NETWORKS AND HETEROGENEOUS MEDIA doi:10.3934/nhm.2012.7.373

Volume **7**, Number **3**, September **2012** pp. **373–384**

STRUCTURAL PROPERTIES OF THE LINE-GRAPHS ASSOCIATED TO DIRECTED NETWORKS

Regino Criado, Julio Flores,
Alejandro J. García del Amo and Miguel Romance

Departamento de Matemática Aplicada
Universidad Rey Juan Carlos
28933 Móstoles (Madrid), Spain

Centro de Tecnología Biomédica
Universidad Politécnica de Madrid
28223 Pozuelo de Alarcón (Madrid), Spain

Abstract. The centrality and efficiency measures of an undirected network G were shown by the authors to be strongly related to the respective measures on the associated line graph $L(G)$. In this note we extend this study to a directed network $\vec{G}$ and its associated directed network $\vec{L}(\vec{G})$. The Bonacich centralities of these two networks are shown to be related in a surprisingly simpler manner than in the non directed case. Efficiency is also considered and the corresponding relations established. In addition, an estimation of the clustering coefficient of $\vec{L}(\vec{G})$ is given in terms of the clustering coefficient of $\vec{G}$, and by means of an example we show that a reverse estimation cannot be expected.

Given a non directed graph G, there is a natural way to obtain from it a directed line graph, namely $\vec{L}(D(G))$, where the directed graph $D(G)$ is obtained from G in the usual way. With this approach the authors estimate some parameters of $\vec{L}(D(G))$ in terms of the corresponding ones in $L(G)$. Particularly, we give an estimation of the norm difference between the centrality vectors of $\vec{L}(D(G))$ and $L(G)$ in terms of the Collatz-Sinogowitz index (which is a measure of the irregularity of G). Analogous estimations are given for the efficiency measures. The results obtained strongly suggest that for a given non directed network G, the directed line graph $\vec{L}(D(G))$ captures more adequately the properties of G than the non directed line graph $L(G)$.

1. **Introduction.** A great progress has been made in the last years to describe the complex structure of real world networks [1, 16, 17, 18, 22]. Complex networks are everywhere and have applications in fields ranging from biology (which include issues such as genetic regulatory networks, metabolic pathways, or protein folding) to the World Wide Web, the Internet and other technological systems. The investigation of such issues must necessarily embrace a diversity of viewpoints that include different complementary aspects of the network structure. The motivation behind our study of line-graphs is to consider the importance that edges have sometimes over nodes in the context of networks and graphs. Directed line graphs are particularly interesting

2000 *Mathematics Subject Classification.* Primary: 05C90, 05C75; Secondary: 68M10, 94C15, 90B18.

Key words and phrases. Complex networks, dual graph, line graph, directed line graph.

This work was supported by the Spanish Government Project MTM2009-13848.

under the point of view of its applications. Some examples of this come from urbanism [10, 11], transport networks [21, 2] or urban traffic [15], where the line graph associated to a given directed network G is considered.

In [8] we showed some relationships between the network's efficiency and the network's Bonacich ([5, 6]) centrality of a network G and the respective measures on the associated line-graph $L(G)$ and the bipartite $B(G)$ associated network (see below for definitions). Some other properties and relationships between the centrality of a network G and the centrality of its line-graph $L(G)$ have been studied in [9]. But these relationships were obtained for undirected networks, and most of the real networks where these results can have applications are directed. So, in the context of urban traffic, when the underlying (primal) graph is considered then intersections (or settlements) are seen as nodes while segments of streets (or directed links between two nodes) are seen as edges. In contrast when the line (dual) graph is considered, segments of streets become nodes, while intersections of streets become links between the corresponding nodes [15].

The main goal of this work is to exhibit some relations arising from the various ways in which line graphs can be obtained from a given directed network. This in turn can be applied to obtaining estimations for several parameters that measure different properties related to the network structure and performance.

In order to investigate such properties, it is necessary to understand the main structure of the underlying network [1, 16] and also to consider other complementary topological aspects.

2. Definitions and notation.

From a schematic point of view, a complex network is a mathematical object $G = (V, E)$ composed by a set of nodes or vertices $V = \{v_1 \ldots, v_n\}$ that are pairwise joined by links or edges $\{\ell_1, \ldots, \ell_m\}$. The adjacency matrix of G is the symmetric $n \times n$ dimensional $(0, 1)$-matrix $A(G) = (a_{ij})$ determined by the conditions:

$$a_{ij} = \begin{cases} 1 & \text{if} \quad \{v_i, v_j\} \in E \\ 0 & \text{if} \quad \{v_i, v_j\} \notin E. \end{cases}$$

The bipartite network $B(G)$ associated to an undirected network $G = (V, E)$ is defined by $B(G) = (V \cup E, E(B(G)))$ whose adjacency matrix is given by

$$A_{B(G)} = \left(\begin{array}{c|c} 0 & I(G) \\ \hline I(G)^t & 0 \end{array} \right),$$

where $I(G)$ is the incidence matrix of G. It is shown that

$$(A_{B(G)})^2 = \left(\begin{array}{c|c} A_G + gr & 0 \\ \hline 0 & A_{L(G)} + 2I_m \end{array} \right),$$

where $A_G + gr$ denotes the matrix obtained by adding to A_G the diagonal matrix (b_{ij}) and b_{ii} is the degree of the vertex v_i while $L(G)$ denotes the line (or dual) network associated to G ([12], p. 26).

The adjacency matrix of a directed network (digraph) $\vec{G} = (V, E)$ is the $n \times n$ dimensional $(0, 1)$-matrix $A_{\vec{G}} = (a_{ij})$ determined by the conditions:

$$a_{ij} = \begin{cases} 1 & \text{if} \quad (v_i, v_j) \in E \\ 0 & \text{if} \quad (v_i, v_j) \notin E. \end{cases}$$

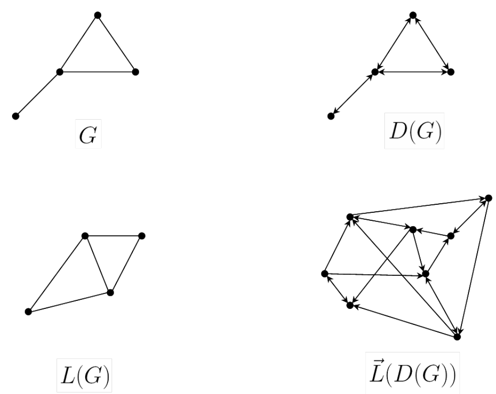

FIGURE 1. An example of the line-graph of an undirected network (on the left) and for a directed network (on the right).

The bipartite network $B(\vec{G})$ associated to $\vec{G}$ is defined by $B(\vec{G}) = (V \cup E, E(B(\vec{G})))$ whose adjacency matrix is given by

$$A_{B(\vec{G})} = \left(\begin{array}{c|c} 0 & H_{\vec{G}} \\ \hline T^t_{\vec{G}} & 0 \end{array} \right),$$

where $H = H_{\vec{G}}$ is the incidence matrix of heads of $\vec{G}$ defined by

$$H_{ij} = \begin{cases} 1 & \text{if} \quad \ell_j = (v_i, -) \\ 0 & \text{otherwise,} \end{cases}$$

and $T = T_{\vec{G}}$ is the incidence matrix of tails of $\vec{G}$ defined by

$$T_{ij} = \begin{cases} 1 & \text{if} \quad \ell_j = (-, v_i) \\ 0 & \text{otherwise} \end{cases}$$

It is also shown that

$$(A_{B(\vec{G})})^2 = \left(\begin{array}{c|c} A_{\vec{G}} & 0 \\ \hline 0 & A_{\vec{L}(\vec{G})}, \end{array} \right)$$

where $\vec{L}(\vec{G})$ denotes the directed line network (line digraph) associated to the directed network $\vec{G}$, i.e., the digraph $\vec{L}(\vec{G})$ whose vertices are the arcs $E(\vec{G})$ of $\vec{G}$, while (e, f) is an arc in $\vec{L}(\vec{G})$ if the end of e coincides with the origin of f. Note (this fact is remarkable in the context of this work) that $(A_{B(\vec{G})})^2$ becomes simpler when we are working on a directed graph.

If now G is an undirected graph, we will denote by $D(G)$ the associated symmetric digraph obtained by replacing each edge of G by an arc pair in which the two arcs are inverse to each other (see figure 1). Observe that $A_G = A_{D(G)}$. Using this idea, there is an alternative definition for the line graph associated with G that has received relatively little attention: the directed line graph $\vec{L}(D(G))$. It is remarkable that in the references [19, 20], by using an alternative definition for the directed line graph $\vec{L}(\vec{G})$, the concept so defined is employed to capture graph-class structure and clustering graphs.

In the sequel, if $\vec{G}$ is a directed graph, we will denote by $U(\vec{G})$ the associated undirected graph obtained by replacing each arc of $\vec{G}$ by an (undirected) edge joining the same nodes (see figure 2). Finally, let us recall that a directed graph is called strongly connected if there is a path from each vertex in the graph to every other vertex.

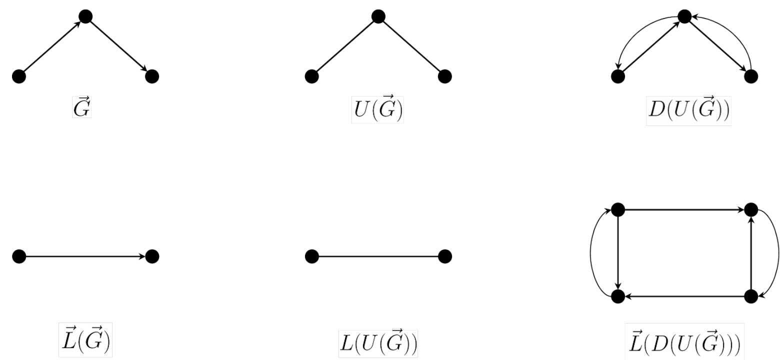

FIGURE 2. An example of line graphs of a directed network G, $U(G)$ and $D(U(G))$.

3. **Some structural properties of line-graph of directed networks.** In [8] the authors showed some relationships between the network's efficiency and the network's Bonacich ([5], [6]) centrality of an undirected network G and the respective measures on the line-graph (dual) $L(G)$ and the bipartite $B(G)$ associated networks. Note that the networks considered there were undirected.

The main goal of this section is to exhibit some relations arising from the various ways in which line graphs can be obtained from a given directed network. This in turn can be applied to obtaining estimations for several parameters that measure different properties related to the network structure and performance.

3.1. **Centrality of directed line graphs.** In order to establish some results related to the structural properties of line-graph of directed networks it is important to recall that the Bonacich centrality of a strongly connected directed complex network $\vec{G}$ is the non-negative normalized eigenvector $c_{\vec{G}} \in \mathbb{R}^n$ associated to the spectral radius of the adjacency matrix of $\vec{G}$ [5, 6, 16]. We also know what $\vec{L}(\vec{G})$ is strongly connected if $\vec{G}$ is strongly connected [14].

As we showed in [8], if we know the Bonacich centrality $c_{L(G)}$, we can recover $c_{B(G)}$ and reciprocally, if G is an undirected network. In addition to this, if G is regular (and undirected), then each of the three centralities can be recovered from any of the other two. Reasoning as in [8] we can establish for strongly connected directed networks the following relations between the Bonacich centralities of $\vec{G}$, $\vec{L}(\vec{G})$ and $B(\vec{G})$:

Theorem 3.1. *Let $\vec{G} = (V, E)$ be a strongly connected directed network with n vertices and m edges. Let $c_{\vec{G}} \in \mathbb{R}^n$, $c_{\vec{L}(\vec{G})} \in \mathbb{R}^m$ and $c_{B(\vec{G})} = (c_1, c_2) \in \mathbb{R}^n \times \mathbb{R}^m$ be the Bonacich centralities of $\vec{G}$, $\vec{L}(\vec{G})$ and $B(\vec{G})$ respectively. Then, if $\|v\|_1 = \sum_{i=1}^{n} |v_i|$ for any arbitrary $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, we have:*

(i) $c_{\vec{G}} = \frac{c_1}{\|c_1\|_1}$ *and* $c_{\vec{L}(\vec{G})} = \frac{c_2}{\|c_2\|_1}$.

(ii) Reciprocally, $c_{B(\vec{G})} = \frac{1}{2}\left(c_{\vec{G}}, c_{\vec{L}(\vec{G})}\right)$ *and*

$$c_{\vec{G}} = \frac{T_{\vec{G}}\, c_{\vec{L}(\vec{G})}}{\|T_{\vec{G}}\, c_{\vec{L}(\vec{G})}\|_1}, \quad c_{\vec{L}(\vec{G})} = \frac{H^t_{\vec{G}}\, c_{\vec{G}}}{\|H^t_{\vec{G}}\, c_{\vec{G}}\|_1}.$$

Proof. (i) Let $c_{B(\vec{G})} = (c_1, c_2) \in \mathbb{R}^n \times \mathbb{R}^m$ the Bonacich centrality of $B(\vec{G})$. Then, if λ is the spectral radius of $B(\vec{G})$,

$$(A^t_{B(\vec{G})})^2 c_{B(\vec{G})} = \left(\begin{array}{c|c} A^t_{\vec{G}} & 0 \\ \hline 0 & A^t_{\vec{L}(\vec{G})} \end{array}\right) \left(\begin{array}{c} c_1 \\ \hline c_2 \end{array}\right) = \left(\begin{array}{c} A^t_{\vec{G}}\, c_1 \\ \hline A^t_{\vec{L}(\vec{G})}\, c_2 \end{array}\right) = \lambda^2 \left(\begin{array}{c} c_1 \\ \hline c_2 \end{array}\right).$$

It follows that c_1 is an eigenvector of $A^t_{\vec{G}}$ and c_2 is an eigenvector of $A^t_{\vec{L}(\vec{G})}$. We finish the proof by observing that $c_1, c_2 > 0$.

(ii) From the unity of the centrality vector and the fact that it has norm one follows $c_{B(\vec{G})} = \frac{1}{2}\left(c_{\vec{G}}, c_{\vec{L}(\vec{G})}\right)$. Now, if λ is the spectral radius of $B(\vec{G})$, we have on the one hand

$$A^t_{B(\vec{G})} c_{B(\vec{G})} = \lambda c_{B(\vec{G})} = \frac{\lambda}{2}\left(c_{\vec{G}}, c_{\vec{L}(\vec{G})}\right),$$

and on the other hand

$$A^t_{B(\vec{G})} c_{B(\vec{G})} = \frac{1}{2} \left(\begin{array}{c|c} 0 & T_{\vec{G}} \\ \hline H^t_{\vec{G}} & 0 \end{array}\right) \left(\begin{array}{c} c_{\vec{G}} \\ \hline c_{\vec{L}(\vec{G})} \end{array}\right) = \frac{1}{2} \left(\begin{array}{c} T_{\vec{G}}\, c_{\vec{L}(\vec{G})} \\ \hline H^t_{\vec{G}}\, c_{\vec{G}} \end{array}\right)$$

Hence the claimed relations follow. □

Example 1. Let us consider the directed graph $\vec{G} = (V, E)$ with 6 nodes and 8 links as it is given in figure 3.

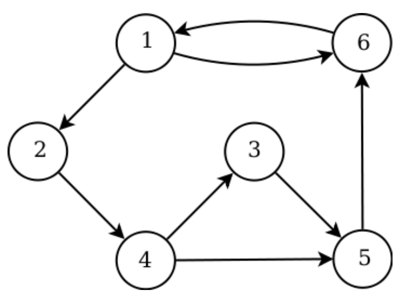

FIGURE 3. A directed graph $\vec{G} = (V, E)$ with 6 nodes and 8 links.

It is easy to check that the line graph $\vec{L}(\vec{G})$ of the previous network is the directed graph with 8 nodes and 10 links given in figure 4.

If we consider $V = \{1, 2, 3, 4, 5, 6\}$ and $E = \{\ell_1, \cdots, \ell_8\}$ defined as

$$\begin{array}{ll} \ell_1 = (1,2), & \ell_5 = (4,3), \\ \ell_2 = (1,6), & \ell_6 = (4,3), \\ \ell_3 = (2,4), & \ell_7 = (5,6), \\ \ell_4 = (3,5), & \ell_8 = (6,1), \end{array}$$

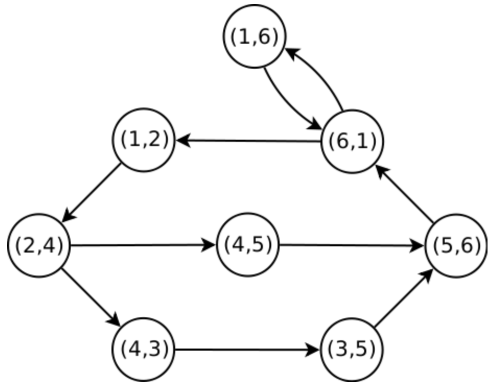

FIGURE 4. The directed linegraph graph $\vec{L}(\vec{G})$ of the network given in figure 3.

then the incidence matrix of heads of $\vec{G}$ is

$$H_{\vec{G}} = (H_{ij}) = \left\{ \begin{array}{ll} 1 & \text{if} \quad \ell_j = (v_i, -) \\ 0 & \text{otherwise,} \end{array} \right\} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

the incidence matrix of tails of $\vec{G}$ is

$$T_{\vec{G}} = (T_{ij}) = \left\{ \begin{array}{ll} 1 & \text{if} \quad \ell_j = (-, v_i) \\ 0 & \text{otherwise} \end{array} \right\} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and therefore the adjacency matrices of $\vec{G}$ and $\vec{L}(\vec{G})$ are

$$A_{\vec{G}} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad A_{\vec{L}(\vec{G})} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

By direct computation we get that

$$c^t_{\vec{G}} = (0.20684, 0.15614,\ 0.08897,\ 0.11787,\ 0.15614,\ 0.27401),$$
$$c^t_{\vec{L}(\vec{G})} = (0.15614,\ 0.15614,\ 0.11787,\ 0.06716,\ 0.08897,\ 0.08897,\ 0.11787).$$

If we use the formulas obtained in theorem 3.1, we get that

$$\left(\frac{H_{\vec{G}}^t c_{\vec{G}}}{\|H_{\vec{G}}^t c_{\vec{G}}\|_1}\right)^t = (0.15614,\ 0.15614,\ 0.11787,\ 0.06716,\ 0.08897,\ 0.08897,\ 0.11787) = c_{\vec{L}(\vec{G})}^t,$$

$$\left(\frac{T_{\vec{G}} c_{\vec{L}(\vec{G})}}{\|T_{\vec{G}} c_{\vec{L}(\vec{G})}\|_1}\right)^t = (0.20684, 0.15614,\ 0.08897,\ 0.11787,\ 0.15614,\ 0.27401) = c_{\vec{G}}^t.$$

3.2. **Efficiency of directed line graphs.** In this subsection we are interested in finding relations between the efficiency of a directed network graph $\vec{G} = (V, E)$, the directed line graph $\vec{L}(\vec{G})$ and the bipartite graph $B(\vec{G})$. We start with the relation between $E(\vec{G})$ and $E(\vec{L}(\vec{G}))$. Recall that the efficiency of a (directed) complex network $\vec{G}$ is the value

$$E(\vec{G}) = \frac{1}{n(n-1)} \sum_{i \neq j} \frac{1}{d_{\vec{G}}(i,j)},$$

where $d_{\vec{G}}(i,j)$ is the distance from node i to node j, i.e., the number of edges in a shortest directed path connecting them (geodesic distance). If there is no path from node i to node j, the distance $d_{\vec{G}}(i,j)$ is considered as infinite.

The efficiency in $\vec{L}(\vec{G})$ is naturally defined

$$E(\vec{L}(\vec{G})) = \frac{1}{m(m-1)} \sum_{\ell,\ell' \in E, \ell \neq \ell'} \frac{1}{d_{\vec{L}(\vec{G})}(\ell,\ell')}.$$

and the efficiency in $B(\vec{G})$ can be expressed as

$$\begin{aligned} E(B(\vec{G})) &= \frac{1}{(n+m)(n+m-1)} \sum_{a,b \in V \cup E, a \neq b} \frac{1}{d_{B(\vec{G})}(a,b)} \\ &= \frac{1}{(n+m)(n+m-1)} \left(\sum_{i,j \in V, i \neq j} \frac{1}{d_{B(\vec{G})}(i,j)} + \sum_{i \in V, (j,k) \in E} \frac{1}{d_{B(\vec{G})}(i,(j,k))} \right. \\ &\qquad \left. + \sum_{(j,k) \in E, i \in V} \frac{1}{d_{B(\vec{G})}((j,k),i)} + \sum_{(i,j) \neq (i',j')} \frac{1}{d_{B(\vec{G})}((i,j)(i',j'))} \right). \end{aligned}$$

It is not difficult to prove the following lemma:

Lemma 3.2. *If $\vec{G} = (V, E)$ is a directed complex network, then for every collection of nodes $i, j, k, i', j' \in V$:*

(i) $d_{\vec{L}(\vec{G})}\big((i,j),\ (i',j')\big) = 1 + d_{\vec{G}}(j,i')$,
(ii) $d_{B(\vec{G})}(i,j) = 2d_{\vec{G}}(i,j)$,
(iii) $d_{B(\vec{G})}(i,(j,k)) = 1 + d_{B(\vec{G})}(i,j) = 1 + 2d_{\vec{G}}(i,j)$,
(iv) $d_{B(\vec{G})}((j,k),i) = 1 + d_{B(\vec{G})}(k,i) = 1 + 2d_{\vec{G}}(k,i)$,
(v) $d_{B(\vec{G})}\big((i,j),(i',j')\big) = 2d_{\vec{L}(\vec{G})}\big((i,j),(i',j')\big) = 2 + 2d_{\vec{G}}(j,i')$.

Now, using the previous lemma, it is not difficult to prove the following result:

Theorem 3.3. *Let $\vec{G} = (V, E)$ be a directed graph with n nodes and $\vec{L}(\vec{G})$ its associated directed line graph. Then*

$$E(\vec{L}(\vec{G})) \leq F(n,m)E(\vec{G}) + G(n,m),$$

where

$$F(n,m) = \frac{n(n-1)}{m(m-1)} \left(\max\{gr_{in}(i) \cdot gr_{out}(j) \mid i,j \in V\}\right)$$

and

$$G(n,m) = \frac{n^2(n-1)}{m(m-1)} \left(\max\{gr_{in}(i) \cdot gr_{out}(i) \mid i \in V\}\right).$$

Reasoning into a similar way, it is possible to get a relationship between $E(B(\vec{G}))$ and $E(\vec{G})$.

Similarly, since $\sigma^{\vec{L}(\vec{G})}_{(i,j),(i',j')} = \sigma^{\vec{G}}_{j,i'}$ if $(i,j) \neq (i',j')$ with $\sigma^{\vec{G}}_{j,i'} = 0$ if $j = i'$, where $\sigma^{\vec{G}}_{(i,j)}$ (respectively, $\sigma^{\vec{L}(\vec{G})}_{(i,j),(i',j')}$) is the number of geodesics from node i to node j in $\vec{G}$ (respectively, the number of geodesics from node (i,j) to node (i',j') in $\vec{L}(\vec{G})$), we can estimate the betweenness centrality of a node (i,j) in $\vec{L}(\vec{G})$ by using the betweenness centrality of i and j in $\vec{G}$.

3.3. **Clustering in directed line graphs.** Recall that a triple in the directed graph $\vec{G} = (V,E)$ is a set of three nodes and two edges while a triangle is a set of three nodes and three edges (multiedges and loops are excluded) The following definition for clustering coefficient is well known:

$$c_c(\vec{G}) = \frac{|\text{number of triangles}|}{|\text{number of triples}|},$$

where $|A|$ stands for the cardinality of the set A.

Theorem 3.4. *Let $\vec{G} = (V,E)$ be a directed graph with n nodes and $\vec{L}(\vec{G})$ its associated directed line graph. Then*

$$c_c(\vec{L}(\vec{G})) \leq \frac{c_c(\vec{G})}{n-3}.$$

Proof. Note that every triple in $\vec{L}(\vec{G})$ falls into one of the following types (see figure 5):

- Type (i), those triples with nodes $\{a,b,c\}$ and edges (a,b) and (b,c).
- Type (ii), those triples with nodes $\{a,b,c\}$ and edges (a,b) and (c,b).
- Type (iii), those triples with nodes $\{a,b,c\}$ and edges (b,a) and (b,c).

Note also that triples of type (i) proceed from subgraphs in $\vec{G}$ of type (i^*), those with four nodes $\{i,j,k,l\}$ and three edges $(i,j) = a$, $(j,k) = b$ and $(k,l) = c$, and reciprocally. Analogously, triples of type (ii) proceed from subgraphs in $\vec{G}$ of type (ii^*), those with four nodes $\{i,j,k,l\}$ and three edges $(i,j) = a$, $(j,k) = b$ and $(l,j) = c$, and reciprocally. Finally, triples of type (iii) proceed from subgraphs in $\vec{G}$ of type (iii^*), those with four nodes $\{i,j,k,l\}$ and three edges $(j,i) = a$, $(l,j) = b$ and $(j,k) = c$, and reciprocally.

Observe that the number of triples in $\vec{G}$ of type (i^{**}), those with three nodes $\{i,j,k\}$ and edges $\big((i,j),(j,k)\big)$, is less or equal than $(n-3)^{-1}$ times the number of subgraphs in $\vec{G}$ of type (i^*) (note that $n-3$ is imposed in order to avoid loops and multiedges). Analogously the number of triples in $\vec{G}$ of type (ii^{**}), those with nodes $\{i,j,k\}$ and edges $\big((i,j),(k,j)\big)$ is less or equal than $(n-3)^{-1}$ times the number of subgraphs in $\vec{G}$ of type (ii^*). Finally the number of triples in $\vec{G}$ of type (iii^{**}),

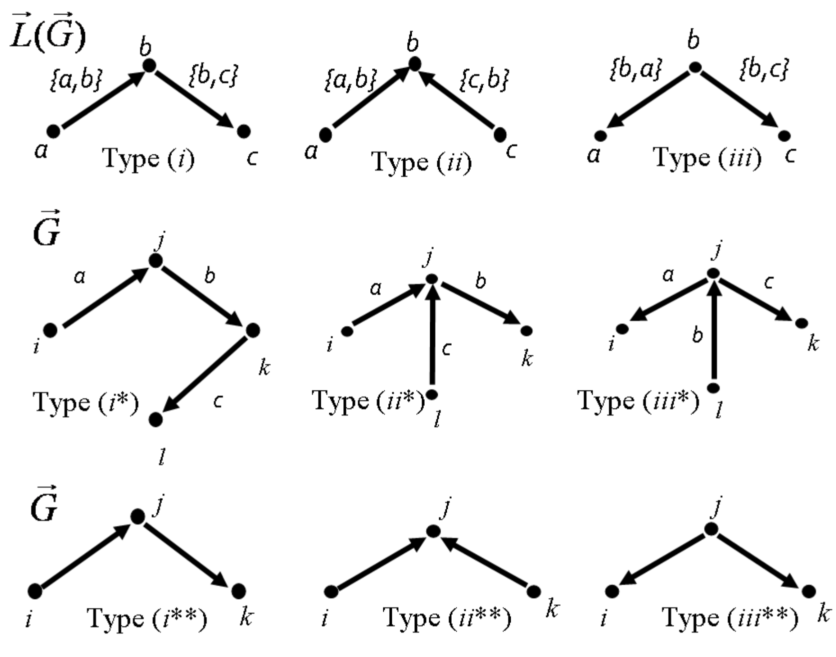

FIGURE 5. Types of triples in a directed line network.

those with nodes $\{i, j, k\}$ and edges $\big((j,i),(j,k)\big)$ is less or equal than $(n-3)^{-1}$ times the number of subgraphs in $\vec{G}$ of type (iii^{*}).

Notice in addition that from the definition of directed line graph every triangle in $\vec{L}(\vec{G})$ must necessarily be oriented in the sense that the degree in and the degree out of every node is one, and it must proceed from a oriented triangle in $\vec{G}$.

Thus we have

$$\begin{aligned}
\frac{1}{c_c(\vec{L}(\vec{G}))} &= \frac{|\text{triples of type}(i)|}{|\text{triangles}|} + \frac{|\text{triples of type}(ii)|}{|\text{triangles}|} + \frac{|\text{triples of type}(iii)|}{|\text{triangles}|} \\
&= \frac{|\text{subgraphs of type}(i^{*})|}{|\text{oriented triangles in } \vec{G}|} + \frac{|\text{subgraphs of type}(i^{**})|}{|\text{oriented triangles in } \vec{G}|} \\
&\qquad + \frac{|\text{subgraphs of type}(i^{***})|}{|\text{oriented triangles in } \vec{G}|} \\
&\geq \frac{(n-3)|\text{triples of type}(i^{**})|}{|\text{oriented triangles in } \vec{G}|} + \frac{(n-3)|\text{triples of type}(ii^{**})|}{|\text{oriented triangles in } \vec{G}|} \\
&\qquad + \frac{(n-3)|\text{triples of type}(iii^{**})|}{|\text{oriented triangles in } \vec{G}|} \\
&= \frac{(n-3)|\text{triples in } \vec{G}|}{|\text{oriented triangles in } \vec{G}|} \geq \frac{(n-3)|\text{triples in } \vec{G}|}{|\text{triangles in } \vec{G}|} = \frac{n-3}{c_c(\vec{G})}.
\end{aligned}$$

□

Remark 1. From the previous result follows that the line graph $\vec{L}(\vec{G})$ of a directed graph $\vec{G}$ with more than 4 nodes cannot be isomorphic to the initial graph $\vec{G}$. In contrast if $\vec{G}$ has only 4 nodes then $\vec{L}(\vec{G})$ can be isomorphic to $\vec{G}$. In this case the inequality in Theorem 3.4 can turn into an equality as 4-3=1. The following example illustrates the situation. Consider $\vec{G}$ with nodes $\{i, j, k, l\}$ and the directed edges (i, j), (j, k), (k, l) and (l, j). It is straightforward to check that $\vec{G}$ and $\vec{L}(\vec{G})$ have the same clustering coefficient as they are isomorphic.

Remark 2. We cannot expect a reverse inequality in Theorem 3.4 as the following example shows. Consider G the triangle of nodes i, j, k and edges (i, j), (k, j) and (k, i). The associated line graph consists of nodes (i, j), (k, j) and (k, i) and the single edge $\big((k, i), (i, j)\big)$. Hence $\vec{L}(\vec{G})$ has clustering coefficient zero while $\vec{G}$ has clustering coefficient one.

4. Interplay between directed and undirected line-graphs for undirected networks.

This section is devoted to analyze some relationships between the centrality and the efficiency of an undirected graph G and the corresponding measures of the directed graph $D(G)$ obtained from G.

4.1. Centrality and irregularity.

In this subsection we intend to exploit some relations between the centralities of an undirected graph G and the directed graph $D(G)$, obtained from G, and the centralities of their line graphs counterparts. In particular, we want to, raughty speaking, gauge the norm of the difference of centralities $c_{L(G)}$ and $c_{\vec{L}(D(G))}$ as a function of the Collatz-Sinogowitz irregularity index of G ([7]).

Note that, according to Theorem 3.1, we know that for the directed graph $D(G)$, we have the equality

$$c_{D(G)} = \frac{H_{D(G)}\, c_{\vec{L}(D(G))}}{\|H_{D(G)}\, c_{\vec{L}(D(G))}\|_1}.$$

After renaming we have the equality $c_{D(G)} = Rc_{\vec{L}(D(G))}$ where $R : \mathbb{R}^m \to \mathbb{R}^n$ is the linear transformation $R(x) = \lambda H_{D(G)}(x)$ with $\lambda = \|H_{D(G)}\, c_{\vec{L}(D(G))}\|_1^{-1}$.

Analogously it was proved in [8] that for the undirected graph G we have the equalities

$$c_G = \frac{I_G\, c_{L(G)}}{\|I_G\, c_{L(G)}\|_1}, \quad c_{L(G)} = \frac{I_G^t c_G}{\|I_G^t\, c_G\|_1}.$$

S ince $A(G) + gr = I_G I_G^t$, it follows after renaming that $c_{G+gr} = Sc_{L(G)}$, where $S : \mathbb{R}^m \to \mathbb{R}^n$ is the linear transformation $S(x) = \eta I(x)$ and $\eta = \|I_G c_{L(G)}\|_1^{-1}$.

Summing up, there are two linear transformations $R, S : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$c_{D(G)} = Rc_{\vec{L}(D(G))}, \quad c_{G+gr} = Sc_{L(G)}$$

In addition, from the fact that G and $D(G)$ share the same adjacency matrix, as previously noticed, it readily follows

$$c_G = c_{D(G)}.$$

Call now $i(G)$ the well known Collatz-Sinogowitz index, $\rho(G) - 2m/n$, which is a measure of the irregularity of the graph G ([7]).

If $i(G) < \dfrac{(\sqrt{2}-1)^2}{(4\sqrt{2}-2)^2 n^2 \sqrt{2m}}\,\delta^2$, where δ is the spectral separation of G, then it was shown in [9] that

$$\|c_{G+gr} - c_G\|_2 \leq \frac{(4-\sqrt{2})\sqrt{2}\,n\sqrt[4]{2m}}{\delta}\,\sqrt{i(G)}.$$

But then from the previous lines the following result is derived

Theorem 4.1. *Following the previous notation,*

$$\|Rc_{\vec{L}(D(G))} - Sc_{L(G)}\|_2 \leq f(i(G)),$$

where $f : [0, \alpha) \to [0, \infty)$ *is defined as*

$$f(t) = \frac{(4-\sqrt{2})\sqrt{2}\,n\sqrt[4]{2m}}{\delta}\,\sqrt{t},$$

with $\alpha = \dfrac{(\sqrt{2}-1)^2}{(4\sqrt{2}-2)^2 n^2 \sqrt{2m}}\,\delta^2$.

In other words, up to dimension, we have evaluated the distance of the centrality vectors of $L(G)$ and $\vec{L}(D(G))$ in terms of the irregularity of G.

4.2. **Efficiency.** In this subsection we are interested in remarking the relationship amongst the efficiency of an undirected network G, the directed graph $D(G)$ and the corresponding line graphs $L(G)$ and $\vec{L}(D(G))$.

As $E(G) = E(D(G))$, by the Theorem 3.5 in [8], taking p as the number of edges of $L(G)$, and using the same expressions for $F(n,m)$ and $G(n,m)$ as in the theorem of the previous section, we can get

$$\begin{aligned} E(\vec{L}(D(G)) &\leq F(n,m)E(D(G)) + G(n,m) \\ &\leq F(n,m)E(G) + G(n,m) \\ &\leq F(n,m)\Big(\frac{1}{n(n-1)}(8m(m-1)E(L(G)) - 15p + 2)\Big) + G(n,m). \end{aligned}$$

5. **Conclusions.** The analytical results proved in the previous sections illustrate that there are strong correlations between the properties of a directed network and its associated (directed) line-graph that extend the same phenomenon that occurs in the undirected case. Furthermore, in the last section, we can conclude that even in the undirected case, the directed version of the line-graph can infer deep properties of an undirected graph. Therefore, if we consider an undirected network G, in addition to the classic (undirected) line graph $L(G)$, the directed line graph $\vec{L}(D(G))$ captures many properties of G even better that $L(G)$ (this is the case of the eigenvector centrality).

Considering the analysis of directed line graphs of directed networks is not superfluous as many real-life applications studied by means of the line-graph are directed in nature (for example, urban street networks [10, 11] and many others). The results obtained here show explicitly the connection between the primal and the dual approach in directed networks and how the properties of a directed line graph and the associated directed line graph are related by means of the (directed) bipartite representation of the network.

As a future task we leave the study of the relationship between the directed line graph of a directed network $\vec{G}$ and the undirected line graph associated to the underlying undirected network $U(\vec{G})$. Indeed, several directed networks in real life (such as urban street networks) have been analyzed via the (undirected) line graph of the underlying undirected network; our goal would be to give rigorous support to the conclusions obtained so far (see [10, 11]) which implicitly have assumed that the primal and dual analysis of directed networks and its underlying directed versions are essentially equivalent.

REFERENCES

[1] R. Albert and A. L. Barabási, *Statistical mechanics of complex networks*, Rev. Mod. Phys., **74** (2002), 47–97.

[2] J. Anez, T. De La Barra and B. Perez, *Dual graph representation of transport networks*, Trans. Res. B, **30** (1996), 209–216.

[3] C. Balbuena, D. Ferrero, X. Marcote and I. Pelayo, *Algebraic properties of a digraph and its line digraph*, J. of Interconnection Networks, **4** (2003), 377–393.

[4] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez and D. U. Hwang, *Complex networks: Structure and dynamics*, Physics Reports, **424** (2006), 175–308.

[5] P. Bonacich, *Factoring and weighing approaches to status scores and clique information*, J. Math. Soc., **2** (1972), 113.

[6] P. Bonacich and P. Lloyd, *Eigenvectors-like measures of centrality for asymetric relations*, Soc. Netw., **23** (2001), 191.

[7] L. Collatz and U. Sinogowitz, *Spektren endlicher grafen*, Abh. Math. Sem. University Hamburg., **21** (1957), 63–77.

[8] R. Criado, J. Flores, A. García del Amo and M. Romance, *Analytical relationships between metric and centrality measures of a network and its dual*, J. Comput. Appl. Math., **235** (2011), 1775–1780.

[9] R. Criado, J. Flores, A. García del Amo and M. Romance, *Centrality and measure of irregularity*, Preprint, 2011.

[10] P. Crucitti, V. Latora and S. Porta, *Centrality in networks of urban streets*, Chaos, **16** (2006), 015113.

[11] P. Crucitti, V. Latora and S. Porta, *Centrality measures in spatial networks of urban streets*, Phys. Rev. E, **73** (2006), 036125.

[12] C. J. L. Gross and J. Yellen, "Handbook of Graph Theory," CRC Press, New Jersey, 2004.

[13] V. Latora and M. Marchiori, *Efficient behavior of small-world networks*, Phys. Rev. Lett., **87** (2001), 198701.

[14] R. L. Hemminger and L. W. Beineke, *Line graphs and line digraphs*, in "Selected Topics in Graph Theory" (eds. L. W. Beineke and R. J. Wilson), Academic Press Inc., (1978), pp. 271305.

[15] M. B. Hua, R. Jianga, R. Wang and Q. S. Wu, *Urban traffic simulated from the dual representation: Flow, crisis and congestion*, Physics Letters A, **373** (2009), 2007–2011.

[16] M. E. and J. Newman, *The structure and function of complex networks*, SIAM Review, **45** (2003), 167–256.

[17] M. E. and J. Newman, "Networks: An Introduction," Oxford Univ. Press, Oxford, 2010.

[18] M. E., J. Newman, A. L. Barabási and D. J. Watts, "The Structure and Dynamics of Networks," Princeton Univ. Press, Princeton, New Jersey, 2006.

[19] P. Ren, R. C. Wilson and E. R. Hancock, *Characteristic polynomial analysis on matrix representations on graphs*, LNCS, **5534/2009** (2009), 243–252.

[20] P. Ren, R. C. Wilson and E. R. Hancock, *Graph characterization via Ihara coefficients*, EEE Trans. on Neural Networks, **22** (2011), 233–245.

[21] D. Volchenkov and Ph. Blanchard, *Transport networks revisited: Why dual graphs?*, arXiv:0710.5494.

[22] S. Wasserman and K. Faust, "Social Networks Analysis," Cambridge Univ. Press, 1994.

Received December 2011; revised May 2012.

E-mail address: regino.criado@urjc.es
E-mail address: julio.flores@urjc.es
E-mail address: alejandro.garciadelamo@urjc.es
E-mail address: miguel.romance@urjc.es