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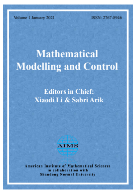

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Research article

A Hopf algebra on (0,1)-matrices

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Abstract: In this paper, we defined that a conjunction multiplication and an unshuffle comultiplication on the vector space spanned by (0,1)-matrices. Then, we proved that the vector space with these two operations is a bialgebra. In fact, it is a graded connected bialgebra, so it is a Hopf algebra.

Keywords: Hopf algebra; (0,1)-matrix; conjunction multiplication; unshuffle comultiplication

1. Introduction

In 1941, the basic concept of Hopf algebra was first introducted by Hopf to study algebraic topology and the properties of algebraic groups [1]. Milnor, Moore, Chase, and Sweedler gave the explicit definition, basic properties, and common symbols of Hopf algebra in the 1960's [2–4]. In 1979, Joni and Rota used Hopf algebra as a tool to study combinatorial objects [5]. After that, more and more Hopf structures on combinatorial objects were found, such as Hopf algebra on permutations [6, 7], Hopf algebra on posets [8], and Hopf algebras on matroids and graphs [9].

In 1995, Malvenuto and Reutenauer first gave a classical Hopf algebra on permutations by shuffle product $\sqcup\!\sqcup$ [10]. On this basis, in 2016, Giraudo and Vialette defined the unshuffle coproduct Δ on permutations [11]. In 2020, Zhao and Li derived a new Hopf algebra on permutations with another shuffle product $\sqcup\!\sqcup_G^*$ from the classical one [12]. In 2020, Aval, Bergeron, and Machacek gave a Hopf algebra on labeled simple graphs with the conjunction product and the unshuffle coproduct without a proof [13]. In 2021, Liu and Li proved that the vector space spanned by permutations with the conjunction product $\bullet$ and the unshuffle coproduct Δ^* is a Hopf algebra [14]. In 2023, Dong and Li proved that the vector space spanned by labeled simple graphs with the conjunction product $\diamond$ and the unshuffle coproduct Δ_* is a Hopf algebra [15].

In fact, matrices are closely related to permutations and graphs. A (0,1)-matrix is a matrix whose entries are all 0 or 1, also called a binary matrix. It is widely used in graph theory [16, 17], combinatorics [18], linear programming [19–21], and computer science [22]. In this paper, we first define a conjunction multiplication and an unshuffle comultiplication on the vector space spanned by (0,1)-matrices. Then, we prove it is a Hopf algebra with these two operations.

This paper is organized as follows: In Section 2, we first recall some basic definitions related to Hopf algebra. Then we define the vector space $\mathcal{M}$ spanned by (0,1)-matrices, the conjunction multiplication $\diamond$ and the unshuffle comultiplication Δ on the vector space. In Section 3, we prove that $(\mathcal{M}, \diamond, \mu)$ is a graded algebra and that $(\mathcal{M}, \Delta, \nu)$ is a graded coalgebra, and the compatibility between the operations $\diamond$ and Δ. Furthermore, $(\mathcal{M}, \diamond, \mu, \Delta, \nu)$ is a Hopf algebra in that $\mathcal{M}$ is graded connected. Lastly, we summarize our main conclusions in Section 4.

2. Preliminaries

2.1. Basic definitions

Here, we recall some basic definitions related to Hopf algebra; see [3] for more details. Let R be a commutative ring and B an R-module.

Define a *multiplication* $\pi : B \otimes_R B \longrightarrow B$ and a *unit* $\mu : R \longrightarrow B$, respectively, satisfying the diagrams in Figure 1, then (B, π, μ) is an *R-algebra*.

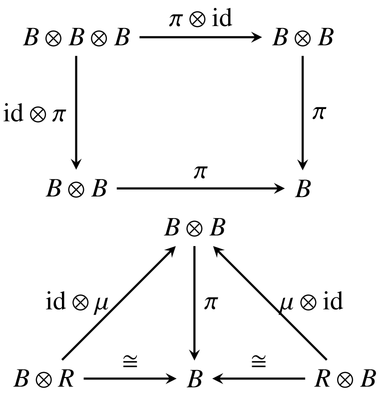

Figure 1. Associative law and unit.

The algebra B is *graded* if there is a direct sum decomposition $B = \bigoplus_{n=0}^{\infty} B_n$ such that the multiplication of homogeneous elements of degrees m and n is homogeneous of degree $m+n$, that is, $\pi(B_m \otimes B_n) \subseteq B_{m+n}$, and $\mu(R) \subseteq B_0$.

Define a *comultiplication* $\Delta : B \longrightarrow B \otimes_R B$ and a *counit* $\nu : B \longrightarrow R$, respectively, satisfying the diagrams in Figure 2, then (B, Δ, ν) is an *R-coalgebra*.

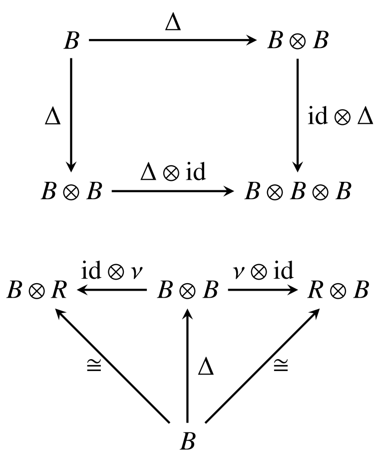

Figure 2. Coassociative law and counit.

The coalgebra B is *graded* if there is a direct sum decomposition $B = \bigoplus_{n=0}^{\infty} B_n$ such that $\Delta(B_n) \subseteq \bigoplus_{0 \le i \le n} (B_i \otimes B_{n-i})$ and $\nu(B_n) = 0$ if $n \geq 1$.

If B is both an R-algebra and an R-coalgebra, and satisfies one of the following equivalent conditions:

(1) Δ and ν are algebra homomorphisms,

(2) π and μ are coalgebra homomorphisms,

then we say the algebra and coalgebra structures on B are *compatible* and $(B, \pi, \mu, \Delta, \nu)$ is an *R-bialgebra*.

If $B = \bigoplus_{n=0}^{\infty} B_n$ is both a graded algebra and a graded coalgebra, and satisfies the compatibility condition, then we say B is a *graded bialgebra*.

If there exists a linear map $\theta : B \longrightarrow B$ satisfying

$$\pi \circ (\theta \otimes id) \circ \Delta = \mu \circ \nu = \pi \circ (id \otimes \theta) \circ \Delta,$$

i.e., the diagram in Figure 3 commutes, then θ is an *antipode*. A bialgebra with an antipode is a *Hopf algebra*.

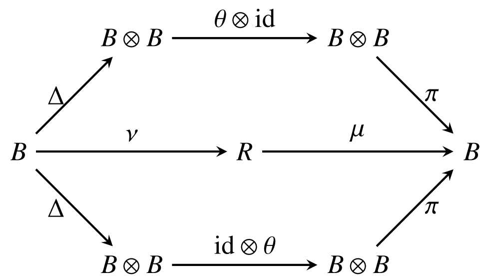

Figure 3. Antipode.

A bialgebra B over a field $\mathbb{K}$ is called *graded connected* if it is graded and satisfies $B_0 = \mathbb{K}1_B$, where $\mathbb{K}$ is a field of characteristic 0. In 2008, Manchon [23] proved that any graded connected bialgebra admits a unique antipode and it is a Hopf algebra.

2.2. The (0, 1)-matrices

An $m \times n$ matrix $A = (a_{ij})_{m \times n}$ is called a (0, 1)-*matrix* if

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s1} & a_{s2} & \cdots & a_{sn} \end{pmatrix}_{m \times n},$$

where a_{ij} is either 0 or 1.

Define

$$[n]=\begin{cases}\{1,2,\ldots,n\}, & n>0,\\ \emptyset, & n=0,\end{cases}$$

and

$$[i,j]=\begin{cases}\{i,i+1,\ldots,j\}, & i\leqslant j,\\ \emptyset, & i>j.\end{cases}$$

Let $I=\{i_1,i_2,\ldots,i_k\}\subseteq[s]$,where $i_1<i_2<\cdots<i_k\leq s$. Similarly, let $J=\{j_1,j_2,\ldots,j_q\}\subseteq[n]$,where $j_1<j_2<\cdots<j_q\leq n$. We take the ${i_1}^{\text{th}}, {i_2}^{\text{th}},\ldots,{i_k}^{\text{th}}$ rows and the ${j_1}^{\text{th}}, {j_2}^{\text{th}},\ldots,{j_q}^{\text{th}}$ columns elements of the matrix A; these entries maintain the same row and column relationship, and shrink into a matrix, which is called the *restriction* of A on $I\times J$, denoted by $A_{I\times J}$. We also call $A_{I\times J}$ a *submatrix* of A. For convenience, when $I=J$, denote $A_{I\times J}=A_I$. In particular, A is the *empty matrix* when I or J is empty, denoted by ϵ.

Example 2.1. *The matrix*

$$A=\begin{pmatrix}0&1&0&0&0&0&1\\0&0&0&0&1&0&0\\0&1&0&1&0&0&0\\0&0&0&0&0&1&0\end{pmatrix}$$

is a 4×7 *(0,1)-matrix. We have*

$$A_{\{1,2\}\times\{1,2,7\}}=\begin{pmatrix}0&1&1\\0&0&0\end{pmatrix}$$

and

$$A_{[3]}=\begin{pmatrix}0&1&0\\0&0&0\\0&1&0\end{pmatrix}.$$

Let $M_n=\left\{A|A=(a_{ij}),\text{ an } n\times n\ (0,1)\text{-matrix}\right\}$ and $\mathcal{M}_n$ be the vector space spanned by M_n over a field $\mathbb{K}$ of characteristic 0, for any non-negative integer n. For example,

$$M_2=\left\{\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix},\begin{pmatrix}1&1\\0&0\end{pmatrix},\begin{pmatrix}1&0\\1&0\end{pmatrix},\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}0&1\\0&1\end{pmatrix},\begin{pmatrix}0&0\\1&1\end{pmatrix},\begin{pmatrix}1&1\\1&0\end{pmatrix},\begin{pmatrix}1&1\\0&1\end{pmatrix},\begin{pmatrix}1&0\\1&1\end{pmatrix},\begin{pmatrix}0&1\\1&1\end{pmatrix},\begin{pmatrix}1&1\\1&1\end{pmatrix}\right\}.$$

In particular, $M_0=\{\epsilon\}$ and $\mathcal{M}_0=\mathbb{K}M_0$. Denote

$$M=\bigcup_{n=0}^{\infty}M_n \quad\text{and}\quad \mathcal{M}=\bigoplus_{n=0}^{\infty}\mathcal{M}_n.$$

Let $I=\{i_1,i_2,\ldots,i_n\}$ be a set of positive integers where $i_1<i_2<\cdots<i_n$. Define a mapping st_I from I to $[|I|]$ by $\mathrm{st}_I(i_a)=a$ for $1\leqslant a\leqslant n$, and call it the *standardization* of I. For x,y in I, $\mathrm{st}_I(x)<\mathrm{st}_I(y)$ if and only if $x<y$. Sometimes, we omit the subscript of the standardization when the set is obvious. Let T be a subset of I, then $\mathrm{st}_I(T)=\{\mathrm{st}_I(x)|x\in T\}$.

2.3. Conjunction multiplication and unshuffle comultiplication

In this section, we construct a conjunction multiplication and an unshuffle comultiplication on the vector space spanned by (0,1)-matrices.

Definition 2.1. *Define the* conjunction multiplication $\diamond$ *on* $\mathcal{M}$ *by*

$$A\diamond B=\begin{pmatrix}A&0\\0&B\end{pmatrix}$$

for A *in* M_m *and* B *in* M_n, *and the* unit μ *from* $\mathbb{K}$ *to* $\mathcal{M}$ *by* $\mu(1)=\epsilon$.

Example 2.2. *For* $A=\begin{pmatrix}1&1\\0&0\end{pmatrix}$, $B=\begin{pmatrix}0&1&1\\0&1&0\\1&0&0\end{pmatrix}$ *and* $C=\begin{pmatrix}1\end{pmatrix}$, *we have*

$$A\diamond B=\begin{pmatrix}1&1&0&0&0\\0&0&0&0&0\\0&0&0&1&1\\0&0&0&1&0\\0&0&1&0&0\end{pmatrix},$$

and

$$B\diamond C=\begin{pmatrix}0&1&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{pmatrix}.$$

Here, we color the entries in $A\diamond B$ *restricted to* A *red and to* B *blue. Similarly, we color the entries in* $B\diamond C$ *restricted to* B *blue and to* C *green.*

Definition 2.2. *Define the* unshuffle comultiplication Δ *on* $\mathcal{M}$ *by*

$$\Delta(A) = \sum_{I\subseteq[n]} A_I \otimes A_{[n]\setminus I},$$

for $A = (a_{ij})$ *in* M_n*, and the* counit ν *from* $\mathcal{M}$ *to* $\mathbb{K}$ *by*

$$\nu(A) = \begin{cases} 1, & A = \epsilon, \\ 0, & otherwise. \end{cases}$$

In particular, $\Delta(\epsilon) = \epsilon \otimes \epsilon$.

Remark 2.1. *We will prove that* Δ *satisfies coassociativity in Theorem 3.2.*

Example 2.3. *For* $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ *and* $B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, *we have*

$$\begin{aligned}\Delta(A) =& \epsilon \otimes \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \\ &+ \begin{pmatrix} 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \otimes \epsilon\end{aligned}$$

and

$$\begin{aligned}\Delta(B) =& \epsilon \otimes \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \end{pmatrix} \\ &+ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes \epsilon.\end{aligned}$$

3. Main theorem

Theorem 3.1. *The vector space* $\mathcal{M}$ *with the conjunction multiplication* $\diamond$ *and the unit* μ *is a graded algebra.*

Proof. For any A in M_m, B in M_n, and C in M_k, we have

$$\begin{aligned}(A \diamond B) \diamond C =& \begin{pmatrix} A & 0_{m\times n} \\ 0_{n\times m} & B \end{pmatrix} \diamond C \\ =& \begin{pmatrix} A & 0_{m\times n} & 0_{m\times k} \\ 0_{n\times m} & B & 0_{n\times k} \\ 0_{k\times m} & 0_{k\times n} & C \end{pmatrix} \\ =& A \diamond \begin{pmatrix} B & 0_{n\times k} \\ 0_{k\times n} & C \end{pmatrix} \\ =& A \diamond (B \diamond C).\end{aligned}$$

So, $\diamond$ is associative. It is easy to prove that the μ is a unit. Then $(\mathcal{M}, \diamond, \mu)$ is an algebra. Obviously, by the definitions of $\diamond$ and μ, we have $\mathcal{M}_m \diamond \mathcal{M}_n \subseteq \mathcal{M}_{m+n}$ for $m, n \geq 0$ and $\mu(\mathbb{K}) \subseteq \mathcal{M}_0$. So $(\mathcal{M}, \diamond, \mu)$ is a graded algebra. □

Example 3.1. *For* $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \end{pmatrix}$ *and* $C = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, *we have*

$$\begin{aligned}(A \diamond B) \diamond C =& \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \diamond C \\ =& \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ =& \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \diamond \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ =& A \diamond (B \diamond C).\end{aligned}$$

Lemma 3.1. *Assume* I *is a set of positive integers,* $J \subseteq I$ *and* $K = \mathrm{st}_I(J)$*. Then*

$$\mathrm{st}_K(\mathrm{st}_I(i)) = \mathrm{st}_J(i),$$

for any i *in* J.

Proof. Denote $I = \{i_1, i_2, \ldots, i_n\}$ and $J = \{i_{j_1}, i_{j_2}, \ldots, i_{j_t}\}$, where $t \leq n$. Suppose $i_1 < i_2 < \cdots < i_n$ and $1 \leq j_1 < j_2 < \cdots < j_t \leq n$. Obviously, $\mathrm{st}_J(i_{j_m}) = m$, $K = \mathrm{st}_I(J) =$

$\{j_1, j_2, \ldots, j_t\}$, and $\mathrm{st}_K(j_m) = m$ for any $i_{j_m} \in J$, where $1 \le m \le t$. Then

$$\mathrm{st}_K(\mathrm{st}_I(i_{j_s})) = \mathrm{st}_K(j_s) = s = \mathrm{st}_J(i_{j_s}),$$

for any $1 \le s \le t$. □

Example 3.2. *If $I = \{3,5,7,8,9\}$ and $J = \{3,7,8\}$, then $st_J(3) = 1, st_J(7) = 2$, $st_J(8) = 3$, $st_I(3) = 1, st_I(7) = 3$, and $st_I(8) = 4$. So $K = st_I(\{3,7,8\}) = \{1,3,4\}$. Futhermore, $st_K(st_I(3)) = 1, st_K(st_I(7)) = 2, st_K(st_I(8)) = 3$.*

Lemma 3.2. *Assume $A = (a_{ij})_{n\times n}$ is a (0,1)-matrix, $J \subseteq I \subseteq [n]$ and $K = \mathrm{st}_I(J)$. Then*

$$(A_I)_K = A_J.$$

Proof. For convenience, we denote A_I as B and $(A_I)_K$ as C, where $B = (b_{ij})$ for $i, j \in [|I|]$ and $C = (c_{ij})$ for $i, j \in [|J|]$. Besides, let $A_J = D = (d_{ij})$. Obviously, C and D are both $|J| \times |J|$ (0,1)-matrices. We just need to show that for each i, j in $[|J|]$, $c_{ij} = d_{ij}$. For c_{ij} in C, there must exist i'' and j'' in K such that $\mathrm{st}_K(i'') = i$ and $\mathrm{st}_K(j'') = j$. Therefore $b_{i''j''} = c_{ij}$. Meanwhile, there must exist i' and j' in J such that $\mathrm{st}_I(i') = i''$ and $\mathrm{st}_I(j') = j''$, therefore $b_{i''j''} = a_{i'j'}$, $\mathrm{st}_K(\mathrm{st}_I(i')) = i$, and $\mathrm{st}_K(\mathrm{st}_I(j')) = j$. By Lemma 3.1, $\mathrm{st}_J(i') = i$ and $\mathrm{st}_J(j') = j$, then $a_{i'j'} = d_{ij}$. Therefore $c_{ij} = d_{ij}$. So

$$(A_I)_K = A_J.$$

□

Example 3.3. *For*

$$A = \begin{pmatrix} 0&0&0&0&0&0&0&0&0\\ 0&1&0&1&0&0&0&0&1\\ 0&1&1&0&0&1&0&0&0\\ 0&0&1&0&0&0&0&0&1\\ 1&0&0&0&1&0&0&1&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0 \end{pmatrix}_{9\times 9},$$

$I = \{2,3,4,6,7,9\}$ and $J = \{2,3,6\}$, then $K = st_I(J) = \{1,2,4\}$. We have

$$A_I = \begin{pmatrix} 1&0&1&0&0&1\\ 1&1&0&1&0&0\\ 0&1&0&0&0&1\\ 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0 \end{pmatrix},$$

and

$$(A_I)_K = \begin{pmatrix} 1&0&0\\ 1&1&1\\ 0&1&0 \end{pmatrix}.$$

On the other hand,

$$A_J = \begin{pmatrix} 1&0&0\\ 1&1&1\\ 0&1&0 \end{pmatrix}.$$

Theorem 3.2. *The vector space $\mathcal{M}$ with the unshuffle comultiplication Δ and the counit ν is a graded coalgebra.*

Proof. Obviously, the empty matrix ϵ satisfies

$$(\Delta \otimes \mathrm{id}) \circ \Delta(\epsilon) = \epsilon \otimes \epsilon \otimes \epsilon = (\mathrm{id} \otimes \Delta) \circ \Delta(\epsilon).$$

For any $A = (a_{ij})$ in M_n, where $n \ge 1$, we have

$$(\Delta \otimes \mathrm{id}) \circ \Delta(A) = (\Delta \otimes \mathrm{id}) \left(\sum_{I \subseteq [n]} A_I \otimes A_{[n]\setminus I} \right). \tag{3.1}$$

Denote A_I by B, then

$$\Delta(B) = \sum_{K \subseteq [|I|]} B_K \otimes B_{[|I|]\setminus K}. \tag{3.2}$$

Let J as a subset in I such that $\mathrm{st}_I(J) = K$, then $\mathrm{st}_I(I\setminus J) = [|I|]\setminus K$. By Lemma 3.2, we have

$$(A_I)_K = A_J$$

and

$$(A_I)_{[|I|]\setminus K} = A_{I\setminus J}.$$

Since K in (3.2) traverses all subsets of $[|I|]$, the corresponding J also traverses all subsets of I. Then (3.2) can be rewritten as

$$\sum_{J \subseteq I} A_J \otimes A_{I\setminus J}. \tag{3.3}$$

Then (3.1) can be rewritten as

$$\sum_{I\subseteq[n],J\subseteq I} A_J\otimes A_{I\backslash J}\otimes A_{[n]\backslash I}. \tag{3.4}$$

Since I and J are arbitrary, (3.4) can be rewritten as

$$\sum_{\substack{I,J,K\subseteq[n]\\ I\cup J\cup K=[n]\\ |I|+|J|+|K|=n}} A_I\otimes A_J\otimes A_K. \tag{3.5}$$

Similarly, we can get that $(\mathrm{id}\otimes\Delta)\circ\Delta(A)$ is also equal to (3.5). Then Δ satisfies coassociativity. It is easy to prove that ν is a counit. So $(\mathcal{M},\Delta,\nu)$ is a coalgebra. Obviously, by the definition of Δ and ν, we have $\Delta(\mathcal{M}_n)\subseteq\bigoplus_{0\le i\le n}(\mathcal{M}_i\otimes\mathcal{M}_{n-i})$ and $\mu(\mathcal{M}_n)=0$ for $n>0$. So $(\mathcal{M},\Delta,\nu)$ is a graded coalgebra. □

Next we prove the compatibility between the conjunction multiplication and the unshuffle comultiplication.

Lemma 3.3. *Let $A=(a_{ij})_{m\times m}$, $B=(b_{ij})_{n\times n}$, $I\subset[m]$, and $J\subseteq[m+1,m+n]$. Then*

$$(A\diamond B)_{I\cup J}=A_I\diamond B_{J'}, \tag{3.6}$$

where $J'=\mathrm{st}_{[m+1,m+n]}(J)$.

Proof. Denote $C=\begin{pmatrix}A & 0\\ 0 & B\end{pmatrix}$. Obviously,

$$C_{[m]}=A \text{ and } C_{[m+1,m+n]}=B. \tag{3.7}$$

Since $I\subset[m]$ and $J\subseteq[m+1,m+n]$, $\max\{I\}<\min\{J\}$. Then

$$C_{I\cup J}=\begin{pmatrix}C_I & 0\\ 0 & C_J\end{pmatrix}.$$

By the definition of $\diamond$,

$$(A\diamond B)_{I\cup J}=C_I\diamond C_J. \tag{3.8}$$

Obviously, $I=\mathrm{st}_{[m]}(I)$. Due to $J'=\mathrm{st}_{[m+1,m+n]}(J)$ and Lemma 3.2, we have

$$(C_{[m]})_I=C_I,\ \ (C_{[m+1,m+n]})_{J'}=C_J.$$

By (3.7), we have

$$A_I=C_I,\ \ B_{J'}=C_{J\times J}.$$

Then (3.8) can be rewritten as (3.6), i.e.,

$$(A\diamond B)_{I\cup J}=A_I\diamond B_{J'}.$$

□

Theorem 3.3. *$(\mathcal{M},\diamond,\mu,\Delta,\nu)$ is a bialgebra.*

Proof. To prove this, we need to prove the following properties:

(1) $\nu(A\diamond B)=\nu(A)\nu(B)$, for any A,B in M.

(2) $\Delta(A\diamond B)=\Delta(A)\diamond\Delta(B)$, for any A,B in M.

That is, the comultiplication Δ and the counit ν are algebra homomorphisms. When $A=\epsilon$ and $B=\epsilon$, $\nu(A\diamond B)=\nu(\epsilon)=1$ and $\nu(A)\nu(B)=1$. So $\nu(A\diamond B)=\nu(A)\nu(B)$. When $A=\epsilon$ or $B=\epsilon$, for convenience, let $A=\epsilon$ and $B\in M_n$, for $n>0$. Then $\nu(A\diamond B)=\nu(B)=0$ and $\nu(A)\nu(B)=0$. So $\nu(A\diamond B)=\nu(A)\nu(B)$. When $A=(a_{ij})_{m\times m}$ and $B=(b_{ij})_{n\times n}$ are non-empty matrices, we denote

$$C=\begin{pmatrix}A & 0\\ 0 & B\end{pmatrix}. \tag{3.9}$$

Then $\nu(A\diamond B)=\nu(C)=0$ and $\nu(A)\nu(B)=0$. So $\nu(A\diamond B)=\nu(A)\nu(B)$. From the above analysis, ν is an algebra homomorphism.

Next, prove (2). From (3.9), have

$$\Delta(C)=\sum_{I\subseteq[m+n]} C_I\otimes C_{[m+n]\backslash I}. \tag{3.10}$$

Denote $I_{11}=I\cap[m]$, $I_{12}=I\cap[m+1,m+n]$, $I_{21}=([m+n]\backslash I)\cap[m]$ and $I_{22}=([m+n]\backslash I)\cap[m+1,m+n]$. Furthermore, denote $\mathrm{st}_{[m+1,m+n]}(I_{12})=J_{12}$ and denote $\mathrm{st}_{[m+1,m+n]}(I_{22})=J_{22}$. By Lemma 3.3,

$$C_I=C_{I_{11}\cup I_{12}}=A_{I_{11}}\diamond B_{J_{12}},$$

since $I_{11}\subseteq[m]$, $I_{12}\subseteq[m+1,m+n]$ and $J_{12}=\mathrm{st}_{[m+1,m+n]}(I_{12})$. Similarly,

$$C_{[m+n]\backslash I}=C_{I_{21}\cup I_{22}}=A_{I_{21}}\diamond B_{J_{22}},$$

since $I_{21}\subseteq[m]$, $I_{22}\subseteq[m+1,m+n]$ and $J_{22}=\mathrm{st}_{[m+1,m+n]}(I_{22})$. Then (3.10) can be rewritten as

$$\sum_{I\subseteq[m+n]} A_{I_{11}}\diamond B_{J_{12}}\otimes A_{I_{21}}\diamond B_{J_{22}}. \tag{3.11}$$

Obviously, when I traverses all subsets of $[m+n]$, I_{11} and I_{21} traverse all disjoint subsets of $[m]$, I_{12} and I_{22} traverse all disjoint subsets of $[m+1,m+n]$, and meanwhile J_{12} and J_{22} travese all disjoint subsets of $[n]$.

Then we rewrite (3.11) as

$$\left(\sum_{\substack{I_{11}\cap I_{21}=\emptyset\\ I_{11}\cup I_{21}=[m]}} A_{I_{11}}\otimes A_{I_{21}}\right)\diamond\left(\sum_{\substack{J_{12}\cap J_{22}=\emptyset\\ J_{12}\cup J_{22}=[n]}} B_{J_{12}}\otimes B_{J_{22}}\right). \qquad (3.12)$$

By the definition of Δ, (3.12) is equal to

$$\Delta(A)\diamond\Delta(B).$$

Therefore,

$$\Delta(A\diamond B)=\Delta(A)\diamond\Delta(B).$$

So $(\mathcal{M},\diamond,\mu,\Delta,\nu)$ is a bialgebra. □

Corollary 3.1. *$(\mathcal{M},\diamond,\mu,\Delta,\nu)$ is a Hopf algebra.*

Proof. By Theorems 3.1–3.3, $(\mathcal{M},\diamond,\mu,\Delta,\nu)$ is a graded connected bialgebra. So it is a Hopf algebra. □

Example 3.4. *For* $A=\begin{pmatrix}1&1\\0&1\end{pmatrix}$ *and* $B=\begin{pmatrix}0&1\\0&1\end{pmatrix}$, *we have*

$$\Delta\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\diamond\begin{pmatrix}0&1\\0&1\end{pmatrix}\right)$$

$$=\Delta\left(\begin{pmatrix}1&1&0&0\\0&1&0&0\\0&0&0&1\\0&0&0&1\end{pmatrix}\right)$$

$$=\epsilon\otimes\begin{pmatrix}1&1&0&0\\0&1&0&0\\0&0&0&1\\0&0&0&1\end{pmatrix}+\begin{pmatrix}1\end{pmatrix}\otimes\begin{pmatrix}1&0&0\\0&0&1\\0&0&1\end{pmatrix}$$

$$+\begin{pmatrix}1\end{pmatrix}\otimes\begin{pmatrix}1&0&0\\0&0&1\\0&0&1\end{pmatrix}+\begin{pmatrix}0\end{pmatrix}\otimes\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$+\begin{pmatrix}1\end{pmatrix}\otimes\begin{pmatrix}1&1&0\\0&1&0\\0&0&0\end{pmatrix}+\begin{pmatrix}1&1\\0&1\end{pmatrix}\otimes\begin{pmatrix}0&1\\0&1\end{pmatrix}$$

$$+\begin{pmatrix}1&0\\0&0\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&0\end{pmatrix}$$

$$+\begin{pmatrix}1&0\\0&0\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&0\end{pmatrix}$$

$$+\begin{pmatrix}0&1\\0&1\end{pmatrix}\otimes\begin{pmatrix}1&1\\0&1\end{pmatrix}+\begin{pmatrix}1&1&0\\0&1&0\\0&0&0\end{pmatrix}\otimes\begin{pmatrix}1\end{pmatrix}$$

$$+\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}\otimes\begin{pmatrix}0\end{pmatrix}+\begin{pmatrix}1&0&0\\0&0&1\\0&0&1\end{pmatrix}\otimes\begin{pmatrix}1\end{pmatrix}$$

$$+\begin{pmatrix}1&0&0\\0&0&1\\0&0&1\end{pmatrix}\otimes\begin{pmatrix}1\end{pmatrix}+\begin{pmatrix}1&1&0&0\\0&1&0&0\\0&0&0&1\\0&0&0&1\end{pmatrix}\otimes\epsilon$$

$$=(\epsilon\diamond\epsilon)\otimes\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\diamond\begin{pmatrix}0&1\\0&1\end{pmatrix}\right)$$

$$+\left(\begin{pmatrix}1\end{pmatrix}\diamond\epsilon\right)\otimes\left(\begin{pmatrix}1\end{pmatrix}\diamond\begin{pmatrix}0&1\\0&1\end{pmatrix}\right)$$

$$+\left(\begin{pmatrix}1\end{pmatrix}\diamond\epsilon\right)\otimes\left(\begin{pmatrix}1\end{pmatrix}\diamond\begin{pmatrix}0&1\\0&1\end{pmatrix}\right)$$

$$+\left(\epsilon\diamond\begin{pmatrix}0\end{pmatrix}\right)\otimes\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\diamond\begin{pmatrix}1\end{pmatrix}\right)$$

$$+\left(\epsilon\diamond\begin{pmatrix}1\end{pmatrix}\right)\otimes\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\diamond\begin{pmatrix}0\end{pmatrix}\right)$$

$$+\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\diamond\epsilon\right)\otimes\left(\epsilon\diamond\begin{pmatrix}0&1\\0&1\end{pmatrix}\right)$$

$$+\left(\begin{pmatrix}1\end{pmatrix}\diamond\begin{pmatrix}0\end{pmatrix}\right)\otimes\left(\begin{pmatrix}1\end{pmatrix}\diamond\begin{pmatrix}1\end{pmatrix}\right)$$

$$+\left(\begin{pmatrix}1\end{pmatrix}\diamond\begin{pmatrix}1\end{pmatrix}\right)\otimes\left(\begin{pmatrix}1\end{pmatrix}\diamond\begin{pmatrix}0\end{pmatrix}\right)$$

$$+\left(\begin{pmatrix}1\end{pmatrix}\diamond\begin{pmatrix}0\end{pmatrix}\right)\otimes\left(\begin{pmatrix}1\end{pmatrix}\diamond\begin{pmatrix}1\end{pmatrix}\right)$$

$$+\left(\begin{pmatrix}1\end{pmatrix}\diamond\begin{pmatrix}1\end{pmatrix}\right)\otimes\left(\begin{pmatrix}1\end{pmatrix}\diamond\begin{pmatrix}0\end{pmatrix}\right)$$

$$+\left(\epsilon\diamond\begin{pmatrix}0&1\\0&1\end{pmatrix}\right)\otimes\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\diamond\epsilon\right)$$

$$+\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\diamond\begin{pmatrix}0\end{pmatrix}\right)\otimes\left(\epsilon\diamond\begin{pmatrix}1\end{pmatrix}\right)$$

$$+\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\diamond\begin{pmatrix}1\end{pmatrix}\right)\otimes\left(\epsilon\diamond\begin{pmatrix}0\end{pmatrix}\right)$$

$$+\left(\begin{pmatrix}1\end{pmatrix}\diamond\begin{pmatrix}0&1\\0&1\end{pmatrix}\right)\otimes\left(\begin{pmatrix}1\end{pmatrix}\diamond\epsilon\right)$$

$$+\left(\begin{pmatrix}1\end{pmatrix}\otimes\begin{pmatrix}0&1\\0&1\end{pmatrix}\right)\otimes\left(\begin{pmatrix}1\end{pmatrix}\diamond\epsilon\right)$$

$$+\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\diamond\begin{pmatrix}0&1\\0&1\end{pmatrix}\right)\otimes(\epsilon\diamond\epsilon)$$

$$=\left(\epsilon\otimes\begin{pmatrix}1&1\\0&1\end{pmatrix}+\begin{pmatrix}1\end{pmatrix}\otimes\begin{pmatrix}1\end{pmatrix}+\begin{pmatrix}1\end{pmatrix}\otimes\begin{pmatrix}1\end{pmatrix}+\begin{pmatrix}1&1\\0&1\end{pmatrix}\otimes\epsilon\right)$$

$$\diamond\left(\epsilon\otimes\begin{pmatrix}0&1\\0&1\end{pmatrix}+\begin{pmatrix}0\end{pmatrix}\otimes\begin{pmatrix}1\end{pmatrix}+\begin{pmatrix}1\end{pmatrix}\otimes\begin{pmatrix}0\end{pmatrix}+\begin{pmatrix}0&1\\0&1\end{pmatrix}\otimes\epsilon\right)$$
$$=\Delta\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)\diamond\Delta\left(\begin{pmatrix}0&1\\0&1\end{pmatrix}\right).$$

4. Conclusions

Let $\mathcal{M}$ be the vector space spanned by $(0,1)$-matrices. First, we define the conjunction multiplication $\diamond$ and the unshuffle comultiplication Δ on $\mathcal{M}$. Then we prove that the conjunction multiplication $\diamond$ satisfies associativity and the unshuffle comultiplication Δ satisfies coassociativity, i.e., $(\mathcal{M},\diamond,\mu)$ is an algebra and $(\mathcal{M},\Delta,\nu)$ is a coalgebra. Lastly, we prove the compatibility between $\diamond$ and Δ, and $(\mathcal{M},\diamond,\mu,\Delta,\nu)$ is a graded connected bialgebra. So $(\mathcal{M},\diamond,\mu,\Delta,\nu)$ is a Hopf algebra. Therefore, the family of combinatorial Hopf algebras has a new member on (0,1)-matrices.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest regarding the publication of this paper.

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