho - \Delta(|\rho|^{m-1}\rho) = 0 \qquad \text{in} \quad (0,+\infty)\times\mathbb{R}^n, \qquad m>1. \tag{1}$$

We associate the energy functional

$$\mathscr{F}(\rho) := \frac{1}{m-1}\int_{\mathbb{R}^n} |\rho(x)|^m \, dx. \tag{2}$$

The porous medium equation is the standard example of nonlinear diffusion with finite speed of propagation. The case of signed equations often appears in the literature. It has been analyzed for instance in [4, 6, 7, 8, 11, 19], for the sake of self-similar solutions and for the study of asymptotic behavior and of the associated positivity set free boundary. The well-posedness theory is well established. One may refer for instance to [3], or to the exhaustive overview contained in the book [17].

In this paper, we are interested in the approach introduced by F. Otto: a conservation equation with velocity vector field being a gradient, is viewed as the gradient flow of an energy functional defined on the space of positive measures with given mass α, endowed with the optimal transport structure. In particular, the theory applies to the "positive" porous medium equation $\partial_t\rho = \Delta\rho^m$ (see [13, 14, 15]), which can be written as

$$\partial_t\rho - div\left(\left(\nabla\frac{m\rho^{m-1}}{m-1}\right)\rho\right) = 0, \tag{3}$$

enlightening the fact that the velocity vector field is gradient of the formal variation of functional $\mathscr{F}$, defined in (2). In order to give rigor to this point of view, one needs to go into the metric-Riemaniann structure of the space $\mathscr{M}_2^\alpha(\mathbb{R}^n)$ of positive

2000 *Mathematics Subject Classification.* 35A15, 35K55, 35K65.

Key words and phrases. Porous media equation, gradient flow, optimal transport, changing sign-solutions, signed transport.

The author is supported by a postdoctoral scholarship of the Fondation Mathématique Jacques Hadamard, he acknowledges hospitality from Paris-sud University. He also acknowledges partial support from the project FP7-IDEAS-ERC-StG Grant #200497 (BioSMA) .

measures with mass α and finite second moment. Such structure comes from optimal transportation theory (see [15]). The quadratic optimal transport distance on $\mathscr{M}_2^\alpha(\mathbb{R}^n)$ is defined as

$$W_2(\mu,\nu) = \inf \left(\int_{\mathbb{R}^n\times\mathbb{R}^n} |x-y|^2 \, d\gamma \right)^{1/2}, \tag{4}$$

where γ varies in the set of measures in the space $\mathscr{M}^\alpha(\mathbb{R}^n\times\mathbb{R}^n)$ having μ and ν as marginals. As $(\mathscr{M}_2^\alpha(\mathbb{R}^n), W_2)$ can be seen as a Riemaniann manifold with a differential structure, it is possible to define a gradient flow of a functional $\mathscr{G}$ giving sense to the standard steepest descent relation

$$\dot{x}(t) = -\nabla\mathscr{G}(x(t)),$$

introducing the suitable notions of tangent vector $\boldsymbol{v} = \dot{x}$ to the curve $t \mapsto x(t)$ and of Fréchet differential of $\mathscr{G}$. For the porous media equation, the role of the tangent vector is played by the velocity field of the continuity equation (3), that is

$$\boldsymbol{v} = -\nabla \frac{\delta\mathscr{F}}{\delta\rho} = -\nabla \frac{m\rho^{m-1}}{m-1}.$$

On the other hand, there is a natural time discretization for this relation in Wasserstein setting, which consists in the following Euler implict scheme. Given $\mu^0 \in \mathscr{M}_2^\alpha(\mathbb{R}^n)$ and a time step $\tau > 0$, we solve recursively

$$\min_{\nu\in\mathscr{M}_2^\alpha(\mathbb{R}^n)} \mathscr{F}(\nu) + \frac{1}{2\tau} W_2^2(\mu^{k-1},\nu).$$

A solution to the scheme is a curve $t \mapsto \mu(t) \in \mathscr{M}_2^\alpha(\mathbb{R}^n)$, obtained as limit $(\tau \to 0)$ of a family of suitable time interpolations $\{\mu_\tau^k(\cdot)\}_{\tau>0,k\in\mathbb{N}}$ of the discrete minimizers. It is shown in [13] that indeed such limit is the unique solution to the porous medium equation in the standard sense of weak solutions, recalled later in the introduction. Notice also that the discrete approach, known as the minimizing movements scheme, does not require any geometric structure to be exploited. Therefore it can be independently used to give another natural notion of gradient flow, which will be equivalent to the "geometric-differential" one under suitable conditions.

The literature originated after the introduction of this point of view, together with the development of the theory for optimal transportation, enlightened the fact that more general nonlinear diffusion equations could be treated in this way. The same for equations involving confinement or interaction potentials (see for instance [1, 5, 20]). One of the restrictions to deal with for the analysis of all the models in this framework is the restriction to positive solutions only. This restriction is automatic, since the optimal transport setting is naturally defined only for positive measures with fixed mass. All the nice interpretation of the space of measures as a Riemaniann manifold, with notions of differential and geodesic, is therefore by now limited to probabilities. On the other hand, many of the models coming from the applications are often describing the evolution of possibly changing-sign quantities. The idea of looking to an optimal transport approach, despite the presence of changing-sign solutions, was first discussed in [2], for the study of an interaction equation appearing in vortices dynamics from Ginzburg-Landau theories. Some difficulties arising from that modeling left the suspect that an optimal transport approach for signed measures could be in general not really suitable, at least for the analysis of partial differential equations.

Aim of this paper is to show that the particular case of nonlinear diffusion equations, like the porous medium equation, can be described within this extended transport framework. Indeed, we can show that the scheme produces the unique solution in the standard sense: there is compactness and no need of introducing a much weaker notion of solution as in the case of [2]. The basic L^p estimates of the porous medium equation are also well understood within the Wasserstein variational setting.

This fact gives a first consistent argument for thinking to this framework as the actual right way of defining a transport cost for real measures, even if it does not carry as many geometric properties on the space as the standard optimal transport problem for probabilities. The result we will show may also be thought as a generalization of the result of Otto in [13] to the signed porous medium equation.

Before sketching the structure of signed transport of measures and stating the main result, we recall some elementary notions about the equation.

Notion of solution. We say that a function $\rho \in L^1_{loc}((0,+\infty)\times\mathbb{R}^n)$ is a weak solution for the signed porous medium equation (1) if

i) $\nabla(|\rho|^{m-1}\rho) \in L^2((0,+\infty)\times\mathbb{R}^n)$,

ii) the equality

$$\int_0^\infty \int_{\mathbb{R}^n} \partial_t\varphi(t,x)\rho(t,x) - \langle \nabla\varphi(t,x), \nabla(|\rho(t,x)|^{m-1}\rho(t,x))\rangle \, dx\, dt = 0$$

holds for any test function $\varphi \in C_0^\infty((0,+\infty)\times\mathbb{R}^n)$.

Moreover, we will say that a function $\rho \in L^1_{loc}((0,+\infty)\times\mathbb{R}^n)$ is a weak solution of the corrisponding Cauchy problem with initial datum $\bar\rho \in L^1(\mathbb{R}^n)$ if, together with the above properties, there is $\rho(t,\cdot)\to\bar\rho(\cdot)$ weakly in $L^1(\mathbb{R}^n)$ as $t\to 0$.

Basic estimates. Assuming more integrability for the initial datum, one obtains also some standard estimates. These are easily understood by means of elementary, formal computations. In particular, if a solution was regular and vanishing enough at infinity, we would have

- The mass conservation. Indeed

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^n}\rho(t,x)\varphi(x)\,dx &= \int_{\mathbb{R}^n}\Delta(|\rho(t,x)|^{m-1}\rho(t,x))\varphi(x)\,dx\\ &= \int_{\mathbb{R}^n}|\rho(t,x)|^{m-1}\rho(t,x)\Delta\varphi(x)\,dx\end{aligned}$$

for any smooth compactly supported function φ. Let $\varphi \to 1$ to obtain $\frac{d}{dt}\int_{\mathbb{R}^n}\rho(t,x)\,dx = 0$.

- The L^p norm dissipation. For $p > 1$, omitting the cutoff procedure,

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n} |\rho(t,x)|^p \, d \\
=&p\int_{\mathbb{R}^n} |\rho(t,x)|^{p-2}\rho(t,x)\partial_t\rho(t,x)\,dx \\
=&p\int_{\mathbb{R}^n} |\rho(t,x)|^{p-2}\rho(t,x)\Delta(|\rho(t,x)|^{m-1}\rho(t,x)) \\
=&-p\int_{\mathbb{R}^n} \langle\nabla(|\rho(t,x)|^{p-2}\rho(t,x)),\nabla(|\rho(t,x)|^{m-1}\rho(t,x))\rangle\,dx \\
=&-pm(p-1)\int_{\mathbb{R}^n} |\rho(t,x)|^{p+m-3}|\nabla\rho(t,x)|^2\,dx \le 0.
\end{aligned} \tag{5}$$

  For $p = 1$ we still obtain the inequality. However, there is some singularity in passing to the limit as p goes to 1, we do not keep a term like the last one as a bound. Indeed, the decay of L^1 norm is due to some term which is concentrated at the interface between positive and negative part of ρ. For instance let $f_k : \mathbb{R} \to [0,1]$ be C^1 functions converging to the Heaviside step, such that $f_k(x) = 0$ for $x \le 0$ and $f_k'(x) \ge 0$ for $x > 0$. We have

$$\begin{aligned}
&\int_{\mathbb{R}^n} f_k(|\rho(t,x)|^{m-1}\rho(t,x))\partial_t\rho(t,x)\,dx \\
=&\int_{\mathbb{R}^n} f_k(|\rho(t,x)|^{m-1}\rho(t,x))\Delta(|\rho(t,x)|^{m-1}\rho(t,x))\,dx \\
=&-\int_{\mathbb{R}^n} f_k'(|\rho(t,x)|^{m-1}\rho(t,x))\,|\nabla|\rho(t,x)|^m|^2\,dx.
\end{aligned}$$

  Now if we let $k \to \infty$, and if ρ^+ is the positive part of ρ, we have $f_k(|\rho|^{m-1}\rho)\partial_t\rho \to \partial_t\rho^+$. On the other hand, $f_k'((|\rho|^{m-1}\rho))$ concentrates where $\rho = 0$ for $k \to \infty$.
- The L^∞ norm dissipation (maximum principle). If the initial datum belongs to $L^\infty(\mathbb{R}^n)$, let us just think to pass to the limit as $p \to \infty$ in the L^p estimate. The signed porous medium equation is a degenerate parabolic problem, one can not use the maximum principle directly.

For a general theory, again we address the reader to the works of J. L. Vázquez, in particular to the monograph [17]. It is well known that there exists a unique weak solution to the initial value problem above, for instance with an $L^1(\mathbb{R}^n)$ initial datum. If moreover the initial datum is in $L^\infty(\mathbb{R}^n)$, the other listed properties hold.

Let us now go towards the statement of the main result. We need to introduce the extended transport framework, already encountered in [2] (see also [12] for an accurate overview).

The main theorem. The main element for introducing a discrete variational framework for the signed porous medium equation, is the definition of a suitable way for transporting signed densities. The standard quadratic transport cost W_2, defined by (4), yields a metric structure on the space of probability measures. The mass conservation for the equation (say normalized to 1) is therefore encoded in the setting of the problem. We should do the same for defining a structure on a suitable set of signed measures: our notion of transport cost should naturally contain the two features of the signed porous medium equation, the conservation of the (signed) mass, but, on the other hand, the dissipation of L^1 norm. A suitable notion, enjoying these properties, is the following relaxation of the standard Wasserstein distance W_2, borrowed from [2, 12]:

$$\mathfrak{W}_2(\mu,\nu) := \inf \left(W_2^2(\nu^+ + \theta, \mu^+) + W_2^2(\nu^- + \theta, \mu^-) \right)^{1/2}.$$

Here μ and ν are two $L^1(\mathbb{R}^n)$ densities, or more generally two measures, satisfying $\mu(\mathbb{R}^n) = \nu(\mathbb{R}^n)$ and having finite total mass and second moments. Moreover, this transport cost is naturally defined with the constraint $|\mu|(\mathbb{R}^n) \geq |\nu|(\mathbb{R}^n)$, the infimum being taken among all positive measures θ, having the same integral of $\mu^+ - \nu^+$ (and $\mu^- - \nu^-$). We leave the details for the next section.

We pass to the formulation of our result. In the sequel, $\lceil \cdot \rceil$ denotes the upper integer part.

Theorem 1.1. *Consider the following approximation scheme. Given $\rho_0 \in L^1 \cap L^\infty(\mathbb{R}^n)$ with finite second moment and $\tau > 0$, define the sequence $\{\rho_\tau^k\}_{k\in\mathbb{N}}$ by letting $\rho_\tau^0 = \rho_0$ and solving recursively the minimization problem*

$$\min \int_{\mathbb{R}^n} |\rho|^m + \frac{1}{2\tau} \mathfrak{W}_2^2(\rho, \rho_\tau^k),$$

where the minimum is taken among all functions $\rho \in L^1(\mathbb{R}^n)$ that satisfy

$$\int_{\mathbb{R}^n} \rho(x)\,dx = \int_{\mathbb{R}^n} \rho_\tau^k(x)\,dx \quad \text{and} \quad \int_{\mathbb{R}^n} |\rho(x)|\,dx \leq \int_{\mathbb{R}^n} |\rho_\tau^k(x)|\,dx.$$

Consider the discrete solution

$$\rho_\tau(t,\cdot) := \rho_\tau^{\lceil t/\tau \rceil}(\cdot). \tag{6}$$

Then, as $\tau \to 0$, for any $T > 0$ we have

$$\rho_\tau \to \rho \quad \text{strongly in} \quad L^1((0,T)\times\mathbb{R}^n),$$

where ρ is the weak solution to the signed porous medium equation (1), starting from ρ_0.

Remark 1.2. It is worth to point out that the analysis we are going through does not extend to the case $m = 1$, that is, to the heat equation, which therefore should be considered as a singular limit. In the standard Wasserstein framework for positive solutions, it is well known after [10] that the heat equation is gradient flow of the entropy functional $\rho \mapsto \int_{\mathbb{R}^n} \rho \log \rho$. However, unlike the case of the energy $\rho \mapsto \int_{\mathbb{R}^n} \rho^m/(m-1)$, the entropy functional does not admit an extension to signed functions, keeping the weak lower semicontinuity (indeed $x \in [0,+\infty) \mapsto x \log x$ may not be defined on $\mathbb{R}$ keeping convexity). But weak lower semicontinuity is needed for working with minimizing movements.

Plan of the paper. In Section 2 we introduce the cost $\mathfrak{W}_2$ for transporting real measures and give a brief discussion about some of its properties. In Section 3 we analyze the discrete minimization scheme of Theorem 1.1, we derive the corresponding Euler-Lagrange equation and discuss the L^p estimates at the discrete level. Finally we show the strong compactness properties of the scheme. In Section 4 we give the proof of Theorem 1.1.

2. The Wasserstein-minimizing movements scheme.

The optimal transport cost. Let $\mathscr{M}^+(\mathbb{R}^n)$ be the set of positive measures over $\mathbb{R}^n$. Let $\alpha \geq 0$ and let $\mathscr{M}_2^\alpha(\mathbb{R}^n)$ be the subset of positive measures μ with finite second moment, and such that $\mu(\mathbb{R}^n) = \alpha$. A transport plan is a measure γ in the product space $\mathscr{M}^\alpha(\mathbb{R}^n) \times \mathscr{M}^\alpha(\mathbb{R}^n)$. The quadratic optimal transport cost between two measures $\mu, \nu \in \mathscr{M}_2^\alpha(\mathbb{R}^n)$ is defined as the infimum in the Kantorovich optimal transport problem, that is, as (4). The theory of optimal transportation (see for instance the monograph [20]) tells us that the infimum is attained and defines a distance in the space $\mathscr{M}_2^\alpha(\mathbb{R}^n)$, the Wasserstein distance. Minimizers are called optimal transport plans, the convex set they form is denoted by $\Gamma_o(\mu,\nu)$. If an optimal transport plan γ is induced by a map $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^n$, that is, $\gamma = (\mathbf{i}, \mathbf{T})_{\#}\mu$, with a little abuse of notation we will write $\mathbf{T} \in \Gamma_o(\mu,\nu)$. This means that

$$W_2^2(\mu,\nu) = \int_{\mathbb{R}^n} |x - \mathbf{T}(x)|^2 \, d\mu(x).$$

When the involved measures possess a density, for instance $\mu = \rho_1\mathcal{L}^n$ and $\nu = \rho_2\mathcal{L}^n$, with some more abuse of notation we will write $\Gamma_o(\rho_1,\rho_2)$ in place of $\Gamma_o(\mu,\nu)$. As well, we will often identify measures and their densities in integrals, besides cutting sometimes the dependence on integration variables and omitting "dx" when integrals are with respect to the Lebesgue measure. Therefore the following notations will mean the same thing: $\int_{\mathbb{R}^n} d\mu(x)$, $\int_{\mathbb{R}^n} \rho_1 \, dx$, $\int_{\mathbb{R}^n} \rho_1$.

Remark 2.1 (Regularity of optimal transport maps)**.** We will always deal with measures with a density. The optimal transport theory shows that optimal transport plans among absolutely continuous measures enjoy different regularity properties, such as the ones listed below. We refer to [1, Section 6.2]. Let μ and ν be two absolutely continuous measures over $\mathbb{R}^n$ and let $\mathbf{T}$ denote the (unique) optimal transport map from μ to ν (the existence of such a map is ensured by absolute continuity of the starting measure only, by Brenier theorem). $\mathbf{T}$ is μ-essentially injective and (approximately) differentiable μ-a.e. in $\mathbb{R}^n$. The (approximate) Jacobian $\nabla\mathbf{T}$ is diagonalizable with positive eigenvalues: there is $\det(\nabla\mathbf{T}) > 0$, μ-a.e. in $\mathbb{R}^n$.

The signed optimal transport cost. In order to deal with signed densities, we introduce the space $\mathscr{M}_2^{\alpha,M}(\mathbb{R}^n)$ of real measures μ having finite second moment (i.e $\int_{\mathbb{R}^n} |x|^2 \, d|\mu|(x) < +\infty$), such that $\mu(\mathbb{R}^n) = \alpha$ and $|\mu|(\mathbb{R}^n) \leq M$. Here $|\mu|$ denotes the total variation measure. We are also making use of the Hahn decomposition in positive and negative parts for a real measure: $\mu = \mu^+ - \mu^-$, where μ^+ and μ^- are orthogonal, so that $|\mu| = \mu^+ + \mu^-$. On the set $\mathscr{M}_2^{\alpha,M}(\mathbb{R}^n)$, we consider the usual narrow topology of measures, given by duality with continuous and bounded functions on $\mathbb{R}^n$.

Given two measures $\mu, \nu \in \mathscr{M}_2^{\alpha,M}(\mathbb{R}^n)$, with the property $|\mu|(\mathbb{R}^n) \geq |\nu|(\mathbb{R}^n)$, we define the following quantity:

$$\mathfrak{W}_2^2(\mu,\nu) := \inf_{\theta \in \mathscr{M}_2^\ell(\mathbb{R}^n)} \left\{ W_2^2(\nu^+ + \theta, \mu^+) + W_2^2(\nu^- + \theta, \mu^-) \right\}, \tag{7}$$

where $\ell = \mu^+(\mathbb{R}^n) - \nu^+(\mathbb{R}^n)$. This kind of transport cost was introduced in [2]. Its basic properties are described in [12]. In particular, $\mathfrak{W}_2$ is not a distance, the triangle inequality being easily shown to fail. We recall the structure of the transportation associated to the cost $\mathfrak{W}_2$. There is a part of mass which gets transported and another one which corresponds to mutual cancellations. Indeed, we may subdivide the plans corresponding to the Wasserstein distances above in four plans as follows:

$$\gamma^{\pm} \in \Gamma_o(\nu^{\pm}, (\mu^{\pm})_\gamma), \qquad \beta^{\pm} \in \Gamma_o(\theta, (\mu^{\pm})_\beta).$$

Plans β correspond to cancellation: they account for interaction among the excess mass θ and two suitable parts $(\mu^+)_\beta$, $(\mu^-)_\beta$ of μ^+ and μ^- respectively. Plans γ correspond to advection towards the target measure ν, so that they involve the remaining parts $(\mu^+)_\gamma$, $(\mu^-)_\gamma$ of μ^+ and μ^- respectively. Notice that $(\mu^+)_\gamma$ and $(\mu^+)_\beta$ sum to μ^+ but are not orthogonal in general, and the same for negative parts. For a detailed description of this splitting of plans, we refer to [12] as well.

Proposition 2.2. *Let $\mu \in \mathscr{M}_2^{\alpha,M}(\mathbb{R}^n)$ with $|\mu|(\mathbb{R}^n) = M$. The map $\nu \mapsto \mathfrak{W}_2^2(\nu, \mu)$ is convex and lower semicontinuous in the narrow topology of $\mathscr{M}_2^{\alpha,M}(\mathbb{R}^n)$.*

Proof. Semicontinuity is proved in [2]. We also recall from [2] that the infimum in (7) is attained, thanks to narrow semicontinuity of the standard Wasserstein distance and to the uniform bounds on second moments of minimizing sequences, yielding tightness.

Let us prove convexity. Let $\nu_1, \nu_2 \in \mathscr{M}_2^{\alpha,M}(\mathbb{R}^n)$, with $M = \int_{\mathbb{R}^n} |\mu|$. Let σ_1, σ_2 be positive measures such that $\sigma_1 - \sigma_2 = \nu_1$ and $\sigma_1 - \nu_1^+$ realizes the infimum in the definition of $\mathfrak{W}_2(\nu_1, \mu)$. The same for ς_1, ς_2 with respect to $\mathfrak{W}_2(\nu_2, \mu)$. The convexity property we are proving is standard for the usual square Wasserstein distance, therefore

$$\begin{aligned}
\mathfrak{W}_2^2((1-\varepsilon)\nu_1 + \varepsilon\nu_2, \mu) \leq & W_2^2((1-\varepsilon)\sigma_1 + \varepsilon\varsigma_1, \mu^+) + W_2^2((1-\varepsilon)\sigma_2 + \varepsilon\varsigma_2, \mu^-) \\
\leq & (1-\varepsilon)W_2^2(\sigma_1, \mu^+) + \varepsilon W_2^2(\varsigma_1, \mu^+) \\
& + (1-\varepsilon)W_2^2(\sigma_2, \mu^-) + \varepsilon W_2^2(\varsigma_2, \mu^-) \\
\leq & (1-\varepsilon)\mathfrak{W}_2^2(\nu_1, \mu) + \varepsilon\mathfrak{W}_2^2(\nu_2, \mu).
\end{aligned}$$

Convexity is not strict, since it is not for W_2 to which $\mathfrak{W}_2$ is reduced for positive measures. □

The $\mathfrak{W}_2$ approximation scheme. In the general measure setting, the minimizing movements scheme for a functional $\mathscr{G} : \mathscr{M}_2^{\alpha,M}(\mathbb{R}^n) \to \mathbb{R}$, with respect to the $\mathfrak{W}_2$ structure, is the following. Let $\mu_0 \in \mathscr{M}_2^{\alpha,M}(\mathbb{R}^n)$ be a given starting point, with $\mathscr{G}(\mu_0) < +\infty$. Let $\tau > 0$. Construct the sequence $\{\mu_\tau^k\}_{k\in\mathbb{N}}$ by solving recursively

$$\min_{\mu \in \mathscr{M}_2^{\kappa,M}(\mathbb{R}^n)} \mathscr{G}(\mu) + \frac{1}{2\tau}\mathfrak{W}_2^2(\mu, \mu_\tau^{k-1}) + \mathbf{1}_\mu\left(\{\nu : |\nu|(\mathbb{R}^n) \leq |\nu_\tau^{k-1}|(\mathbb{R}^n)\}\right), \tag{8}$$

where $\mathbf{1}_\mu(A)$ is 0 if $\mu \in A$ and $+\infty$ otherwise. We then define the piecewise constant interpolation of discrete minimizers, that is, the family $\mu_\tau(t, x) := \mu_\tau^{\lceil t/\tau \rceil}(x)$ as in (6), where $\lceil \cdot \rceil$ denotes upper integer part, with $\mu_\tau(0, \cdot) := \mu_0$. A curve $t \in (0, +\infty) \mapsto \mu(t) \in \mathscr{M}_2^{\alpha,M}(\mathbb{R}^n)$ is a minimizing movement for $\mathscr{G}$ if for any t it is a narrow limit of $\mu_\tau(t, \cdot)$ as $\tau \to 0$. Therefore Theorem 1.1 says that minimizing movements of $\mathscr{F}$, with respect to $\mathfrak{W}_2$, solve the signed porous medium equation (and that a stronger convergence holds indeed).

3. **Analysis of the approximation scheme.** We begin with the analysis of a single discrete minimizer of the scheme. Therefore we let $\bar{\rho} \in L^1(\mathbb{R}^n)$ and we

consider the problem

$$\min_{\rho\in L^1(\mathbb{R}^n),\,\int\rho=\int\bar\rho} \mathscr{F}(\rho)+\frac{1}{2\tau}\mathfrak{W}_2^2(\rho,\bar\rho)+\mathbf{1}_\rho\left(\left\{\textstyle\int_{\mathbb{R}^n}|\varrho|\le\int_{\mathbb{R}^n}|\bar\rho|\right\}\right). \tag{9}$$

Proposition 3.1. *Let $\bar\rho\in L^1\cap L^m(\mathbb{R}^n)$ (so that $\mathscr{F}(\bar\rho)<+\infty$). Let $\int_{\mathbb{R}^n}|\bar\rho(x)||x|^2\,dx<+\infty$. The minimization problem (9) admits a unique minimizer $\rho\in L^m(\mathbb{R}^n)$ which satisfies*

- *i)* $2\tau\mathscr{F}(\rho)+\mathfrak{W}_2^2(\rho,\bar\rho)\le 2\tau\mathscr{F}(\bar\rho)$,
- *ii)* $|\rho|^m\in W^{1,1}(\mathbb{R}^n)$,
- *iii)* $\tau\nabla|\rho|^m+(\mathbf{i}-\mathbf{T}_+)\rho^++(\mathbf{i}-\mathbf{T}_-)\rho^-=0$ *a.e. in* $\mathbb{R}^n$, *where* $\mathbf{T}_+\in\Gamma_o(\rho^+,\bar\rho_\gamma^+)$ *and* $\mathbf{T}_-\in\Gamma_o(\rho^-,\bar\rho_\gamma^-)$.

Proof. Let (ρ_l) be a minimizing sequence: we have the semicontinuity (in the narrow topology of measures) of $\mathscr{F}(\rho_l)$ and $\mathfrak{W}_2(\rho_l,\bar\rho)$, by Proposition 2.2. (ρ_l) is tight since the transport cost term bounds its second moments. Existence then follows by direct method. The map $\rho\mapsto\mathscr{F}(\rho)$ is strictly convex, whereas the other two terms in (9) are convex, so that there exists a unique minimizer $\rho\in L^m(\mathbb{R}^n)$. Moreover, *i)* follows easily by testing (9) with $\bar\rho$.

Let us consider the first variation of the internal energy functional. We may obtain

$$\frac{d}{d\varepsilon}\mathscr{F}(S_\varepsilon(\rho))\bigg|_{\varepsilon=0}=\int_{\mathbb{R}^n}\langle\boldsymbol{\xi},\nabla|\rho|^m\rangle,$$

where the variation is along a smooth vector field: $S_\varepsilon(\rho)$ above is the density of $(\mathbf{i}+\varepsilon\boldsymbol{\xi})_\#(\rho\mathcal{L}^n)$ with respect to $\mathcal{L}^n$ and $\boldsymbol{\xi}\in C_0^\infty(\mathbb{R}^n;\mathbb{R}^n)$. Indeed, the change of variables formula, together with the elementary expansion $\det(\nabla(\mathbf{i}+\varepsilon\boldsymbol{\xi}))=1+\varepsilon\operatorname{div}\boldsymbol{\xi}+o(\varepsilon)$ entails

$$\frac{1}{m-1}\int_{\mathbb{R}^n}|S_\varepsilon(\rho)|^m-|\rho|^m=-\varepsilon\int_{\mathbb{R}^n}|\rho|^m\operatorname{div}\boldsymbol{\xi}+o(\varepsilon)$$

On the other hand, let $\gamma_+\in\Gamma_o(\rho^+,\bar\rho_\gamma^+)$ and $\gamma_-\in\Gamma_o(\rho^-,\bar\rho_\gamma^-)$. Estimating $\mathfrak{W}_2$ with the transport cost associated to the plans $(\mathbf{i}+\varepsilon\boldsymbol{\xi},\mathbf{i})_\#\gamma_+$ and $(\mathbf{i}+\varepsilon\boldsymbol{\xi},\mathbf{i})_\#\gamma_-$ one gets

$$\mathfrak{W}_2^2(S_\varepsilon(\rho),\bar\rho)-\mathfrak{W}_2^2(\rho,\bar\rho)\le\varepsilon\int_{\mathbb{R}^n\times\mathbb{R}^n}\langle\boldsymbol{\xi}(x),x-y\rangle\,d(\gamma_++\gamma_-)(x,y)+o(\varepsilon).$$

By minimality of ρ and arbitrariness of $\boldsymbol{\xi}$, we find the Euler Lagrange equation which has to be satisfied by the solution ρ:

$$\int_{\mathbb{R}^n}\langle\nabla|\rho|^m,\boldsymbol{\xi}\rangle=\frac{1}{\tau}\int_{\mathbb{R}^n\times\mathbb{R}^n}\langle\boldsymbol{\xi}(x),y-x\rangle\,d(\gamma_++\gamma_-)(x,y). \tag{10}$$

The infinitesimal version is

$$\nabla|\rho|^m=\frac{1}{\tau}\pi^1_\#((y-x)(\gamma_++\gamma_-)), \tag{11}$$

This equality holds in distribution sense. However, the plans γ_+ and γ_- are induced by maps $\mathbf{T}_+$ and $\mathbf{T}_-$ respectively, so that

$$\begin{aligned}\pi^1_\#((y-x)(\gamma_++\gamma_-))&=\pi^1_\#((y-x)(\mathbf{i},\mathbf{T}_+)_\#\rho^++(y-x)(\mathbf{i},\mathbf{T}_-)_\#\rho^-)\\&=(\mathbf{T}_+-\mathbf{i})\rho^++(\mathbf{T}_-+\mathbf{i})\rho^-.\end{aligned}$$

Since $(\mathbf{i}-\mathbf{T}_+)\rho^+\in L^1(\mathbb{R}^n)$ and $(\mathbf{i}-\mathbf{T}_-)\rho^-\in L^1(\mathbb{R}^n)$, *ii)* and *iii)* also follow. □

Remark 3.2. Since $\nabla|\rho|^m = \nabla(\rho^+)^m + \nabla(\rho^-)^m = (\rho^+)^{m-1}\nabla\rho^+ + (\rho^-)^{m-1}\nabla\rho^-$, the equation of point *iii)* in the previous proposition can be conveniently split into

$$\begin{aligned} -\nabla(|\rho^+|^m) &= \frac{1}{\tau}(\mathbf{i} - \mathbf{T}_+)\rho^+, \\ -\nabla(|\rho^-|^m) &= \frac{1}{\tau}(\mathbf{i} - \mathbf{T}_-)\rho^-, \end{aligned} \tag{12}$$

from which we also deduce

$$-\nabla(|\rho|^{m-1}\rho) = \frac{1}{\tau}\pi^1_{\#}((x-y)(\gamma_+ - \gamma_-)). \tag{13}$$

Let us discuss the L^p estimates of the porous medium equation at the discrete level. We have already noticed that the solution to (1) is characterized by the nonincreasing property for $t \mapsto \|\rho(t,\cdot)\|_p$. We have

Proposition 3.3 (L^p estimates)**.** *Let $p \in [1, +\infty]$. Let $\bar\rho \in L^1 \cap L^{p\vee m}(\mathbb{R}^n)$, let $\bar\rho$ have finite second moment and let ρ be the solution of (9), starting from $\bar\rho$. Then there holds*

$$\int_{\mathbb{R}^n} |\rho|^p \le \int_{\mathbb{R}^n} |\bar\rho|^p.$$

Proof. Notice that the result is trivial if $p = 1$ or $p = m$. In general, let $p > 1$. If we reason separately on positive and negative parts, we may take advantage of the already established theory for functionals like $\mathscr{F}$ in standard Wasserstein setting $(\mathscr{M}_2^\alpha(\mathbb{R}^n), W_2)$; if we consider positive parts for instance, α will be the mass transported by $\mathbf{T}_+$ in this case. Therefore, we can give a quick proof making use of the results describing the differential of variational integrals along transports (Wasserstein differential). On $\mathscr{M}_2^\alpha(\mathbb{R}^n)$, consider a functional $\rho \mapsto \int_{\mathbb{R}^n}\varphi(\rho)\,dx$ (extended with value $+\infty$ for measures without a density) where the convex function $\varphi : [0, +\infty) \to \mathbb{R}$ also satisfies the displacement convexity property: $x \mapsto x^n\varphi(x^{-n})$ is convex and nonincreasing. Then, by the general theory developed in [1, Chapter 9-10], and in particular combining Proposition 9.3.9 and Theorem 10.4.6 therein, we have

$$\int_{\mathbb{R}^n} \varphi((\mathbf{T})_{\#}\rho) - \varphi(\rho) \ge \int_{\mathbb{R}^n} \left\langle \frac{\nabla(\rho\varphi'(\rho) - \varphi(\rho))}{\rho}, \mathbf{T} - \mathbf{i} \right\rangle \rho,$$

where $\mathbf{T}$ is the optimal transport map from ρ to any reference measure. Indeed, the vector $\nabla(\rho\varphi'(\rho) - \varphi(\rho))/\rho$ has to be understood as the Wasserstein gradient of the functional at point $\rho\mathcal{L}^n$, and the above relation has the meaning of convexity of such functional with respect to variations along optimal transport maps (a property which requires the displacement convexity of φ, as discussed in [1, § 9.3]).

We apply the inequality to the displacement convex function $\varphi(x) = x^p/(p-1)$, which also satisfies $x\varphi'(x) - \varphi(x) = x^p$, and for the transport map $\mathbf{T}$ we choose the map $\mathbf{T}_+ \in \Gamma_o(\rho^+, \bar\rho^+_\gamma)$. We obtain, since $\bar\rho^+_\gamma \le \bar\rho^+$,

$$\frac{1}{p-1}\int_{\mathbb{R}^n} (\bar\rho^+)^p - (\rho^+)^p \ge \frac{1}{p-1}\int_{\mathbb{R}^n} ((\mathbf{T}_+)_{\#}\rho^+)^p - (\rho^+)^p \ge \int_{\mathbb{R}^n} \langle \nabla(\rho^+)^p, \mathbf{T}_+ - \mathbf{i}\rangle. \tag{14}$$

But (12) gives $\mathbf{T}_+ - \mathbf{i} = \tau\nabla(\rho^+)^m/\rho^+$, therefore the right hand side above is

$$\int_{\mathbb{R}^n} \langle \nabla(\rho^+)^p, \mathbf{T}_+ - \mathbf{i}\rangle = pm\tau\int_{\mathbb{R}^n} (\rho^+)^{p+m-3}|\nabla(\rho^+)|^2 \ge 0,$$

yielding

$$\int_{\mathbb{R}^n} (\rho^+)^p + pm(p-1)\tau \int_{\mathbb{R}^n} (\rho^+)^{p+m-3} |\nabla(\rho^+)|^2 \leq \int_{\mathbb{R}^n} (\bar{\rho}^+)^p. \tag{15}$$

The very same for negative parts and the result follows for $p < \infty$. The case $p = \infty$ is then obtained in the limit $p \to \infty$ if $\bar{\rho} \in L^\infty(\mathbb{R}^n)$. □

Remark 3.4. Inequality (15) is the discrete counterpart of the standard estimate (5). About the case $p = 1$, we can see at the discrete level that the decrease of the L^1 norm is due to the interaction between positive and negative parts at the interface. Suppose that $\bar{\rho}^+$ and $\bar{\rho}^-$ have distant supports (say $dist(\operatorname{supp} \bar{\rho}^+, \operatorname{supp} \bar{\rho}^-) = R > 0$). Let ρ be the solution to (9). If τ is small enough we have $\|\rho\|_1 = \|\bar{\rho}\|_1$. Indeed, suppose by contradiction that $\|\rho\|_1 < \|\bar{\rho}\|_1$. This means that some mass has been cancelled passing from $\bar{\rho}$ to ρ. But if a portion σ^+ of $\bar{\rho}^+$ of mass Q is transported to a corresponding subdensity σ^- of $\bar{\rho}^-$ (suppose for instance that σ^+ and σ^- are characteristic functions of small balls), the cost for this cancellation (which is subdivided in the transports of β^+ and β^-) is at least $QR^2/4$, as seen by comparison with the cost for transporting Dirac deltas in the corresponding centers of mass, since projection decreases the Wasserstein distance. On the other hand, these parts of mass could have been transported and not cancelled, at a distance $r \ll R$, towards a competitor $\tilde{\rho}$ having energy $\mathscr{F}(\rho + \sigma^+ + \sigma^-)$. Neglecting r^2 in comparison with R^2, considering $\tilde{\rho}$ instead of ρ we have an overall gain in transport cost of at least $QR^2/4$ and a loss in energy of $\tau(\mathscr{F}(\rho + \sigma^+ + \sigma^-) - \mathscr{F}(\rho))$. Using the fact that $\sigma^\pm \leq \rho^\pm$ and the uniform L^∞ estimates of the previous proposition, the difference $|\mathscr{F}(\rho + \sigma^+ + \sigma^-) - \mathscr{F}(\rho)|$ can be estimated by KQ, where K is a constant depending only on m and $\bar{\rho}$. Therefore the above loss in energy is at most $KQ\tau$, which is smaller than $QR^2/4$ for τ small enough with respect to R^2. This co ntradicts the minimality of ρ.

Corollary 3.5. *Let $p \in [1, +\infty]$. Let $\alpha := \frac{p-1}{p+1}$. If $\bar{\rho}$ is as in* Proposition 3.3*, we have $|\rho|^m \in W^{1,1+\alpha}(\mathbb{R}^n)$ and*

$$\left(\int_{\mathbb{R}^n} |\nabla |\rho|^m|^{\alpha+1} \right)^{\frac{p+1}{p}} \leq \frac{1}{\tau^2} \mathfrak{W}_2^2(\rho, \bar{\rho}) \left(\int_{\mathbb{R}^n} |\bar{\rho}|^p \right)^{\frac{1}{p}}. \tag{16}$$

Proof. We consider the relation (11), multiply by $\boldsymbol{\xi} := |\nabla|\rho|^m|^{\alpha-1}\nabla|\rho|^m$ and integrate, obtaining

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \langle \nabla|\rho|^m, \boldsymbol{\xi} \rangle \, dx \right| &\leq \frac{1}{\tau} \int_{\mathbb{R}^2 \times \mathbb{R}^2} |x-y||\boldsymbol{\xi}(x)| \, d(\gamma_+ + \gamma_-)(x,y) \\ &\leq \frac{1}{\tau} \mathfrak{W}_2(\rho, \bar{\rho}) \left(\int_{\mathbb{R}^n} |\rho||\boldsymbol{\xi}|^2 \, dx \right)^{\frac{1}{2}}. \end{aligned}$$

Therefore

$$\int_{\mathbb{R}^n} |\nabla|\rho|^m|^{\alpha+1} \leq \frac{1}{\tau} \mathfrak{W}_2(\rho, \bar{\rho}) \left(\int_{\mathbb{R}^n} |\rho|^p \right)^{\frac{1}{2p}} \left(\int_{\mathbb{R}^n} |\nabla|\rho|^m|^{\alpha+1} \right)^{\frac{p-1}{2p}}$$

and the result follows applying the L^p estimate coming from Proposition 3.3. □

Let us now consider the recursive approximation scheme, which is (8), for functional $\mathscr{F}$. We are going to show some compactness properties of the scheme, which will follow from the regularity arising from the Euler-Lagrange equation above. We recall the basic estimate which immediately follows from the formulation of the variational problem (8), that is

$$\sum_{k=0}^{\infty} \mathfrak{W}_2^2(\rho_\tau^{k+1}, \rho_\tau^k) \le 2\tau \mathscr{F}(\rho_0). \tag{17}$$

Moreover, notice that Proposition 3.3 gives an estimate which does not depend on τ, and in particular at the recursive level translates into $\|\rho_\tau^{k+1}\|_p \le \|\rho_\tau^k\|_p \le \|\rho_0\|_p$ for any k and for any τ. Considering the family $\{\rho_\tau(\cdot,\cdot)\}_{\tau>0}$ of piecewise constant interpolations of the discrete minimizers, defined as $\rho_\tau(t,\cdot) = \rho_\tau^{\lceil t/\tau \rceil}(\cdot)$, we have therefore $\|\rho_\tau(t,\cdot)\|_p \le \|\rho_0\|_p$ for any $t \ge 0$. The first compactness property we obtain for the family $\{\rho_\tau(\cdot,\cdot)\}_{\tau>0}$ is the following.

Proposition 3.6. *Let $\rho_0 \in L^1 \cap L^\infty(\mathbb{R}^n)$. Let the second moment of ρ_0 be finite. There holds*

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^n} |\nabla |\rho_\tau(t,x)|^m|^2 \, dx \, dt &= \int_0^\infty \int_{\mathbb{R}^n} |\nabla(|\rho_\tau(t,x)|^{m-1}\rho_\tau(t,x))|^2 \, dx \, dt \\ &\le 2\|\rho_0\|_\infty \mathscr{F}(\rho_0).\end{aligned} \tag{18}$$

Proof. Making use of (16) for each k (with $p = +\infty$), we have

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^n} |\nabla |\rho_\tau(t,x)|^m|^2 \, dx \, dt &= \sum_{k=0}^\infty \int_{k\tau}^{(k+1)\tau} \int_{\mathbb{R}^n} |\nabla |\rho_\tau(t,x)|^m|^2 \, dx \, dt \\ &= \sum_{k=0}^\infty \tau \int_{\mathbb{R}^n} |\nabla |\rho_\tau^k(x)|^m|^2 \, dx \\ &\le \|\rho_0\|_\infty \sum_{k=0}^\infty \frac{1}{\tau} \mathfrak{W}_2^2(\rho_\tau^k, \rho_\tau^{k+1}),\end{aligned}$$

and the result follows from (17). ☐

We have a gradient control from Proposition 3.6, and for compactness in space we also need a tail control. This comes from the uniform estimate on moments, which is a standard fact for Wasserstein minimizing movements and can be straightforwardly proven also for the signed scheme. We do not need a sharp estimate, it is enough to show that second moments remain bounded as $\tau \to 0$.

Proposition 3.7. *Let $\rho_0 \in L^1 \cap L^m(\mathbb{R}^n)$ have finite second moment. There holds*

$$\limsup_{\tau \to 0} \int_{\mathbb{R}^n} |\rho_\tau(t,x)||x|^2 \le 4e\mathscr{F}(\rho_0)t + e\int_{\mathbb{R}^n} |\rho_0||x|^2. \tag{19}$$

Proof. For $t, \tau > 0$, if $(\gamma_\tau^k)^+ \in \Gamma_o((\rho_\tau^k)^+, (\rho_\tau^{k-1})_\gamma^+)$, estimating $|x|^2$ with Young inequality as $|x|^2 \le (1 + t/\tau)|x-y|^2 + (1+\tau/t)|x|^2$ it is easy to compute

$$\begin{aligned}\int_{\mathbb{R}^n} (\rho_\tau^k)^+ |x|^2 &= \int_{\mathbb{R}^n \times \mathbb{R}^n} |x|^2 \, d(\gamma_\tau^k)^+ \\ &\le \left(1 + \frac{t}{\tau}\right) W_2^2((\rho_\tau^k)^+, (\rho_\tau^{k-1})_\gamma^+) + \left(1 + \frac{\tau}{t}\right) \int_{\mathbb{R}^n} (\rho_\tau^{k-1})_\gamma^+ |x|^2 \\ &\le \left(1 + \frac{t}{\tau}\right) \mathfrak{W}_2^2(\rho_\tau^k, \rho_\tau^{k-1}) + \left(1 + \frac{\tau}{t}\right) \int_{\mathbb{R}^n} (\rho_\tau^{k-1})^+ |x|^2.\end{aligned} \tag{20}$$

The last inequality follows from $(\rho_\tau^k)_\gamma^+ \le (\rho_\tau^k)^+$ and from the fact that $W_2^2((\rho_\tau^k)^+, (\rho_\tau^{k-1})_\gamma^+)$ is the transportation cost of only one of the four plans for which $\mathfrak{W}_2^2(\rho_\tau^k,\rho_\tau^{k-1})$ accounts. Notice that (20) holds for any $k\geq 1$. If we apply it recursively we obtain

$$\int_{\mathbb{R}^n}(\rho_\tau^k)^+|x|^2 \leq \sum_{j=0}^{k-1}\left(1+\frac{\tau}{t}\right)^j\left(1+\frac{t}{\tau}\right)\mathfrak{W}_2^2(\rho_\tau^{k-j},\rho_\tau^{k-j-1})+\left(1+\frac{\tau}{t}\right)^k\int_{\mathbb{R}^n}(\rho_0)^+|x|^2$$
$$\leq\left(1+\frac{\tau}{t}\right)^k\left(1+\frac{t}{\tau}\right)\sum_{j=0}^{k-1}\mathfrak{W}_2^2(\rho_\tau^{k-j},\rho_\tau^{k-j-1})+\left(1+\frac{\tau}{t}\right)^k\int_{\mathbb{R}^n}(\rho_0)^+|x|^2.$$

We make use of this inequality for $k=\lceil t/\tau\rceil$, and by (17) we get

$$\int_{\mathbb{R}^n}\left(\rho_\tau^{\lceil t/\tau\rceil}\right)^+|x|^2\leq\left(1+\frac{\tau}{t}\right)^{\frac{t}{\tau}+1}\left(2\tau\mathscr{F}(\rho_0)+2t\mathscr{F}(\rho_0)+\int_{\mathbb{R}^n}(\rho_0)^+|x|^2\right).$$

Summing the analogous inequality for negative parts and taking the limit as $\tau\to 0$ yields the thesis. □

Remark 3.8. Our aim is to prove compactness of the family $\{\rho_\tau(\cdot,\cdot)\}_{\tau>0}$ in the strong $L^1((0,T)\times\mathbb{R}^n)$ topology. In particular, we will complement (18) with an integral equicontinuity estimate in time, which was already known for the case of positive densities (see [13]). We would also like to stress that strong compactness will be crucial to pass to the limit in a consistent way, since we need to deal with positive and negative parts.

Lemma 3.9. *Let ρ_0 be as in* Proposition 3.6. *There holds*

$$\int_0^\infty\int_{\mathbb{R}^n}|\rho_\tau(t+s,x)-\rho_\tau(t,x)|^{m+1}\,dx\,dt\leq K_1 s, \tag{21}$$

where K_1 is a positive constants depending only on m and on the initial datum.

Proof. We consider the transport within two generic steps ρ_τ^k, ρ_τ^{k+1} of the discrete scheme. It is described by the usual four plans:

$$\begin{aligned}(\gamma_\tau^{k+1})^+\in\Gamma_o\left((\rho_\tau^{k+1})^+,(\rho_\tau^k)^+_\gamma\right),\qquad&(\gamma_\tau^{k+1})^-\in\Gamma_o\left((\rho_\tau^{k+1})^-,(\rho_\tau^k)^-_\gamma\right),\\(\beta_\tau^{k+1})^+\in\Gamma_o\left(\theta_\tau^{k+1},(\rho_\tau^k)^+_\beta\right),\qquad&(\beta_\tau^{k+1})^-\in\Gamma_o\left(\theta_\tau^{k+1},(\rho_\tau^k)^-_\beta\right).\end{aligned}\tag{22}$$

Here θ_τ^{k+1} is supposed to be the auxiliary measure solving the minimization problem (7) which defines $\mathfrak{W}_2(\rho_\tau^{k+1},\rho_\tau^k)$, and accounting for mutual cancellations. We set

$$\gamma_\tau^{k+1}:=(\gamma_\tau^{k+1})^+-(\gamma_\tau^{k+1})^-+(\beta_\tau^{k+1})^+-(\beta_\tau^{k+1})^-. \tag{23}$$

This way, fixing a function $\zeta\in W^{1,2}\cap L^\infty(\mathbb{R}^n)$ there holds

$$\begin{aligned}&\int_{\mathbb{R}_n}\zeta(x)(\rho_\tau^{k+1}(x)-\rho_\tau^k(x))\,dx\\&=\int_{\mathbb{R}^n\times\mathbb{R}^n}(\zeta(x)-\zeta(y))\,d\gamma_\tau^{k+1}(x,y)\\&=\int_0^1\int_{\mathbb{R}^n\times\mathbb{R}^n}\langle\nabla\zeta((1-t)x+ty),x-y\rangle\,d\gamma_\tau^{k+1}(x,y)\,dt\\&\leq\left(\int_0^1\int_{\mathbb{R}^n\times\mathbb{R}^n}|\nabla\zeta((1-t)x+ty)|^2\,d\gamma_\tau^{k+1}\,dt\right)^{\frac12}\left(\int_0^1\int_{\mathbb{R}^n\times\mathbb{R}^n}|x-y|^2\gamma_\tau^{k+1}\,dt\right)^{\frac12}.\end{aligned}$$

The former estimate can not be established separately for positive and negative parts of the discrete minimizer ρ_τ^k, due to the presence of a part in excess, corresponding to the mass which is cancelled step by step. Therefore one needs to subdivide $(\rho_\tau^k)^+$ as $(\rho_\tau^k)^+_\gamma + (\rho_\tau^k)^+_\beta$, as done in (22). Similarly for negative parts. Therefore the subsequent estimate reads

$$\begin{aligned}
&\int_{\mathbb{R}_n} \zeta(x)\left((\rho_\tau^{k+1})^+ - (\rho_\tau^k)^+\right)\,dx\\
&= \int_{\mathbb{R}^n} \zeta(x)\left((\rho_\tau^{k+1})^+ - (\rho_\tau^k)^+_\gamma\right)\,dx - \int_{\mathbb{R}^n} \zeta(x)(\rho_\tau^k)^+_\beta\,dx\\
&\le \left(\int_0^1\int_{\mathbb{R}^n\times\mathbb{R}^n} |\nabla\zeta((1-t)x+ty)|^2\,d(\gamma_\tau^{k+1})^+\,dt\right)^{\frac{1}{2}} \mathfrak{W}_2(\rho_\tau^{k+1},\rho_\tau^k) - \int_{\mathbb{R}^n}\zeta(x)(\rho_\tau^k)^+_\beta\,dx,
\end{aligned}$$

where we also exploited the estimate $\int_{\mathbb{R}^n}|x-y|^2\,d(\gamma_\tau^{k+1})^+ \le \mathfrak{W}_2^2(\rho_\tau^{k+1},\rho_\tau^k)$. The analogous control holds for negative parts. Moreover, if we are to consider l steps from k to $k+l$, we get

$$\begin{aligned}
\int_{\mathbb{R}^n}\left((\rho_\tau^{k+l})^+ - (\rho_\tau^k)^+\right)\zeta(x)\,dx &= \sum_{j=1}^{l}\int_{\mathbb{R}^n}\left((\rho_\tau^{k+j})^+ - (\rho_\tau^{k+j-1})^+\right)\zeta(x)\,dx\\
&\le \sum_{j=1}^{l}\left(\int_0^1\int_{\mathbb{R}^n\times\mathbb{R}^n}|\nabla\zeta((1-t)x+ty)|^2\,d(\gamma_\tau^{k+j})^+\,dt\right)^{\frac{1}{2}}\mathfrak{W}_2(\rho_\tau^{k+j},\rho_\tau^{k+j-1})\\
&\quad - \sum_{j=1}^{l}\int_{\mathbb{R}^n}(\rho_\tau^{k+j-1})^+_\beta\zeta(x)\,dx.
\end{aligned} \tag{24}$$

The last term is bounded taking into account that the total sum of the part of mass which gets cancelled during the different time steps is bounded by the mass of the initial datum:

$$\sum_{k=0}^{\infty}\int_{\mathbb{R}^n}(\rho_\tau^k)^+_\beta(x)\,dx \le M. \tag{25}$$

Notice that, letting $K := \|\rho_0\|_\infty$,

$$\begin{aligned}
&\int_0^1\int_{\mathbb{R}^n\times\mathbb{R}^n}|\nabla\zeta((1-t)x+ty)|^2\,d(\gamma_\tau^{k+j})^+\\
&= \int_0^1\int_{\mathbb{R}^n\times\mathbb{R}^n}|\nabla\zeta|^2\,d\left(((1-t)\pi^1+t\pi^2)_\#(\gamma_\tau^{k+j})^+\right) \le K\int_{\mathbb{R}^n}|\nabla\zeta|^2.
\end{aligned} \tag{26}$$

Indeed, the argument to prove such an estimate is similar to the one of Proposition 3.3, making use of the estimation for the variation along transports for integral functionals of the form $\rho\mapsto\int_{\mathbb{R}^n}\varphi(d\rho/d\mathcal{L}^n)\,d\mathcal{L}^n$, when φ is displacement convex. We have to think to the map $t\in[0,1]\mapsto((1-t)\pi^1+t\pi^2)_\#(\gamma_\tau^{k+j})^+\in\mathscr{M}_2^\ell(\mathbb{R}^n)$ as the geodesic interpolation among the marginals of plan $(\gamma_\tau^{k+j})^+$ (here $\ell = \|(\rho_\tau^{k+j})^+\|_1$). The integral functional is convex along such a geodesic, in the space $\mathscr{M}_2^\ell(\mathbb{R}^n)$ endowed with the Wasserstein distance W_2, if φ is displacement convex (see [1, Proposition 9.3.9]). Choose now $\varphi(x) := x$ for $x\le K$, extended with value $+\infty$ otherwise. We know by Proposition 3.3 that $\|\rho_\tau^{k+j}\|_\infty \le \|\rho_\tau^{k+j-1}\|_\infty \le K$. Then the mentioned convexity property implies

$$\int_{\mathbb{R}^n}\varphi\left(((1-t)\pi^1+t\pi^2)_\#(\gamma_\tau^{k+j})^+\right) \le (1-t)\int_{\mathbb{R}^n}\varphi((\rho_\tau^{k+j})^+)^+ + t\int_{\mathbb{R}^n}\varphi((\rho_\tau^{k+j-1})^+_\gamma) \le M$$

where we used the general L^1 bound. The left hand side is finite, implying that $\|((1-t)\pi^1+t\pi^2)_\#(\gamma_\tau^{k+j})^+\|_\infty \le K$ for any $t\in(0,1)$, and (26) follows.

In particular, if $\zeta = |(\rho_\tau^{k+l})^+|^m - |(\rho_\tau^k)^+|^m$, making use of (16) and $|\nabla|(\rho_\tau^k)^+|^m| \le |\nabla|\rho_\tau^k|^m|$ (which comes from (12)), we have

$$\int_0^1\int_{\mathbb{R}^n\times\mathbb{R}^n}|\nabla\zeta((1-t)x+ty)|^2\,d(\gamma_\tau^{k+j})^+ \le 2K\int_{\mathbb{R}^n}\left|\nabla|\rho_\tau^{k+l}|^m\right|^2 + 2K\int_{\mathbb{R}^n}\left|\nabla|\rho_\tau^k|^m\right|^2$$
$$\le \frac{2K}{\tau^2}\left(\mathfrak{W}_2^2(\rho_\tau^{k+l},\rho_\tau^{k+l-1}) + \mathfrak{W}_2^2(\rho_\tau^k,\rho_\tau^{k-1})\right).$$

Therefore, with this choice of ζ, having the uniform control $|\zeta(x)|\le 2K^m$ which comes from Proposition 3.3, from (24) we deduce

$$\begin{aligned}&\int_{\mathbb{R}^n}\left(|(\rho_\tau^{k+l})^+|^m - |(\rho_\tau^k)^+|^m\right)\left((\rho_\tau^{k+l})^+ - (\rho_\tau^k)^+\right)\\ &\le \left(\frac{\sqrt{2K}}{\tau}\mathfrak{W}_2(\rho_\tau^{k+l},\rho_\tau^{k+l-1}) + \frac{\sqrt{2K}}{\tau}\mathfrak{W}_2(\rho_\tau^k,\rho_\tau^{k-1})\right)\sum_{j=1}^l \mathfrak{W}_2(\rho_\tau^{k+j},\rho_\tau^{k+j-1}) \\ &\quad + 2K^m\sum_{j=1}^l\int_{\mathbb{R}^n}(\rho_\tau^{k+j-1})_\beta^+\,dx.\end{aligned} \tag{27}$$

Let $A_{\tau,k,l} := \sqrt{2K}\,\mathfrak{W}_2(\rho_\tau^{k+l},\rho_\tau^{k+l-1}) + \sqrt{2K}\,\mathfrak{W}_2(\rho_\tau^k,\rho_\tau^{k-1})$ and notice that, by (17), there is

$$\sum_{k=1}^\infty (A_{\tau,k,l})^2 \le 16K\tau\mathscr{F}(\rho_0). \tag{28}$$

Now, for $s>0$, let $l = \left\lceil\frac{t+s}{\tau}\right\rceil - \left\lceil\frac{t}{\tau}\right\rceil$. Therefore $\rho_\tau(t+s,x) = \rho_\tau^{k+l}(x)$ when $\rho_\tau(t,x) = \rho_\tau^k(x)$. Collecting the inequalities (25), (27) and (28), and using the elementary inequality $|a-b|^m \le |a^m - b^m|$, holding for any positive numbers a,b, we find

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}^n}|\rho_\tau^+(t+s,x)-\rho_\tau^+(t,x)|^{m+1}\\ &\le \int_0^\infty\int_{\mathbb{R}^n}\left(\rho_\tau^+(t+s,x)-\rho_\tau^+(t,x)\right)\left(\rho_\tau^+(t+s,x))^m - (\rho_\tau^+(t,x))^m\right)\\ &= \sum_{k=1}^\infty \tau\int_{\mathbb{R}^n}\left((\rho_\tau^{k+l})^+ - (\rho_\tau^k)^+\right)\left(|(\rho_\tau^{k+1})^+|^m - |(\rho_\tau^k)^+|^m\right)\\ &\le \sum_{k=1}^\infty\left\{A_{\tau,k,l}\sum_{j=1}^l\mathfrak{W}_2(\rho_\tau^{k+l},\rho_\tau^{k+l-1})\right\} + \sum_{k=1}^\infty 2\tau K^m\sum_{j=1}^l\int_{\mathbb{R}^n}(\rho_\tau^{k+j-1})_\beta^+\\ &\le \left(\sum_{k=1}^\infty (A_{\tau,k,l})^2\right)^{\frac12}\left(\sum_{k=1}^\infty\left(\sum_{j=1}^l\mathfrak{W}_2(\rho_\tau^{k+l},\rho_\tau^{k+l-1})\right)^2\right)^{\frac12} + \sum_{j=1}^l 2MK^m\tau\\ &\le 4\sqrt{K\tau\mathscr{F}(\rho_0)}\left(l\sum_{j=1}^l\sum_{k=0}^\infty\mathfrak{W}_2^2(\rho_\tau^{k+l},\rho_\tau^{k+l-1})\right)^{\frac12} + 2MK^m\tau l\\ &\le 4\sqrt{2K}\mathscr{F}(\rho_0)\tau l + 2MK^m\tau l = \left(4\sqrt{2K}\mathscr{F}(\rho_0) + 2MK^m\right)s.\end{aligned}$$

The same reasoning yields this equicontinuity estimate in time for negative parts of $\rho_\tau(\cdot,\cdot)$, and then for $\rho_\tau(\cdot,\cdot)$ itself. □

4. **Proof of the main result.** Let the initial datum ρ_0 belong to $L^1\cap L^\infty(\mathbb{R}^n)$. We have proved the two equicontinuity estimates (18) and (21), which give, together with the tail control (19), the compactness in $L^1((0,T)\times\mathbb{R}^n)$ for any $T>0$ for sequences $(\rho_{\tau_n}(\cdot,\cdot))$, for $\tau_n\to 0$ (and in L^p for any $p<+\infty$, since we have the uniform L^∞ bound from Proposition 3.3). We take a limit, and we are in the position to prove our main result.

Proof of Theorem 1.1. Let $\rho(\cdot,\cdot)$ be a $L^1((0,T)\times\mathbb{R}^n)$ limit for the family $\rho_\tau(\cdot,\cdot)$ (along a suitable sequence, still denoted by ρ_τ). By the global $L^\infty((0,+\infty)\times\mathbb{R}^n)$ bounds for $\rho_\tau(\cdot,\cdot)$ there is also convergence in $L^m((0,T)\times\mathbb{R}^n)$. Therefore, from (18) we also deduce that $\nabla(|\rho_\tau|^{m-1}\rho_\tau)\to\nabla(|\rho|^{m-1}\rho)$ weakly in $L^2((0,+\infty)\times\mathbb{R}^n)$. In particular ρ satisfies property *i)* in the definition of solution.

We may reason directly on the discrete solution $\rho_\tau(t,x)$, letting φ be a smooth test function compactly supported in $(0,+\infty)\times\mathbb{R}^n$, and deriving in the sense of distributions. We obtain, letting γ_τ^k be the plans defined by (23),

$$\begin{aligned}
&-\int_0^\infty\int_{\mathbb{R}^n}\partial_t\varphi(t,x)\rho_\tau(t,x)\,dx\,dt\\
&=\int_0^\infty\int_{\mathbb{R}^n}\varphi(t,x)\partial_t\rho_\tau(t,x)\,dx\,dt\\
&=\int_0^\infty\int_{\mathbb{R}^n}\varphi(t,x)\sum_{k=0}^\infty\delta_{k\tau}(t)(\rho_\tau^{k+1}(x)-\rho_\tau^k(x))\,dx\,dt\\
&=\sum_{k=0}^\infty\left(\int_{\mathbb{R}^n\times\mathbb{R}^n}\langle\nabla\varphi(k\tau,x),x-y\rangle\,d\gamma_\tau^{k+1}(x,y)+\mathbf{R}_\tau^{k+1}\right).
\end{aligned}\tag{29}$$

Here $\mathbf{R}_\tau^k$ corresponds to the remainder term in the Taylor expansion, therefore, using (17), we have the estimate

$$\sum_{k=1}^\infty\mathbf{R}_\tau^k\le\sum_{k=1}^\infty\left|\sup_{\mathbb{R}^n}\nabla^2\varphi\right|\int_{\mathbb{R}^n}|x-y|^2\,d\gamma_\tau^k(x,y)\le 2\tau\mathscr{F}(\rho_0)\left|\sup_{\mathbb{R}^n}\nabla^2\varphi\right|.$$

This shows that $\sum_k\mathbf{R}_\tau^k=o(1)$ for $\tau\to 0$. On the other hand, about the parts $(\beta_\tau^k)_+,(\beta_\tau^k)_-$ of γ_τ^k, using (17), (25) and Cauchy-Schwarz, we find the estimate

$$\begin{aligned}
&\sum_{k=0}^\infty\int_{\mathbb{R}^n\times\mathbb{R}^n}\langle\varphi(k\tau,x),x-y\rangle\,d((\beta_\tau^{k+1})_+-(\beta_\tau^{k+1})_-)\\
&\le\left(\sup_{\mathbb{R}^n}|\nabla\varphi|\right)\sum_{k=0}^\infty\int_{\mathbb{R}^n\times\mathbb{R}^n}|x-y|\,d((\beta_\tau^{k+1})_+-(\beta_\tau^{k+1})_-)\\
&\le\sqrt{2\mathscr{F}(\rho_0)M\tau}\sup_{\mathbb{R}^n}|\nabla\varphi|,
\end{aligned}$$

therefore this contribution is also infinitesimal for $\tau\to 0$. The remaining part in the right hand side of (29) is an integral with respect to the plan $(\gamma_\tau^{k+1})^+-(\gamma_\tau^{k+1})^-$, hence it can be written using (13) as

$$\begin{aligned}
&\sum_{k=0}^{\infty}\int_{\mathbb{R}^n\times\mathbb{R}^n}\langle\nabla\varphi(k\tau,x),x-y\rangle\,d((\gamma_\tau^{k+1})^+-(\gamma_\tau^{k+1})^-)(x,y)\\
&=\sum_{k=0}^{\infty}\tau\int_{\mathbb{R}^n}\langle\nabla\varphi(k\tau,x),\nabla(\rho_\tau^{k+1}(x)|\rho_\tau^{k+1}(x)|^{m-1})\rangle\,dx\\
&=\sum_{k=0}^{\infty}\int_{k\tau}^{(k+1)\tau}dt\int_{\mathbb{R}^n}\langle\nabla\varphi(k\tau,x),\nabla(\rho_\tau^{k+1}(x)|\rho_\tau^{k+1}(x)|^{m-1})\rangle\,dx\\
&=\sum_{k=0}^{\infty}\int_{k\tau}^{(k+1)\tau}\int_{\mathbb{R}^n}\langle\nabla\varphi(\tau\lceil t/\tau\rceil-\tau,x),\nabla(\rho_\tau(t,x)|\rho_\tau(t,x)|^{m-1})\rangle\,dx\,dt\\
&=\int_0^{\infty}\int_{\mathbb{R}^n}\langle\nabla\varphi(\tau\lceil t/\tau\rceil-\tau,x),\nabla(\rho_\tau(t,x)|\rho_\tau(t,x)|^{m-1})\rangle\,dx\,dt.
\end{aligned}$$

Therefore we end up with

$$\begin{aligned}
&-\int_0^{\infty}\int_{\mathbb{R}^n}\partial_t\varphi(t,x)\rho_\tau(t,x)\,dx\,dt\\
&=\int_0^{\infty}\int_{\mathbb{R}^n}\langle\nabla\varphi(\tau\lceil t/\tau\rceil-\tau,x),\nabla(\rho_\tau(t,x)|\rho_\tau(t,x)|^{m-1})\rangle\,dx\,dt+o(1).
\end{aligned}$$

With the convergence properties of $\rho_\tau(\cdot,\cdot)$, we finally obtain

$$-\int_0^{\infty}\int_{\mathbb{R}^n}\partial_t\varphi(t,x)\rho(t,x)\,dx\,dt=\int_0^{\infty}\int_{\mathbb{R}^n}\langle\nabla\varphi(t,x),\nabla(|\rho(t,x)|^{m-1}\rho(t,x))\rangle\,dx\,dt.$$

The initial datum is got and therefore ρ is the unique weak solution to Cauchy problem for the signed porous medium equation.

Acknowledgments. The author wishes to thank Nicola Gigli for fruitful discussions.

REFERENCES

[1] L. Ambrosio, N. Gigli and G. Savaré, "Gradient Flows in Metric Spaces and in the Spaces of Probability Measures," Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2005.

[2] L. Ambrosio, E. Mainini and S. Serfaty, *Gradient flow of the Chapman-Rubinstein-Schatzman model for signed vortices*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **28** (2011), 217–246.

[3] D. G. Aronson, *The porous medium equation*, in "Nonlinear Diffusion Problems" (eds. A. Fasano and M. Primicerio), Lecture Notes in Math. 1224, Springer, Berlin, (1986), 1–46.

[4] M. Bertsch and D. Hilhorst, *The interface between regions where $u < 0$ and $u > 0$ in the porous medium equation*, Appl. Anal., **41** (1991), 111–130.

[5] J. A. Carrillo, R. J. McCann and C. Villani, *Contractions in the 2-Wasserstein length space and thermalization of granular media*, Arch. Ration. Mech. Anal., **179** (2006), 217–263.

[6] A. Friedman and S. Kamin, *The asymptotic behavior of gas in an n-dimensional porous medium*, Trans. Amer. Math. Soc., **262** (1980), 551–563.

[7] J. Hulshof, *Similarity solutions of the porous medium equation with sign changes*, J. Math. Anal. Appl., **157** (1991), 75–111.

[8] J. Hulshof, J. R. King and M. Bowen, *Intermediate asymptotics of the porous medium equation with sign changes*, Adv. Differential Equations, **6** (2001), 1115–1152.

[9] J. Hulshof and J. L. Vázquez, *The dipole solution for the porous medium equation in several space dimensions*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), **20** (1993), 193–217.

[10] R. Jordan, D. Kinderlehrer and F. Otto, *The variational formulation of the Fokker-Planck equation*, SIAM J. Math. Anal., **29** (1998), 1–17.

[11] S. Kamin and J. L. Vázquez, *Asymptotic behaviour of solutions of the porous medium equation with changing sign*, SIAM J. Math. Anal., **22** (1991), 34–45.

[12] E. Mainini, *A description of transport cost for signed measures*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), **390** (2011), 147–181.

[13] F. Otto, *Dynamics of labyrinthine pattern formation in magnetic fluids: A mean-field theory*, Arch. Rational Mech. Anal., **141** (1998), 63–103.

[14] F. Otto, *Evolution of microstructure in unstable porous media flow: A relaxational approach*, Comm. Pure Appl. Math., **52** (1999), 873–915.

[15] F. Otto, *The geometry of dissipative evolution equations: the porous medium equation*, Comm. Partial Differential Equations, **26** (2001), 101–174.

[16] C. J. van Duijn, S. M. Gomes and H. F. Zhang, *On a class of similarity solutions of the equation* $u_t = (|u|^{m-1}u_x)_x$ *with* $m > -1$, IMA J. Appl. Math., **41** (1988), 147–163.

[17] J. L. Vázquez, "The Porous Medium Equation," Mathematical Theory, Oxford Mathematical Monographs, Oxford, 2007.

[18] J. L. Vázquez, *Asymptotic beahviour for the porous medium equation posed in the whole space*, Dedicated to Philippe Bénilan. J. Evol. Equ., **3** (2003), 67–118.

[19] J. L. Vázquez, *New self-similar solutions of the porous medium equation and the theory of solutions of changing sign*, Nonlinear Anal., **15** (1990), 931–942.

[20] C. Villani, "Optimal Transport, Old and New," Springer-Verlag, 2008.

Received December 2011; revised July 2012.

E-mail address: edoardo.mainini@math.u-psud.fr, edoardo.mainini@unipv.it