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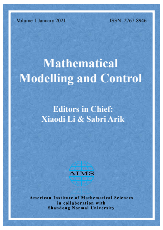


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***Research article***

# Efficiency conditions in new interval-valued control models via modified $T$-objective functional approach and saddle-point criteria

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**Abstract:** The paper studied efficiency conditions in new interval-valued control models via a modified $T$-objective functional approach and saddle-point type criteria. More precisely, first, by considering the necessary optimality conditions associated to single-objective variational control models with interval values, we formulated the necessary efficiency conditions for a new control model, denoted by ($P$), with multiple cost interval-valued functionals. Thereafter, by considering the notions of $T$-convexity and $T$-pseudoconvexity, with $T$ as a sublinear functional (with respect to the sixth and seventh variable), we formulated a characterization result of saddle-point for a Lagrange functional associated with ($P$). Further, we established sufficient efficiency conditions for ($P$) via the modified $T$-objective functional approach. Finally, under suitable generalized convexity assumptions, we stated the connection between a Lower-Upper-efficient solution (in short, $LU$-efficient solution) for ($P$) and a saddle-point for the Lagrange functional associated with the modified control model.

**Keywords:** control; saddle-point; interval-valued functional

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## 1. Introduction

In the last decades, optimization problems have become an active field with several important implications in engineering, management sciences, and economics, especially concerning multi-objective optimization. Optimization problems containing concrete objects are categorized as deterministic optimization problems. These types of problems refer to those in which the parameters, constraints, and objective functions are precisely defined and do not involve uncertainty. In contrast to deterministic optimization, where all parameters are assumed to be fixed and known exactly, stochastic optimization acknowledges the inherent uncertainty in real-world scenarios. Modeling uncertainty through probability distributions allows for a more realistic representation of optimization problems in various fields.

On the other hand, convexity plays a key role in several aspects of optimization theory. It serves as a fundamental principle in optimization theory, providing important insights into global optimality conditions. In additional hypotheses, it ensures that local minima are also global minima, thereby simplifying the search for optimal solutions. However, in many practical situations, strict convexity assumptions may not hold, leading to the need for more flexible frameworks. Generalized convex functions, including approximate convex functions (see Ngai et al. [1]), extend the concept of convexity to accommodate a broader range of problem structures and conditions. By relaxing the strict convexity requirement, these generalized convex functions offer more inclusive conditions for global optimality while still maintaining important stability properties. The fundamentals as well as the applications of generalized convexity have been well documented in [2–6].

Hanson [7], following Berkovitz [8], explored and extended for the first time the relationship between classical variational problems and optimization problems. Throughout time, these types of problems have been widely studied. Bector and Husain [9] obtained duality results for a properly efficient solution under convexity assumptions applying the duality method of the standard vector optimization problem to a multi-objective variational problem. Craven [10] established that the necessary Kuhn–Tucker type conditions for multi-objective variational problems are sufficient if the objective functions are pseudo-convex and the constraints are quasi-convex. Throughout time, the variational problem was extended in a multi-dimensional framework, and it is known as the multi-dimensional (multi-time) variational problem. The multi-temporal equations have been introduced for the first time in physics by Dirac et al. [11], where for each particle, an individual time is introduced. Also, the term *many-time* appears explicitly in Tomonaga [12]. The *multi-time* term was used also in mathematics by Friedman [13], Yurchuk [14], Prepeliţă [15], Treanţă [16, 17]. The concept of multi-time has major implications and is essential for designing systems that adapt and react effectively to dynamic environments. Starting from the theoretical and application aspects, these optimization problems have been intensively analyzed in the last few years. Many works on variational problems have focused on looking for best solutions procedures in robust control and interval analysis (see, for instance, Moore [18], Ishibuch and Tanaka [19], Ahmad et al. [20]). Great importance in the literature is given in formulating new methods such that the solvability of the initial mathematical programming problem is described by optimal points of the associated problem. Such types of techniques have been also formulated to obtain new saddle-point criteria in several classes of optimization problems. Antczak [21] proposed one of such methods and introduced a modified objective function in order to investigate a class of smooth nonconvex multi-objective optimization problems. Also, Jha et al. [22] studied saddle-point and efficiency criteria associated with some multi-objective variational problems with interval values.

In this paper, in accordance with Egudo [23], Preda [24], Mishra and Mukherjee [25], and following Jha et al. [22], we study the efficiency conditions in new interval-valued control models via a modified $T$-objective functional approach and saddle-point criteria. More precisely, firstly, by considering the necessary optimality conditions associated to single-objective variational control models with interval values (see Treanţă [26]), we formulate the necessary efficiency conditions for a new control model, denoted by $(P)$, with multiple cost interval-valued functionals. Thereafter, by considering the notions of $T$-convexity and $T$-pseudoconvexity, with $T$ as a sublinear functional (with respect to the sixth and seventh variable), associated with controlled integral-type functionals, we formulate a characterization result of saddle-point for a Lagrange functional associated with $(P)$. Further, we establish sufficient efficiency conditions for $(P)$ via the modified $T$-objective functional approach. Finally, under suitable generalized convexity assumptions, we state the connection between an $LU$-efficient point of $(P)$ and a saddle-point for the Lagrange functional associated with the modified control model. The main novelty elements provided by this study are the following: (1) defining the new notions of $T$-convexity and (strictly) $T$-pseudoconvexity for controlled simple integral functionals (in the abovementioned references, these concepts have been considered only for uncontrolled functionals); (2) introducing new Lagrange-type functionals and the corresponding saddle-point criteria for the control problems under study and the associated modified models; (3) providing innovative proofs for the main derived results.

This article continues with the next 5 sections, as follows. In Section 2, we introduce important preliminaries, address the definition of *T-convexity* for a controlled integral functional with interval values, and formulate the necessary efficiency conditions for ($P$). In Section 3, by using the concept of *saddle-point* for a *Lagrange-type functional* associated with ($P$), we formulate a characterization result for $LU$-efficient solutions of ($P$). Section 4 includes the sufficient efficiency conditions for ($P$) via the modified $T$-objective functional approach. In this section, we constructed the multi-objective variational control problem $\left(\mathrm{P}_T(\bar{\xi}, \bar{c})\right)$, with the modified $T$-objective functional corresponding to ($P$). In Section 5, under suitable generalized convexity assumptions, we state the equivalence between an $LU$-efficient point for ($P$) and an $LU$-$T$-saddle-point for the Lagrange functional associated with $\left(\mathrm{P}_T(\bar{\xi}, \bar{c})\right)$. Section 6 provides the conclusions of this paper.

## 2. Notations and preliminaries

For $A = [a_1, a_2]$, a real interval, let us consider the spaces $V = \{\xi : A \mapsto \mathbb{R}^n \text{— } \xi \text{ is piecewise smooth function}\}$ and $W = \{c : A \mapsto \mathbb{R}^m \text{— } c \text{ is continuous function}\}$. Suppose $\varphi : A \times V \times V \times W \mapsto \mathbb{R}^n$, $\varphi = \varphi(\tau, \xi, \dot{\xi}, c)$ is a continuously differentiable function with respect to its arguments. The first-order partial derivatives of $\varphi^i$, $i = \overline{1, n}$, with respect to $\xi$, $\dot{\xi} := \frac{d\xi}{d\tau}$, and $c$, are denoted by $\varphi^i_\xi$, $\varphi^i_{\dot{\xi}}$, $\varphi^i_c$, respectively, and defined as $\varphi^i_\xi = \left(\frac{\partial \varphi^i}{\partial s_1}, \ldots, \frac{\partial \varphi^i}{\partial s_n}\right)$, $\varphi^i_{\dot{\xi}} = \left(\frac{\partial \varphi^i}{\partial \dot{\xi}_1}, \ldots, \frac{\partial \varphi^i}{\partial \dot{\xi}_n}\right)$, and $\varphi^i_c = \left(\frac{\partial \varphi^i}{\partial u_1}, \ldots, \frac{\partial \varphi^i}{\partial u_m}\right)$, respectively. For $\xi = (s_1, s_2) \in \mathbb{R}^{2n}$ and $\Gamma_n := \{1, \ldots, n\}$, we consider:

(i) $s_1 = s_2 \Leftrightarrow s_{1i} = s_{2i},\ i \in \Gamma_n$;

(ii) $s_1 < s_2 \Leftrightarrow s_{1i} < s_{2i},\ i \in \Gamma_n$;

(iii) $s_1 \leqq s_2 \Leftrightarrow s_{1i} \leq s_{1i},\ i \in \Gamma_n$;

(iv) $s_1 \leq s_2 \Leftrightarrow s_1 \leqq s_1$ and $s_1 \neq s_2$.

Let $\Delta = \left\{\left[M^L, M^U\right] : M^L \leq M^U, M^L, M^U \in \mathbb{R}\right\}$ and

$$R^n_\Delta = \{M = (M_1, \cdots, M_n) \in \mathbb{R}^n \mid M_i \in \Delta, i \in \Gamma_n\},$$

involving that $\Delta = R_\Delta$.

Further, for $M, N \in \Delta$ and $\alpha \in \mathbb{R}$, we consider:

$$M + N = \left[M^L + N^L, M^U + N^U\right],$$
$$M + \alpha = \left[M^L + \alpha, M^U + \alpha\right],$$

and

$$M\alpha = \begin{cases} \left[M^L\alpha, M^U\alpha\right], & \alpha \geq 0 \\ \left[M^U\alpha, M^L\alpha\right], & \alpha < 0. \end{cases}$$

It is obvious that $-M = -\left[M^L, M^U\right] = \left[-M^U, -M^L\right]$, implying that

$$M - N = \left[M^L, M^U\right] - \left[N^L, N^U\right]$$
$$= \left[M^L, M^U\right] + \left[-N^U, -N^L\right] = \left[M^L - N^U, M^U - N^L\right].$$

In this paper, for $M, N \in \Delta$, we use the following conventions for interval inequalities:

$$M \preceq_{LU} N \Longleftrightarrow M^L \leq N^L,\ M^U \leq N^U;$$

$$M \prec_{LU} N \Longleftrightarrow \begin{cases} M^L < N^L, \\ M^U < N^U, \end{cases} \begin{cases} M^L \leq N^L, \\ M^U < N^U, \end{cases} \begin{cases} M^L < N^L, \\ M^U \leq N^U. \end{cases}$$

In a similar manner, for $M, N \in R^n_\Delta$, we consider: $M \leqq N$ if, and only if, $M_i \preceq_{LU} N_i, i \in \Gamma_n$, and $M \leq N$ if, and only if, $M_i \preceq_{LU} N_i, i \in \Gamma_n$, and $M_{i^*} \prec_{LU} N_{i^*}$, for at least one $i^* \in \Gamma_n$.

In the next definitions, we introduce the notion of an interval-valued functional generated by controlled simple integrals, and, also, the notion of sublinear functional with respect to the sixth and seventh variable (see, for instance, Mishra and Mukherjee [25], Treanţă [27], Ahmad et al. [20], Jha et al. [22]).

**Definition 2.1.** A controlled simple integral functional

$$\mathcal{T} : V \times W \to R_\Delta,\ \mathcal{T}(\xi, c) := \int_{a_1}^{a_2} f(\tau, \xi(\tau), c(\tau)) d\tau,$$

with $f : A \times V \times W \mapsto R_\Delta$, $f = [f^L, f^U]$, is said to be an *interval-valued functional* if

$$\int_{a_1}^{a_2} f(\tau, \xi(\tau), c(\tau)) d\tau$$
$$= \left[\int_{a_1}^{a_2} f^L(\tau, \xi(\tau), c(\tau)) d\tau, \int_{a_1}^{a_2} f^U(\tau, \xi(\tau), c(\tau)) d\tau\right],$$

$$\int_{a_1}^{a_2} f^L(\tau, \xi(\tau), c(\tau)) d\tau$$
$$\leq \int_{a_1}^{a_2} f^U(\tau, \xi(\tau), c(\tau)) d\tau,\ \forall (\xi, c) \in V \times W,$$

where $\mathcal{T}^L(\xi, c) := \int_{a_1}^{a_2} f^L(\tau, \xi(\tau), c(\tau)) d\tau$, $\mathcal{T}^U(\xi, c) := \int_{a_1}^{a_2} f^U(\tau, \xi(\tau), c(\tau)) d\tau$ are the lower and upper real-valued integral functionals.

**Definition 2.2.** The functional $T : A \times (V \times W)^2 \times \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}$ is said to be *sublinear with respect to the sixth and seventh variable* if, for all $(\xi, c), (b, v) \in V \times W$, the following relations hold:

(i) $T(\tau, \xi, c, b, v; \theta_1 + \theta_2, \gamma) \leq T(\tau, \xi, c, b, v; \theta_1, \gamma) + T(\tau, \xi, c, b, v; \theta_2, \gamma)$,

(ii) $T(\tau, \xi, c, b, v; \theta, \gamma_1 + \gamma_2) \leq T(\tau, \xi, c, b, v; \theta, \gamma_1) + T(\tau, \xi, c, b, v; \theta, \gamma_2)$,

(iii) $T(\tau, \xi, c, b, v; a\theta, \gamma) = aT(\tau, \xi, c, b, v; \theta, \gamma)$,

(iv) $T(\tau, \xi, c, b, v; \theta, a\gamma) = |a| T(\tau, \xi, c, b, v; \theta, \gamma)$, for any $a \in \mathbb{R},\ \theta, \theta_1, \theta_2 \in \mathbb{R}^n,\ \gamma, \gamma_1, \gamma_2 \in \mathbb{R}^m$.

Taking into account conditions (iii) and (iv), we notice $T(\tau, \xi, c, b, v; 0, \gamma) = 0$, $T(\tau, \xi, c, b, v; \theta, 0) = 0$. In the following, for the sake of simplicity, we consider $T(\tau, \xi, b; \theta, \gamma)$ in place of $T(\tau, \xi, c, b, v; \theta, \gamma)$.

In this study, we investigate the following interval-valued control model:

$$(P) \qquad \min_{(\xi,c)} \left( \int_{a_1}^{a_2} \pi^1(\tau, \xi, c) d\tau, \cdots, \int_{a_1}^{a_2} \pi^r(\tau, \xi, c) d\tau \right)$$

subject to

$$p(\tau, \xi, c) \leqq 0,\ \tau \in A,$$

$$z(\tau, \xi, c) = \dot{\xi},\ \tau \in A,$$

$$\xi(a_1) = \sigma,\ \xi(a_2) = \omega,$$

where

$$\int_{a_1}^{a_2} \pi^i(\tau, \xi, c) d\tau := \left[ \int_{a_1}^{a_2} \pi^{iL}(\tau, \xi, c) d\tau, \int_{a_1}^{a_2} \pi^{iU}(\tau, \xi, c) d\tau \right],$$

for $i \in \Gamma_r$, and $p : A \times V \times W \to \mathbb{R}^q,\ z : A \times V \times W \to \mathbb{R}^n$ are $C^1$-class functionals. Further, $F$ denotes the set of all feasible solutions to $(P)$, that is

$$F = \{(\xi, c) \in V \times W \mid p(\tau, \xi, c) \leqq 0, z(\tau, \xi, c) = \dot{\xi},$$

$$\tau \in A, \xi(a_1) = \sigma, \xi(a_2) = \omega\}.$$

Let $D := \frac{d}{d\tau}$ denote the differential operator. To introduce the concept of *T-convexity* associated with the interval-valued controlled integral functional, let us first give its definition for a controlled simple integral functional with real values (see, for example, Mishra and Mukherjee [25], Jha et al. [22], and the references therein).

**Definition 2.3.** A controlled simple integral functional with real values $\Phi(\xi, c) := \int_{a_1}^{a_2} \phi(\tau, \xi, c) d\tau$ is called to be *T-convex at* $(b, v) \in V \times W$ if, for all $(\xi, c) \in V \times W$, the inequality

$$\int_{a_1}^{a_2} \phi(\tau, \xi, c) d\tau - \int_{a_1}^{a_2} \phi(\tau, b, v) d\tau$$

$$\geq \int_{a_1}^{a_2} T\left(\tau, \xi, b; \phi_\xi(\tau, b, v) - D\phi_{\dot{\xi}}(\tau, b, v), \phi_c(\tau, b, v)\right) d\tau$$

is satisfied.

**Definition 2.4.** A controlled simple integral functional with interval values $\mathcal{T}(\xi, c) := \int_{a_1}^{a_2} f(\tau, \xi, c) d\tau$ is called *T-convex at* $(b, v) \in V \times W$ if $\mathcal{T}^L(\xi, c) := \int_{a_1}^{a_2} f^L(\tau, \xi, c) d\tau$ and $\mathcal{T}^U(\xi, c) := \int_{a_1}^{a_2} f^U(\tau, \xi, c) d\tau$ are $T$-convex at $(b, v) \in V \times W$.

**Definition 2.5.** A controlled simple integral functional with real values $\Phi(\xi, c) := \int_{a_1}^{a_2} \phi(\tau, \xi, c) d\tau$ is called to be *(strictly) T-pseudoconvex at* $(b, v) \in V \times W$ if, for all $(\xi, c) \in V \times W$, the inequality

$$\int_{a_1}^{a_2} T\left(\tau, \xi, b; \phi_\xi(\tau, b, v) - D\phi_{\dot{\xi}}(\tau, b, v), \phi_c(\tau, b, v)\right) d\tau \geq 0$$

implies

$$\int_{a_1}^{a_2} \phi(\tau, \xi, c) d\tau \geq (>) \int_{a_1}^{a_2} \phi(\tau, b, v) d\tau,$$

or, equivalently,

$$\int_{a_1}^{a_2} \phi(\tau, \xi, c) d\tau < (\leq) \int_{a_1}^{a_2} \phi(\tau, b, v) d\tau$$

implies

$$\int_{a_1}^{a_2} T\left(\tau, \xi, b; \phi_\xi(\tau, b, v) - D\phi_{\dot{\xi}}(\tau, b, v), \phi_c(\tau, b, v)\right) d\tau < 0.$$

**Definition 2.6.** A controlled simple integral functional with interval values $\mathcal{T}(\xi, c) := \int_{a_1}^{a_2} f(\tau, \xi, c) d\tau$ is called *T-pseudoconvex at* $(b, v) \in V \times W$ if $\mathcal{T}^L(\xi, c) := \int_{a_1}^{a_2} f^L(\tau, \xi, c) d\tau$ and $\mathcal{T}^U(\xi, c) := \int_{a_1}^{a_2} f^U(\tau, \xi, c) d\tau$ are $T$-pseudoconvex at $(b, v) \in V \times W$. If at least one of the real-valued controlled integral functionals $\mathcal{T}^L(\xi, c)$ and

$\mathcal{T}^U(\xi, c)$ is strictly $T$-pseudoconvex at $(b, v) \in V \times W$, then the interval-valued controlled integral functional $\mathcal{T}(\xi, c) := \int_{a_1}^{a_2} f(\tau, \xi, c)d\tau$ is called *strictly $T$-pseudoconvex at* $(b, v) \in V \times W$.

**Remark 2.1.** In the following, we assume the hypothesis $T(\tau, \xi, \xi; \cdot, \cdot) = 0$.

**Definition 2.7.** The pair $(\bar{\xi}, \bar{c}) \in F$ is named the *$LU$-efficient solution of* $(P)$ if there exists no $(\xi, c) \in F$ satisfying

$$\int_{a_1}^{a_2} \pi^i(\tau, \xi, c)d\tau \preceq_{LU} \int_{a_1}^{a_2} \pi^i(\tau, \bar{\xi}, \bar{c})d\tau, \ \forall i \in \Gamma_r,$$
$$\int_{a_1}^{a_2} \pi^{i^*}(\tau, \xi, c)d\tau \prec_{LU} \int_{a_1}^{a_2} \pi^{i^*}(\tau, \bar{\xi}, \bar{c})d\tau, \text{ for some } i^* \in \Gamma_r.$$

**Definition 2.8.** The pair $(\bar{\xi}, \bar{c}) \in F$ is named the *weak $LU$-efficient solution* of $(P)$ if there exists no $(\xi, c) \in F$ satisfying

$$\int_{a_1}^{a_2} \pi^i(\tau, \xi, c)d\tau \prec_{LU} \int_{a_1}^{a_2} \pi^i(\tau, \bar{\xi}, \bar{c})d\tau, \ \forall i \in \Gamma_r.$$

By considering the necessary optimality conditions associated to single-objective variational control models with interval values (see Treanţă [26]), we formulate the necessary efficiency conditions for $(P)$.

**Theorem 2.1.** *If $(\bar{\xi}, \bar{c}) \in F$ is a normal $LU$-efficient solution for $(P)$, then there exist $\bar{\lambda}^{iL}, \bar{\lambda}^{iU} \in \mathbb{R}, i \in \Gamma_r$, and $\bar{\nu}(\tau) \in^q$ and $\bar{\mu}(\tau) \in \mathbb{R}^n$ satisfying*

$$\bar{\lambda}^{iL}\frac{\partial \pi^{iL}}{\partial \xi}(\tau, \bar{\xi}, \bar{c}) + \bar{\lambda}^{iU}\frac{\partial \pi^{iU}}{\partial \xi}(\tau, \bar{\xi}, \bar{c}) + \bar{\nu}(\tau)\frac{\partial p}{\partial \xi}(\tau, \bar{\xi}, \bar{c}) + \bar{\mu}(\tau)\frac{\partial z}{\partial \xi}(\tau, \bar{\xi}, \bar{c})$$
$$= D\Big(\bar{\lambda}^{iL}\frac{\partial \pi^{iL}}{\partial \dot{\xi}}(\tau, \bar{\xi}, \bar{c}) + \bar{\lambda}^{iU}\frac{\partial \pi^{iU}}{\partial \dot{\xi}}(\tau, \bar{\xi}, \bar{c}) + \bar{\nu}(\tau)\frac{\partial p}{\partial \dot{\xi}}(\tau, \bar{\xi}, \bar{c}) - \bar{\mu}(\tau)\Big), \tag{2.1}$$

$$\bar{\lambda}^{iL}\frac{\partial \pi^{iL}}{\partial c}(\tau, \bar{\xi}, \bar{c}) + \bar{\lambda}^{iU}\frac{\partial \pi^{iU}}{\partial c}(\tau, \bar{\xi}, \bar{c})$$
$$+\bar{\nu}(\tau)\frac{\partial p}{\partial c}(\tau, \bar{\xi}, \bar{c}) + \bar{\mu}(\tau)\frac{\partial z}{\partial c}(\tau, \bar{\xi}, \bar{c}) = 0, \tag{2.2}$$

$$\bar{\nu}(\tau)p(\tau, \bar{\xi}, \bar{c}) = 0, \tag{2.3}$$

$$\bar{\lambda}^{iL}, \bar{\lambda}^{iU} \geq 0, \ \bar{\nu}(\tau) \geqq 0, \tag{2.4}$$

*for all $\tau \in A$, except at discontinuities.*

*Proof.* The proof of this theorem follows the same line as in Treanţă [26], hence we have omitted it. □

**Definition 2.9.** An $LU$-efficient solution $(\bar{\xi}, \bar{c}) \in F$ for $(P)$ is named *normal $LU$-efficient solution* if $\lambda = (\lambda^L, \lambda^U) > 0$.

## 3. $LU$-saddle-point criteria associated with $(P)$

In this section, by considering a feasible point in $(P)$, we formulate a characterization result of *saddle-point* for a *Lagrange functional* associated with $(P)$.

**Definition 3.1.** The *Lagrange functional* associated with $(P)$, denoted by $\mathcal{K}(\xi, c, \bar{\lambda}, \nu, \mu)$, is defined as follows:

$$\mathcal{K}(\xi, c, \bar{\lambda}, \nu, \mu) = \int_{a_1}^{a_2} \Big(\bar{\lambda}^{iL}\pi^{iL}(\tau, \xi, c) + \bar{\lambda}^{iU}\pi^{iU}(\tau, \xi, c)\Big)d\tau$$
$$+ \int_{a_1}^{a_2} \{\nu(\tau)p(\tau, \xi, c) + \mu(\tau)\big[z(\tau, \xi, c) - \dot{\xi}\big]\}d\tau.$$

**Definition 3.2.** We say that $(\bar{\xi}, \bar{c}, \bar{\lambda}, \bar{\nu}, \bar{\mu})$ is the *$LU$-saddle-point of the Lagrange functional* associated with $(P)$ if the following two inequalities hold:

(i) $\mathcal{K}(\bar{\xi}, \bar{c}, \bar{\lambda}, \bar{\nu}, \bar{\mu}) \leq \mathcal{K}(\xi, c, \bar{\lambda}, \bar{\nu}, \bar{\mu}), \ \forall (\xi, c) \in F$,

(ii) $\mathcal{K}(\bar{\xi}, \bar{c}, \bar{\lambda}, \nu, \mu) \leq \mathcal{K}(\bar{\xi}, \bar{c}, \bar{\lambda}, \bar{\nu}, \bar{\mu}), \ \forall \nu \in \mathbb{R}^q_+, \ \forall \mu \in \mathbb{R}^n$.

**Theorem 3.1.** *Assume that $(\bar{\xi}, \bar{c}) \in F$ satisfies Eqs (2.1)–(2.3) for $\bar{\lambda}^{iL}, \bar{\lambda}^{iU} \in \mathbb{R}, \bar{\nu} \in \mathbb{R}^q_+, \bar{\mu} \in \mathbb{R}^n$, and the functionals*

$$\int_{a_1}^{a_2} \pi^{iL}(\tau, \xi, c)d\tau, \int_{a_1}^{a_2} \pi^{iU}(\tau, \xi, c)d\tau, i \in \Gamma_r,$$

$$\int_{a_1}^{a_2} \bar{\nu}(\tau)p(\tau, \xi, c)d\tau$$

*and $\int_{a_1}^{a_2} \bar{\mu}(\tau)\big[z(\tau, \xi, c) - \dot{\xi}\big]d\tau$ are $T$-convex at $(\bar{\xi}, \bar{c}) \in F$. Then $(\bar{\xi}, \bar{c}, \bar{\lambda}, \bar{\nu}, \bar{\mu})$ is an $LU$-saddle-point for the Lagrange functional associated with $(P)$.*

*Proof.* Consider the point $(\bar{\xi}, \bar{c}, \bar{\lambda}, \bar{\nu}, \bar{\mu})$ satisfies Eqs (2.1)–(2.3) in Theorem 2.1. Now, by hypothesis, we have

$$\int_{a_1}^{a_2} \pi^{iL}(\tau, \xi, c)d\tau - \int_{a_1}^{a_2} \pi^{iL}(\tau, \bar{\xi}, \bar{c})d\tau$$
$$\geq \int_{a_1}^{a_2} T\left(\tau, \xi, \bar{\xi}; \pi^{iL}_{\xi}(\tau, \bar{\xi}, \bar{c}) - D\pi^{iL}_{\dot{\xi}}(\tau, \bar{\xi}, \bar{c}), \pi^{iL}_{c}(\tau, \bar{\xi}, \bar{c})\right)d\tau, \tag{3.1}$$

$$\int_{a_1}^{a_2} \pi^{iU}(\tau, \xi, c)d\tau - \int_{a_1}^{a_2} \pi^{iU}(\tau, \bar{\xi}, \bar{c})d\tau$$
$$\geq \int_{a_1}^{a_2} T\left(\tau, \xi, \bar{\xi}; \pi^{iU}_{\xi}(\tau, \bar{\xi}, \bar{c}) - D\pi^{iU}_{\dot{\xi}}(\tau, \bar{\xi}, \bar{c}), \pi^{iU}_{c}(\tau, \bar{\xi}, \bar{c})\right)d\tau, \tag{3.2}$$

$$\int_{a_1}^{a_2} \bar{\nu}(\tau)p(\tau, \xi, c)d\tau - \int_{a_1}^{a_2} \bar{\nu}(\tau)p(\tau, \bar{\xi}, \bar{c})d\tau$$
$$\geq \int_{a_1}^{a_2} T(\tau, \xi, \bar{\xi}; \bar{\nu}(\tau)p_{\xi}(\tau, \bar{\xi}, \bar{c}) - D\bar{\nu}(\tau)p_{\dot{\xi}}(\tau, \bar{\xi}, \bar{c}), \bar{\nu}(\tau)p_c(\tau, \bar{\xi}, \bar{c}))d\tau, \tag{3.3}$$

$$\int_{a_1}^{a_2} \bar{\mu}(\tau)\left[z(\tau,\xi,c)-\dot{\xi}\right]d\tau-\int_{a_1}^{a_2} \bar{\mu}(\tau)\left[z(\tau,\bar{\xi},\bar{c})-\dot{\bar{\xi}}\right]d\tau$$

$$\geq \int_{a_1}^{a_2} T(\tau,\xi,\bar{\xi};\bar{\mu}(\tau)z_{\xi}(\tau,\bar{\xi},\bar{c})+D\bar{\mu}(\tau),\bar{\mu}(\tau)z_c(\tau,\bar{\xi},\bar{c}))d\tau. \tag{3.4}$$

Multiplying Eqs (3.1) and (3.2) by $\bar{\lambda}^{iL},\bar{\lambda}^{iU}$, respectively, and adding them together with Eqs (3.3) and (3.4), it follows that

$$\int_{a_1}^{a_2}\left(\bar{\lambda}^{iL}\pi^{iL}(\tau,\xi,c)+\bar{\lambda}^{iU}\pi^{iU}(\tau,\xi,c)+\bar{\nu}(\tau)p(\tau,\xi,c)+\bar{\mu}(\tau)\left[z(\tau,\xi,c)-\dot{\xi}\right]\right)d\tau$$

$$-\int_{a_1}^{a_2}\left(\bar{\lambda}^{iL}\pi^{iL}(\tau,\bar{\xi},\bar{c})+\bar{\lambda}^{iU}\pi^{iU}(\tau,\bar{\xi},\bar{c})+\bar{\nu}(\tau)p(\tau,\bar{\xi},\bar{c})+\bar{\mu}(\tau)\left[z(\tau,\bar{\xi},\bar{c})-\dot{\bar{\xi}}\right]\right)d\tau$$

$$\geq \int_{a_1}^{a_2} T(\tau,\xi,\bar{\xi};\bar{\lambda}^{iL}\pi_{\xi}^{iL}(\tau,\bar{\xi},\bar{c})+\bar{\lambda}^{iU}\pi_{\xi}^{iU}(\tau,\bar{\xi},\bar{c})+\bar{\nu}(\tau)p_{\xi}(\tau,\bar{\xi},\bar{c})+\bar{\mu}(\tau)z_{\xi}(\tau,\bar{\xi},\bar{c})$$

$$-D\left(\bar{\lambda}^{iL}\pi_{\dot{\xi}}^{iL}(\tau,\bar{\xi},\bar{c})+\bar{\lambda}^{iU}\pi_{\dot{\xi}}^{iU}(\tau,\bar{\xi},\bar{c})+\bar{\nu}(\tau)p_{\dot{\xi}}(\tau,\bar{\xi},\bar{c})-\bar{\mu}(\tau)\right),$$

$$\bar{\lambda}^{iL}\pi_c^{iL}(\tau,\bar{\xi},\bar{c})+\bar{\lambda}^{iU}\pi_c^{iU}(\tau,\bar{\xi},\bar{c})+\bar{\nu}(\tau)p_c(\tau,\bar{\xi},\bar{c})+\bar{\mu}(\tau)z_c(\tau,\bar{\xi},\bar{c}))d\tau.$$

Now, by using the necessary efficiency conditions given in Theorem 2.1 and by considering the sublinearity associated with $T$, it results in

$$\int_{a_1}^{a_2}\left(\bar{\lambda}^{iL}\pi^{iL}(\tau,\xi,c)+\bar{\lambda}^{iU}\pi^{iU}(\tau,\xi,c)+\bar{\nu}(\tau)p(\tau,\xi,c)\right.$$
$$\left.+\bar{\mu}(\tau)\left[z(\tau,\xi,c)-\dot{\xi}\right]\right)d\tau$$
$$\geq \int_{a_1}^{a_2}\left(\bar{\lambda}^{iL}\pi^{iL}(\tau,\bar{\xi},\bar{c})+\bar{\lambda}^{iU}\pi^{iU}(\tau,\bar{\xi},\bar{c})+\bar{\nu}(\tau)p(\tau,\bar{\xi},\bar{c})\right.$$
$$\left.+\bar{\mu}(\tau)\left[z(\tau,\bar{\xi},\bar{c})-\dot{\bar{\xi}}\right]\right)d\tau,$$

equivalent with

$$\mathcal{K}(\xi,c,\bar{\lambda},\bar{\nu},\bar{\mu})\geq\mathcal{K}(\bar{\xi},\bar{c},\bar{\lambda},\bar{\nu},\bar{\mu}),\ \forall(\xi,c)\in F. \tag{3.5}$$

Further, from Eq (2.4) and the feasibility property of $(\bar{\xi},\bar{c})$, it follows that

$$\int_{a_1}^{a_2}\nu(\tau)p(\tau,\bar{\xi},\bar{c})d\tau\leq 0,$$

and, by using Eq (2.3), the inequality given above yields

$$\int_{a_1}^{a_2}\nu(\tau)p(\tau,\bar{\xi},\bar{c})d\tau\leq\int_{a_1}^{a_2}\bar{\nu}(\tau)p(\tau,\bar{\xi},\bar{c})d\tau.$$

Adding the term $\int_{a_1}^{a_2}[\bar{\lambda}^{iL}\pi^{iL}(\tau,\bar{\xi},\bar{c})+\bar{\lambda}^{iU}\pi^{iU}(\tau,\bar{\xi},\bar{c})+\mu(\tau)\left(z(\tau,\bar{\xi},\bar{c})-\dot{\bar{\xi}}\right)]d\tau$ on both sides of the above inequality, we get

$$\int_{a_1}^{a_2}\left(\bar{\lambda}^{iL}\pi^{iL}(\tau,\bar{\xi},\bar{c})+\bar{\lambda}^{iU}\pi^{iU}(\tau,\bar{\xi},\bar{c})+\nu(\tau)p(\tau,\bar{\xi},\bar{c})\right.$$
$$\left.+\mu(\tau)\left(z(\tau,\bar{\xi},\bar{c})-\dot{\bar{\xi}}\right)\right)d\tau$$
$$\leq\int_{a_1}^{a_2}\left(\bar{\lambda}^{iL}\pi^{iL}(\tau,\bar{\xi},\bar{c})+\bar{\lambda}^{iU}\pi^{iU}(\tau,\bar{\xi},\bar{c})+\bar{\nu}(\tau)p(\tau,\bar{\xi},\bar{c})\right.$$
$$\left.+\bar{\mu}(\tau)\left(z(\tau,\bar{\xi},\bar{c})-\dot{\bar{\xi}}\right)\right)d\tau,$$

which again, by considering the definition of the Lagrange functional, we get

$$\mathcal{K}(\bar{\xi},\bar{c},\bar{\lambda},\nu,\mu)\leq\mathcal{K}(\bar{\xi},\bar{c},\bar{\lambda},\bar{\nu},\bar{\mu}),\quad\forall(\nu,\mu)\in\mathbb{R}_+^q\times\mathbb{R}^n. \tag{3.6}$$

Therefore, the relations Eqs (3.5) and (3.6) involve that $(\bar{\xi},\bar{c},\bar{\lambda},\bar{\nu},\bar{\mu})$ is an $LU$-saddle-point of the Lagrange functional associated with $(P)$. □

## 4. Sufficient efficiency conditions for $(P)$ via modified $T$-objective functional approach

In this section, we formulate sufficient efficiency conditions for $(P)$ via the modified $T$-objective functional approach. In this regard, consider $(\bar{\xi},\bar{c})$ is a feasible solution (arbitrary given) of $(P)$. The multi-objective control model $\left(\mathrm{P}_T(\bar{\xi},\bar{c})\right)$ corresponding to $(P)$, with the modified $T$-objective functional, is defined as follows:

$$\left(\mathrm{P}_T(\bar{\xi},\bar{c})\right)\quad\min_{(\xi,c)}\left(H^1(\xi,c),\ldots,H^r(\xi,c)\right)$$

subject to

$$p(\tau,\xi,c)\leqq 0,\ \tau\in A,$$
$$z(\tau,\xi,c)=\dot{\xi},\ \tau\in A,$$
$$\xi(a_1)=\sigma,\ \xi(a_2)=\omega,$$

where

$$H^i(\xi,c)$$
$$:=\Big[\int_{a_1}^{a_2}T\left(\tau,\xi,\bar{\xi};\pi_{\xi}^{iL}(\tau,\bar{\xi},\bar{c})-D\pi_{\dot{\xi}}^{iL}(\tau,\bar{\xi},\bar{c}),\pi_c^{iL}(\tau,\bar{\xi},\bar{c})\right)d\tau,$$
$$\int_{a_1}^{a_2}T\left(\tau,\xi,\bar{\xi};\pi_{\xi}^{iU}(\tau,\bar{\xi},\bar{c})-D\pi_{\dot{\xi}}^{iU}(\tau,\bar{\xi},\bar{c}),\pi_c^{iU}(\tau,\bar{\xi},\bar{c})\right)d\tau\Big],$$

with $T,\pi^{iL},\pi^{iU},p,z$ considered as the same functionals given in Section 2.

**Remark 4.1.** We notice the feasible solution set of $\left(\mathrm{P}_T(\bar{\xi},\bar{c})\right)$ matches that of $(P)$.

**Definition 4.1.** The pair $(\bar{\xi}, \bar{c}) \in F$ is named the *LU-efficient solution of* $\left(\mathrm{P}_T(\bar{\xi}, \bar{c})\right)$ if there exists no $(b, v) \in F$ satisfying

$$H^i(b, v) \preceq_{LU} H^i(\bar{\xi}, \bar{c}), \forall i \in \Gamma_r,$$
$$H^{i^*}(b, v) \prec_{LU} H^{i^*}(\bar{\xi}, \bar{c}), \text{ for at least one } i^* \in \Gamma_r.$$

Next, by *LU*-efficiency and weaker hypotheses, we present the equivalence between $(P)$ and $\left(\mathrm{P}_T(\bar{\xi}, \bar{c})\right)$.

**Theorem 4.1.** *Consider that* $(\bar{\xi}, \bar{c})$ *is an LU-efficient solution of* $\left(\mathrm{P}_T(\bar{\xi}, \bar{c})\right)$ *and*

$$\int_{a_1}^{a_2} \left(\bar{\lambda}^L \pi^L(\tau, \xi, c) + \bar{\lambda}^U \pi^U(\tau, \xi, c)\right) d\tau$$

*is strictly T-pseudoconvex at* $(\bar{\xi}, \bar{c}) \in F$. *Then,* $(\bar{\xi}, \bar{c})$ *is also an LU-efficient solution to* $(P)$.

*Proof.* Contrary to the result, we consider that $(\bar{\xi}, \bar{c})$ is not an *LU*-efficient solution to $(P)$. Then, the inequalities

$$\left[\int_{a_1}^{a_2} \pi^{iL}(\tau, b, v) d\tau, \int_{a_1}^{a_2} \pi^{iU}(\tau, b, v) d\tau\right]$$
$$\preceq_{LU} \left[\int_{a_1}^{a_2} \pi^{iL}(\tau, \bar{\xi}, \bar{c}) d\tau, \int_{a_1}^{a_2} \pi^{iU}(\tau, \bar{\xi}, \bar{c}) d\tau\right], i \in \Gamma_r,$$
$$\left[\int_{a_1}^{a_2} \pi^{i^*L}(\tau, b, v) d\tau, \int_{a_1}^{a_2} \pi^{i^*U}(\tau, b, v) d\tau\right]$$
$$\prec_{LU} \left[\int_{a_1}^{a_2} \pi^{i^*L}(\tau, \bar{\xi}, \bar{c}) d\tau, \int_{a_1}^{a_2} \pi^{i^*U}(\tau, \bar{\xi}, \bar{c}) d\tau\right],$$

for at least one $i^* \in \Gamma_r$ hold.

By considering the partial order relation in the second section, the above relations imply

$$\int_{a_1}^{a_2} \pi^{iL}(\tau, b, v) d\tau \leq \int_{a_1}^{a_2} \pi^{iL}(\tau, \bar{\xi}, \bar{c}) d\tau,$$
$$\int_{a_1}^{a_2} \pi^{iU}(\tau, b, v) d\tau \leq \int_{a_1}^{a_2} \pi^{iU}(\tau, \bar{\xi}, \bar{c}) d\tau, \ i \in \Gamma_r,$$

and

$$\begin{cases} \int_{a_1}^{a_2} \pi^{i^*L}(\tau, b, v) d\tau < \int_{a_1}^{a_2} \pi^{i^*L}(\tau, \bar{\xi}, \bar{c}) d\tau, \\ \int_{a_1}^{a_2} \pi^{i^*U}(\tau, b, v) d\tau < \int_{a_1}^{a_2} \pi^{i^*U}(\tau, \bar{\xi}, \bar{c}) d\tau, \end{cases}$$

$$\begin{cases} \int_{a_1}^{a_2} \pi^{i^*L}(\tau, b, v) d\tau \leq \int_{a_1}^{a_2} \pi^{i^*L}(\tau, \bar{\xi}, \bar{c}) d\tau, \\ \int_{a_1}^{a_2} \pi^{i^*U}(\tau, b, v) d\tau < \int_{a_1}^{a_2} \pi^{i^*U}(\tau, \bar{\xi}, \bar{c}) d\tau, \end{cases}$$

$$\begin{cases} \int_{a_1}^{a_2} \pi^{i^*L}(\tau, b, v) d\tau < \int_{a_1}^{a_2} \pi^{i^*L}(\tau, \bar{\xi}, \bar{c}) d\tau, \\ \int_{a_1}^{a_2} \pi^{i^*U}(\tau, b, v) d\tau \leq \int_{a_1}^{a_2} \pi^{i^*U}(\tau, \bar{\xi}, \bar{c}) d\tau, \end{cases}$$

for at least one $i^* \in \Gamma_r$.

Multiplying the inequalities given above by $\bar{\lambda}^{iL}, \bar{\lambda}^{iU}, \forall i \in \Gamma_r$, and $\bar{\lambda}^{i^*L}, \bar{\lambda}^{i^*U}$ and adding them, for at least one $i^* \in \Gamma_r$, we get

$$\int_{a_1}^{a_2} \left(\bar{\lambda}^{iL} \pi^{iL}(\tau, b, v) + \bar{\lambda}^{iU} \pi^{iU}(\tau, b, v)\right) d\tau$$
$$\leq \int_{a_1}^{a_2} \left(\bar{\lambda}^{iL} \pi^{iL}(\tau, \bar{\xi}, \bar{c}) + \bar{\lambda}^{iU} \pi^{iU}(\tau, \bar{\xi}, \bar{c})\right) d\tau, \ \forall i \in \Gamma_r,$$
$$\int_{a_1}^{a_2} \left(\bar{\lambda}^{i^*L} \pi^{i^*L}(\tau, b, v) + \bar{\lambda}^{i^*U} \pi^{i^*U}(\tau, b, v)\right) d\tau$$
$$< \int_{a_1}^{a_2} \left(\bar{\lambda}^{i^*L} \pi^{i^*L}(\tau, \bar{\xi}, \bar{c}) + \bar{\lambda}^{i^*U} \pi^{i^*U}(\tau, \bar{\xi}, \bar{c})\right) d\tau,$$

for at least one $i^* \in \Gamma_r$.

Since $\int_{a_1}^{a_2} \left(\bar{\lambda}^L \pi^L(\tau, \xi, c) + \bar{\lambda}^U \pi^U(\tau, \xi, c)\right) d\tau$ is strictly $T$-pseudoconvex at $(\bar{\xi}, \bar{c})$, therefore, the above inequalities involve

$$\int_{a_1}^{a_2} T(\tau, b, \bar{\xi}; \left(\bar{\lambda}^{iL} \pi_{\xi}^{iL}(\tau, \bar{\xi}, \bar{c}) + \bar{\lambda}^{iU} \pi_{\xi}^{iU}(\tau, \bar{\xi}, \bar{c})\right.$$
$$-D\left(\bar{\lambda}^{iL} \pi_{\dot{\xi}}^{iL}(\tau, \bar{\xi}, \bar{c}) + \bar{\lambda}^{iU} \pi_{\dot{\xi}}^{iU}(\tau, \bar{\xi}, \bar{c})\right),$$
$$\left.\bar{\lambda}^{iL} \pi_c^{iL}(\tau, \bar{\xi}, \bar{c}) + \bar{\lambda}^{iU} \pi_c^{iU}(\tau, \bar{\xi}, \bar{c})\right) d\tau \leq 0, \forall i \in \Gamma_r,$$
$$\int_{a_1}^{a_2} T(\tau, b, \bar{\xi}; \left(\bar{\lambda}^{i^*L} \pi_{\xi}^{i^*L}(\tau, \bar{\xi}, \bar{c}) + \bar{\lambda}^{i^*U} \pi_{\xi}^{i^*U}(\tau, \bar{\xi}, \bar{c})\right.$$
$$-D\left(\bar{\lambda}^{i^*L} \pi_{\dot{\xi}}^{i^*L}(\tau, \bar{\xi}, \bar{c}) + \bar{\lambda}^{i^*U} \pi_{\dot{\xi}}^{i^*U}(\tau, \bar{\xi}, \bar{c})\right),$$
$$\left.\bar{\lambda}^{i^*L} \pi_c^{i^*L}(\tau, \bar{\xi}, \bar{c}) + \bar{\lambda}^{i^*U} \pi_c^{i^*U}(\tau, \bar{\xi}, \bar{c})\right) d\tau < 0,$$

for at least one $i^* \in \Gamma_r$. By using Remark 2.1, the above inequalities can be written

$$\int_{a_1}^{a_2} T(\tau, b, \bar{\xi}; \left(\bar{\lambda}^{iL} \pi_{\xi}^{iL}(\tau, \bar{\xi}, \bar{c}) + \bar{\lambda}^{iU} \pi_{\xi}^{iU}(\tau, \bar{\xi}, \bar{c})\right.$$
$$-D\left(\bar{\lambda}^{iL} \pi_{\dot{\xi}}^{iL}(\tau, \bar{\xi}, \bar{c}) + \bar{\lambda}^{iU} \pi_{\dot{\xi}}^{iU}(\tau, \bar{\xi}, \bar{c})\right),$$
$$\left.\bar{\lambda}^{iL} \pi_c^{iL}(\tau, \bar{\xi}, \bar{c}) + \bar{\lambda}^{iU} \pi_c^{iU}(\tau, \bar{\xi}, \bar{c})\right) d\tau$$
$$\leq \int_{a_1}^{a_2} T(\tau, \bar{\xi}, \bar{\xi}; \left(\bar{\lambda}^{iL} \pi_{\xi}^{iL}(\tau, \bar{\xi}, \bar{c}) + \bar{\lambda}^{iU} \pi_{\xi}^{iU}(\tau, \bar{\xi}, \bar{c})\right.$$
$$-D\left(\bar{\lambda}^{iL} \pi_{\dot{\xi}}^{iL}(\tau, \bar{\xi}, \bar{c}) + \bar{\lambda}^{iU} \pi_{\dot{\xi}}^{iU}(\tau, \bar{\xi}, \bar{c})\right),$$
$$\left.\bar{\lambda}^{iL} \pi_c^{iL}(\tau, \bar{\xi}, \bar{c}) + \bar{\lambda}^{iU} \pi_c^{iU}(\tau, \bar{\xi}, \bar{c})\right) d\tau, \forall i \in \Gamma_r,$$
$$\int_{a_1}^{a_2} T(\tau, b, \bar{\xi}; \left(\bar{\lambda}^{i^*L} \pi_{\xi}^{i^*L}(\tau, \bar{\xi}, \bar{c}) + \bar{\lambda}^{i^*U} \pi_{\xi}^{i^*U}(\tau, \bar{\xi}, \bar{c})\right.$$

$$-D\Big(\bar{\lambda}^{i^*L}\pi_{\dot{\xi}}^{i^*L}(\tau,\bar{\xi},\bar{c})+\bar{\lambda}^{i^*U}\pi_{\dot{\xi}}^{i^*U}(\tau,\bar{\xi},\bar{c})\Big),$$

$$\bar{\lambda}^{i^*L}\pi_c^{i^*L}(\tau,\bar{\xi},\bar{c})+\bar{\lambda}^{i^*U}\pi_c^{i^*U}(\tau,\bar{\xi},\bar{c})\Big)d\tau,$$

$$<\int_{a_1}^{a_2} T(\tau,\bar{\xi},\bar{\xi};\big(\bar{\lambda}^{i^*L}\pi_{\xi}^{i^*L}(\tau,\bar{\xi},\bar{c})+\bar{\lambda}^{i^*U}\pi_{\xi}^{i^*U}(\tau,\bar{\xi},\bar{c})$$

$$-D\Big(\bar{\lambda}^{i^*L}\pi_{\dot{\xi}}^{i^*L}(\tau,\bar{\xi},\bar{c})+\bar{\lambda}^{i^*U}\pi_{\dot{\xi}}^{i^*U}(\tau,\bar{\xi},\bar{c})\Big),$$

$$\bar{\lambda}^{i^*L}\pi_c^{i^*L}(\tau,\bar{\xi},\bar{c})+\bar{\lambda}^{i^*U}\pi_c^{i^*U}(\tau,\bar{\xi},\bar{c})\Big)d\tau,$$

for at least one $i^* \in \Gamma_r$, involving that $(b,v)$ is an *LU*-efficient solution for $\big(\mathrm{P}_T(\bar{\xi},\bar{c})\big)$. We obtain a contradiction, and this completes the proof. □

Next, we formulate and prove the reciprocal of Theorem 4.1.

**Theorem 4.2.** *Assume that $(\bar{\xi},\bar{c})$ is an LU-efficient solution for (P) at which Eqs (2.1)–(2.4) in Theorem 2.1 are fulfilled. Then, $(\bar{\xi},\bar{c})$ is an LU-efficient solution for $\big(\mathrm{P}_T(\bar{\xi},\bar{c})\big)$ if the functionals $\int_{a_1}^{a_2}\bar{\nu}(\tau)p(\tau,\xi,c)d\tau$ and $\int_{a_1}^{a_2}\bar{\mu}(\tau)[z(\tau,\xi,c)-\dot{\xi}]d\tau$ are T-convex at $(\bar{\xi},\bar{c})\in F$.*

*Proof.* Consider $(\bar{\xi},\bar{c})$ satisfies Eqs (2.1)–(2.4) in Theorem 2.1, and, on the contrary, we suppose $(\bar{\xi},\bar{c})$ is not an *LU*-efficient solution for $\big(\mathrm{P}_T(\bar{\xi},\bar{c})\big)$. Then, we have $(b,v)\in F$ satisfying

$$\Big[\int_{a_1}^{a_2} T\Big(\tau,b,\bar{\xi};\pi_{\xi}^{iL}(\tau,\bar{\xi},\bar{c})-D\pi_{\dot{\xi}}^{iL}(\tau,\bar{\xi},\bar{c}),\pi_c^{iL}(\tau,\bar{\xi},\bar{c})\Big)d\tau,$$

$$\int_{a_1}^{a_2} T\Big(\tau,b,\bar{\xi};\pi_{\xi}^{iU}(\tau,\bar{\xi},\bar{c})-D\pi_{\dot{\xi}}^{iU}(\tau,\bar{\xi},\bar{c}),\pi_c^{iU}(\tau,\bar{\xi},\bar{c})\Big)d\tau\Big]$$

$$\preceq_{LU}\Big[\int_{a_1}^{a_2} T\Big(\tau,\bar{\xi},\bar{\xi};\pi_{\xi}^{iL}(\tau,\bar{\xi},\bar{c})-D\pi_{\dot{\xi}}^{iL}(\tau,\bar{\xi},\bar{c}),\pi_c^{iL}(\tau,\bar{\xi},\bar{c})\Big)d\tau,$$

$$\int_{a_1}^{a_2} T\Big(\tau,\bar{\xi},\bar{\xi};\pi_{\xi}^{iU}(\tau,\bar{\xi},\bar{c})-D\pi_{\dot{\xi}}^{iU}(\tau,\bar{\xi},\bar{c}),,\pi_c^{iU}(\tau,\bar{\xi},\bar{c})\Big)d\tau\Big],\forall i\in\Gamma_r,$$

$$\Big[\int_{a_1}^{a_2} T\Big(\tau,b,\bar{\xi};\pi_{\xi}^{i^*L}(\tau,\bar{\xi},\bar{c})-D\pi_{\dot{\xi}}^{i^*L}(\tau,\bar{\xi},\bar{c}),\pi_c^{i^*L}(\tau,\bar{\xi},\bar{c})\Big)d\tau,$$

$$\int_{a_1}^{a_2} T\Big(\tau,b,\bar{\xi};\pi_{\xi}^{i^*U}(\tau,\bar{\xi},\bar{c})-D\pi_{\dot{\xi}}^{i^*U}(\tau,\bar{\xi},\bar{c}),\pi_c^{i^*U}(\tau,\bar{\xi},\bar{c})\Big)d\tau\Big]$$

$$\prec_{LU}\Big[\int_{a_1}^{a_2} T\Big(\tau,\bar{\xi},\bar{\xi};\pi_{\xi}^{i^*L}(\tau,\bar{\xi},\bar{c})-D\pi_{\dot{\xi}}^{i^*L}(\tau,\bar{\xi},\bar{c}),\pi_c^{i^*L}(\tau,\bar{\xi},\bar{c})\Big)d\tau,$$

$$\int_{a_1}^{a_2} T\Big(\tau,\bar{\xi},\bar{\xi};\pi_{\xi}^{i^*U}(\tau,\bar{\xi},\bar{c})-D\pi_{\dot{\xi}}^{i^*U}(\tau,\bar{\xi},\bar{c}),,\pi_c^{i^*U}(\tau,\bar{\xi},\bar{c})\Big)d\tau\Big],$$

for some $i^*\in\Gamma_r$.

Using Remark 2.1 in the inequality given above, we obtain

$$\Big[\int_{a_1}^{a_2} T\Big(\tau,b,\bar{\xi};\pi_{\xi}^{iL}(\tau,\bar{\xi},\bar{c})-D\pi_{\dot{\xi}}^{iL}(\tau,\bar{\xi},\bar{c}),\pi_c^{iL}(\tau,\bar{\xi},\bar{c})\Big)d\tau,$$

$$\int_{a_1}^{a_2} T\Big(\tau,b,\bar{\xi};\pi_{\xi}^{iU}(\tau,\bar{\xi},\bar{c})-D\pi_{\dot{\xi}}^{iU}(\tau,\bar{\xi},\bar{c}),\pi_c^{iU}(\tau,\bar{\xi},\bar{c})\Big)d\tau\Big]$$

$$\preceq_{LU}[0,0],\ \forall i\in\Gamma_r,$$

$$\Big[\int_{a_1}^{a_2} T\Big(\tau,b,\bar{\xi};\pi_{\xi}^{i^*L}(\tau,\bar{\xi},\bar{c})-D\pi_{\dot{\xi}}^{i^*L}(\tau,\bar{\xi},\bar{c}),\pi_c^{i^*L}(\tau,\bar{\xi},\bar{c})\Big)d\tau,$$

$$\int_{a_1}^{a_2} T\Big(\tau,b,\bar{\xi};\pi_{\xi}^{i^*U}(\tau,\bar{\xi},\bar{c})-D\pi_{\dot{\xi}}^{i^*U}(\tau,\bar{\xi},\bar{c}),\pi_c^{i^*U}(\tau,\bar{\xi},\bar{c})\Big)d\tau\Big]$$

$$\prec_{LU}[0,0],$$

for some $i^*\in\Gamma_r$. Multiplying the above inequality by $\bar{\lambda}^{iL},\bar{\lambda}^{iU}\geq 0$ and adding them, we get

$$\int_{a_1}^{a_2} T\Big(\tau,b,\bar{\xi};\bar{\lambda}^{iL}\pi_{\xi}^{iL}(\tau,\bar{\xi},\bar{c})+\bar{\lambda}^{iU}\pi_{\xi}^{iU}(\tau,\bar{\xi},\bar{c})$$

$$-D\Big(\bar{\lambda}^{iL}\pi_{\dot{\xi}}^{iL}(\tau,\bar{\xi},\bar{c})+\bar{\lambda}^{iU}\pi_{\dot{\xi}}^{iU}(\tau,\bar{\xi},\bar{c})\Big),$$

$$\bar{\lambda}^{iL}\pi_c^{iL}(\tau,\bar{\xi},\bar{c})+\bar{\lambda}^{iU}\pi_c^{iL}(\tau,\bar{\xi},\bar{c})\Big)d\tau<0. \tag{4.1}$$

Applying that $\int_{a_1}^{a_2}\bar{\nu}(\tau)p(\tau,\xi,c)d\tau$ and $\int_{a_1}^{a_2}\bar{\mu}(\tau)[z(\tau,\xi,c)-\dot{\xi}]d\tau$ are $T$-convex at $(\bar{\xi},\bar{c})\in F$, therefore, by the definition, we have

$$\int_{a_1}^{a_2}\bar{\nu}(\tau)p(\tau,b,v)d\tau-\int_{a_1}^{a_2}\bar{\nu}(\tau)p(\tau,\bar{\xi},\bar{c})d\tau$$
$$\geq\int_{a_1}^{a_2} T(\tau,b,\bar{\xi};\bar{\nu}(\tau)p_{\xi}(\tau,\bar{\xi},\bar{c})-D\bar{\nu}(\tau)p_{\dot{\xi}}(\tau,\bar{\xi},\bar{c}),\bar{\nu}(\tau)p_c(\tau,\bar{\xi},\bar{c}))d\tau,$$
$$\int_{a_1}^{a_2}\bar{\mu}(\tau)\Big[z(\tau,b,v)-\dot{b}\Big]d\tau-\int_{a_1}^{a_2}\bar{\mu}(\tau)\Big[z(\tau,\bar{\xi},\bar{c})-\dot{\bar{\xi}}\Big]d\tau$$
$$\geq\int_{a_1}^{a_2} T(\tau,b,\bar{\xi};\bar{\mu}(\tau)z_{\xi}(\tau,\bar{\xi},\bar{c})+D\bar{\mu}(\tau),\bar{\mu}(\tau)z_c(\tau,\bar{\xi},\bar{c}))d\tau.$$

Since $(b,v)\in F$, and by necessary efficiency conditions given in Theorem 2.1, we obtain

$$\int_{a_1}^{a_2} T(\tau,b,\bar{\xi};\bar{\nu}(\tau)p_{\xi}(\tau,\bar{\xi},\bar{c})$$

$$-D\bar{\nu}(\tau)p_{\dot{\xi}}(\tau,\bar{\xi},\bar{c}),\bar{\nu}(\tau)p_c(\tau,\bar{\xi},\bar{c}))d\tau\leq 0, \tag{4.2}$$

$$\int_{a_1}^{a_2} T(\tau,b,\bar{\xi};\bar{\mu}(\tau)z_{\xi}(\tau,\bar{\xi},\bar{c})+D\bar{\mu}(\tau),\bar{\mu}(\tau)z_c(\tau,\bar{\xi},\bar{c}))d\tau\leq 0. \tag{4.3}$$

By considering Eqs (4.1)–(4.3), we get

$$\int_{a_1}^{a_2} T(\tau, b, \bar{\xi}; \bar{\lambda}^{iL}\pi_{\xi}^{iL}(\tau,\bar{\xi},\bar{c}) + \bar{\lambda}^{iU}\pi_{\xi}^{iU}(\tau,\bar{\xi},\bar{c}) + \bar{\nu}(\tau)p_{\xi}(\tau,\bar{\xi},\bar{c}) + \bar{\mu}(\tau)z_{\xi}(\tau,\bar{\xi},\bar{c})$$

$$-D\left(\bar{\lambda}^{iL}\pi_{\dot{\xi}}^{iL}(\tau,\bar{\xi},\bar{c}) + \bar{\lambda}^{iU}\pi_{\dot{\xi}}^{iU}(\tau,\bar{\xi},\bar{c}) + \bar{\nu}(\tau)p_{\dot{\xi}}(\tau,\bar{\xi},\bar{c}) - \bar{\mu}(\tau)\right),$$

$$\bar{\lambda}^{iL}\pi_c^{iL}(\tau,\bar{\xi},\bar{c}) + \bar{\lambda}^{iU}\pi_c^{iU}(\tau,\bar{\xi},\bar{c}) + \bar{\nu}(\tau)p_c(\tau,\bar{\xi},\bar{c}) + \bar{\mu}(\tau)z_c(\tau,\bar{\xi},\bar{c}))d\tau < 0,$$

which contradicts the necessary efficiency conditions and the sublinearity property of $T$. □

## 5. $LU$-saddle-point criteria for $\left(\mathrm{P}_T(\bar{\xi},\bar{c})\right)$

In this part of our study, under suitable generalized convexity assumptions, we establish the connection between an $LU$-efficient solution for $(P)$ and a saddle-point of the Lagrange functional associated to $\left(\mathrm{P}_T(\bar{\xi},\bar{c})\right)$. Consequently, by keeping in mind the notion of the Lagrange function associated with variational problems, we introduce the notions of *Lagrange functional* and *LU-T-saddle-point* for $\left(\mathrm{P}_T(\bar{\xi},\bar{c})\right)$.

**Definition 5.1.** A *Lagrange functional associated with* $\left(\mathrm{P}_T(\bar{\xi},\bar{c})\right)$, denoted by $\mathcal{K}_T(\xi, c, \bar{\lambda}, \nu, \mu)$, is defined as follows:

$$\mathcal{K}_T(\xi, c, \bar{\lambda}, \nu, \mu) = \int_{a_1}^{a_2} T\Big(\tau, \xi, \bar{\xi}; \bar{\lambda}^{iL}\pi_{\xi}^{iL}(\tau,\bar{\xi},\bar{c}) + \bar{\lambda}^{iU}\pi_{\xi}^{iU}(\tau,\bar{\xi},\bar{c})$$

$$-D\left(\bar{\lambda}^{iL}\pi_{\dot{\xi}}^{iL}(\tau,\bar{\xi},\bar{c}) + \bar{\lambda}^{iU}\pi_{\dot{\xi}}^{iU}(\tau,\bar{\xi},\bar{c})\right), \bar{\lambda}^{iL}\pi_c^{iL}(\tau,\bar{\xi},\bar{c}) + \bar{\lambda}^{iU}\pi_c^{iU}(\tau,\bar{\xi},\bar{c})\Big)d\tau$$

$$+\int_{a_1}^{a_2} \nu(\tau)p(\tau,\xi,c)d\tau + \int_{a_1}^{a_2} \mu(\tau)[z(\tau,\xi,c) - \dot{\xi}]d\tau.$$

**Definition 5.2.** We say $(\bar{\xi}, \bar{c}, \bar{\lambda}, \bar{\nu}, \bar{\mu})$ is named the *LU-T-saddle-point* of the Lagrange functional associated to $\left(\mathrm{P}_T(\bar{\xi},\bar{c})\right)$ if:

(i) $\mathcal{K}_T(\bar{\xi}, \bar{c}, \bar{\lambda}, \nu, \mu) \le \mathcal{K}_T(\bar{\xi}, \bar{c}, \bar{\lambda}, \bar{\nu}, \bar{\mu}), \quad \forall \nu \in \mathbb{R}_+^q,\ \mu \in \mathbb{R}^n$,

(ii) $\mathcal{K}_T(\bar{\xi}, \bar{c}, \bar{\lambda}, \bar{\nu}, \bar{\mu}) \le \mathcal{K}_T(\xi, c, \bar{\lambda}, \bar{\nu}, \bar{\mu}), \quad \forall (\xi, c) \in F$

hold.

The next result, based on the above new concepts, introduces new sufficient efficiency conditions for a feasible point to become an $LU$-efficient solution for $(P)$.

**Theorem 5.1.** *Consider $(\bar{\xi}, \bar{c})$ is a feasible solution and the functional*

$$\int_{a_1}^{a_2} \left(\bar{\lambda}^{iL}\pi^{iL}(\tau,\xi,c) + \bar{\lambda}^{iU}\pi^{iU}(\tau,\xi,c)\right)d\tau, \quad i \in \Gamma_r$$

*is strictly $T$-pseudoconvex at $(\bar{\xi}, \bar{c})$. The point $(\bar{\xi}, \bar{c})$ is an LU-efficient solution for $(P)$ if $(\bar{\xi}, \bar{c}, \bar{\lambda}, \bar{\nu}, \bar{\mu})$ is an LU-T-saddle-point of the Lagrangian associated to $\left(\mathrm{P}_T(\bar{\xi},\bar{c})\right)$.*

*Proof.* The point $(\bar{\xi}, \bar{c}, \bar{\lambda}, \bar{\nu}, \bar{\mu})$ being an $LU$-T-saddle point in $\left(\mathrm{P}_T(\bar{\xi},\bar{c})\right)$, we have

$$\mathcal{K}_T(\bar{\xi}, \bar{c}, \bar{\lambda}, \nu, \mu) \le \mathcal{K}_T(\bar{\xi}, \bar{c}, \bar{\lambda}, \bar{\nu}, \bar{\mu}), \quad \forall \nu \in \mathbb{R}_+^q, \mu \in \mathbb{R}^n,$$

equivalent with

$$\int_{a_1}^{a_2} T\Big(\tau, \xi, \bar{\xi}; \bar{\lambda}^{iL}\pi_{\xi}^{iL}(\tau,\bar{\xi},\bar{c}) + \bar{\lambda}^{iU}\pi_{\xi}^{iU}(\tau,\bar{\xi},\bar{c})$$

$$-D\left(\bar{\lambda}^{iL}\pi_{\dot{\xi}}^{iL}(\tau,\bar{\xi},\bar{c}) + \bar{\lambda}^{iU}\pi_{\dot{\xi}}^{iU}(\tau,\bar{\xi},\bar{c})\right),$$

$$\bar{\lambda}^{iL}\pi_c^{iL}(\tau,\bar{\xi},\bar{c}) + \bar{\lambda}^{iU}\pi_c^{iU}(\tau,\bar{\xi},\bar{c})\Big)d\tau$$

$$+\int_{a_1}^{a_2} \nu(\tau)p(\tau,\bar{\xi},\bar{c})d\tau + \int_{a_1}^{a_2} \mu(\tau)[z(\tau,\bar{\xi},\bar{c}) - \dot{\bar{\xi}}]d\tau$$

$$\le \int_{a_1}^{a_2} T\Big(\tau, \xi, \bar{\xi}; \bar{\lambda}^{iL}\pi_{\xi}^{iL}(\tau,\bar{\xi},\bar{c}) + \bar{\lambda}^{iU}\pi_{\xi}^{iU}(\tau,\bar{\xi},\bar{c})$$

$$-D\left(\bar{\lambda}^{iL}\pi_{\dot{\xi}}^{iL}(\tau,\bar{\xi},\bar{c}) + \bar{\lambda}^{iU}\pi_{\dot{\xi}}^{iU}(\tau,\bar{\xi},\bar{c})\right),$$

$$\bar{\lambda}^{iL}\pi_c^{iL}(\tau,\bar{\xi},\bar{c}) + \bar{\lambda}^{iU}\pi_c^{iU}(\tau,\bar{\xi},\bar{c})\Big)d\tau$$

$$+\int_{a_1}^{a_2} \bar{\nu}(\tau)p(\tau,\bar{\xi},\bar{c})d\tau + \int_{a_1}^{a_2} \bar{\mu}(\tau)[z(\tau,\bar{\xi},\bar{c}) - \dot{\bar{\xi}}]d\tau.$$

Since $(\bar{\xi}, \bar{c}) \in F$, we obtain

$$\int_{a_1}^{a_2} \bar{\nu}(\tau)p(\tau,\bar{\xi},\bar{c})d\tau = 0, \int_{a_1}^{a_2} \bar{\mu}(\tau)[z(\tau,\bar{\xi},\bar{c}) - \dot{\bar{\xi}}]d\tau = 0. \tag{5.1}$$

Contrary to the result, we assume that $(\bar{\xi}, \bar{c})$ is not an $LU$-efficient solution of $(P)$. Then, we get that

$$\left[\int_{a_1}^{a_2} \pi^{iL}(\tau,b,\nu)d\tau, \int_{a_1}^{a_2} \pi^{iU}(\tau,b,\nu)d\tau\right]$$

$$\preceq_{LU} \left[\int_{a_1}^{a_2} \pi^{iL}(\tau,\bar{\xi},\bar{c})d\tau, \int_{a_1}^{a_2} \pi^{iU}(\tau,\bar{\xi},\bar{c})d\tau\right], i \in \Gamma_r,$$

$$\left[\int_{a_1}^{a_2} \pi^{i^*L}(\tau,b,\nu)d\tau, \int_{a_1}^{a_2} \pi^{i^*U}(\tau,b,\nu)d\tau\right]$$

$$\prec_{LU} \left[\int_{a_1}^{a_2} \pi^{i^*L}(\tau,\bar{\xi},\bar{c})d\tau, \int_{a_1}^{a_2} \pi^{i^*U}(\tau,\bar{\xi},\bar{c})d\tau\right],$$

for at least one $i^* \in \Gamma_r$, holds. By considering the partial order relation given in the second section, the above relations imply $\int_{a_1}^{a_2}\pi^{iL}(\tau,b,v)d\tau \le \int_{a_1}^{a_2}\pi^{iL}(\tau,\bar{\xi},\bar{c})d\tau$, $\int_{a_1}^{a_2}\pi^{iU}(\tau,b,v)d\tau \le \int_{a_1}^{a_2}\pi^{iU}(\tau,\bar{\xi},\bar{c})d\tau$, $i\in\Gamma_r$, and

$$\begin{cases}\int_{a_1}^{a_2}\pi^{i^*L}(\tau,b,v)d\tau < \int_{a_1}^{a_2}\pi^{i^*L}(\tau,\bar{\xi},\bar{c})d\tau,\\ \int_{a_1}^{a_2}\pi^{i^*U}(\tau,b,v)d\tau < \int_{a_1}^{a_2}\pi^{i^*U}(\tau,\bar{\xi},\bar{c})d\tau,\end{cases}$$

$$\begin{cases}\int_{a_1}^{a_2}\pi^{i^*L}(\tau,b,v)d\tau \le \int_{a_1}^{a_2}\pi^{i^*L}(\tau,\bar{\xi},\bar{c})d\tau,\\ \int_{a_1}^{a_2}\pi^{i^*U}(\tau,b,v)d\tau < \int_{a_1}^{a_2}\pi^{i^*U}(\tau,\bar{\xi},\bar{c})d\tau,\end{cases}$$

$$\begin{cases}\int_{a_1}^{a_2}\pi^{i^*L}(\tau,b,v)d\tau < \int_{a_1}^{a_2}\pi^{i^*L}(\tau,\bar{\xi},\bar{c})d\tau,\\ \int_{a_1}^{a_2}\pi^{i^*U}(\tau,b,v)d\tau \le \int_{a_1}^{a_2}\pi^{i^*U}(\tau,\bar{\xi},\bar{c})d\tau,\end{cases}$$

for at least one $i^*\in\Gamma_r$.

Multiplying the above inequalities by $\bar{\lambda}^{iL},\bar{\lambda}^{iU},\forall i\in\Gamma_r$, and $\bar{\lambda}^{i^*L},\bar{\lambda}^{i^*U}$, and adding them, for at least one $i^*\in\Gamma_r$, we get

$$\int_{a_1}^{a_2}\left(\bar{\lambda}^{iL}\pi^{iL}(\tau,b,v)+\bar{\lambda}^{iU}\pi^{iU}(\tau,b,v)\right)d\tau$$
$$\le \int_{a_1}^{a_2}\left(\bar{\lambda}^{iL}\pi^{iL}(\tau,\bar{\xi},\bar{c})+\bar{\lambda}^{iU}\pi^{iU}(\tau,\bar{\xi},\bar{c})\right)d\tau,\ \forall i\in\Gamma_r,$$
$$\int_{a_1}^{a_2}\left(\bar{\lambda}^{i^*L}\pi^{i^*L}(\tau,b,v)+\bar{\lambda}^{i^*U}\pi^{i^*U}(\tau,b,v)\right)d\tau$$
$$< \int_{a_1}^{a_2}\left(\bar{\lambda}^{i^*L}\pi^{i^*L}(\tau,\bar{\xi},\bar{c})+\bar{\lambda}^{i^*U}\pi^{i^*U}(\tau,\bar{\xi},\bar{c})\right)d\tau,$$

for at least one $i^*\in\Gamma_r$.

Since $\int_{a_1}^{a_2}\left(\bar{\lambda}^L\pi^L(\tau,\xi,c)+\bar{\lambda}^U\pi^U(\tau,\xi,c)\right)d\tau$ is strictly $T$-pseudoconvex at $(\bar{\xi},\bar{c})$, therefore, the above inequalities involve

$$\int_{a_1}^{a_2}T\Big(\tau,b,\bar{\xi};\bar{\lambda}^{iL}\pi^{iL}_{\xi}(\tau,\bar{\xi},\bar{c})+\bar{\lambda}^{iU}\pi^{iU}_{\xi}(\tau,\bar{\xi},\bar{c})$$
$$-D\left(\bar{\lambda}^{iL}\pi^{iL}_{\dot{\xi}}(\tau,\bar{\xi},\bar{c})+\bar{\lambda}^{iU}\pi^{iU}_{\dot{\xi}}(\tau,\bar{\xi},\bar{c})\right),$$
$$\bar{\lambda}^{iL}\pi^{iL}_{c}(\tau,\bar{\xi},\bar{c})+\bar{\lambda}^{iU}\pi^{iU}_{c}(\tau,\bar{\xi},\bar{c})\Big)d\tau<0,\forall i\in\Gamma_r.$$

By using Remark 2.1, according to Eq (5.1), the above inequalities become

$$\int_{a_1}^{a_2}T\Big(\tau,b,\bar{\xi};\bar{\lambda}^{iL}\pi^{iL}_{\xi}(\tau,\bar{\xi},\bar{c})+\bar{\lambda}^{iU}\pi^{iU}_{\xi}(\tau,\bar{\xi},\bar{c})-D\left(\bar{\lambda}^{iL}\pi^{iL}_{\dot{\xi}}(\tau,\bar{\xi},\bar{c})+\bar{\lambda}^{iU}\pi^{iU}_{\dot{\xi}}(\tau,\bar{\xi},\bar{c})\right),$$
$$\bar{\lambda}^{iL}\pi^{iL}_{c}(\tau,\bar{\xi},\bar{c})+\bar{\lambda}^{iU}\pi^{iU}_{c}(\tau,\bar{\xi},\bar{c})\Big)d\tau+\int_{a_1}^{a_2}\bar{\nu}(\tau)p(\tau,b,v)d\tau+\int_{a_1}^{a_2}\bar{\mu}(\tau)[z(\tau,b,v)-\dot{b}]d\tau$$
$$<\int_{a_1}^{a_2}T\Big(\tau,\bar{\xi},\bar{\xi};\bar{\lambda}^{iL}\pi^{iL}_{\xi}(\tau,\bar{\xi},\bar{c})+\bar{\lambda}^{iU}\pi^{iU}_{\xi}(\tau,\bar{\xi},\bar{c})-D\left(\bar{\lambda}^{iL}\pi^{iL}_{\dot{\xi}}(\tau,\bar{\xi},\bar{c})+\bar{\lambda}^{iU}\pi^{iU}_{\dot{\xi}}(\tau,\bar{\xi},\bar{c})\right),$$
$$\bar{\lambda}^{iL}\pi^{iL}_{c}(\tau,\bar{\xi},\bar{c})+\bar{\lambda}^{iU}\pi^{iU}_{c}(\tau,\bar{\xi},\bar{c})\Big)d\tau+\int_{a_1}^{a_2}\bar{\nu}(\tau)p(\tau,\bar{\xi},\bar{c})d\tau+\int_{a_1}^{a_2}\bar{\mu}(\tau)[z(\tau,\bar{\xi},\bar{c})-\dot{\bar{\xi}}]d\tau,$$
$$\forall i\in\Gamma_r,$$

which is a contradiction to (ii) (see Definition 5.2). This completes the proof. □

The next theorem is the reciprocal result of Theorem 5.1.

**Theorem 5.2.** *Consider $(\bar{\xi},\bar{c})\in F$ is an LU-efficient solution for $(P)$ at which Eqs (2.1)–(2.4) are fulfilled together with $\bar{\lambda},\bar{\nu},\bar{\mu}$. Then, $(\bar{\xi},\bar{c},\bar{\lambda},\bar{\nu},\bar{\mu})$ is an LU-T-saddle-point for $\left(\mathrm{P}_T(\bar{\xi},\bar{c})\right)$, provided $\int_{a_1}^{a_2}\bar{\nu}(\tau)p(\tau,\xi,c)d\tau$ and $\int_{a_1}^{a_2}\bar{\mu}(\tau)[z(\tau,\xi,c)-\dot{\xi}]d\tau$ are $T$-convex at $(\bar{\xi},\bar{c})\in F$.*

*Proof.* As $(\bar{\xi},\bar{c})$ is an $LU$-efficient solution for $(P)$, the relations Eqs (2.1)–(2.3) are fulfilled together with $\bar{\lambda},\bar{\nu},\bar{\mu}$. From Eq (2.1), it follows that

$$\sum_{i=1}^{r}\bar{\lambda}^{iL}\frac{\partial\pi^{iL}}{\partial\xi}(\tau,\bar{\xi},\bar{c})+\sum_{i=1}^{r}\bar{\lambda}^{iU}\frac{\partial\pi^{iU}}{\partial\xi}(\tau,\bar{\xi},\bar{c})+\bar{\nu}(\tau)\frac{\partial p}{\partial\xi}(\tau,\bar{\xi},\bar{c})+\bar{\mu}(\tau)\frac{\partial z}{\partial\xi}(\tau,\bar{\xi},\bar{c})$$
$$=D\Big(\sum_{i=1}^{r}\bar{\lambda}^{iL}\frac{\partial\pi^{iL}}{\partial\dot{\xi}}(\tau,\bar{\xi},\bar{c})+\sum_{i=1}^{r}\bar{\lambda}^{iU}\frac{\partial\pi^{iU}}{\partial\dot{\xi}}(\tau,\bar{\xi},\bar{c})+\bar{\nu}(\tau)\frac{\partial p}{\partial\dot{\xi}}(\tau,\bar{\xi},\bar{c})-\bar{\mu}(\tau)\Big),$$

that is,

$$\sum_{i=1}^{r}\bar{\lambda}^{iL}\frac{\partial\pi^{iL}}{\partial\xi}(\tau,\bar{\xi},\bar{c})+\sum_{i=1}^{r}\bar{\lambda}^{iU}\frac{\partial\pi^{iU}}{\partial\xi}(\tau,\bar{\xi},\bar{c})$$
$$-D\left(\sum_{i=1}^{r}\bar{\lambda}^{iL}\frac{\partial\pi^{iL}}{\partial\dot{\xi}}(\tau,\bar{\xi},\bar{c})+\sum_{i=1}^{r}\bar{\lambda}^{iU}\frac{\partial\pi^{iU}}{\partial\dot{\xi}}(\tau,\bar{\xi},\bar{c})\right)$$
$$=D\left(\bar{\nu}(\tau)\frac{\partial p}{\partial\dot{\xi}}(\tau,\bar{\xi},\bar{c})-\bar{\mu}(\tau)\right)-\bar{\nu}(\tau)\frac{\partial p}{\partial\xi}(\tau,\bar{\xi},\bar{c})-\bar{\mu}(\tau)\frac{\partial z}{\partial\xi}(\tau,\bar{\xi},\bar{c}). \tag{5.2}$$

From Eq (2.2), we have

$$\sum_{i=1}^{r}\bar{\lambda}^{iL}\frac{\partial\pi^{iL}}{\partial c}(\tau,\bar{\xi},\bar{c})+\sum_{i=1}^{r}\bar{\lambda}^{iU}\frac{\partial\pi^{iU}}{\partial c}(\tau,\bar{\xi},\bar{c})$$
$$+\bar{\nu}(\tau)\frac{\partial p}{\partial c}(\tau,\bar{\xi},\bar{c})+\bar{\mu}(\tau)\frac{\partial z}{\partial c}(\tau,\bar{\xi},\bar{c})=0. \tag{5.3}$$

Let $(\xi,c)\in F$ be an arbitrary feasible point. Since $\int_{a_1}^{a_2}\bar{\nu}(\tau)p(\tau,\xi,c)d\tau$ and $\int_{a_1}^{a_2}\bar{\mu}(\tau)\left[z(\tau,\xi,c)-\dot{\xi}\right]d\tau$ are $T$-convex at $(\bar{\xi},\bar{c})$, we get

$$\int_{a_1}^{a_2}\bar{\nu}(\tau)p(\tau,\xi,c)d\tau-\int_{a_1}^{a_2}\bar{\nu}(\tau)p(\tau,\bar{\xi},\bar{c})d\tau$$
$$\ge\int_{a_1}^{a_2}T(\tau,\xi,\bar{\xi};\bar{\nu}(\tau)p_{\xi}(\tau,\bar{\xi},\bar{c})-D\bar{\nu}(\tau)p_{\dot{\xi}}(\tau,\bar{\xi},\bar{c}),\bar{\nu}(\tau)p_c(\tau,\bar{\xi},\bar{c}))d\tau$$

and

$$\int_{a_1}^{a_2} \bar{\mu}(\tau)\left[z(\tau,\xi,c) - \dot{\xi}\right] d\tau - \int_{a_1}^{a_2} \bar{\mu}(\tau)\left[z(\tau,\bar{\xi},\bar{c}) - \dot{\bar{\xi}}\right] d\tau$$

$$\geq \int_{a_1}^{a_2} T(\tau,\xi,\bar{\xi};\bar{\mu}(\tau)z_{\xi}(\tau,\bar{\xi},\bar{c}) + D\bar{\mu}(\tau), \bar{\mu}(\tau)z_c(\tau,\bar{\xi},\bar{c}))d\tau,$$

which, by using Eqs (5.2) and (5.3), yields

$$\int_{a_1}^{a_2} \bar{\nu}(\tau)p(\tau,\xi,c)d\tau - \int_{a_1}^{a_2} \bar{\nu}(\tau)p(\tau,\bar{\xi},\bar{c})d\tau$$

$$+ \int_{a_1}^{a_2} \bar{\mu}(\tau)\left[z(\tau,\xi,c) - \dot{\xi}\right] d\tau - \int_{a_1}^{a_2} \bar{\mu}(\tau)\left[z(\tau,\bar{\xi},\bar{c}) - \dot{\bar{\xi}}\right] d\tau$$

$$\geq - \int_{a_1}^{a_2} T\Big(\tau,\xi,\bar{\xi};\bar{\lambda}^{iL}\pi_{\xi}^{iL}(\tau,\bar{\xi},\bar{c}) + \bar{\lambda}^{iU}\pi_{\xi}^{iU}(\tau,\bar{\xi},\bar{c})$$

$$-D\left(\bar{\lambda}^{iL}\pi_{\dot{\xi}}^{iL}(\tau,\bar{\xi},\bar{c}) + \bar{\lambda}^{iU}\pi_{\dot{\xi}}^{iU}(\tau,\bar{\xi},\bar{c})\right), \bar{\lambda}^{iL}\pi_{c}^{iL}(\tau,\bar{\xi},\bar{c}) + \bar{\lambda}^{iU}\pi_{c}^{iU}(\tau,\bar{\xi},\bar{c})\Big)d\tau.$$

Since the sublinear functional $T$ vanishes with respect to the same arguments, therefore, we have

$$\int_{a_1}^{a_2} T\Big(\tau,\xi,\bar{\xi};\bar{\lambda}^{iL}\pi_{\xi}^{iL}(\tau,\bar{\xi},\bar{c}) + \bar{\lambda}^{iU}\pi_{\xi}^{iU}(\tau,\bar{\xi},\bar{c}) - D\left(\bar{\lambda}^{iL}\pi_{\dot{\xi}}^{iL}(\tau,\bar{\xi},\bar{c}) + \bar{\lambda}^{iU}\pi_{\dot{\xi}}^{iU}(\tau,\bar{\xi},\bar{c})\right),$$

$$\bar{\lambda}^{iL}\pi_{c}^{iL}(\tau,\bar{\xi},\bar{c}) + \bar{\lambda}^{iU}\pi_{c}^{iU}(\tau,\bar{\xi},\bar{c})\Big)d\tau + \int_{a_1}^{a_2} \bar{\nu}(\tau)p(\tau,\xi,c)d\tau + \int_{a_1}^{a_2} \bar{\mu}(\tau)[z(\tau,\xi,c) - \dot{\xi}]d\tau$$

$$\geq \int_{a_1}^{a_2} T\Big(\tau,\bar{\xi},\bar{\xi};\bar{\lambda}^{iL}\pi_{\xi}^{iL}(\tau,\bar{\xi},\bar{c}) + \bar{\lambda}^{iU}\pi_{\xi}^{iU}(\tau,\bar{\xi},\bar{c}) - D\left(\bar{\lambda}^{iL}\pi_{\dot{\xi}}^{iL}(\tau,\bar{\xi},\bar{c}) + \bar{\lambda}^{iU}\pi_{\dot{\xi}}^{iU}(\tau,\bar{\xi},\bar{c})\right),$$

$$\bar{\lambda}^{iL}\pi_{c}^{iL}(\tau,\bar{\xi},\bar{c}) + \bar{\lambda}^{iU}\pi_{c}^{iU}(\tau,\bar{\xi},\bar{c})\Big)d\tau + \int_{a_1}^{a_2} \bar{\nu}(\tau)p(\tau,\bar{\xi},\bar{c})d\tau$$

$$+ \int_{a_1}^{a_2} \bar{\mu}(\tau)[z(\tau,\bar{\xi},\bar{c}) - \dot{\bar{\xi}}]d\tau, \forall i \in \Gamma_r,$$

which, by definition of the Lagrange functional, yields

$$\mathcal{K}_T(\bar{\xi},\bar{c},\bar{\lambda},\bar{\nu},\bar{\mu}) \leq \mathcal{K}_T(\xi,c,\bar{\lambda},\bar{\nu},\bar{\mu}), \ \forall (\xi,c) \in F. \tag{5.4}$$

Now, as $(\bar{\xi},\bar{c})$ is a feasible solution, we get

$$\int_{a_1}^{a_2} \bar{\nu}(\tau)p(\tau,\bar{\xi},\bar{c})d\tau = 0, \ \int_{a_1}^{a_2} \bar{\mu}(\tau)[z(\tau,\bar{\xi},\bar{c}) - \dot{\bar{\xi}}]d\tau = 0.$$

Again, by Remark 2.1, in the next inequality we have

$$\int_{a_1}^{a_2} T\Big(\tau,\bar{\xi},\bar{\xi};\bar{\lambda}^{iL}\pi_{\xi}^{iL}(\tau,\bar{\xi},\bar{c}) + \bar{\lambda}^{iU}\pi_{\xi}^{iU}(\tau,\bar{\xi},\bar{c}) - D\left(\bar{\lambda}^{iL}\pi_{\dot{\xi}}^{iL}(\tau,\bar{\xi},\bar{c}) + \bar{\lambda}^{iU}\pi_{\dot{\xi}}^{iU}(\tau,\bar{\xi},\bar{c})\right),$$

$$\bar{\lambda}^{iL}\pi_{c}^{iL}(\tau,\bar{\xi},\bar{c}) + \bar{\lambda}^{iU}\pi_{c}^{iU}(\tau,\bar{\xi},\bar{c})\Big)d\tau + \int_{a_1}^{a_2} \nu(\tau)p(\tau,\bar{\xi},\bar{c})d\tau + \int_{a_1}^{a_2} \mu(\tau)[z(\tau,\bar{\xi},\bar{c}) - \dot{\bar{\xi}}]d\tau$$

$$\leq \int_{a_1}^{a_2} T\Big(\tau,\bar{\xi},\bar{\xi};\bar{\lambda}^{iL}\pi_{\xi}^{iL}(\tau,\bar{\xi},\bar{c}) + \bar{\lambda}^{iU}\pi_{\xi}^{iU}(\tau,\bar{\xi},\bar{c}) - D\left(\bar{\lambda}^{iL}\pi_{\dot{\xi}}^{iL}(\tau,\bar{\xi},\bar{c}) + \bar{\lambda}^{iU}\pi_{\dot{\xi}}^{iU}(\tau,\bar{\xi},\bar{c})\right),$$

$$\bar{\lambda}^{iL}\pi_{c}^{iL}(\tau,\bar{\xi},\bar{c}) + \bar{\lambda}^{iU}\pi_{c}^{iU}(\tau,\bar{\xi},\bar{c})\Big)d\tau + \int_{a_1}^{a_2} \bar{\nu}(\tau)p(\tau,\bar{\xi},\bar{c})d\tau + \int_{a_1}^{a_2} \bar{\mu}(\tau)[z(\tau,\bar{\xi},\bar{c}) - \dot{\bar{\xi}}]d\tau,$$

which, by Lagrangian's definition, yields

$$\mathcal{K}_T(\bar{\xi},\bar{c},\bar{\lambda},\nu,\mu) \leq \mathcal{K}_T(\bar{\xi},\bar{c},\bar{\lambda},\bar{\nu},\bar{\mu}), \quad \forall \nu \in \mathbb{R}_+^q, \ \mu \in \mathbb{R}^n. \tag{5.5}$$

Thus, by Eqs (5.4) and (5.5) it follows that $(\bar{\xi},\bar{c},\bar{\lambda},\bar{\nu},\bar{\mu})$ is an $LU$-$T$-saddle-point for $\left(\mathrm{P}_T(\bar{\xi},\bar{c})\right)$. □

## 6. Conclusions

Various efficiency conditions in new multiple-objective interval-valued control models via modified $T$-objective functional approach and saddle-point criteria have been formulated and studied. In this regard, by using new concepts and notions related to our control model, a connection between $LU$-efficient solutions of the considered problem $(P)$ and the saddle-points of the Lagrange-type functionals associated with the modified problems has been established.

## Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare no conflicts of interest.

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