NETWORKS AND HETEROGENEOUS MEDIA
doi:10.3934/nhm.2012.7.503

Volume **7**, Number **3**, September **2012**
pp. **503–524**

CONVERGENCE OF MSFEM APPROXIMATIONS FOR ELLIPTIC, NON-PERIODIC HOMOGENIZATION PROBLEMS

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(Communicated by Alfio Quarteroni)

Abstract. In this work, we are concerned with the convergence of the multiscale finite element method (MsFEM) for elliptic homogenization problems, where we do not assume a certain periodic or stochastic structure, but an averaging assumption which in particular covers periodic and ergodic stochastic coefficients. We also give a result on the convergence in the case of an arbitrary coupling between grid size H and a parameter ϵ. ϵ is an indicator for the size of the fine scale which converges to zero. The findings of this work are based on the homogenization results obtained in [B. Schweizer and M. Veneroni, The needle problem approach to non-periodic homogenization, Netw. Heterog. Media, 6 (4), 2011].

1. **Introduction.** This contribution is dedicated to the numerical analysis of the multiscale finite element method (MsFEM) for elliptic homogenization problems. This method, originally developed by Hou and Wu [31], is constructed to solve partial differential equations, where the coefficient functions are rapidly oscillating. Typically, standard methods fail to directly solve such types of equations, since resolving the oscillatory structure requires a tremendous computational demand. Therefore it is necessary to propose alternative methods, so called multiscale methods, which are capable of determining the average effect of the micro-structure on the effective macroscopic behavior, without resolving all the fine-scale details. One example for a multiscale method is the heterogeneous multiscale finite element method (HMM) introduced by E and Engquist [11], where the fine scale behavior of the solution is reconstructed in small cells around quadrature points to pass an averaged information to a discrete coarse-scale problem (c.f. [11, 12, 4, 13, 3, 44, 26, 27]). Another example is the variational multiscale method (VMM), based on the works of Hughes *et al.* [33, 34]. Here, the solution space is split into a direct sum of a coarse scale space and a fine scale space. Then fine scale equations are formally solved in dependency of the residual of the coarse scale solution (c.f. [35, 36, 43, 42, 37, 38]). Another approach, based on the construction of a suitable two-scale finite element space, is the two-scale finite element method by Matache and Schwab [40, 41, 45]

2000 *Mathematics Subject Classification.* Primary: 35J15, 65G99, 74Q15; Secondary: 65N30.
Key words and phrases. Elliptic homogenization, MsFEM, convergence, multiscale methods.
This work was supported by the Deutsche Forschungsgemeinschaft under the contract number OH 98/4-1.

or the Sparse Multiscale FEM by Hoang and Schwab [29, 28] that is based on a discretization of a multiscale homogenized equation by means of sparse grids.

As already mentioned, this work is dedicated to the multiscale finite element method (MsFEM), whose central idea is to determine a set of fine-scale finite element basis functions, which are obtained by solving suitable fine-scale problems for every grid element of a given triangulation. These basis functions incorporate the required information about the micro-structure of the problem, so that the remaining (discrete) global problem is low dimensional, but still yields accurate approximations of the exact solution of the original equation. Lately, the MsFEM was proposed for a large variety of applications, such as for two phase flow in porous media [14], for stochastic porous media flow [1] and applications to uncertainty quantification [10], for solving optimal control problems governed by elliptic homogenization problems [39], for mechanical problems of heterogeneous materials in elasticity [47], for elliptic interface problems with high contrast coefficients [8] and for solving high-contrast problems using local spectral basis functions [22]. Mixed multiscale finite element methods using limited global information were proposed by Aarnes, Efendiev and Jiang [2]. A detailed survey on the topic of MsFEM is given in the book by Efendiev and Hou [15]. In the following we are concerned with the numerical analysis of MsFEM approximations for linear elliptic problems, i.e., find $u^\epsilon \in \mathring{H}^1(\Omega)$, such that

$$-\nabla \cdot (A^\epsilon \nabla u_\epsilon) \quad = \quad f \qquad \text{in } \Omega.$$

Here, ϵ is a parameter which characterises the fine scale of the problem, i.e. the smaller ϵ, the faster the micro-scale oscillations of the matrix A^ϵ which varies on a scale of size $\mathrm{O}(\epsilon)$. A typical structure might be $A^\epsilon(x) = A(\frac{x}{\epsilon})$ where A is a 1-periodic function.

There are several contributions dealing with the convergence of MsFEM approximations for this type of problems. First a-priori error estimates in the L^2 and in the H^1-norm were obtained by Hou, Wu and Cai [31, 32] in the periodic setting. The convergence of a nonconforming multiscale finite element method in the periodic setting was investigated in [23]. The mentioned work also includes the analysis of an oversampling technique to reduce the resonance error which appears when there is a mismatch between the mesh-size and the wavelength of the fine-scale oscillations. The resonance error becomes apparent in the derived a-priori estimates, which contain terms of order $\frac{\epsilon}{H}$, where H denotes the grid size. The convergence of the MsFEM for nonlinear elliptic problems was treated by Efendiev, Hou and Ginting [16] and by Chen and Savchuk [7], again, under the assumption of periodicity. An analysis for the MsFEM for random homogenization problems was also given by Chen and Savchuk [7]. The case of a multiscale finite element method with nonconforming elements for elliptic (random and periodic) homogenization problems was treated by Chen, Cui, Savchuk and Yu [6].

There are also several works by Efendiev and Pankov in which they can show convergence (up to a subsequence) of the coarse scale part of MsFEM approximations to the homogenized solution. Nonlinear elliptic homogenization problems are treated in [17, 18] and nonlinear parabolic homogenization problems in [19, 20, 21]. These contributions are in the general setting of G-convergence, however, the proof of corrector convergence (i.e. an accurate approximation of the solution gradient) still requires the assumption of ergodic stochastic coefficients.

In this work we present a convergence study for MsFEM approximations (including the convergence to the correct solution gradient) in a general setting which does not assume a certain periodic or ergodic stochastic structure as required in previous works. Still, these cases are included in the analysis. For $2d$ and $3d$ we get that the H^1-error in fact converges to zero for either $\lim_{H\to 0}\lim_{\epsilon\to 0}$ or $\lim_{\epsilon\to 0}\lim_{H\to 0}$ (i.e. the limits are obtained one after the other). Furthermore, we also treat the case of an arbitrary coupling of ϵ and mesh size H. The finding is, that for the $1d$ case, any sequence of MsFEM approximations with $(H,\epsilon)\to 0$ is convergent to the correct solution. Even resonance errors average out by an intrinsic homogenization process, once ϵ and H become small enough. This case has not yet been studied to the best of our knowledge. We also find out, that the mentioned $1d$ result does not hold for any other space dimension. The analysis in this contribution is based on the homogenization theory presented by Schweizer and Veneroni [46] under the assumption that the family of coefficients A^ϵ *allows averaging*. For the subsequent work, this is the only assumption that we make on the type of the fine-scale structure of A^ϵ. Since we are working in a general framework, it is not possible to state explicit orders for the speed of convergence in ϵ.

Outline: In Section 2 we introduce the setting of this paper and we state the definition of the MsFEM for elliptic homogenization problems. In Section 3 we state the major assumption on the structure of our elliptic multiscale problem and we state the associated homogenization results obtained by Schweizer and Veneroni [46]. Furthermore, we present the two main results of this contribution concerning the convergence of the MsFEM. A discussion and the proofs of these two theorems are given in Section 4 for the $2d$ and $3d$ case and in Section 5 for the $1d$ case.

2. Setting and definitions.

The following definitions and assumption are assumed for all the subsequent sections. $\Omega\subset\mathbb{R}^d$ denotes a d-dimensional, bounded Lipschitz domain with polygonal boundary and with $d\in\{1,2,3\}$. In the following we assume that $(A^\epsilon)_{\epsilon>0}\in L^\infty(\Omega,\mathbb{R}^{d\times d})$ is a family of coefficient functions which is uniformly elliptic in ϵ, i.e. there exist constants $\alpha,\beta\in\mathbb{R}_{>0}$, such that:

$$\alpha|\gamma|^2\le A^\epsilon(x)\gamma\cdot\gamma\le\beta|\gamma|^2\quad\forall\gamma\in\mathbb{R}^d,\text{ and almost everywhere in }\Omega.$$

For the source term f, we demand $f\in L^2(\Omega)$. Moreover, we define

$$\mathring{H}^1(\Omega):=\overline{\mathring{C}^\infty(\Omega)}^{\|\cdot\|_{H^1(\Omega)}},$$

where $\mathring{C}^\infty(\Omega)$ denotes the space of infinitely differentiable functions with compact support in Ω and where $\|u\|_{H^1(\Omega)}:=\|u\|_{L^2(\Omega)}+\|\nabla u\|_{L^2(\Omega)}$. Analogously we define the seminorm on H^1 by $|u|_{H^1(\Omega)}:=\|\nabla u\|_{L^2(\Omega)}$. Furthermore, we introduce for simplicity:

$$\mathcal{A}^\epsilon(\Phi,\Psi):=\int_\Omega A^\epsilon(x)\nabla\Phi(x)\cdot\nabla\Psi(x)\,dx\ \text{ and }\ \mathcal{F}(\Phi):=\int_\Omega f(x)\Phi(x)\,dx.$$

In the following, we consider the problem to find $u^\epsilon\in\mathring{H}^1(\Omega)$ with

$$\mathcal{A}^\epsilon(u^\epsilon,\Phi)=\mathcal{F}(\Phi)\quad\forall\Phi\in\mathring{H}^1(\Omega). \tag{1}$$

In particular, we are interested in the case of ϵ becoming extremely small.

For discretizing the problem, let $\mathcal{T}_H(\Omega)$ be a regular simplicial partition of Ω. The elements of $\mathcal{T}_H(\Omega)$ are denoted by T and the barycenter of $T\in\mathcal{T}_H(\Omega)$ is denoted by x_T. For the diameter of an element $T \in \mathcal{T}_H(\Omega)$ we use H_T :=diam(T) and $H := \sup_{T\in\mathcal{T}_H(\Omega)} H_T$. The usual finite element space of continuous, piecewise linear functions is given by

$$V_H(\Omega) \quad := \quad \{\Phi_H \in \mathring{H}^1(\Omega) \cap C^0(\Omega) \ |\Phi_{H_{|T}} \in \mathbb{P}^1(T) \ \forall T \in \mathcal{T}_H(\Omega)\}.$$

Here, $\mathbb{P}^1(T)$ denotes the space of polynomials of degree 1 on T.

Let N denote the dimension of $V_H(\Omega)$ and let $\{\Phi_i| \ 1 \le i \le N\}$ denote the usual Lagrange basis of $V_H(\Omega)$. We define the MsFEM solution space by

$$V_H^\epsilon(\Omega) := \mathrm{span}\{\Phi_i^\epsilon| \ 1 \le i \le N\},$$

where for every $T \in \mathcal{T}_H(\Omega)$, $\Phi_i^\epsilon \in \mathring{H}^1(\Omega)$ is the solution of

$$\int_T A^\epsilon(x)\nabla\Phi_i^\epsilon(x)\cdot\nabla\phi(x) = 0 \qquad \forall\phi \in \mathring{H}^1(T),$$

and with $\Phi_i^\epsilon = \Phi_i$ on ∂T. Due to continuity, this yields a conforming set of basis functions, i.e. $V_H^\epsilon(\Omega) \subset H^1(\Omega)$ (c.f. the book of Efendiev and Hou [15]). Now, we can define the MsFEM solution u_H^ϵ:

Definition 2.1 (MsFEM)**.** The MsFEM approximation $u_H^\epsilon \in V_H^\epsilon(\Omega)$ of u^ϵ solves

$$\int_\Omega A^\epsilon(x)\nabla u_H^\epsilon \cdot \nabla\Phi_H^\epsilon(x)\,dx = \int_\Omega f(x)\Phi_H^\epsilon(x)\,dx$$

for all $\Phi_H^\epsilon \in V_H^\epsilon(\Omega)$. We note that u_H^ϵ is a H^1-approximation of u^ϵ.

3. **Homogenization and main results.** In this section we introduce the homogenization result obtained by Schweizer and Veneroni [46] under the assumption that the family of coefficients A^ϵ *allows averaging*, which is defined below. On the basis of this result, we are concerned with the convergence of a sequence of MsFEM approximations. The corresponding main results are presented at the end of this section.

We start with the following assumption, initially introduced in [46] for the needle problem approach to non-periodic homogenization:

Assumption 1. We assume that A^ϵ allows averaging of the constitutive relation with the matrix $A^0 \in \mathbb{R}^{d\times d}$, i.e. for every simplex $T \subset \Omega$, every $\xi \in \mathbb{R}^d$, every $b \in \mathbb{R}$ we have

$$\lim_{\epsilon\to 0} \fint_T A^\epsilon \nabla v^\epsilon = A^0\xi \tag{2}$$

where $v^\epsilon \in H^1(T)$ is defined as the solution of

$$\int_T A^\epsilon(x)\nabla v^\epsilon(x)\cdot\nabla\phi(x) = 0 \qquad \forall\phi \in \mathring{H}^1(T)$$

and with $v^\epsilon(x) = \xi\cdot x + b$ on ∂T.

For instance, this assumption covers the periodic setting (i.e. $A^\epsilon(x) = A(\frac{x}{\epsilon})$, with a $[0,1]^d$-periodic matrix A) or the case of ergodic stochastic coefficients. In the periodic case, the convergence in (2) is directly obtained via weak convergence of $A^\epsilon\nabla v^\epsilon$ and in the case of ergodic stochastic coefficients, we refer to the appendix of [46].

The following homogenization result was obtained by Schweizer and Veneroni [46]:

Theorem 3.1. *Let $u^\epsilon \in \mathring{H}^1(\Omega)$ denote the solution of equation (1). Under the general assumptions of Section 2 and if A^ϵ allows averaging of the constitutive relation with the matrix $A^0 \in \mathbb{R}^{n\times n}$, we obtain that the sequence $(u^\epsilon)_\epsilon$ of solutions satisfies:*

$$u^\epsilon \rightharpoonup u^0 \quad \text{weakly in } H^1(\Omega),$$
$$A^\epsilon \nabla u^\epsilon \rightharpoonup A^0 \nabla u^0 \quad \text{weakly in } L^2(\Omega),$$

where $u^0 \in \mathring{H}^1(\Omega)$ is the solution of

$$\int_\Omega A^0 \nabla u^0 \cdot \nabla \Phi = \int_\Omega f\, \Phi \quad \forall \Phi \in \mathring{H}^1(\Omega) \tag{3}$$

Remark 1. In [46], Theorem 3.1 is only stated for $d = 2, 3$. However, it is easy to verify, that it also holds for $d = 1$. In this case, the existence of a homogenized matrix A^* and a homogenized solution u^* can be obtained in the very general setting of H-convergence. To verify $A^* = A^0$, we can use that the H-limit A^* is equal to the inverse of the weak-$*$ L^∞-limit of $(A^\epsilon)^{-1}$. A simple computation yields that is identical to A^0. The same argument is also used in Section 5, where it is elaborated with more details.

Remark 2. The homogenized matrix $A^0 \in \mathbb{R}^{d\times d}$ introduced in Theorem 3.1 is elliptic with the same constant $\alpha > 0$ as A^ϵ, i.e.:

$$A^0\gamma \cdot \gamma \geq \alpha|\gamma|^2 \quad \forall \gamma \in \mathbb{R}^d.$$

This is a simple conclusion if we observe that Theorem 3.1 implies H-convergence of A^ϵ to A^0. Compactness results of H-convergent sequences guarantee that if the whole sequence $(A^\epsilon)_{\epsilon>0}$ is uniformly elliptic with the same constant α, the condition also holds for the limit A^0. See for instance Theorem 13.4 in [9].

Definition 3.2. We define $u_H^{0,\epsilon} \in V_H(\Omega)$, the *coarse scale part* of u_H^ϵ, by

$$u_H^{0,\epsilon} := \sum_{i=1}^N \alpha_i^\epsilon \Phi_i,$$

where $(\Phi_i)_i$ denotes the Lagrange basis of $V_H(\Omega)$ and $\alpha^\epsilon \in \mathbb{R}^N$ denotes the coefficient vector of u_H^ϵ in $V_H^\epsilon(\Omega)$, i.e. $u_H^\epsilon = \sum_{i=1}^N \alpha_i^\epsilon \Phi_i^\epsilon$.

Now, we can state the main results of this contribution. The first theorem treats the case $d = 2, 3$ and, in particular, shows that under the given assumptions, the sequence of MsFEM approximations captures the fine-scale oscillations of the exact solution u^ϵ, i.e. we have $\lim_{H\to 0}\lim_{\epsilon\to 0}\|u^\epsilon - u_H^\epsilon\|_{H^1(\Omega)} = 0$:

Theorem 3.3 (Convergence in 2- and 3-d)**.** *Let $d = 2, 3$ and let $u^\epsilon \in \mathring{H}^1(\Omega)$ denote the solution of equation (1) and let u_H^ϵ denote the MsFEM solution from Definition 2.1. If Ω is a convex domain and if A^ϵ allows averaging in the sense of Assumption 1, we obtain the following estimates:*

$$\limsup_{\epsilon\to 0} \|u^\epsilon - u_H^\epsilon\|_{H^1(\Omega)} \leq CH \quad \text{and} \quad \lim_{\epsilon\to 0} \|u^\epsilon - u_H^\epsilon\|_{L^2(\Omega)} \leq CH^2,$$
$$\lim_{\epsilon\to 0} \|u^0 - u_H^{0,\epsilon}\|_{H^1(\Omega)} \leq CH \quad \text{and} \quad \lim_{\epsilon\to 0} \|u^0 - u_H^{0,\epsilon}\|_{L^2(\Omega)} \leq CH^2.$$

Here, C denotes a constant independent of ϵ and H.

In general, i.e. if Ω is possibly not convex, there exists some $s \in [\frac{3}{2}, 2]$, where $u^0 \in \mathring{H}^1(\Omega) \cap H^s(\Omega)$, and we get:

$$\limsup_{\epsilon\to 0} \|u^\epsilon - u_H^\epsilon\|_{H^1(\Omega)} \leq CH^{s-1} \quad \text{and} \quad \lim_{\epsilon\to 0} \|u^\epsilon - u_H^\epsilon\|_{L^2(\Omega)} \leq CH^s,$$
$$\lim_{\epsilon\to 0} \|u^0 - u_H^{0,\epsilon}\|_{H^1(\Omega)} \leq CH^{s-1} \quad \text{and} \quad \lim_{\epsilon\to 0} \|u^0 - u_H^{0,\epsilon}\|_{L^2(\Omega)} \leq CH^{s-1}.$$

For instance, if $n = 2$, we have $s = 1 + \frac{\pi}{\omega}$, where $\omega > \pi$ denotes the largest interior angle of an opening (i.e. of a re-entrant corner).

The proof of this theorem is given in Section 4.

For completeness, we also state a well known result which holds for any space dimension and which guarantees convergence for $H \to 0$. For convenience of the reader it is also stated in Section 4 (see Proposition 1):

Remark 3. Let $u^\epsilon \in \mathring{H}^1(\Omega)$ denote the solution of equation (1) and let u_H^ϵ denote the MsFEM solution from Definition 2.1, then we also have

$$\lim_{H\to 0} \|u^\epsilon - u_H^\epsilon\|_{H^1(\Omega)} = 0.$$

In the following theorem, the case $d = 1$ is treated. Here we observe that, independently of how we couple $\epsilon \to 0$ and $H \to 0$, we get convergence for any sequence of MsFEM approximations u_H^ϵ to the same limit. The reason for this is that the MsFEM-problem behaves like a homogenized (or averaged) problem, once ϵ and H get sufficiently small (i.e. the MsFEM problem behaves like a discretization of a homogenized/averaged equation). In Section 5, we go into detail. In the following theorem, we also note the interesting case of $0 < \frac{H}{\epsilon}$ =const< 1, which typically yields a rapidly oscillating coarse scale part $u_H^{0,\epsilon}$.

Theorem 3.4. *Assume $d = 1$, let $u^\epsilon \in \mathring{H}^1(\Omega)$ denote the solution of (1) and u_H^ϵ the MsFEM approximation from Definition 2.1. If we assume that A^ϵ allows averaging in the sense of Assumption 1, we get the following convergence for any sequence $H(\epsilon)$ with $H(\epsilon) \to 0$ for $\epsilon \to 0$:*

$$\lim_{\epsilon\to 0} \|u_{H(\epsilon)}^\epsilon - u^\epsilon\|_{L^2(\Omega)} = 0.$$

The theorem is a conclusion from Theorem 5.3, which is proved in Section 5.

Remark 4. Note that Theorem 3.4 does not hold for higher dimensions. If we couple H and ϵ by a fixed ratio, we might obtain convergence to a wrong approximation. Even if $\frac{\epsilon}{H} = r \ll 1$, there might be always a (possibly extremely small) remainder. This is also discussed at the end of Section 4.

4. Convergence of the MsFEM for $d = 2, 3$.

In this section, we are essentially concerned with proving Theorem 3.3. We therefore assume $d = 2, 3$.

We start this section with introducing a new formulation of the MsFEM problem, which is more convenient for our purposes. First, we define the discrete multiscale operator R^ϵ which transforms a basis function into a multiscale basis function.

Definition 4.1 (Discrete Multiscale Operator)**.** For $\Phi \in \mathring{H}^1(\Omega)$ and $T \in \mathcal{T}_H$, the local multiscale correction $Q_T^\epsilon(\Phi) \in \mathring{H}^1(T)$ is the solution of the following problem:

$$\int_T A^\epsilon(x)\nabla Q_T^\epsilon(\Phi)(x) \cdot \nabla\phi(x) = -\int_T A^\epsilon(x)\nabla\Phi(x) \cdot \nabla\phi(x) \qquad \forall \phi \in \mathring{H}^1(T).$$

The global multiscale correction $Q^\epsilon(\Phi) \in \mathring{H}^1(\Omega)$ is given piecewise for every $T \in \mathcal{T}_H$ by the local parts:

$$Q^\epsilon(\Phi)(x) := Q^\epsilon_T(\Phi)(x) \quad \text{for } x \in T \in \mathcal{T}_H.$$

Furthermore we define $R^\epsilon(\Phi) := \Phi + Q^\epsilon(\Phi)$. Note that $R^\epsilon(\Phi) \in \mathring{H}^1(\Omega)$, since $Q^\epsilon(\Phi) \in \mathring{H}^1(T)$ for all $T \in \mathcal{T}_H(\Omega)$.

Remark 5. Observe that we have the relation $R^\epsilon(\Phi_i) = \Phi_i^\epsilon$, since for every i, for every $T \in \mathcal{T}_H(\Omega)$ and every $\phi \in \mathring{H}^1(T)$:

$$\begin{aligned}&\int_T A^\epsilon(x)\nabla R^\epsilon(\Phi_i)(x) \cdot \nabla\phi(x)\,dx \\ &\quad = \int_T A^\epsilon(x)\nabla Q^\epsilon(\Phi_i)(x) \cdot \nabla\phi(x)\,dx + \int_T A^\epsilon(x)\nabla\Phi_i(x) \cdot \nabla\phi(x)\,dx = 0\end{aligned}$$

and $R^\epsilon(\Phi_i) = Q^\epsilon(\Phi_i) + \Phi_i = \Phi_i$ on ∂T, which is exactly the definition of Φ_i^ϵ.

Next, we define the MsFEM bilinear form and the MsFEM right hand side functional:

Definition 4.2. To describe the MsFEM, we define the multiscale bilinear form $\mathcal{A}^\epsilon_H$ on $V_H(\Omega) \times V_H(\Omega)$ by:

$$\mathcal{A}^\epsilon_H(\Phi_H, \Psi_H) := \int_\Omega A^\epsilon(x)\nabla R^\epsilon(\Phi_H)(x) \cdot \nabla R^\epsilon(\Psi_H)(x)\,dx.$$

and the associated right hand side functional by:

$$\mathcal{F}^\epsilon_H(\Phi_H) := \int_\Omega f(x) R^\epsilon(\Phi_H)(x)\,dx \quad \text{for } \Phi_H \in V_H(\Omega).$$

The following is a direct conclusion from the preceding definition and from Remark 5:

Conclusion 1 (Reformulation of the MsFEM)**.** If $u_H^{0,\epsilon} \in V_H(\Omega)$ solves

$$\mathcal{A}^\epsilon_H(u_H^{0,\epsilon}, \Phi_H) = \mathcal{F}^\epsilon_H(\Phi_H) \tag{4}$$

for all $\Phi_H \in V_H(\Omega)$, then we have $u_H^\epsilon = R^\epsilon(u_H^{0,\epsilon})$, where u_H^ϵ denotes the MsFEM solution. In particular, we obtain

$$\int_T A^\epsilon(x)\nabla u_H^\epsilon(x) \cdot \nabla\phi(x) = 0 \qquad \forall \phi \in \mathring{H}^1(T). \tag{5}$$

The next theorem proves that $\mathcal{A}^\epsilon_H$ is a coercive bilinear form:

Theorem 4.3 (Ellipticity of $\mathcal{A}^\epsilon_H$)**.** *The bilinearform $\mathcal{A}^\epsilon_H$ is uniformly coercive on $V_H(\Omega) \times V_H(\Omega)$ with the same constant $\alpha > 0$ assumed for A^ϵ, i.e. we have independent of ϵ and H:*

$$\alpha|\Phi_H|^2_{H^1(\Omega)} \leq \mathcal{A}^\epsilon_H(\Phi_H, \Phi_H).$$

Proof. Let us fix $T \in \mathcal{T}_H$ and $\Phi_H \in V_H$. For the affine function $g(x) := \Phi_H(x)_{|T} = \xi \cdot x + b$ (i.e. $\xi := \nabla\Phi_H(x_T)$ and $b := \Phi_H(x_T) - \nabla\Phi_H(x_T) \cdot x_T$) we consider the problem: find $w = u + g \in H^1(T)$ with $u \in \mathring{H}^1(T)$ and

$$\int_T \nabla u \cdot \nabla\phi = 0 \qquad \forall \phi \in \mathring{H}^1(T).$$

From the Lax-Milgram theorem we get the unique solution $u = w - g = 0$, i.e. $w(x) = g(x) = \Phi_H(x)_{|T}$. This is equivalent to w minimizing the corresponding energy functional. We get:

$$\begin{aligned}\int_T |\nabla \Phi_H|^2 &= \int_T |\nabla w|^2 \\ &= \min\left\{ \int_T |\nabla v|^2 |\ v \in H^1(T),\ v = g \text{ on } \partial T \right\} \\ &\leq \int_T |\nabla(\Phi_H + Q^\epsilon(\Phi_H))|^2,\end{aligned}$$

where we used that $\Phi_H + Q^\epsilon(\Phi_H)$ is admissible because of $\Phi_H = g_{|T}$ and $Q^\epsilon(\Phi_H) \in \mathring{H}^1(T)$. Exploiting this inequality we get:

$$\begin{aligned}\mathcal{A}_H^\epsilon(\Phi_H, \Phi_H) &= \sum_{T\in\mathcal{T}_H} \int_T A^\epsilon(x)\nabla R^\epsilon(\Phi_H)(x) \cdot \nabla R^\epsilon(\Phi_H)(x)\, dx \\ &\geq \sum_{T\in\mathcal{T}_H} \int_T \alpha |\nabla \Phi_H(x) + \nabla Q^\epsilon(\Phi_H)(x)|^2\, dx \\ &\geq \sum_{T\in\mathcal{T}_H} \int_T \alpha |\nabla \Phi_H(x)|^2\, dx = \alpha |\Phi_H|^2_{H^1(\Omega)}.\end{aligned}$$

□

Theorem 4.3 guarantees uniform boundedness of the coarse scale part of u_H^ϵ:

Remark 6. Let $u_H^{0,\epsilon}$ denote the solution of problem (4), then we have:

$$\|u_H^{0,\epsilon}\|_{H^1(\Omega)} \leq \frac{\sqrt{c_p}}{\alpha} \|f\|_{L^2(\Omega)}.$$

Here, c_p denotes the constant from the Poincaré-inequality. This is a direct consequence of Theorem 4.3, which gives us

$$\alpha \|u_H^{0,\epsilon}\|^2_{H^1(\Omega)} \leq \mathcal{A}_H^\epsilon(u_H^{0,\epsilon}, u_H^{0,\epsilon}) = \mathcal{F}_H^\epsilon(u_H^{0,\epsilon}) \leq \|f\|_{L^2(\Omega)} \|u_H^\epsilon\|_{L^2(\Omega)} \leq \frac{c_p}{\alpha} \|f\|^2_{L^2(\Omega)},$$

where we used $\int_\Omega A^\epsilon(x)\nabla u_H^\epsilon \cdot \nabla u_H^\epsilon(x)\, dx = \int_\Omega f(x) u_H^\epsilon(x)\, dx$ in the last step.

Before we can deal with the H^1-convergence of a sequence of MsFEM approximations, we require a stabilization result and a compensated compactness result, both obtained by Schweizer and Veneroni in [46]. Furthermore, we need some additional definitions to state the mentioned results properly. We start with the stabilization:

Lemma 4.4 (Stabilization)**.** *Suppose that the general assumptions of this section are fulfilled. In particular, we assume that $A^\epsilon \in L^\infty(\Omega, \mathbb{R}^{d\times d})$ allows averaging with the matrix A^0 according to Assumption 1. Let be $T \in \mathcal{T}_H(\Omega)$, $\xi \in \mathbb{R}^d$, $b \in \mathbb{R}$ and let $(v^\epsilon)_{\epsilon>0} \subset H^1(T)$ be a sequence of weak solutions of*

$$\int_T A^\epsilon(x)\nabla v^\epsilon(x) \cdot \nabla \phi(x) = 0 \qquad \forall \phi \in \mathring{H}^1(T)$$

with the boundary condition $v^\epsilon(x) = \xi \cdot x + b$ on ∂T. Then we obtain the following convergence for the sequence v^ϵ:

$$\begin{aligned} v^\epsilon &\rightharpoonup v \qquad \text{weakly in } H^1(T) \text{ and} \\ A^\epsilon \nabla v^\epsilon &\rightharpoonup A^0 \nabla v \quad \text{weakly in } L^2(T, \mathbb{R}^d),\end{aligned}$$

where v is linear with $\nabla v \equiv \xi$.

The proof of this Lemma can be found in [46], Proposition 2.7.

Conclusion 2. Let R^ϵ denote the multiscale operator of Definition 4.1. Then we have for every $\Phi_H \in V_H(\Omega)$ and every $\phi \in \mathring{H}^1(T)$:

$$\int_T A^\epsilon \nabla R^\epsilon(\Phi_H) \cdot \nabla \phi = \int_T A^\epsilon \left(\nabla Q^\epsilon(\Phi_H) + \nabla \Phi_H \right) \cdot \nabla \phi = 0$$

and $R^\epsilon(\Phi_H)(x) = \Phi_H(x) = \Phi_H(x_T) + (x - x_T) \cdot \nabla \Phi_H(x_T)$ on ∂T. Therefore, using Lemma 4.4 (which yields an affine weak limit v_H) and the above boundary condition, we obtain

$$\begin{aligned} R^\epsilon(\Phi_H) &\rightharpoonup \Phi_H && \text{weakly in } H^1(T) \text{ and} \\ A^\epsilon \nabla R^\epsilon(\Phi_H) &\rightharpoonup A^0 \nabla \Phi_H && \text{weakly in } L^2(T, \mathbb{R}^d). \end{aligned}$$

In the next step, we show ellipticity of the homogenized matrix A^0. The proof is similar to the proof of Theorem 4.3, but we need a little more tools, in particular, the already mentioned compensated compactness. For this purpose, we need to introduce some definitions, which can be also found in [46]: see Definition 3.1 for points of typical average, Definition 3.3 for typical segments and Definition 3.7 for $2d$ adapted grids.

Definition 4.5 (Points of typical average)**.** Let $(\epsilon_k)_{k\in\mathbb{N}}$ denote a sequence with $\epsilon_k \to 0$. Then, $x \in \Omega$ is called a *a point with typical average* for u^ϵ and $(\epsilon_k)_{k\in\mathbb{N}}$, if there exists a subsequence $(\epsilon_{k_l})_{l\in\mathbb{N}}$ of $(\epsilon_k)_{k\in\mathbb{N}}$, real numbers c_x and M_x, such that:

$$\fint_{B_{k_l^{-1}}(x)} |\nabla u^{\epsilon_{k_l}}(z)|^2 \, d\mathcal{L}^d(z) \le M_x \quad \forall l \in \mathbb{N}, \tag{6}$$

$$\fint_{B_{k_l^{-1}}(x)} u^{\epsilon_{k_l}}(z) \, d\mathcal{L}^d(z) \to c_x \quad \text{for } l \to \infty. \tag{7}$$

Here, $B_{k_l^{-1}}(x)$ denotes the ball of radius k_l^{-1} and with center x and $\mathcal{L}^d$ denotes the d-dimensional Lebesgue measure. A subsequence $(\epsilon_{k_l})_{l\in\mathbb{N}}$ of $(\epsilon_k)_{k\in\mathbb{N}}$ is called a *good sequence* for the point x if (6) and (7) are fulfilled for this sequence.

Definition 4.6 (Typical segments)**.** Let $\Omega \subset \mathbb{R}^d$ denote a domain, $(u^\epsilon)_\epsilon$ a bounded sequence in $H^1(\Omega)$ and $(\epsilon_k)_{k\in\mathbb{N}}$ a sequence with $\epsilon_k \to 0$. $\Gamma = [x, y]$ is called a *typical segment* if x and y are points of typical average and if there exists a subsequence $(\epsilon_{k_l})_{l\in\mathbb{N}}$ of $(\epsilon_k)_{k\in\mathbb{N}}$ and a positive real number M_Γ with

$$\|u^{\epsilon_{k_l}}_{|\Gamma}\|^2_{L^2(\Gamma)} + \|\nabla_\tau u^{\epsilon_{k_l}}_{|\Gamma}\|^2_{L^2(\Gamma)} \le M_\Gamma$$

and where $(\epsilon_{k_l})_{l\in\mathbb{N}}$ is a good subsequence for x and y. A subsequence with these properties is called a *good subsequence* for the segment Γ. Here, ∇_τ denotes the (weak) tangential gradient (along Γ). For regular functions u this yields $\nabla_\tau u(x) = \nabla u(x) - (n(x) \cdot \nabla u(x)) n(x)$ with unit outer normal n (i.e. the projection of the gradient ∇u onto the tangent space at $x \in \Gamma$).

For the sake of simplicity, we restrict ourselves to presenting the definition of an *adapted grid* only for the case $d = 2$. In higher dimensions, additional definitions are required which we leave out for the convenience of the reader. For further details we refer to the work of Schweizer and Veneroni [46].

Definition 4.7 (Adapted grid for $d = 2$)**.** Let $Q \subset \mathbb{R}^2$ denote a bounded Lipschitz domain, $(u^\epsilon)_\epsilon$ a bounded sequence in $H^1(Q)$ and $(\epsilon_k)_{k\in\mathbb{N}}$ a sequence with $\epsilon_k \to 0$. For fixed $h > 0$, we call a family $\mathcal{T}_h$ of simplices an *adapted grid for* $(u^\epsilon)_\epsilon$ if the boundaries of all these simplices are typical segments and if there is one subsequence $(\epsilon_{k_l})_{l\in\mathbb{N}}$ of $(\epsilon_k)_{k\in\mathbb{N}}$ that is a good subsequence for all the segments.

Now, we are finally prepared to state the compensated compactness result, which is given in [46], Theorem 4.8.

Theorem 4.8. *Assume that we have space dimension 2 or 3. Let $Q \subset \mathbb{R}^d$ be a bounded Lipschitz domain with polygonal boundary and $(w^\epsilon)_\epsilon$ a bounded sequence in $H^1(Q)$. Then, for any $h > 0$ there exists a triangulation $\mathcal{T}_h$ of Q (i.e. the maximum diameter of a grid element is less than h) that is an adapted grid for $(w^\epsilon)_\epsilon$. Furthermore, if $w^\epsilon \rightharpoonup w$ weakly in $H^1(Q)$ and if $(q^\epsilon)_\epsilon$ is a sequence in $L^2(Q;\mathbb{R}^d)$ with the following properties:*

$$\begin{aligned} q^\epsilon &\rightharpoonup q \quad \text{weakly in } L^2(Q);\\ \nabla\cdot q^\epsilon &\to b \quad \text{strongly in } H^{-1}(S), \text{ for all } S \in \mathcal{T}_h, \end{aligned}$$

then the following result holds true:

$$\lim_{\epsilon\to 0}\int_Q q^\epsilon(x)\cdot\nabla w^\epsilon(x)\,dx = \int_Q q(x)\cdot\nabla w(x)\,dx.$$

Now, we can prove H^1-convergence of the coarse scale part of u_H^ϵ to the homogenized solution u^0:

Theorem 4.9. *Let $u_H^0 \in V_H(\Omega)$ denote the finite element approximation of the homogenized solution of problem (3), i.e. u_H^0 solves*

$$\int_\Omega A^0\nabla u_H^0\cdot\nabla\Phi_H = \int_\Omega f\Phi_H$$

for all $\Phi_H \in V_H(\Omega)$. If furthermore $u_H^{0,\epsilon} \in V_H(\Omega)$ denotes the coarse scale part of the MsFEM solution u_H^ϵ, i.e. if $u_H^{0,\epsilon}$ solves (4), then we have:

$$u_H^{0,\epsilon} \overset{\epsilon\to 0}{\longrightarrow} u_H^0 \qquad \text{strongly in } H^1(\Omega).$$

Proof. Using Remark 6, we see that $(u_H^{0,\epsilon})_\epsilon$ is a bounded sequence in the finite dimensional Hilbert space $V_H(\Omega)$. Therefore, there exists a subsequence $(u_H^{0,\epsilon_k})_{k\in\mathbb{N}}$ of $(u_H^{0,\epsilon})_\epsilon$ and a function $\tilde{u}_H^0 \in V_H(\Omega)$, so that

$$\|u_H^{0,\epsilon_k} - \tilde{u}_H^0\|_{H^1(\Omega)} \overset{k\to\infty}{\longrightarrow} 0.$$

Due to the definitions of $Q^\epsilon(u_H^{0,\epsilon}) \in \mathring{H}^1(T)$ and $Q^\epsilon(\tilde{u}_H^0) \in \mathring{H}^1(T)$, we get

$$\int_T A^\epsilon\nabla Q^\epsilon(u_H^{0,\epsilon})\cdot\nabla\left(Q^\epsilon(u_H^{0,\epsilon}) - Q^\epsilon(\tilde{u}_H^0)\right) = -\int_T A^\epsilon\nabla u_H^{0,\epsilon}\cdot\nabla\left(Q^\epsilon(u_H^{0,\epsilon}) - Q^\epsilon(\tilde{u}_H^0)\right)$$

and

$$\int_T A^\epsilon\nabla Q^\epsilon(\tilde{u}_H^0)\cdot\nabla\left(Q^\epsilon(u_H^{0,\epsilon}) - Q^\epsilon(\tilde{u}_H^0)\right) = -\int_T A^\epsilon\nabla \tilde{u}_H^0\cdot\nabla\left(Q^\epsilon(u_H^{0,\epsilon}) - Q^\epsilon(\tilde{u}_H^0)\right).$$

Combining this and using the ellipticity of A^ϵ, we get

$$\alpha|Q^\epsilon(\tilde{u}_H^0) - Q^\epsilon(u_H^{0,\epsilon})|_{H^1(T)} \le \beta|\tilde{u}_H^0 - u_H^{0,\epsilon}|_{H^1(T)}.$$

And therefore:

$$\begin{aligned}|R^{\epsilon_k}(\tilde{u}_H^0) - R^{\epsilon_k}(u_H^{0,\epsilon_k})|_{H^1(T)} &\leq |Q^{\epsilon_k}(\tilde{u}_H^0) - Q^{\epsilon_k}(u_H^{0,\epsilon_k})|_{H^1(T)} + |\tilde{u}_H^0 - u_H^{0,\epsilon_k}|_{H^1(T)} \\ &\leq \left(1 + \frac{\beta}{\alpha}\right) |\tilde{u}_H^0 - u_H^{0,\epsilon_k}|_{H^1(T)} \stackrel{k\to\infty}{\longrightarrow} 0.\end{aligned}$$

This yields the following strong convergence:

$$|R^{\epsilon_k}(\tilde{u}_H^0) - R^{\epsilon_k}(u_H^{0,\epsilon_k})|_{H^1(\Omega)} \stackrel{k\to\infty}{\longrightarrow} 0. \tag{8}$$

Furthermore, by means of Conclusion 2:

$$A^\epsilon \nabla R^\epsilon(\tilde{u}_H^0) \rightharpoonup A^0 \nabla \tilde{u}_H^0 \text{ weakly in } L^2(T, \mathbb{R}^d). \tag{9}$$

Now, we can identify the limit equation that is fulfilled by $\tilde{u}_H^0$. Recall Definition 2.1 and the relation $u_H^\epsilon = R^\epsilon(u_H^{0,\epsilon})$. First we observe

$$\begin{aligned}&\int_T A^\epsilon(x) \nabla u_H^\epsilon(x) \cdot \nabla R^\epsilon(\Phi_H)(x)\, dx \\ &= \int_T A^\epsilon(x) \nabla u_H^\epsilon(x) \cdot \nabla Q^\epsilon(\Phi_H)(x)\, dx + \int_T A^\epsilon(x) \nabla u_H^\epsilon(x) \cdot \nabla \Phi_H(x)\, dx \\ &= \int_T A^\epsilon(x) \nabla u_H^\epsilon(x) \cdot \nabla \Phi_H(x)\, dx,\end{aligned}$$

due to (5) with $Q^\epsilon(\Phi_H) \in \mathring{H}^1(T)$. Therefore

$$\int_\Omega A^\epsilon(x) \nabla u_H^\epsilon(x) \cdot \nabla R^\epsilon(\Phi_H)(x)\, dx = \int_\Omega A^\epsilon(x) \nabla u_H^\epsilon(x) \cdot \nabla \Phi_H(x)\, dx$$

for all $\Phi_H \in V_H(\Omega)$. Together with the definition of the MsFEM this gives us:

$$\begin{aligned}0 &= \int_\Omega A^{\epsilon_k} \nabla u_H^{\epsilon_k} \cdot \nabla R^{\epsilon_k}(\Phi_H) - \int_\Omega f R^{\epsilon_k}(\Phi_H) \\ &= \int_\Omega A^{\epsilon_k} \nabla u_H^{\epsilon_k} \cdot \nabla \Phi_H - \int_\Omega f R^{\epsilon_k}(\Phi_H) \\ &= \int_\Omega A^{\epsilon_k} \nabla\Big(R^{\epsilon_k}(u_H^{0,\epsilon_k}) - R^{\epsilon_k}(\tilde{u}_H^0)\Big) \cdot \nabla \Phi_H + \int_\Omega A^{\epsilon_k} \nabla R^{\epsilon_k}(\tilde{u}_H^0) \cdot \nabla \Phi_H - \int_\Omega f R^{\epsilon_k}(\Phi_H) \\ &\stackrel{k\to\infty}{\longrightarrow} \int_\Omega A^0 \nabla \tilde{u}_H^0 \cdot \nabla \Phi_H - \int_\Omega f \Phi_H.\end{aligned}$$

Here we used (8), (9) and Conclusion 2. Note that

$$\int_\Omega A^\epsilon \left(\nabla R^\epsilon(u_H^{0,\epsilon}) - \nabla R^\epsilon(\tilde{u}_H^0)\right) \cdot \nabla \Phi_H \to 0$$

because of

$$\begin{aligned}&\int_\Omega A^{\epsilon_k} \left(\nabla R^{\epsilon_k}(u_H^{0,\epsilon_k}) - \nabla R^{\epsilon_k}(\tilde{u}_H^0)\right) \cdot \nabla \Phi_H \\ &\leq \beta |R^{\epsilon_k}(\tilde{u}_H^0) - R^{\epsilon_k}(u_H^{0,\epsilon_k})|_{H^1(\Omega)} |\Phi_H|_{H^1(\Omega)} \to 0.\end{aligned}$$

So $\tilde{u}_H^0 \in V_H(\Omega)$ solves

$$\int_\Omega A^0 \nabla \tilde{u}_H^0 \cdot \nabla \Phi_H - \int_\Omega f \Phi_H = 0 \quad \forall \Phi_H \in V_H(\Omega).$$

Because of Remark 2, we know that the problem above yields a unique solution. Due to this uniqueness, we obtain that any subsequence must converge to the same limit. So we have convergence of the whole sequence $(u_H^{0,\epsilon})_{\epsilon>0}$ and in particular $\tilde{u}_H^0 = u_H^0$. □

Conclusion 3. From the proof of the last theorem, we conclude:

$$u_H^\epsilon - R^\epsilon(u_H^0) \overset{\epsilon\to 0}{\longrightarrow} 0 \quad \text{strongly in } H^1(\Omega).$$

This is a direct consequence of the Poincaré-inequality and equation (8) if we recall $u_H^\epsilon = R^\epsilon(u_H^{0,\epsilon})$, $\tilde{u}_H^0 = u_H^0$ and that we have convergence for the whole sequence.

Finally, we can deal with the proof of the first main result, namely Theorem 3.3:

Proof of Theorem 3.3. Let $d = 2, 3$ and let u_H^0 denote the FEM-approximation of the homogenized solution u^0. If we split the MsFEM-error into

$$\|u^\epsilon - u_H^\epsilon\|_{H^1(\Omega)} \leq \|u^\epsilon - R^\epsilon(u_H^0)\|_{H^1(\Omega)} + \|R^\epsilon(u_H^0) - u_H^\epsilon\|_{H^1(\Omega)},$$

it remains to estimate $\|u^\epsilon - R^\epsilon(u_H^0)\|_{H^1(\Omega)}$, since we already have $\|R^\epsilon(u_H^0) - u_H^\epsilon\|_{H^1(\Omega)} \to 0$ by Conclusion 3.

We start with the L^2-norm. Note that we need to treat the L^2 and the H^1-error separately to get an optimal order of convergence for the L^2-part. First, we recall that u^ϵ converges to u^0 weakly in $H^1(\Omega)$ (by Theorem 3.1) and $R^\epsilon(u_H^0)$ converges to u_H^0 also weakly in $H^1(\Omega)$ (by Conclusion 2). Since Ω has a Lipschitz boundary we can use the Sobolev embedding theorem to see that both sequences must converge strongly in $L^2(\Omega)$. This gives us:

$$\lim_{\epsilon\to 0} \|u^\epsilon - u_H^\epsilon\|_{L^2(\Omega)} = \lim_{\epsilon\to 0} \|u^\epsilon - R^\epsilon(u_H^0)\|_{L^2(\Omega)} = \|u^0 - u_H^0\|_{L^2(\Omega)}. \tag{10}$$

Next, we treat the H^1-seminorm, which requires the compensated compactness stated in Theorem 4.8. We estimate:

$$\begin{aligned}
\alpha|u^\epsilon - R^\epsilon(u_H^0)|^2_{H^1(\Omega)} &\leq \int_\Omega A^\epsilon(\nabla u^\epsilon - \nabla R^\epsilon(u_H^0)) \cdot (\nabla u^\epsilon - \nabla R^\epsilon(u_H^0)) \\
&= \int_\Omega A^\epsilon \nabla u^\epsilon \cdot \nabla u^\epsilon - \int_\Omega A^\epsilon \nabla u^\epsilon \cdot \nabla R^\epsilon(u_H^0) \\
&\quad - \int_\Omega A^\epsilon \nabla R^\epsilon(u_H^0) \cdot \nabla u^\epsilon + \int_\Omega A^\epsilon \nabla R^\epsilon(u_H^0) \cdot \nabla R^\epsilon(u_H^0) \\
&= \int_\Omega f u^\epsilon + \int_\Omega A^\epsilon \nabla R^\epsilon(u_H^0) \cdot \nabla R^\epsilon(u_H^0) \\
&\quad - \int_\Omega A^\epsilon \nabla u^\epsilon \cdot \nabla R^\epsilon(u_H^0) - \int_\Omega A^\epsilon \nabla R^\epsilon(u_H^0) \cdot \nabla u^\epsilon.
\end{aligned}$$

Now, we let $\epsilon \to 0$ in the various parts of the right hand side. Due to $u^\epsilon \rightharpoonup u^0$ in $H^1(\Omega)$, we have

$$\int_\Omega f(x) u^\epsilon(x) dx \to \int_\Omega f(x) u^0(x) dx.$$

For the second summand we can use Assumption 1, which says that A^ϵ allows averaging:

$$\begin{aligned}
&\int_\Omega A^\epsilon \nabla R^\epsilon(u_H^0) \cdot (\nabla u_H^0 + \nabla Q^\epsilon(u_H^0)) \\
&= \int_\Omega A^\epsilon \nabla R^\epsilon(u_H^0) \cdot \nabla u_H^0 \\
&= \sum_{T \in \mathcal{T}_H} \left(\int_T A^\epsilon \nabla R^\epsilon(u_H^0) \right) \cdot \nabla u_H^0(x_T) \\
&\to \sum_{T \in \mathcal{T}_H} \left(\int_T A^0 \nabla u_H^0 \right) \cdot \nabla u_H^0(x_T) \\
&= \sum_{T \in \mathcal{T}_H} \int_T A^0 \nabla u_H^0 \cdot \nabla u_H^0 \\
&= \int_\Omega A^0 \nabla u_H^0 \cdot \nabla u_H^0 \, dx
\end{aligned}$$

For the third summand, we apply Theorem 4.8 with $Q = T$ to obtain an adapted grid $\mathcal{T}_h(T)$ of T. Let us define $q^\epsilon := A^\epsilon \nabla u^\epsilon$ which is a sequence in $L^2(T, \mathbb{R}^d)$ with weak limit $A^0 \nabla u^0$. For $\nabla \cdot q^\epsilon$ we obtain for every $S \in \mathcal{T}_h(T)$

$$(\nabla \cdot q^\epsilon)(\phi) = \int_S A^\epsilon \nabla u^\epsilon \cdot \nabla \phi = 0 \quad \forall \phi \in \mathring{H}^1(S).$$

So $\nabla \cdot q^\epsilon = 0 \in H^{-1}(S)$ for all ϵ. Additionally, with Conclusion 2, we have

$$R^\epsilon(u_H^0) \rightharpoonup u_H^0 \qquad \text{weakly in } H^1(T).$$

We therefore have that the assumptions of Theorem 4.8 are fulfilled and we get:

$$\int_T A^\epsilon \nabla u^\epsilon \cdot \nabla R^\epsilon(u_H^0) \to \int_T A^0 \nabla u^0 \cdot \nabla u_H^0.$$

This yields:

$$\begin{aligned}
&\int_\Omega A^\epsilon \nabla u^\epsilon \cdot \nabla R^\epsilon(u_H^0) = \sum_{T \in \mathcal{T}_H} \int_T A^\epsilon \nabla u^\epsilon \cdot \nabla R^\epsilon(u_H^0) \\
&\to \sum_{T \in \mathcal{T}_H} \int_T A^0 \nabla u^0 \cdot \nabla u_H^0 = \int_\Omega A^0 \nabla u^0 \cdot \nabla u_H^0.
\end{aligned}$$

For the last summand we proceed analogously, using again Conclusion 2. Here, we define $q^\epsilon := A^\epsilon \nabla R^\epsilon(u_H^0)$ to exploit Theorem 4.8. Let $\mathcal{T}_h(T)$ be an adapted grid for T, then we also have from the definition of R^ϵ:

$$(\nabla \cdot q^\epsilon)(\phi) = \int_S A^\epsilon \nabla R^\epsilon(u_H^0) \cdot \nabla \phi = \int_T A^\epsilon \nabla R^\epsilon(u_H^0) \cdot \nabla \phi = 0$$

for all $S \in \mathcal{T}_h(T)$ and for all $\phi \in \mathring{H}^1(S)$. So Theorem 4.8 applies again and we get:

$$\begin{aligned}
&\int_\Omega A^\epsilon \nabla R^\epsilon(u_H^0) \cdot \nabla u^\epsilon = \sum_{T \in \mathcal{T}_H} \int_T A^\epsilon \nabla R^\epsilon(u_H^0) \cdot \nabla u^\epsilon \\
&\to \sum_{T \in \mathcal{T}_H} \int_T A^0 \nabla u_H^0 \cdot \nabla u^0 = \int_\Omega A^0 \nabla u_H^0 \cdot \nabla u^0.
\end{aligned}$$

Combining the various parts yields:

$$\begin{aligned}\limsup_{\epsilon\to 0} |u^\epsilon - u^\epsilon_H|^2_{H^1(\Omega)} &= \limsup_{\epsilon\to 0} |u^\epsilon - R^\epsilon(u^0_H)|^2_{H^1(\Omega)} \\ &\leq \frac{1}{\alpha}\int_\Omega A^0(\nabla u^0 - \nabla u^0_H)\cdot(\nabla u^0 - \nabla u^0_H) \\ &\leq \frac{\beta}{\alpha}|u^0 - u^0_H|^2_{H^1(\Omega)}.\end{aligned}$$

In summary, what we have shown is the following:

$$\lim_{\epsilon\to 0}\|u^\epsilon - u^\epsilon_H\|_{L^2(\Omega)} = \|u^0 - u^0_H\|_{L^2(\Omega)};\ \ \limsup_{\epsilon\to 0}|u^\epsilon - u^\epsilon_H|_{H^1(\Omega)} \leq C|u^0 - u^0_H|_{H^1(\Omega)}.$$

But u^0_H denotes the FEM approximation of the homogenized solution u^0 so we can use standard estimates to control $\|u^0 - u^0_H\|_{H^m(\Omega)}$, for $m = 0, 1$. If $u^0 \in \mathring{H}^1(\Omega)\cap H^{s_0}(\Omega)$, with $1 \leq s_0 \leq 2$, we get with standard interpolation estimates:

$$\|u^0 - u^0_H\|_{L^2(\Omega)} \leq CH^{s_0} \quad \text{and} \quad |u^0 - u^0_H|_{H^1(\Omega)} \leq CH^{s_0-1}.$$

Since Ω is a bounded polygonal domain and since A^0 is constant, we even know that $\frac{3}{2} \leq s_0 \leq 2$ (c.f. [25], e.g. chapter 6) and therefore at least:

$$\|u^0 - u^0_H\|_{L^2(\Omega)} \leq CH^{\frac{3}{2}} \quad \text{and} \quad |u^0 - u^0_H|_{H^1(\Omega)} \leq CH^{\frac{1}{2}}.$$

In the case of $n = 2$, s_0 can specified by means of the largest interior angle of an opening. Here we have $s_0 = 1 + \frac{\pi}{\omega}$, where $\omega > \pi$ denotes the largest angle of a re-entrant corner (c.f. [5]). Finally, if Ω is a convex domain, we obtain full H^2-regularity for u^0 (c.f. [24], chapter 3.2) and therefore an optimal second order convergence for the L^2-error and linear convergence for the H^1-error.

The remaining estimates for $\|u^0 - u^{0,\epsilon}_H\|_{L^2(\Omega)}$ and $|u^0 - u^{0,\epsilon}_H|_{H^1(\Omega)}$ are obtained in the same way, using Theorem 4.9 which immediately yields

$$\lim_{\epsilon\to 0}\|u^0 - u^{0,\epsilon}_H\|_{H^1(\Omega)} = \|u^0 - u^0_H\|_{H^1(\Omega)}.$$

This ends the proof. □

In Remark 3 we already mentioned the $H^1(\Omega)$-convergence of the MsFEM approximations in H, i.e. $u^\epsilon - u^\epsilon_H \overset{H\to 0}{\longrightarrow} 0$ strongly in H^1. This is a known result which can be for instance found in the book of Efendiev and Hou, [15]. For the sake of completeness, we also state the result in our framework. A proof can be found e.g. in [15].

Proposition 1. *Let $u^\epsilon \in \mathring{H}^1(\Omega)$ denote the solution equation (1) and let u^ϵ_H denote the MsFEM solution from Definition 2.1, then the error is bounded independent of H and ϵ:*

$$\|u^\epsilon - u^\epsilon_H\|_{H^1(\Omega)} \leq C\|f\|_{L^2(\Omega)},$$

furthermore, the limits in ϵ and the limits in H are both equal to zero. In particular, we have:

$$\begin{aligned}&\limsup_{\epsilon\to 0}|u^\epsilon - u^\epsilon_H|_{H^1(\Omega)} \leq CH \ \text{ and} \\ &\qquad |u^\epsilon - u^\epsilon_H|_{H^1(\Omega)} \leq C|u^\epsilon - I_H(u^\epsilon)|_{H^1(\Omega)} + CH\|f\|_{L^2(\Omega)},\end{aligned}$$

$I_H : \mathring{H}^1(\Omega) \to V_H(\Omega)$ defines an arbitrary interpolation operator. Note that the quality of the second estimate depends on the regularity of u^ϵ, whereas the first estimate is independent of the regularity.

The estimates above suggest to ask for convergence if we choose a coupling such as $\frac{\epsilon}{H} \approx$ const. The answer is, that we cannot formulate general results, since the sequence of MsFEM approximations is still convergent, but typically not to the correct approximation. In the following, we give an easy example with scalar diffusion, where A^ϵ is constant in one direction and periodic in the other direction:

Model Problem. Let $\Omega :=]0, \frac{4}{5}[^2$. Find $u^\epsilon \in \mathring{H}^1(\Omega)$ with

$$-\nabla \cdot (A^\epsilon \nabla u^\epsilon) = 1 \quad \text{in } \Omega.$$

and where $A^\epsilon(x) := A(\frac{x}{\epsilon})$ is given by $A(y_1, y_2) := (1.01 + \cos(2\pi y_1))\mathrm{Id}$.

In this special case it is easy to compute that the homogenized matrix is given by

$$A^0 = \begin{pmatrix} \left(\int_0^1 (1.01 + \cos(2\pi y))^{-1} \, dy\right)^{-1} & 0 \\ 0 & 1.01 \end{pmatrix}$$

(see for instance the book of Cioranescu and Donato [9]).

Now let us define the 'MsFEM Matrix' $A^{0,\epsilon}$ by

$$A_{ij}^{0,\epsilon}(x) := \fint_T A^\epsilon(e_i + \nabla Q^\epsilon(v_i)) \cdot e_j \quad \text{for } x \in T$$

and where $v_i(x) := e_i \cdot x$. We easily see

$$\int_\Omega A^\epsilon \nabla R^\epsilon(u_H^{0,\epsilon}) \cdot \nabla R^\epsilon(\Phi_H) = \int_\Omega A^{0,\epsilon} \nabla u_H^{0,\epsilon} \cdot \nabla \Phi_H.$$

Now, let us assume we start with a triangulation $\mathcal{T}_{H_0}(\Omega)$ that separates Ω into two right triangles. Then we create $\mathcal{T}_{H_{i+1}}(\Omega)$ by two uniform refinements of $\mathcal{T}_{H_i}(\Omega)$, so that $H_{i+1} = \frac{H_i}{2}$. For $\mathcal{T}_{H_i}$, ϵ_i shall be equal to the length of a cathetus of an element of $\mathcal{T}_{H_i}(\Omega)$. With this strategy, we get a coupling of H_i and ϵ_i. With the transformation formula, it is easy to check that any element T which is a shifted and scaled version of the reference element yields the same value for $A_{ij}^{0,\epsilon}$ on T. But due to $\cos(2\pi(1 - x_1)) = \cos(2\pi x_1)$, we get that A^{0,ϵ_i} also takes the same value on every triangle T that is rotated at 180°. All in all, we get the same value for every triangle T and for every of the above triangulations $\mathcal{T}_{H_i}(\Omega)$. So we have $A^{0,\epsilon_i}(x) = \tilde{A}$ for some matrix $\tilde{A} \in \mathbb{R}^{d \times d}$, for every x and every ϵ_i. This implies that $u_{H_i}^{\epsilon_i,0}$ is independent of ϵ_i and H_i. An easy computation yields

$$\tilde{A} \approx \begin{pmatrix} 0.667 & 0 \\ 0 & 1.01 \end{pmatrix}, \quad \text{whereas} \quad A^0 \approx \begin{pmatrix} 0.144 & 0 \\ 0 & 1.01 \end{pmatrix}$$

which is why $\|u_H^{0,\epsilon} - u_0\|$ is constant and does not converge. Also if H does not hit the period ϵ and if we do not construct a constant matrix $A^{0,\epsilon}$, we can still observe a stagnation of the error as depicted in Table 1.

However, it can be shown for periodic (c.f. [32]) or some cases of random homogenization problems (c.f. [7]) that we have the following a-priori error estimate:

$$\|u^0 - u_H^{0,\epsilon}\|_{L^2(\Omega)} \leq C \left(H + \left(\frac{\epsilon}{H}\right)^{\frac{1}{2}} \right)$$

TABLE 1. *In the table we can see a stagnation of the error for a coupling of H and ϵ by the fixed ratio $\frac{\epsilon}{H} = 1.28$. The local problems for computing the multiscale basis functions are solved with sufficient accuracy ($h = 2^{-6} H_T$ for a triangulation $\mathcal{T}_h(T)$ of T).*

ϵ	H	$\|u^0 - u_H^{0,\epsilon}\|_{L^2(\Omega)}$
0.256	0.2	0.01685
0.128	$0.2 \cdot 2^{-1}$	0.01493
0.064	$0.2 \cdot 2^{-2}$	0.01318
0.032	$0.2 \cdot 2^{-3}$	0.01296
0.016	$0.2 \cdot 2^{-4}$	0.01294
0.008	$0.2 \cdot 2^{-5}$	0.01293
0.004	$0.2 \cdot 2^{-6}$	0.01293
0.002	$0.2 \cdot 2^{-7}$	0.01293

This at least yields smallness of the error if $\epsilon \ll H$, but we still might only deal with convergence to the homogenized solution, up to a small reminder of size $\sqrt{\delta}$, where $\delta = \frac{\epsilon}{H}$. Note that, due to our very general assumptions (which say nothing about the speed of convergence in (2)), we can not derive a-priori error estimates with explicit orders in ϵ in our framework. However, in comparison to $d > 1$, it is possible to give a clear answer to the question of convergence of the MsFEM solutions for $H(\epsilon) \to 0$ for the $1d$ case. This is done in the subsequent section.

5. **Convergence of the MsFEM for $d = 1$.** In this section, we are concerned with the convergence of a sequence of MsFEM approximations in one space dimension. Here, it is possible to partially generalize the results from the preceding section, in the sense that we can show L^2-convergence without any restriction on the way of coupling H and ϵ. In particular we do neither assume $\frac{H}{\epsilon} \to 0$ nor $\frac{\epsilon}{H} \to 0$. The proof is accomplished by deriving explicit formulas for computing the MsFEM approximations. It turns out that there occurs a homogenization process for the MsFEM solutions within the original homogenization process in ϵ. Now, we introduce the setting of this section:

Definition 5.1. In this section, $I := [a, b] \subset \mathbb{R}$ denotes an interval with partition $a = x_0 < x_1 < x_2 < ... < x_N = b$, where $N \in \mathbb{N}_{>0}$. Furthermore, we define the mesh size by $H_i := x_{i+1} - x_i$ and the maximum by $H := \max\{H_i | 1 \leq i \leq N\}$. In this spirit, $\mathcal{T}_H(I)$ corresponds with this triangulation. The MsFEM approximation u_H^ϵ is given by Definition 2.1 and the multiscale correction operator Q^ϵ by Definition 4.1.

We start with a lemma that is required for computing $Q^\epsilon(\mathrm{id})$:

Lemma 5.2. *Let $[c_0, c_1] \subset I$ be an interval and $w \in \mathring{H}^1(c_0, c_1)$ the solution of*

$$\int_{c_0}^{c_1} A^\epsilon w' \phi' = -\int_{c_0}^{c_1} A^\epsilon \phi' \qquad \forall \phi \in \mathring{H}^1(c_0, c_1).$$

Then we have

$$w(x) = v(x) - g(x)\frac{v(c_1)}{g(c_1)},$$

where

$$v(x) := \int_{c_0}^{x} \frac{A^\epsilon(c_1) - A^\epsilon(y)}{A^\epsilon(y)}\, dy \quad \text{and} \quad g(x) := \int_{c_0}^{x} \frac{A^\epsilon(c_1)}{A^\epsilon(y)}\, dy.$$

In particular, we get

$$A^\epsilon(x)(w'(x)+1) = \left(⨍_{c_0}^{c_1} \frac{1}{A^\epsilon(y)}\, dy \right)^{-1}$$

Proof. It is easy to verify that w is the unique solution of the above problem. For the last statement, we calculate:

$$\begin{aligned} w'(x)+1 &= \frac{A^\epsilon(c_1)}{A^\epsilon(x)} - \frac{A^\epsilon(c_1)}{A^\epsilon(x)} \frac{v(c_1)}{g(c_1)} \\ &= \frac{A^\epsilon(c_1)}{A^\epsilon(x)} - \frac{A^\epsilon(c_1)}{A^\epsilon(x)} \frac{\int_{c_0}^{c_1} \frac{A^\epsilon(c_1)}{A^\epsilon(y)}\, dy - (c_1 - c_0)}{\int_{c_0}^{c_1} \frac{A^\epsilon(c_1)}{A^\epsilon(y)}\, dy} \\ &= \frac{(c_1 - c_0)}{A^\epsilon(x)} \frac{1}{\int_{c_0}^{c_1} \frac{1}{A^\epsilon(y)}\, dy}. \end{aligned}$$

Multiplying this term with $A^\epsilon(x)$ gives the desired result. □

By means of Lemma 5.2, we can restate the MsFEM in a more explicit way:

Conclusion 4. In one space dimension ($d = 1$), we have explicit formulas for the description of the MsFEM problem to determine the approximation $u_H^\epsilon \in V_H^\epsilon(I)$. In particular, we have that $u_H^{0,\epsilon} \in V_H(I)$ is the solution of

$$\sum_{n=0}^{N-1} H_i^2 \left(\int_{x_i}^{x_{i+1}} \frac{1}{A^\epsilon(y)}\, dy \right)^{-1} (u_H^{0,\epsilon})'(x_{i+\frac{1}{2}})\ \Phi_H'(x_{i+\frac{1}{2}}) = \int_I f \Phi_H \quad \forall \Phi_H \in V_H(I),$$

where $x_{i+\frac{1}{2}} = 2^{-1}(x_{i+1} + x_i)$. Furthermore, to compute $u_H^\epsilon = u_H^{0,\epsilon} + Q^\epsilon(u_H^{0,\epsilon})$, we have

$$Q^\epsilon(\Phi_H)(x) = Q_i^\epsilon(x) \cdot \Phi_H'(x) \quad \text{for } x \in [x_i, x_{i+1}] \text{ and } \Phi_H \in V_H(I),$$

where $Q_i^\epsilon(x) = v_i^\epsilon(x) - g_i^\epsilon(x) \frac{v_i^\epsilon(x_{i+1})}{g_i^\epsilon(x_{i+1})}$ and

$$v_i^\epsilon(x) := \int_{x_i}^{x} \frac{A^\epsilon(x_{i+1}) - A^\epsilon(y)}{A^\epsilon(y)}\, dy \text{ and } g_i^\epsilon(x) := \int_{x_i}^{x} \frac{A^\epsilon(x_{i+1})}{A^\epsilon(y)}\, dy.$$

Proof. This conclusion is a direct consequence of Lemma 5.2. We have:

$$\begin{aligned} \int_I A^\epsilon \nabla R^\epsilon(u_H^{0,\epsilon}) \cdot \nabla R^\epsilon(\Phi_H) &= \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} A^\epsilon(\nabla u_H^{0,\epsilon} + \nabla Q^\epsilon(u_H^{0,\epsilon})) \cdot \nabla \Phi_H \\ &= \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} \left(⨍_{x_i}^{x_{i+1}} A^\epsilon(1 + \nabla Q^\epsilon(id)) \right) \nabla u_H^{0,\epsilon} \cdot \nabla \Phi_H \\ &= \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} \left(⨍_{x_i}^{x_{i+1}} \frac{1}{A^\epsilon} \right)^{-1} \nabla u_H^{0,\epsilon} \cdot \nabla \Phi_H \\ &= \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} A^{0,\epsilon} \nabla u_H^{0,\epsilon} \cdot \nabla \Phi_H, \end{aligned}$$

where we defined

$$A^{0,\epsilon}(x) := \left(\fint_{x_i}^{x_{i+1}} \frac{1}{A^\epsilon(y)}\, dy \right)^{-1} \quad \text{for } x \in [x_i, x_{i+1}).$$

□

Using the previous results, we are prepared to formulate the main result of this section, namely the independence of the L^2-limit of the MsFEM solution on the choice of how to couple ϵ and H. Again we note that the following result does not hold for $d > 1$ as discussed in the previous section.

Theorem 5.3. *Let $d = 1$. If we assume that A^ϵ allows averaging in the sense of Assumption 1, we get that the L^2-limit of u_H^ϵ is independent of the coupling of ϵ and H. This means, for any sequence of tuples (H_n, ϵ_n) with $H_n \to 0$ and $\epsilon_n \to 0$, we have*

$$\|u_{H_n}^{\epsilon_n} - u^{\epsilon_n}\|_{L^2(I)} + \|u_{H_n}^{0,\epsilon_n} - u^0\|_{L^2(I)} \to 0 \qquad \text{for } n \to \infty,$$

where u^0 denotes the homogenized solution introduced in problem (3). Note the special case $H = r \cdot \epsilon$, with $r \in \mathbb{R}_{>0}$.

Proof. Let H and ϵ be coupled in an arbitrary way with '$H \to 0 \Leftrightarrow \epsilon \to 0$'. We start in the spirit of Conclusion 4 and define $A^{0,\epsilon}(x) := \left(\fint_{x_i}^{x_{i+1}} \frac{1}{A^\epsilon(y)}\, dy \right)^{-1}$ for $x \in [x_i, x_{i+1})$. Let us consider the problem to find $u^{0,\epsilon} \in \mathring{H}^1(I)$ with

$$\int_I A^{0,\epsilon} \nabla u^{0,\epsilon} \cdot \nabla \Phi = \int_I f \Phi \quad \forall \Phi \in \mathring{H}^1(I).$$

We can regard this as a new homogenization problem with a sequence of positive $L^\infty(I)$-coefficients $A^{0,\epsilon}$ (this is possible since $H = H(\epsilon)$). In the setting of H-convergence, we obtain that there exists a subsequence (for simplicity still denoted by $A^{0,\epsilon}$) with associated H-limit A^*. This limit is identical to the inverse of the weak-$*$ L^∞-limit of $(A^{0,\epsilon})^{-1}$ (c.f. Hornung [30]). In particular, we get

$$\lim_{\epsilon \to 0} \int_I (A^{0,\epsilon})^{-1} \phi = \int_I A^* \phi \quad \forall \phi \in L^1(I). \tag{11}$$

Now, let $I_0 = [a_0, b_0]$ denote an arbitrary subinterval of I and let $w \in \mathring{H}^1(I_0)$ denote the solution of

$$\int_{I_0} A^\epsilon w' \phi' = -\int_{I_0} A^\epsilon \phi' \qquad \forall \phi \in \mathring{H}^1(I_0).$$

By choosing ϕ as the indicator function of $I_0 = [a_0, b_0]$ in (11), we get with Lemma 5.2:

$$\begin{aligned}
\int_{I_0} A^* &= \left(\lim_{\epsilon \to 0} \int_{I_0} (A^{0,\epsilon})^{-1} \right)^{-1} \\
&= \left(\lim_{\epsilon \to 0} \sum_{[x_i, x_{i+1}] \subset I_0} (x_{i+1} - x_i) \left(\fint_{x_i}^{x_{i+1}} \frac{1}{A^\epsilon(y)}\, dy \right) + O(\epsilon) \right)^{-1} \\
&= \lim_{\epsilon \to 0} \left(\int_{I_0} \frac{1}{A^\epsilon(y)}\, dy \right)^{-1} = \lim_{\epsilon \to 0} \int_{I_0} A^\epsilon(y)(w'(y) + 1)\, dy \\
&= |I_0| A^0,
\end{aligned}$$

TABLE 2. *In this table, we can see the error between homogenized solution and MsFEM approximation for $H = 0.1$, but for different values of ϵ. For $\epsilon = H$, we get the best approximation, for $\epsilon \neq H$ we are dealing with a significant resonance error. Here, we denote the relative error by* $\|u^0 - u_H^{0,\epsilon}\|_{L^2(I)}^{rel} := \frac{\|u^0 - u_H^{0,\epsilon}\|_{L^2(I)}}{\|u^0\|_{L^2(I)}}$.

ϵ	H	$\|u^0 - u_H^{0,\epsilon}\|_{L^2(I)}^{rel}$
0.1	0.1	$3.1 \cdot 10^{-5}$
0.099	0.1	$2.2 \cdot 10^{-2}$
0.09	0.1	$2.6 \cdot 10^{-2}$
0.08	0.1	$1.9 \cdot 10^{-2}$
0.07	0.1	$1.8 \cdot 10^{-2}$

where we used Assumption 1 in the last step. Since I_0 was arbitrary, we get $A^* = A^0$ and we obtain:

$$\|u^{0,\epsilon} - u^0\|_{L^2(I)} \to 0 \quad \text{for } \epsilon \to 0.$$

Note that the above subsequence can be replaced by the whole sequence in ϵ, since A^* is unique. Finally we obtain:

$$\|u_H^{0,\epsilon} - u^0\|_{L^2(I)} \leq \|u^{0,\epsilon} - u_H^{0,\epsilon}\|_{L^2(I)} + \|u^{0,\epsilon} - u^0\|_{L^2(I)} \leq C\frac{\beta}{\alpha}H + \|u^{0,\epsilon} - u^0\|_{L^2(I)} \to 0.$$

The convergence of $\|u_H^\epsilon - u^\epsilon\|_{L^2(I)} \to 0$ is an easy conclusion, where we use that $\|u^\epsilon - u^0\|_{L^2(I)} \to 0$ and $\|u_H^{0,\epsilon} - u_H^\epsilon\|_{L^2(I)} \to 0$. Here, the second part can be obtained by a simple computation using $|Q_i^\epsilon| \leq CH_i$. □

The above result in Theorem 5.3 can be illustrated in a numerical experiment. Let us therefore consider the following model problem:

Model Problem. Find $u^\epsilon \in \mathring{H}^1(0,1)$ with

$$\frac{\mathrm{d}}{\mathrm{dx}}\left(A^\epsilon(x)\frac{\mathrm{d}}{\mathrm{dx}}u^\epsilon(x)\right) = 1 \quad \text{in } I := (0,1)$$

and where $A^\epsilon(x) := A(\frac{x}{\epsilon})$ is given by $A(y) := (2 + \cos(2\pi y))^{-1}$.

In the above problem, we have explicit formulas for A^0, u^0 and u^ϵ. In particular, we get $A^0 = 2^{-1}$, $u_0(x) = x - x^2$ and

$$u^\epsilon(x) = 2x - x^2 - \frac{\epsilon^2}{4\pi^2}\cos(2\pi\frac{x}{\epsilon}) + \frac{\epsilon^2}{4\pi^2} - \left(2x + \frac{\epsilon}{2\pi}\sin(2\pi\frac{x}{\epsilon})\right)\frac{1 + \frac{\epsilon^2}{4\pi^2}(1 - \cos(\frac{2\pi}{\epsilon}))}{2 + \frac{\epsilon}{2\pi}\sin(\frac{2\pi}{\epsilon})}.$$

For the problem above, it is possible to derive exact formulas for computing the MsFEM approximations u_H^ϵ and $u_H^{0,\epsilon}$. Using these results we can compute the L^2-error between u^0 and $u_H^{0,\epsilon}$ for fixed $H = 0.1$ but various values of ϵ. The errors are depicted in Table 2. First of all, we see that the typical resonance error becomes significant even for very small discrepancies between ϵ and H. For $(H,\epsilon) = (0.1, 0.099)$ the error is a thousand times larger than for $(H,\epsilon) = (0.1, 0.1)$. However, Theorem 5.3 predicts that the effects of the resonance error are 'homogenized', if $H(\epsilon)$ becomes small enough. In fact, this is exactly what we can see in Table 3, where we observe a nice linear convergence. This is complementary to the $2d$-case

TABLE 3. *Depiction of various errors. In this table we can see that effects of a resonance error start to average out, once* ϵ *and* H *become small enough. Here, we denote* $\|u^0 - u_H^{0,\epsilon}\|_{L^2(I)}^{rel} := \frac{\|u^0-u_H^{0,\epsilon}\|_{L^2(I)}}{\|u^0\|_{L^2(I)}}$ *and* $\|u^\epsilon - u_H^\epsilon\|_{L^2(I)}^{rel} := \frac{\|u^\epsilon-u_H^\epsilon\|_{L^2(I)}}{\|u^\epsilon\|_{L^2(I)}}$.

ϵ	H	$\frac{\|u^0-u^\epsilon\|_{L^2(I)}}{\|u^\epsilon\|_{L^2(I)}}$	$\|u^0 - u_H^{0,\epsilon}\|_{L^2(I)}^{rel}$	$\|u^\epsilon - u_H^{0,\epsilon}\|_{L^2(I)}^{rel}$	$\|u^\epsilon - u_H^\epsilon\|_{L^2(I)}^{rel}$
$7\cdot10^{-1}$	1	$1.34\cdot10^{-1}$	$2.42\cdot10^{-2}$	$1.24\cdot10^{-1}$	$6.08\cdot10^{-2}$
$7\cdot10^{-2}$	10^{-1}	$2.09\cdot10^{-2}$	$1.83\cdot10^{-2}$	$1.36\cdot10^{-2}$	$9.94\cdot10^{-4}$
$7\cdot10^{-3}$	10^{-2}	$1.85\cdot10^{-3}$	$1.56\cdot10^{-3}$	$1.40\cdot10^{-3}$	$1.08\cdot10^{-5}$
$7\cdot10^{-4}$	10^{-3}	$1.46\cdot10^{-4}$	$1.08\cdot10^{-4}$	$1.41\cdot10^{-4}$	$3.67\cdot10^{-7}$
$7\cdot10^{-5}$	10^{-4}	$2.12\cdot10^{-5}$	$1.88\cdot10^{-5}$	$1.40\cdot10^{-5}$	$4.56\cdot10^{-8}$
$7\cdot10^{-6}$	10^{-5}	$1.86\cdot10^{-6}$	$1.58\cdot10^{-6}$	$1.40\cdot10^{-6}$	$1.41\cdot10^{-8}$

example at the end of Section 4. This might be also relevant for applications, in which there is no explicit knowledge about the size of ϵ.

6. **Conclusion.** In this work we dealt with the convergence of the H^1-error between MsFEM approximations and the exact solution of an elliptic homogenization problem. This was established without assuming a certain periodic or stochastic structure of the problem. Furthermore, we were, in particular, dealing with the case of $1d$-problems, to observe that the convergence does not depend on the coupling between ϵ and the grid size H, whereas the result cannot be generalized to other dimensions. Numerical experiments were given to emphasize our results.

Acknowledgments. We would like to thank the referees very much for their helpful comments, ideas and valuable suggestions which led to great improvements of the content.

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Received November 2011; revised May 2012.

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