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Research article

Dynamics of a stochastic hybrid delay one-predator-two-prey model with harvesting and jumps in a polluted environment

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Abstract: This paper concerns the dynamics of a stochastic, hybrid delay, one-predator-two-prey model with harvesting and Lévy jumps in a polluted environment. Under some basic assumptions, sufficient conditions of stochastic persistence in the mean and extinction of each species are obtained, as well as the existence of optimal harvesting strategy (OHS). Our results show that both time delays and environmental noises affect the survival state of the species. Moreover, the accurate expressions for the optimal harvesting effort (OHE) and the maximum of expectation of sustainable yield (MESY) are given. Finally, some numerical simulations are provided to support our results.

Keywords: stochastic predator-prey model; Markovian switching; time delay; Lévy jump

1. Introduction

The classical Lotka-Volterra model with one predator and two competing preys under the catch-per-unit-effort (CPUE) harvesting hypothesis ([1,2]) can be expressed as follows:

$$\begin{cases} \dot{x}_1(t) = x_1(t)\left[r_1 - h_1 - a_{11}x_1(t) - a_{12}x_2(t) - a_{13}x_3(t)\right], \\ \dot{x}_2(t) = x_2(t)\left[r_2 - h_2 - a_{21}x_1(t) - a_{22}x_2(t) - a_{23}x_3(t)\right], \\ \dot{x}_3(t) = x_3(t)\left[-r_3 - h_3 + a_{31}x_1(t) + a_{32}x_2(t) - a_{33}x_3(t)\right], \end{cases} \tag{1.1}$$

where $x_i(t)$ is the population density of species i at time t, r_1 and r_2 are the intrinsic growth rates of two preys, r_3 is the mortality rate of the predator, a_{ii} denotes the intra-specific competition rate of species i, a_{12} and a_{21} are the inter-specific competition rates, a_{13}, a_{23} are the capture rates, a_{31}, a_{32} are the food conversion rates, and $h_i \geq 0$ is the harvesting effort of species i ($i = 1, 2, 3$).

On the one hand, "time delays occur so often that to ignore them is to ignore reality" because any species in nature will not always react at once to variation on their own population size or on that of the interacting species, though they will preferably do so after a time lag [3,4]. Hence, it is crucial to consider the effect of time delay on the population dynamics, and incorporating a time delay into ecosystems makes them much more realistic than those without a time delay [5–7].

On the other hand, the deterministic system has its limitation in the mathematical modeling of ecosystems since the parameters involved in the system are unable of capturing the influence of environmental noises [8]. Hence, it is of a great theoretical and practical significance to study the effect of environmental noise on the population dynamics. There are three common types of environmental noises, namely Gaussian white noise, telegraph noise, and Lévy noise. Introducing Gaussian white noises into the corresponding deterministic model is one common way to characterize environmental noises [9–12]. Additionally, telegraph noise should be taken into account since parameters in ecosystems often switch because of environmental changes; for example, the population may suffer sudden catastrophic shocks [13], the growth rates of some species often vary according to the changes in rainfall [14], the growth rates of some species in dry season are much different from those in rainy season [15], and these changes can be described by a continuous-time

Markovian chain with a finite-state space, instead of white noises [13–15]. Besides, ecosystems may be subject to some sudden discontinuous environmental perturbations, such as earthquakes, typhoons, and infectious diseases, which can be described by Lévy jumps [16–19].

Meanwhile, the environment has been along with the rapid development of the economy. As more and more toxic substances and pollutants enter the ecosystem, the quality of the environment gradually declines, which results in many species becoming extinct and others on the brink of extinction [20]. Therefore, the environmental pollution has become an important problem that the world has to face [21, 22]. In addition, unreasonable capture can easily lead to species extinction, ecological damage, and so on. Hence, harvesting is one of the important processes in the management of population dynamics [23–26]. Motivated by the above discussions, in this paper, we consider the dynamics of the following stochastic, hybrid delay, one-predator-two-prey model with harvesting and jumps in a polluted environment:

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[r_1(\rho(t)) - r_{11}C_1(t) - h_1 - \mathcal{D}_{11}(x_1)(t)\right. \\ \qquad \left. -\mathcal{D}_{12}(x_2)(t) - \mathcal{D}_{13}(x_3)(t)\right]\mathrm{d}t + \mathcal{S}_1(t,\rho(t))x_1(t), \\ \mathrm{d}x_2(t) = x_2(t)\left[r_2(\rho(t)) - r_{22}C_2(t) - h_2 - \mathcal{D}_{21}(x_1)(t)\right. \\ \qquad \left. -\mathcal{D}_{22}(x_2)(t) - \mathcal{D}_{23}(x_3)(t)\right]\mathrm{d}t + \mathcal{S}_2(t,\rho(t))x_2(t), \\ \mathrm{d}x_3(t) = x_3(t)\left[-r_3(\rho(t)) - r_{33}C_3(t) - h_3 + \mathcal{D}_{31}(x_1)(t)\right. \\ \qquad \left. +\mathcal{D}_{32}(x_2)(t) - \mathcal{D}_{33}(x_3)(t)\right]\mathrm{d}t + \mathcal{S}_3(t,\rho(t))x_3(t), \\ \mathrm{d}C_i(t) = \left[k_iC_E(t) - (g_i + m_i)\,C_i(t)\right]\mathrm{d}t, \; i = 1,2,3, \\ \mathrm{d}C_E(t) = \left[-hC_E(t) + u(t)\right]\mathrm{d}t, \end{cases} \tag{1.2}$$

where

$$\mathcal{D}_{ji}(x_i)(t) = a_{ji}x_i(t) + \int_{-\tau_{ji}}^{0} x_i(t+\theta)\mathrm{d}\mu_{ji}(\theta),$$

$$\mathcal{S}_i\left(t,\rho(t)\right) = \sigma_i(\rho(t))\mathrm{d}W_i(t) + \int_{\mathbb{Z}} \gamma_i(\mu,\rho(t))\widetilde{N}(\mathrm{d}t,\mathrm{d}\mu),$$

$W_i(t)$ are standard Wiener processes defined on a complete probability space $(\Omega,\mathcal{F},P)$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ that satisfies the usual conditions, $\rho(t)$ is a continuous time Markov chain with finite state space

$$\mathbb{S} = \{1,2,...,S\},$$

N is a Poisson counting measure with the characteristic measure λ on a measurable subset

$$\mathbb{Z} \subseteq [0,+\infty)$$

with

$$\lambda(\mathbb{Z}) < +\infty$$

and

$$\widetilde{N}(\mathrm{d}t,\mathrm{d}\mu) = N(\mathrm{d}t,\mathrm{d}\mu) - \lambda(\mathrm{d}\mu)\mathrm{d}t,$$

$\gamma_j(\mu,\rho(t))$ are bounded functions, $\int_{-\tau_{ji}}^{0} x_i(t+\theta)\mathrm{d}\mu_{ji}(\theta)$ are Lebesgue-Stieltjes integrals, $\tau_{ji} > 0$ are delays,

$$\tau = \max\left\{\tau_{ji}\right\},$$

$\mu_{ji}(\theta)$, $\theta \in [-\tau, 0]$ are nondecreasing bounded variation functions. For other parameters in system (1.2), see [27, Table 1].

The rest of this paper is arranged as follows. In Section 2, we study the existence and uniqueness of global positive solution to systems (1.2). The sufficient conditions for stochastic persistence in the mean and extinction of each species are obtained in Section 3. In Section 4, the sufficient conditions for the existence of the optimal harvesting strategy (OHS) are established. Furthermore, we provide the accurate expressions of the optimal harvesting effort (OHE) and the maximum of expectation of sustainable yield (MESY). In Section 5, some numerical simulations are provided to verify the theoretical results. Finally, some brief conclusions and discussions are shown in Section 6.

2. Existence and uniqueness of global positive solution

In this paper, we have four fundamental assumptions for system (1.2).

Assumption 2.1. *[28–30] $W_1(t)$, $W_2(t)$, $W_3(t)$, $\rho(t)$, and N are mutually independent, and $\rho(t)$ is irreducible with one unique stationary distribution*

$$\pi = (\pi_1, \pi_2, ..., \pi_S).$$

Assumption 2.2. *[31, 32] $r_j(i) > 0$, $a_{jk} > 0$, and there exist*

$$\gamma_j^*(i) \geq \gamma_{j*}(i) > -1,$$

such that

$$\gamma_{j*}(i) \leq \gamma_j(\mu,i) \leq \gamma_j^*(i) \;\; (\mu \in \mathbb{Z}), \;\; \forall i \in \mathbb{S}, \;\; j,k = 1,2,3.$$

Hence, for any constant $p > 0$, there exists $C_j(p) > 0$ such that

$$\max_{i\in\mathbb{S}}\left\{\int_{\mathbb{Z}}\left[\ln\left(1+\gamma_j(\mu,i)\right)\right]^2\lambda(\mathrm{d}\mu)\right\}\le C_j(p)<+\infty.$$

Remark 2.1. *Assumption 2.2 implies that the intensities of Lévy jumps are not too big to ensure that the solution will not explode in a finite time (see, e.g., [10, 18, 19]).*

Assumption 2.3.

$$0<k_i\le g_i+m_i,\quad i=1,2,3,$$
$$\sup_{t\in\mathbb{R}_+}u(t)\le h.$$

Remark 2.2. *Assumption 2.3 means*

$$0\le C_i(t)<1$$

and

$$0\le C_E(t)<1,$$

which must be satisfied to be realistic because $C_i(t)$ and $C_E(t)$ are concentrations of the toxicant ($i = 1,2,3$) (see [33, Lemma 2.1]).

Assumption 2.4. *The limit of $u(t)$ when $t\to+\infty$ exists, i.e.,*

$$\lim_{t\to+\infty}u(t)\triangleq u^E.$$

Lemma 2.1. *[2, Lemma 4.2] If Assumption 2.4 holds, then*

$$\lim_{t\to+\infty}C_E(t)=\frac{u^E}{h},$$
$$\lim_{t\to+\infty}t^{-1}\int_0^t C_i(s)\mathrm{d}s=\frac{k_iu^E}{(g_i+m_i)h}\triangleq C_i^E,\quad i=1,2,3.$$

Theorem 2.1. *For any initial condition*

$$\phi\in C\left([-\tau,0],\mathbb{R}_+^3\right),$$

system (1.2) has a unique global solution on $t\in[-\tau,+\infty)$ a.s. Moreover, for any constant $p>0$, there exists $K_i(p)>0$ such that

$$\sup_{t\in\mathbb{R}_+}\mathbb{E}\left[x_i^p(t)\right]\le K_i(p),\quad i=1,2,3.$$

Proof. The proof is standard; hence, it is omitted (see e.g., [34]). □

3. Stochastic persistence and extinction

Denote

$$\begin{cases}
B_i(\cdot)=r_i(\cdot)-\dfrac{\sigma_i^2(\cdot)}{2}-\displaystyle\int_{\mathbb{Z}}\left[\gamma_i(\mu,\cdot)-\ln(1+\gamma_i(\mu,\cdot))\right]\lambda(\mathrm{d}\mu),\\
\qquad (i=1,2),\\
B_3(\cdot)=r_3(\cdot)+\dfrac{\sigma_3^2(\cdot)}{2}+\displaystyle\int_{\mathbb{Z}}\left[\gamma_3(\mu,\cdot)-\ln(1+\gamma_3(\mu,\cdot))\right]\lambda(\mathrm{d}\mu),\\
\Sigma_j=\displaystyle\sum_{i=1}^{S}\pi_iB_j(i)-r_{jj}C_j^E,\ (j=1,2),\\
\Sigma_3=-\displaystyle\sum_{i=1}^{S}\pi_iB_3(i)-r_{33}C_3^E,\\
\Xi_j=\Sigma_j-h_j,\ A_{ij}=a_{ij}+\displaystyle\int_{-\tau_{ij}}^0\mathrm{d}\mu_{ij}(\theta),\ (i,j=1,2,3),\\
\mathbf{\Xi}=\begin{pmatrix}\Xi_1\\ \Xi_2\\ \Xi_3\end{pmatrix},\quad \mathbf{A_0}=\begin{pmatrix}A_{11}&0&0\\0&A_{22}&0\\-A_{31}&-A_{32}&A_{33}\end{pmatrix},\\
\mathbf{A}=\begin{pmatrix}A_{11}&A_{12}&A_{13}\\A_{21}&A_{22}&A_{23}\\-A_{31}&-A_{32}&A_{33}\end{pmatrix}.
\end{cases}$$

Assume that

$$\Theta=\det(\mathbf{A})>0.$$

Let $\mathbf{A_j}$ be $\mathbf{A}$ with column j replaced by $\mathbf{\Xi}$ and

$$\Theta_j=\det\left(\mathbf{A_j}\right).$$

For $\mathbf{A}$ and $\mathbf{A_k}$, denote the complement minor of the $(i,j)-th$ element by M_{ij}^{Θ} and $M_{ij}^{\Theta_k}$, respectively ($i,j,k=1,2,3$). Denote

$$X(\infty)=\lim_{t\to+\infty}X(t),\quad \overline{X(\infty)}=\lim_{t\to+\infty}t^{-1}\int_0^t X(s)\mathrm{d}s,$$
$$\overline{\mathbf{X}^{\mathrm{T}}(\infty)}=\lim_{t\to+\infty}t^{-1}\left(\int_0^t X_1(s)\mathrm{d}s,\int_0^t X_2(s)\mathrm{d}s,\int_0^t X_3(s)\mathrm{d}s\right).$$

Lemma 3.1. *[35] Denote*

$$o(t)=\left\{f(t)\mid\lim_{t\to+\infty}\tfrac{f(t)}{t}=0\right\}.$$

Suppose

$$Z(t)\in C(\Omega\times[0,+\infty),\mathbb{R}_+).$$

(i) *If there exists constant $\delta_0>0$ such that for $t\gg1$,*

$$\ln Z(t)\le\delta t-\delta_0\int_0^t Z(s)\mathrm{d}s+o(t),\tag{3.1}$$

then,

$$\begin{cases} \limsup_{t\to+\infty} t^{-1}\int_0^t Z(s)\mathrm{d}s \le \frac{\delta}{\delta_0},\ a.s. & (\delta \ge 0); \\ \lim_{t\to+\infty} Z(t) = 0,\ a.s. & (\delta < 0). \end{cases} \tag{3.2}$$

(ii) *If there exist constants $\delta > 0$ and $\delta_0 > 0$ such that for $t \gg 1$,*

$$\ln Z(t) \ge \delta t - \delta_0 \int_0^t Z(s)\mathrm{d}s + o(t), \tag{3.3}$$

then,

$$\liminf_{t\to+\infty} t^{-1}\int_0^t Z(s)\mathrm{d}s \ge \frac{\delta}{\delta_0},\ a.s. \tag{3.4}$$

First, let us consider the following stochastic delay differential equation (SDDE):

$$\begin{cases} \mathrm{d}X_i(t) = X_i(t)\left[r_i(\rho(t)) - r_{ii}C_i(t) - h_i - \mathcal{D}_{ii}(X_i)(t)\right]\mathrm{d}t \\ \qquad + \mathcal{S}_i(t,\rho(t))X_i(t),\ (i = 1,2), \\ \mathrm{d}X_3(t) = X_3(t)\left[-r_3(\rho(t)) - r_{33}C_3(t) - h_3 + \mathcal{D}_{31}(X_1)(t)\right. \\ \qquad \left.+\mathcal{D}_{32}(X_2)(t) - \mathcal{D}_{33}(X_3)(t)\right]\mathrm{d}t + \mathcal{S}_3(t,\rho(t))X_3(t), \\ \mathrm{d}C_i(t) = \left[k_iC_E(t) - (g_i + m_i)C_i(t)\right]\mathrm{d}t,\ i = 1,2,3, \\ \mathrm{d}C_E(t) = \left[-hC_E(t) + u(t)\right]\mathrm{d}t. \end{cases} \tag{3.5}$$

Lemma 3.2. *For system (3.5), the following hold:*

($\mathfrak{a}$) *If* $\Xi_1 < 0$, $\Xi_2 < 0$, *then* $\overline{\mathbf{X}^{\mathrm{T}}(\infty)} = (0,0,0)$.

($\mathfrak{b}$) *If* $\Xi_1 < 0$, $\Xi_2 \ge 0$, $\Xi_3 + \frac{A_{32}}{A_{22}}\Xi_2 < 0$, *then*

$$\overline{\mathbf{X}^{\mathrm{T}}(\infty)} = \left(0, \frac{\Xi_2}{A_{22}}, 0\right).$$

($\mathfrak{c}$) *If* $\Xi_1 < 0$, $\Xi_2 \ge 0$, $\Xi_3 + \frac{A_{32}}{A_{22}}\Xi_2 \ge 0$, *then*

$$\overline{\mathbf{X}^{\mathrm{T}}(\infty)} = \left(0, \frac{\Xi_2}{A_{22}}, A_{33}^{-1}\left(\Xi_3 + \frac{A_{32}}{A_{22}}\Xi_2\right)\right).$$

($\mathfrak{d}$) *If* $\Xi_1 \ge 0$, $\Xi_2 < 0$, $\Xi_3 + \frac{A_{31}}{A_{11}}\Xi_1 < 0$, *then*

$$\overline{\mathbf{X}^{\mathrm{T}}(\infty)} = \left(\frac{\Xi_1}{A_{11}}, 0, 0\right).$$

($\mathfrak{e}$) *If* $\Xi_1 \ge 0$, $\Xi_2 < 0$, $\Xi_3 + \frac{A_{31}}{A_{11}}\Xi_1 \ge 0$, *then*

$$\overline{\mathbf{X}^{\mathrm{T}}(\infty)} = \left(\frac{\Xi_1}{A_{11}}, 0, A_{33}^{-1}\left(\Xi_3 + \frac{A_{31}}{A_{11}}\Xi_1\right)\right).$$

($\mathfrak{f}$) *If* $\Xi_1 \ge 0$, $\Xi_2 \ge 0$, $\Xi_3 + \frac{A_{31}}{A_{11}}\Xi_1 + \frac{A_{32}}{A_{22}}\Xi_2 < 0$, *then*

$$\overline{\mathbf{X}^{\mathrm{T}}(\infty)} = \left(\frac{\Xi_1}{A_{11}}, \frac{\Xi_2}{A_{22}}, 0\right).$$

($\mathfrak{g}$) *If* $\Xi_1 \ge 0$, $\Xi_2 \ge 0$, $\Xi_3 + \frac{A_{31}}{A_{11}}\Xi_1 + \frac{A_{32}}{A_{22}}\Xi_2 \ge 0$, *then*

$$\overline{\mathbf{X}^{\mathrm{T}}(\infty)} = \left(\frac{\Xi_1}{A_{11}}, \frac{\Xi_2}{A_{22}}, A_{33}^{-1}\left(\Xi_3 + \frac{A_{31}}{A_{11}}\Xi_1 + \frac{A_{32}}{A_{22}}\Xi_2\right)\right). \tag{3.6}$$

Proof. Thanks to [34, Lemma 2.3], for $j = 1,2$,

$$X_j(\infty) = 0\ a.s.\ \left(\Xi_j < 0\right);\ \overline{X_j(\infty)} = \frac{\Xi_j}{A_{jj}},\ a.s.\ \left(\Xi_j \ge 0\right). \tag{3.7}$$

By Itô's formula,

$$\ln \mathbf{X}(t) = \mathbf{\Xi}\, t - \mathbf{A_0}\int_0^t \mathbf{X}(s)\mathrm{d}s + \begin{pmatrix} -\mathcal{T}_{11}(X_1)(t) \\ -\mathcal{T}_{22}(X_2)(t) \\ \mathcal{T}_{31}(X_1)(t) + \mathcal{T}_{32}(X_2)(t) - \mathcal{T}_{33}(X_3)(t) \end{pmatrix} + \mathbf{o(t)}, \tag{3.8}$$

where

$$\ln \mathbf{X}(t) = \begin{pmatrix} \ln X_1(t) \\ \ln X_2(t) \\ \ln X_3(t) \end{pmatrix},$$

$$\mathbf{o(t)} = o(t)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

$$\int \mathbf{X}(s)\mathrm{d}s = \begin{pmatrix} \int X_1(s)\mathrm{d}s \\ \int X_2(s)\mathrm{d}s \\ \int X_3(s)\mathrm{d}s \end{pmatrix},$$

$$\mathcal{T}_{ji}(X_i)(t) = \int_{-\tau_{ji}}^0 \int_\theta^0 X_i(s)\mathrm{d}s\mathrm{d}\mu_{ji}(\theta) - \int_{-\tau_{ji}}^0 \int_{t+\theta}^t X_i(s)\mathrm{d}s\mathrm{d}\mu_{ji}(\theta).$$

Case 1: $\Xi_1 < 0$, $\Xi_2 < 0$. Based on system (3.8), for $\forall\varepsilon \in (0,1)$ and $t \gg 1$,

$$\ln X_3(t) \le (\Xi_3 + \varepsilon)\, t - a_{33}\int_0^t X_3(s)\mathrm{d}s, \tag{3.9}$$

which implies that $X_3(\infty) = 0$, a.s.

Case 2: $\Xi_1 < 0$, $\Xi_2 \ge 0$. Consider the following SDDE:

$$\begin{cases} \mathrm{d}X_i(t) = X_i(t)\left[r_i(\rho(t)) - r_{ii}C_i(t) - h_i - \mathcal{D}_{ii}(X_i)(t)\right]\mathrm{d}t \\ \qquad + \mathcal{S}_i(t,\rho(t))X_i(t),\ (i = 1,2), \\ \mathrm{d}\widetilde{X_3(t)} = \widetilde{X_3(t)}\left[-r_3(\rho(t)) - r_{33}C_3(t) - h_3 + \mathcal{D}_{31}(X_1)(t)\right. \\ \qquad \left.+\mathcal{D}_{32}(X_2)(t) - a_{33}\widetilde{X_3(t)}\right]\mathrm{d}t + \mathcal{S}_3(t,\rho(t))\widetilde{X_3(t)}, \\ \mathrm{d}C_i(t) = \left[k_iC_E(t) - (g_i + m_i)C_i(t)\right]\mathrm{d}t,\ i = 1,2,3, \\ \mathrm{d}C_E(t) = \left[-hC_E(t) + u(t)\right]\mathrm{d}t. \end{cases} \tag{3.10}$$

Thanks to the comparison theorem for SDDE,

$$X_3(t) \le \widetilde{X_3(t)},$$

a.s. By Itô's formula,

$$\ln \widetilde{X_3(t)} = \left(\Xi_3 + \frac{A_{32}}{A_{22}}\Xi_2\right)t - a_{33}\int_0^t \widetilde{X_3(s)}\mathrm{d}s + o(t). \tag{3.11}$$

In view of Lemma 3.1, for an arbitrary $\gamma > 0$,

$$\lim_{t\to+\infty} t^{-1}\int_{t-\gamma}^t X_i(s)\mathrm{d}s = 0, \ \ a.s.\ (i = 1, 2, 3). \tag{3.12}$$

According to (3.12) and system (3.8),

$$\ln X_3(t) = \left(\Xi_3 + \frac{A_{32}}{A_{22}}\Xi_2\right)t - A_{33}\int_0^t X_3(s)\mathrm{d}s + o(t). \tag{3.13}$$

Thanks to Lemma 3.1, we obtain the desired assertions ($\mathfrak{b}$) and ($\mathfrak{c}$).

Case 3: $\Xi_1 \ge 0, \Xi_2 < 0$. By Itô's formula,

$$\ln \widetilde{X_3(t)} = \left(\Xi_3 + \frac{A_{31}}{A_{11}}\Xi_1\right)t - a_{33}\int_0^t \widetilde{X_3(s)}\mathrm{d}s + o(t). \tag{3.14}$$

Thanks to Lemma 3.1, (3.12) is true for an arbitrary $\gamma > 0$. Therefore,

$$\ln X_3(t) = \left(\Xi_3 + A_{31}\frac{\Xi_1}{A_{11}}\right)t - A_{33}\int_0^t X_3(s)\mathrm{d}s + o(t). \tag{3.15}$$

According to Lemma 3.1, we obtain the desired assertions ($\mathfrak{d}$) and ($\mathfrak{e}$).

Case 4: $\Xi_1 \ge 0, \Xi_2 \ge 0$. By Itô's formula,

$$\ln \widetilde{X_3(t)} = \left(\Xi_3 + \frac{A_{31}}{A_{11}}\Xi_1 + \frac{A_{32}}{A_{22}}\Xi_2\right)t - a_{33}\int_0^t \widetilde{X_3(s)}\mathrm{d}s + o(t). \tag{3.16}$$

Based on Lemma 3.1, (3.12) is also true for an arbitrary $\gamma > 0$. Thus,

$$\ln X_3(t) = \left(\Xi_3 + \frac{A_{31}}{A_{11}}\Xi_1 + \frac{A_{32}}{A_{22}}\Xi_2\right)t - A_{33}\int_0^t X_3(s)\mathrm{d}s + o(t). \tag{3.17}$$

From Lemma 3.1, we obtain the desired assertions ($\mathfrak{f}$) and ($\mathfrak{g}$). □

Lemma 3.3. *System (1.2) satisfies the following:*

$$\limsup_{t\to+\infty} t^{-1}\ln x_i(t) \le 0, \ \ a.s.\ \ (i = 1, 2, 3).$$

Proof. From Lemma 3.2, system (3.5) satisfies the following:

$$\lim_{t\to+\infty} t^{-1}\ln X_i(t) = 0, \ \ \text{a.s.}\ \ (i = 1, 2, 3).$$

By the stochastic comparison theorem, we obtain the desired assertion. □

Theorem 3.1. *For system (1.2), the following holds:*
(i) *If* $\Xi_1 < 0, \Xi_2 < 0$, *then*

$$\overline{\mathbf{x}^{\mathrm{T}}(\infty)} = (0, 0, 0).$$

(ii) *If* $\Xi_1 < 0, \Xi_2 \ge 0, M_{11}^{\Theta_3} > 0$, *then*

$$\overline{\mathbf{x}^{\mathrm{T}}(\infty)} = \left(0, \frac{M_{11}^{\Theta_2}}{M_{11}^{\Theta}}, \frac{M_{11}^{\Theta_3}}{M_{11}^{\Theta}}\right).$$

(iii) *If* $\Xi_1 \ge 0, \Xi_2 < 0, M_{22}^{\Theta_3} > 0$, *then*

$$\overline{\mathbf{x}^{\mathrm{T}}(\infty)} = \left(\frac{M_{22}^{\Theta_1}}{M_{22}^{\Theta}}, 0, \frac{M_{22}^{\Theta_3}}{M_{22}^{\Theta}}\right).$$

($\mathfrak{iv}$) *If* $\Xi_1 \ge 0, \Xi_2 \ge 0, M_{32}^{\Theta} \ge 0, M_{33}^{\Theta_2} \le 0, M_{22}^{\Theta_3} \le 0$, *then*

$$\overline{\mathbf{x}^{\mathrm{T}}(\infty)} = \left(\frac{\Xi_1}{A_{11}}, 0, 0\right).$$

($\mathfrak{v}$) *If* $\Xi_1 \ge 0, \Xi_2 \ge 0, M_{31}^{\Theta} \ge 0, M_{33}^{\Theta_1} \le 0, M_{11}^{\Theta_3} \le 0$, *then*

$$\overline{\mathbf{x}^{\mathrm{T}}(\infty)} = \left(0, \frac{\Xi_2}{A_{22}}, 0\right).$$

($\mathfrak{vi}$) *If* $\Xi_1 \ge 0, \Xi_2 \ge 0, M_{31}^{\Theta} \le 0, M_{32}^{\Theta} \ge 0, \Theta_i > 0\ (i = 1, 2)$, $\Theta_3 < 0, M_{33}^{\Theta_1} \ge 0, M_{33}^{\Theta_2} \ge 0$, *then*

$$\overline{\mathbf{x}^{\mathrm{T}}(\infty)} = \left(\frac{M_{33}^{\Theta_1}}{M_{33}^{\Theta}}, \frac{M_{33}^{\Theta_2}}{M_{33}^{\Theta}}, 0\right).$$

($\mathfrak{vii}$) *If* $\Xi_1 \ge 0, \Xi_2 \ge 0, M_{13}^{\Theta} \ge 0, M_{23}^{\Theta} \le 0, M_{31}^{\Theta} \le 0$, $M_{32}^{\Theta} \ge 0, \Theta_i > 0\ (i = 1, 2, 3)$, *then*

$$\overline{\mathbf{x}^{\mathrm{T}}(\infty)} = \left(\frac{\Theta_1}{\Theta}, \frac{\Theta_2}{\Theta}, \frac{\Theta_3}{\Theta}\right). \tag{3.18}$$

($\mathfrak{viii}$) *If* $\Xi_1 \ge 0, \Xi_2 \ge 0, M_{31}^{\Theta} \le 0, \Theta_1 < 0, M_{11}^{\Theta_3} > 0$, *then*

$$\overline{\mathbf{x}^{\mathrm{T}}(\infty)} = \left(0, \frac{M_{11}^{\Theta_2}}{M_{11}^{\Theta}}, \frac{M_{11}^{\Theta_3}}{M_{11}^{\Theta}}\right).$$

($\mathfrak{ix}$) *If* $\Xi_1 \ge 0, \Xi_2 \ge 0, M_{32}^{\Theta} \ge 0, \Theta_2 < 0, M_{22}^{\Theta_3} > 0$, *then*

$$\overline{\mathbf{x}^{\mathrm{T}}(\infty)} = \left(\frac{M_{22}^{\Theta_1}}{M_{22}^{\Theta}}, 0, \frac{M_{22}^{\Theta_3}}{M_{22}^{\Theta}}\right).$$

Proof. Thanks to Lemma 3.2, for $\forall\gamma > 0$,

$$\lim_{t\to+\infty} t^{-1}\int_{t-\gamma}^{t} x_i(s)\mathrm{d}s = 0, \quad a.s. \quad (i = 1, 2, 3). \tag{3.19}$$

By Itô's formula,

$$\ln \mathbf{x}(t) = \mathbf{\Xi} t - \mathbf{A}\int_0^t \mathbf{x}(s)\mathrm{d}s + \mathbf{o(t)}. \tag{3.20}$$

$\mathfrak{Case}$ (i): $\Xi_1 < 0, \Xi_2 < 0$. From Lemma 3.2 ($\mathfrak{a}$),

$$\overline{\mathbf{x}^{\mathrm{T}}(\infty)} = (0, 0, 0).$$

$\mathfrak{Case}$ (ii): $\Xi_1 < 0$, $\Xi_2 \geq 0$, $M_{11}^{\Theta_3} > 0$. From Lemma 3.2, $x_1(\infty) = 0$, a.s. Compute the following:

$$A_{32}\ln x_2(t) + A_{22}\ln x_3(t) = M_{11}^{\Theta_3} t - M_{11}^{\Theta}\int_0^t x_3(s)\mathrm{d}s + o(t). \tag{3.21}$$

Thanks to Lemma 3.1, we deduce the following:

$$\liminf_{t\to+\infty} t^{-1}\int_0^t x_3(s)\mathrm{d}s \geq \frac{M_{11}^{\Theta_3}}{M_{11}^{\Theta}}, \quad a.s. \tag{3.22}$$

Based on systems (3.20) and (3.22), for $\forall\varepsilon \in (0, 1)$ and $t \gg 1$,

$$\ln x_2(t) \leq \left(A_{22}\frac{M_{11}^{\Theta_2}}{M_{11}^{\Theta}} + \varepsilon\right)t - A_{22}\int_0^t x_2(s)\mathrm{d}s, \tag{3.23}$$

which implies

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x_2(s)\mathrm{d}s \leq \frac{M_{11}^{\Theta_2}}{M_{11}^{\Theta}}, \quad a.s. \tag{3.24}$$

Based on Eqs (3.20) and (3.24), for $\forall\varepsilon \in (0, 1)$ and $t \gg 1$,

$$\ln x_3(t) \leq \left(A_{33}\frac{M_{11}^{\Theta_3}}{M_{11}^{\Theta}} + \varepsilon\right)t - A_{33}\int_0^t x_3(s)\mathrm{d}s, \tag{3.25}$$

which implies

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x_3(s)\mathrm{d}s \leq \frac{M_{11}^{\Theta_3}}{M_{11}^{\Theta}}, \quad a.s. \tag{3.26}$$

Combining (3.22) with (3.26) yields

$$\overline{x_3(\infty)} = \frac{M_{11}^{\Theta_3}}{M_{11}^{\Theta}},$$

a.s. Based on systems (3.20) and (3.26), for $\forall\varepsilon \in (0, 1)$ and $t \gg 1$,

$$\ln x_2(t) \geq \left(A_{22}\frac{M_{11}^{\Theta_2}}{M_{11}^{\Theta}} - \varepsilon\right)t - A_{22}\int_0^t x_2(s)\mathrm{d}s, \tag{3.27}$$

which implies

$$\liminf_{t\to+\infty} t^{-1}\int_0^t x_2(s)\mathrm{d}s \geq \frac{M_{11}^{\Theta_2}}{M_{11}^{\Theta}}, \quad a.s. \tag{3.28}$$

Combining (3.24) with (3.28) yields

$$\overline{x_2(\infty)} = \frac{M_{11}^{\Theta_2}}{M_{11}^{\Theta}},$$

a.s.

$\mathfrak{Case}$ (iii): $\Xi_1 \geq 0$, $\Xi_2 < 0$, $M_{22}^{\Theta_3} > 0$. From Lemma 3.2, $x_2(\infty) = 0$, a.s. Compute the following:

$$A_{33}\ln x_1(t) - A_{13}\ln x_3(t) = M_{22}^{\Theta_1} t - M_{22}^{\Theta}\int_0^t x_1(s)\mathrm{d}s + o(t). \tag{3.29}$$

Thanks to Lemmas 3.1 and 3.3, we deduce the following:

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s \leq \frac{M_{22}^{\Theta_1}}{M_{22}^{\Theta}}, \quad a.s. \tag{3.30}$$

Based on systems (3.20) and (3.30), for $\forall\varepsilon \in (0, 1)$ and $t \gg 1$,

$$\ln x_3(t) \leq \left(A_{33}\frac{M_{22}^{\Theta_3}}{M_{22}^{\Theta}} + \varepsilon\right)t - A_{33}\int_0^t x_3(s)\mathrm{d}s, \tag{3.31}$$

which implies

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x_3(s)\mathrm{d}s \leq \frac{M_{22}^{\Theta_3}}{M_{22}^{\Theta}}, \quad a.s. \tag{3.32}$$

Based on Eqs (3.20) and (3.32), for $\forall\varepsilon \in (0, 1)$ and $t \gg 1$,

$$\ln x_1(t) \geq \left(A_{11}\frac{M_{22}^{\Theta_1}}{M_{22}^{\Theta}} - \varepsilon\right)t - A_{11}\int_0^t x_1(s)\mathrm{d}s, \tag{3.33}$$

which implies

$$\liminf_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s \geq \frac{M_{22}^{\Theta_1}}{M_{22}^{\Theta}}, \quad a.s. \tag{3.34}$$

Combining (3.30) with (3.34) yields

$$\overline{x_1(\infty)} = \frac{M_{22}^{\Theta_1}}{M_{22}^{\Theta}},$$

a.s. Based on systems (3.20) and (3.34), for $\forall\varepsilon \in (0,1)$ and $t \gg 1$,

$$\ln x_3(t) \geq \left(A_{33}\frac{M_{22}^{\Theta_3}}{M_{22}^{\Theta}} - \varepsilon\right)t - A_{33}\int_0^t x_3(s)\mathrm{d}s, \quad (3.35)$$

which implies

$$\liminf_{t\to+\infty} t^{-1}\int_0^t x_3(s)\mathrm{d}s \geq \frac{M_{22}^{\Theta_3}}{M_{22}^{\Theta}}, \quad a.s. \quad (3.36)$$

Combining (3.32) with (3.36) yields

$$\overline{x_3(\infty)} = \frac{M_{22}^{\Theta_3}}{M_{22}^{\Theta}},$$

a.s.

𝔆𝔞𝔰𝔢 (𝔦𝔳): $\Xi_1 \geq 0$, $\Xi_2 \geq 0$, $M_{32}^{\Theta} \geq 0$, $M_{33}^{\Theta_2} \leq 0$, $M_{22}^{\Theta_3} \leq 0$. Based on system (3.20), we compute the following:

$$\begin{aligned} &A_{11}\ln x_2(t) - A_{21}\ln x_1(t) \\ =&M_{33}^{\Theta_2}t - M_{33}^{\Theta}\int_0^t x_2(s)\mathrm{d}s - M_{32}^{\Theta}\int_0^t x_3(s)\mathrm{d}s + o(t). \end{aligned} \quad (3.37)$$

By Lemma 3.3, for $\forall\varepsilon \in (0,1)$ and $t \gg 1$,

$$A_{11}\ln x_2(t) \leq \left(M_{33}^{\Theta_2} + \varepsilon\right)t - M_{33}^{\Theta}\int_0^t x_2(s)\mathrm{d}s, \quad (3.38)$$

which implies $x_2(\infty) = 0$, a.s. Based on system (3.20), for $\forall\varepsilon \in (0,1)$ and $t \gg 1$,

$$\ln x_1(t) \leq (\Xi_1 + \varepsilon)t - A_{11}\int_0^t x_1(s)\mathrm{d}s. \quad (3.39)$$

Thanks to Lemma 3.1, we deduce

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s \leq \frac{\Xi_1}{A_{11}}, \quad a.s. \quad (3.40)$$

Substituting $x_2(\infty) = 0$ and (3.40) into system (3.20), for $\forall\varepsilon \in (0,1)$ and $t \gg 1$,

$$\ln x_3(t) \leq \left(\frac{M_{22}^{\Theta_3}}{A_{11}} + \varepsilon\right)t - A_{33}\int_0^t x_3(s)\mathrm{d}s, \quad (3.41)$$

which implies $x_3(\infty) = 0$, a.s. Substituting $x_2(\infty) = 0$ and $x_3(\infty) = 0$ into system (3.20), for $\forall\varepsilon \in (0,1)$ and $t \gg 1$,

$$\ln x_1(t) \geq (\Xi_1 - \varepsilon)t - A_{11}\int_0^t x_1(s)\mathrm{d}s. \quad (3.42)$$

Thanks to Lemma 3.1, we deduce

$$\liminf_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s \geq \frac{\Xi_1}{A_{11}}, \quad a.s. \quad (3.43)$$

Combining (3.40) with (3.43) yields

$$\overline{x_1(\infty)} = \frac{\Xi_1}{A_{11}},$$

a.s.

𝔆𝔞𝔰𝔢 (𝔳): $\Xi_1 \geq 0$, $\Xi_2 \geq 0$, $M_{31}^{\Theta} \geq 0$, $M_{33}^{\Theta_1} \leq 0$, $M_{11}^{\Theta_3} \leq 0$. Based on system (3.20), we compute the following:

$$\begin{aligned} &A_{22}\ln x_1(t) - A_{12}\ln x_2(t) \\ =& M_{33}^{\Theta_1}t - M_{33}^{\Theta}\int_0^t x_2(s)\mathrm{d}s - M_{31}^{\Theta}\int_0^t x_3(s)\mathrm{d}s + o(t). \end{aligned} \quad (3.44)$$

By Lemma 3.3, for $\forall\varepsilon \in (0,1)$ and $t \gg 1$,

$$A_{22}\ln x_1(t) \leq \left(M_{33}^{\Theta_1} + \varepsilon\right)t - M_{33}^{\Theta}\int_0^t x_1(s)\mathrm{d}s. \quad (3.45)$$

which implies $x_1(\infty) = 0$, a.s. Based on system (3.20), for $\forall\varepsilon \in (0,1)$ and $t \gg 1$,

$$\ln x_2(t) \leq (\Xi_2 + \varepsilon)t - A_{22}\int_0^t x_2(s)\mathrm{d}s. \quad (3.46)$$

Thanks to Lemma 3.1, we deduce

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x_2(s)\mathrm{d}s \leq \frac{\Xi_2}{A_{22}}, \quad a.s. \quad (3.47)$$

Substituting $x_1(\infty) = 0$ and (3.47) into system (3.20), for $\forall\varepsilon \in (0,1)$ and $t \gg 1$,

$$\ln x_3(t) \leq \left(\frac{M_{11}^{\Theta_3}}{A_{22}} + \varepsilon\right)t - A_{33}\int_0^t x_3(s)\mathrm{d}s, \quad (3.48)$$

which implies $x_3(\infty) = 0$, a.s. Substituting $x_1(\infty) = 0$ and $x_3(\infty) = 0$ into system (3.20), for $\forall\varepsilon \in (0,1)$ and $t \gg 1$,

$$\ln x_2(t) \geq (\Xi_2 - \varepsilon)t - A_{22}\int_0^t x_2(s)\mathrm{d}s. \quad (3.49)$$

Thanks to Lemma 3.1, we deduce

$$\liminf_{t\to+\infty} t^{-1}\int_0^t x_2(s)\mathrm{d}s \geq \frac{\Xi_2}{A_{22}}, \quad a.s. \quad (3.50)$$

Combining (3.47) with (3.50) yields

$$\overline{x_2(\infty)} = \frac{\Xi_2}{A_{22}},$$

a.s.

$\mathfrak{Case}$ (vi): $\Xi_1 \geq 0$, $\Xi_2 \geq 0$, $M_{31}^{\Theta} \leq 0$, $M_{32}^{\Theta} \geq 0$, $\Theta_i > 0$ (where $i = 1, 2$), $\Theta_3 < 0$, $M_{33}^{\Theta_1} \geq 0$, $M_{33}^{\Theta_2} \geq 0$. Compute the following:

$$\begin{cases} M_{11}^{\Theta} \ln x_1(t) - M_{21}^{\Theta} \ln x_2(t) + M_{31}^{\Theta} \ln x_3(t) \\ = \Theta_1 t - \Theta \int_0^t x_1(s) \mathrm{d}s + o(t), \\ M_{22}^{\Theta} \ln x_2(t) - M_{12}^{\Theta} \ln x_1(t) - M_{32}^{\Theta} \ln x_3(t) \\ = \Theta_2 t - \Theta \int_0^t x_2(s) \mathrm{d}s + o(t). \end{cases} \tag{3.51}$$

According to Lemmas 3.1 and 3.3 and (3.51), we obtain

$$\limsup_{t\to+\infty} t^{-1} \int_0^t x_i(s) \mathrm{d}s \leq \frac{\Theta_i}{\Theta}, \quad a.s. \quad (i = 1, 2). \tag{3.52}$$

According to (3.52) and system (3.20), for $\forall \varepsilon \in (0, 1)$ and $t \gg 1$,

$$\ln x_3(t) \leq \left(A_{33} \frac{\Theta_3}{\Theta} + \varepsilon \right) t - A_{33} \int_0^t x_3(s) \mathrm{d}s, \tag{3.53}$$

which implies $x_3(\infty) = 0$, a.s. Compute the following:

$$\begin{cases} \ln x_1(t) = \Xi_1 t - A_{11} \int_0^t x_1(s) \mathrm{d}s - A_{12} \int_0^t x_2 s \mathrm{d}s + o(t), \\ \ln x_2(t) = \Xi_2 t - A_{21} \int_0^t x_1(s) \mathrm{d}s - A_{22} \int_0^t x_2 s \mathrm{d}s + o(t). \end{cases} \tag{3.54}$$

Based on system (3.54), we compute the following:

$$\begin{cases} A_{22} \ln x_1(t) - A_{12} \ln x_2(t) = M_{33}^{\Theta_1} t - M_{33}^{\Theta} \int_0^t x_1(s) \mathrm{d}s + o(t), \\ A_{11} \ln x_2(t) - A_{21} \ln x_1(t) = M_{33}^{\Theta_2} t - M_{33}^{\Theta} \int_0^t x_2(s) \mathrm{d}s + o(t). \end{cases} \tag{3.55}$$

By Lemma 3.3, for $\forall \varepsilon \in (0, 1)$ and $t \gg 1$,

$$\begin{cases} A_{22} \ln x_1(t) \leq \left(M_{33}^{\Theta_1} + \varepsilon \right) t - M_{33}^{\Theta} \int_0^t x_1(s) \mathrm{d}s + o(t), \\ A_{11} \ln x_2(t) \leq \left(M_{33}^{\Theta_2} + \varepsilon \right) t - M_{33}^{\Theta} \int_0^t x_1(s) \mathrm{d}s + o(t). \end{cases} \tag{3.56}$$

In view of Lemma 3.1, we deduce

$$\begin{cases} \limsup_{t\to+\infty} t^{-1} \int_0^t x_1(s) \mathrm{d}s \leq \frac{M_{33}^{\Theta_1}}{M_{33}^{\Theta}}, \quad a.s., \\ \limsup_{t\to+\infty} t^{-1} \int_0^t x_2(s) \mathrm{d}s \leq \frac{M_{33}^{\Theta_2}}{M_{33}^{\Theta}}, \quad a.s. \end{cases} \tag{3.57}$$

According to systems (3.54) and (3.57), for $\forall \varepsilon \in (0, 1)$ and $t \gg 1$,

$$\begin{cases} \ln x_2(t) \geq \left(A_{22} \frac{M_{33}^{\Theta_2}}{M_{33}^{\Theta}} - \varepsilon \right) t - A_{22} \int_0^t x_2(s) \mathrm{d}s, \\ \ln x_1(t) \geq \left(A_{11} \frac{M_{33}^{\Theta_1}}{M_{33}^{\Theta}} - \varepsilon \right) t - A_{11} \int_0^t x_1(s) \mathrm{d}s, \end{cases} \tag{3.58}$$

which implies

$$\begin{cases} \liminf_{t\to+\infty} t^{-1} \int_0^t x_2(s) \mathrm{d}s \geq \frac{M_{33}^{\Theta_2}}{M_{33}^{\Theta}}, \quad a.s., \\ \liminf_{t\to+\infty} t^{-1} \int_0^t x_1(s) \mathrm{d}s \geq \frac{M_{33}^{\Theta_1}}{M_{33}^{\Theta}}, \quad a.s. \end{cases} \tag{3.59}$$

Combining (3.57) with (3.59) yields

$$\overline{x_i(\infty)} = \frac{M_{33}^{\Theta_i}}{M_{33}^{\Theta}},$$

a.s. $(i = 1, 2)$.

$\mathfrak{Case}$ (vii): $\Xi_1 \geq 0$, $\Xi_2 \geq 0$, $M_{13}^{\Theta} \geq 0$, $M_{23}^{\Theta} \leq 0$, $M_{31}^{\Theta} \leq 0$, $M_{32}^{\Theta} \geq 0$, $\Theta_i > 0$ $(i = 1, 2, 3)$. Based on system (3.20), we compute the following:

$$\begin{aligned} & M_{13}^{\Theta} \ln x_1(t) - M_{23}^{\Theta} \ln x_2(t) + M_{33}^{\Theta} \ln x_3(t) \\ & = \Theta_3 t - \Theta \int_0^t x_3(s) \mathrm{d}s + o(t). \end{aligned} \tag{3.60}$$

Thanks to (3.60) and Lemmas 3.1 and 3.3, we derive

$$\liminf_{t\to+\infty} t^{-1} \int_0^t x_3(s) \mathrm{d}s \geq \frac{\Theta_3}{\Theta}, \quad a.s. \tag{3.61}$$

Compute the following:

$$\begin{cases} M_{11}^{\Theta} \ln x_1(t) - M_{21}^{\Theta} \ln x_2(t) + M_{31}^{\Theta} \ln x_3(t) \\ = \Theta_1 t - \Theta \int_0^t x_1(s) \mathrm{d}s + o(t), \\ M_{22}^{\Theta} \ln x_2(t) - M_{12}^{\Theta} \ln x_1(t) - M_{32}^{\Theta} \ln x_3(t) \\ = \Theta_2 t - \Theta \int_0^t x_2(s) \mathrm{d}s + o(t). \end{cases} \tag{3.62}$$

According to Lemmas 3.1 and 3.3 and (3.62), we obtain

$$\limsup_{t\to+\infty} t^{-1} \int_0^t x_i(s) \mathrm{d}s \leq \frac{\Theta_i}{\Theta}, \quad a.s. \quad (i = 1, 2). \tag{3.63}$$

According to (3.63) and system (3.20), for $\forall \varepsilon \in (0, 1)$ and $t \gg 1$,

$$\ln x_3(t) \leq \left(A_{33} \frac{\Theta_3}{\Theta} + \varepsilon \right) t - A_{33} \int_0^t x_3(s) \mathrm{d}s, \tag{3.64}$$

which implies

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x_3(s)\mathrm{d}s \le \frac{\Theta_3}{\Theta},\quad a.s. \tag{3.65}$$

Combining (3.61) with (3.65) yields

$$\overline{x_3(\infty)} = \frac{\Theta_3}{\Theta},$$

a.s. Based on Eqs (3.63) and (3.65), and system (3.20), for $\forall\varepsilon \in (0,1)$ and $t \gg 1$,

$$\ln x_i(t) \ge \left(A_{ii}\frac{\Theta_i}{\Theta} - \varepsilon\right)t - A_{ii}\int_0^t x_i(s)\mathrm{d}s \quad (i = 1,2), \tag{3.66}$$

which implies

$$\liminf_{t\to+\infty} t^{-1}\int_0^t x_i(s)\mathrm{d}s \ge \frac{\Theta_i}{\Theta},\quad a.s. \quad (i = 1,2). \tag{3.67}$$

Combining (3.63) with (3.67) yields

$$\overline{x_i(\infty)} = \frac{\Theta_i}{\Theta},$$

a.s. $(i = 1,2)$.

$\mathfrak{Cases}\,(\mathfrak{viii})$: $\Xi_1 \ge 0$, $\Xi_2 \ge 0$, $M_{31}^{\Theta} \le 0$, $M_{11}^{\Theta_3} > 0$, $\Theta_1 < 0$. From Lemma 3.1 and system (3.62), we obtain $x_1(\infty) = 0$, a.s. Based on system (3.20), we compute the following:

$$\begin{cases} \ln x_2(t) = \Xi_2 t - A_{22}\int_0^t x_2(s)\mathrm{d}s - A_{23}\int_0^t x_3(s)\mathrm{d}s + o(t), \\ \ln x_3(t) = \Xi_3 t + A_{32}\int_0^t x_2(s)\mathrm{d}s - A_{33}\int_0^t x_3(s)\mathrm{d}s + o(t). \end{cases} \tag{3.68}$$

Based on system (3.68), we obtain the following:

$$\begin{cases} A_{33}\ln x_2(t) - A_{23}\ln x_3(t) = M_{11}^{\Theta_2} t - M_{11}^{\Theta}\int_0^t x_2(s)\mathrm{d}s + o(t), \\ A_{22}\ln x_3(t) + A_{32}\ln x_2(t) = M_{11}^{\Theta_3} t - M_{11}^{\Theta}\int_0^t x_3(s)\mathrm{d}s + o(t). \end{cases} \tag{3.69}$$

Thanks to Lemma 3.1 and system (3.69), we obtain

$$\begin{cases} \limsup_{t\to+\infty} t^{-1}\int_0^t x_2(s)\mathrm{d}s \le \dfrac{M_{11}^{\Theta_2}}{M_{11}^{\Theta}},\quad a.s., \\ \liminf_{t\to+\infty} t^{-1}\int_0^t x_3(s)\mathrm{d}s \ge \dfrac{M_{11}^{\Theta_3}}{M_{11}^{\Theta}},\quad a.s. \end{cases} \tag{3.70}$$

Based on system (3.68) and (3.70), for $\forall\varepsilon \in (0,1)$ and $t \gg 1$,

$$\ln x_3(t) \le \left(A_{33}\frac{M_{11}^{\Theta_3}}{M_{11}^{\Theta}} + \varepsilon\right) - A_{33}\int_0^t x_3(s)\mathrm{d}s, \tag{3.71}$$

which implies

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x_3(s)\mathrm{d}s \le \frac{M_{11}^{\Theta_3}}{M_{11}^{\Theta}},\quad a.s. \tag{3.72}$$

Combining (3.70) with (3.72) yields

$$\overline{x_3(\infty)} = \frac{M_{11}^{\Theta_3}}{M_{11}^{\Theta}},$$

a.s. Based on system (3.68) and (3.72), for $\forall\varepsilon \in (0,1)$ and $t \gg 1$,

$$\ln x_2(t) \ge \left(A_{22}\frac{M_{11}^{\Theta_2}}{M_{11}^{\Theta}} + \varepsilon\right) - A_{22}\int_0^t x_2(s)\mathrm{d}s, \tag{3.73}$$

which implies

$$\liminf_{t\to+\infty} t^{-1}\int_0^t x_2(s)\mathrm{d}s \ge \frac{M_{11}^{\Theta_2}}{M_{11}^{\Theta}},\quad a.s. \tag{3.74}$$

Combining (3.70) with (3.74) yields

$$\overline{x_2(\infty)} = \frac{M_{11}^{\Theta_2}}{M_{11}^{\Theta}},$$

a.s.

$\mathfrak{Cases}\,(\mathfrak{ix})$: $\Xi_1 \ge 0$, $\Xi_2 \ge 0$, $M_{32}^{\Theta} \ge 0$, $M_{22}^{\Theta_3} > 0$, $\Theta_2 < 0$. By Lemma 3.1 and system (3.62), $x_2(\infty) = 0$, a.s. Based on system (3.20), we obtain the following:

$$\begin{cases} \ln x_1(t) = \Xi_1 t - A_{11}\int_0^t x_1(s)\mathrm{d}s - A_{13}\int_0^t x_3(s)\mathrm{d}s + o(t), \\ \ln x_3(t) = \Xi_3 t + A_{31}\int_0^t x_1(s)\mathrm{d}s - A_{33}\int_0^t x_3(s)\mathrm{d}s + o(t). \end{cases} \tag{3.75}$$

Thanks to system (3.75), we compute the following:

$$\begin{cases} A_{33}\ln x_1(t) - A_{13}\ln x_3(t) = M_{22}^{\Theta_1} t - M_{22}^{\Theta}\int_0^t x_1(s)\mathrm{d}s + o(t), \\ A_{11}\ln x_3(t) + A_{31}\ln x_1(t) = M_{22}^{\Theta_3} t - M_{22}^{\Theta}\int_0^t x_3(s)\mathrm{d}s + o(t). \end{cases} \tag{3.76}$$

From Lemma 3.1 and system (3.76), we obtain

$$\begin{cases} \limsup_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s \le \dfrac{M_{22}^{\Theta_1}}{M_{22}^{\Theta}},\quad a.s., \\ \liminf_{t\to+\infty} t^{-1}\int_0^t x_3(s)\mathrm{d}s \ge \dfrac{M_{22}^{\Theta_3}}{M_{22}^{\Theta}},\quad a.s. \end{cases} \tag{3.77}$$

Based on Eqs (3.75) and (3.77), for $\forall\varepsilon \in (0,1)$ and $t \gg 1$,

$$\ln x_3(t) \le \left(A_{33}\frac{M_{22}^{\Theta_3}}{M_{22}^{\Theta}} + \varepsilon\right) - A_{33}\int_0^t x_3(s)\mathrm{d}s, \tag{3.78}$$

which implies

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x_3(s)\mathrm{d}s \le \frac{M_{22}^{\Theta_3}}{M_{22}^{\Theta}}, \quad a.s. \tag{3.79}$$

Combining (3.77) with (3.79) yields

$$\overline{x_3(\infty)} = \frac{M_{22}^{\Theta_3}}{M_{22}^{\Theta}},$$

a.s. Based on system (3.75) and (3.79), for $\forall\varepsilon \in (0,1)$ and $t \gg 1$,

$$\ln x_1(t) \ge \left(A_{11}\frac{M_{22}^{\Theta_1}}{M_{22}^{\Theta}} + \varepsilon\right) - A_{11}\int_0^t x_1(s)\mathrm{d}s, \tag{3.80}$$

which implies

$$\liminf_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s \ge \frac{M_{22}^{\Theta_1}}{M_{22}^{\Theta}}, \quad a.s. \tag{3.81}$$

Combining (3.77) with (3.81) yields

$$\overline{x_1(\infty)} = \frac{M_{22}^{\Theta_1}}{M_{22}^{\Theta}},$$

a.s. The proof is completed. □

4. Optimal harvesting strategy

Now, let us consider the optimal harvesting problem of system (1.2). Denote the following:

$$\mathbf{H} = (h_1, h_2, h_3)^{\mathrm{T}} \in \overline{\mathbb{R}_+^3} \Leftrightarrow h_i \in \overline{\mathbb{R}_+} \ (i = 1, 2, 3).$$

Our goal is to find the OHE $\mathbf{H}^*$ such that (**i**) all species are not extinct and (**ii**) the expectation of sustained yield

$$Y(\mathbf{H}) = \lim_{t\to+\infty} \mathbb{E}\left[\mathbf{H}^{\mathrm{T}}\mathbf{x}(t)\right]$$

is maximal. Denote the following:

$$\mathbf{D} = \begin{pmatrix} 2M_{11}^{\Theta} & -\left(M_{12}^{\Theta} + M_{21}^{\Theta}\right) & M_{13}^{\Theta} + M_{31}^{\Theta} \\ -\left(M_{12}^{\Theta} + M_{21}^{\Theta}\right) & 2M_{22}^{\Theta} & -\left(M_{23}^{\Theta} - M_{32}^{\Theta}\right) \\ M_{13}^{\Theta} + M_{31}^{\Theta} & -\left(M_{23}^{\Theta} - M_{32}^{\Theta}\right) & 2M_{33}^{\Theta} \end{pmatrix}.$$

Let

$$\mathbf{\Sigma} = (\Sigma_1, \Sigma_2, \Sigma_3)^{\mathrm{T}}, \quad \mathbf{\Delta} = (|\mathbf{\Delta_1}|, |\mathbf{\Delta_2}|, |\mathbf{\Delta_3}|)^{\mathrm{T}},$$

where $\mathbf{\Delta_j}$ is $\mathbf{A}$ with column j replaced by $\mathbf{\Sigma}$. Thanks to Cramer's Rule, if $|\mathbf{D}| \ne 0$, then

$$\mathbf{DH} = \mathbf{\Delta}$$

has a unique solution which is given by

$$\mathbf{H_0^*} = \left(h_1^*, h_2^*, h_3^*\right)^{\mathrm{T}} = |\mathbf{D}|^{-1}\left(|\mathbf{D_1}|, |\mathbf{D_2}|, |\mathbf{D_3}|\right)^{\mathrm{T}},$$

where $\mathbf{D_j}$ is $\mathbf{D}$ with column j replaced by $\mathbf{\Delta}$ ($j = 1, 2, 3$).

Theorem 4.1. *Define the following:*

$$Y^*(\mathbf{H}) = -\frac{1}{2}\mathbf{H}^{\mathrm{T}}\mathbf{DH} + \mathbf{H}^{\mathrm{T}}\mathbf{\Delta}.$$

(i) *If*

$$\Xi_1|_{h_1=h_1^*} \ge 0, \quad \Xi_2|_{h_2=h_2^*} \ge 0,$$

$M_{13}^{\Theta} \ge 0$, $M_{23}^{\Theta} \le 0$, $M_{31}^{\Theta} \le 0$, $M_{32}^{\Theta} \ge 0$ *and*

$$\begin{aligned} &\Theta_i\left(\mathbf{H_0^*}\right)|_{\mathbf{H_0^*}\in\overline{\mathbb{R}_+^3}} > 0 \ (i = 1, 2, 3), \\ &4M_{11}^{\Theta}M_{22}^{\Theta} - \left(M_{12}^{\Theta} + M_{21}^{\Theta}\right)^2 > 0, \ |\mathbf{D}| > 0, \end{aligned} \tag{4.1}$$

then the **OHS** *exists. Moreover,*

$$\mathbf{H}^* = \mathbf{H_0^*}$$

and

$$\mathbf{MESY} = \Theta^{-1}Y^*(\mathbf{H}^*).$$

(ii) *If one of the following conditions holds, then the* **OHS** *does not exist:*

(1) $\Xi_1|_{h_1=h_1^*} < 0$ *or* $\Xi_2|_{h_2=h_2^*} < 0$*;*

(2) $\Xi_1|_{h_1=h_1^*} \ge 0$, $\Xi_2|_{h_2=h_2^*} \ge 0$ *and one of the following conditions holds:*

(2.1) $M_{32}^{\Theta} \ge 0$, $M_{33}^{\Theta_2}|_{h_1=h_1^*,h_2=h_2^*} \le 0$, $M_{22}^{\Theta_3}|_{h_1=h_1^*,h_3=h_3^*} \le 0$*;*

(2.2) $M_{31}^{\Theta} \ge 0$, $M_{33}^{\Theta_1}|_{h_1=h_1^*,h_2=h_2^*} \le 0$, $M_{11}^{\Theta_3}|_{h_2=h_2^*,h_3=h_3^*} \le 0$*;*

(2.3) $M_{31}^{\Theta} \le 0$, $M_{32}^{\Theta} \ge 0$, $M_{33}^{\Theta_1}|_{h_1=h_1^*,h_2=h_2^*} \ge 0$, $M_{33}^{\Theta_2}|_{h_1=h_1^*,h_2=h_2^*} \ge 0$, $\Theta_i|_{\mathbf{H}=\mathbf{H_0^*}} > 0$ ($i = 1, 2$), $\Theta_3|_{\mathbf{H}=\mathbf{H_0^*}} < 0$*;*

(2.4) $M_{31}^{\Theta} \le 0$, $M_{11}^{\Theta_3}|_{h_2=h_2^*,h_3=h_3^*} > 0$, $\Theta_1|_{\mathbf{H}=\mathbf{H_0^*}} < 0$*;*

(2.5) $M_{32}^{\Theta} \ge 0$, $M_{22}^{\Theta_3}|_{h_1=h_1^*,h_3=h_3^*} > 0$, $\Theta_2|_{\mathbf{H}=\mathbf{H_0^*}} < 0$*;*

(3) $\mathbf{H_0^*} \notin \overline{\mathbb{R}_+^3}$*;*

(4) $|\mathbf{D}| < 0$ *or* $4M_{11}^{\Theta}M_{22}^{\Theta} - \left(M_{12}^{\Theta} + M_{21}^{\Theta}\right)^2 < 0$.

Proof. According to [36, Theorem 3.1.1], $(\mathbf{x}(t), \rho(t))^{\mathrm{T}}$ has the following invariant measure:

$$\nu(\cdot \times \cdot) \in \mathbb{R}^3_+ \times \mathbb{S}.$$

From [37, Theorem 3.1], $\nu(\cdot \times \cdot)$ is unique. Thanks to [38, Theorem 3.2.6], $\nu(\cdot \times \cdot)$ is ergodic. Hence,

$$\overline{x_i(\infty)} = \sum_{k=1}^{S} \int_{\mathbb{R}^3_+} \theta_i \nu(\mathrm{d}\theta_1, \mathrm{d}\theta_2, \mathrm{d}\theta_3, k), \quad a.s. \quad (i = 1, 2, 3). \tag{4.2}$$

Let

$$\mathcal{U} = \Big\{ \mathbf{H} \in \overline{\mathbb{R}^3_+} \mid \Xi_1 \geq 0, \Xi_2 \geq 0, M^{\Theta}_{13} \geq 0, M^{\Theta}_{23} \leq 0, M^{\Theta}_{31} \leq 0, M^{\Theta}_{32} \geq 0, \Theta_i > 0 \ (i = 1, 2, 3) \Big\}.$$

From (vii) in Theorem 3.1, for every $\mathbf{H} \in \mathcal{U}$, we have

$$\overline{\mathbf{x}^{\mathrm{T}}(\infty)} = \left(\frac{\Theta_1}{\Theta}, \frac{\Theta_2}{\Theta}, \frac{\Theta_3}{\Theta} \right), \quad a.s. \tag{4.3}$$

This completes the proof. □

Proof of (i). $\mathcal{U} \neq \emptyset$. Based on (4.3), for every

$$\mathbf{H} \in \mathcal{U}, \quad \overline{\mathbf{H}^{\mathrm{T}} \mathbf{x}(\infty)} = \Theta^{-1} Y^* (\mathbf{H}).$$

Let $\varrho(\cdot \times \cdot)$ be the stationary probability density of system (1.2); then,

$$Y(\mathbf{H}) = \lim_{t \to +\infty} \mathbb{E}\left[\mathbf{H}^{\mathrm{T}} \mathbf{x}(t)\right] = \sum_{k=1}^{S} \int_{\mathbb{R}^3_+} \mathbf{H}^{\mathrm{T}} \theta \varrho(\theta, k) \,\mathrm{d}\theta. \tag{4.4}$$

Note that system (1.2) has a unique ergodic invariant measure $\nu(\cdot \times \cdot)$, and that there exists a one-to-one correspondence between $\varrho(\cdot \times \cdot)$ and $\nu(\cdot \times \cdot)$. We deduce the following:

$$\sum_{k=1}^{S} \int_{\mathbb{R}^3_+} \mathbf{H}^{\mathrm{T}} \theta \varrho(\theta, k) \mathrm{d}\theta = \sum_{k=1}^{S} \int_{\mathbb{R}^3_+} \mathbf{H}^{\mathrm{T}} \theta \nu(\mathrm{d}\theta, k). \tag{4.5}$$

Thanks to (4.2)–(4.5), we deduce the following:

$$Y(\mathbf{H}) = \Theta^{-1} Y^*(\mathbf{H}).$$

Solving

$$\frac{\mathrm{d}Y^*(\mathbf{H})}{\mathrm{d}\mathbf{H}} = \mathbf{0}$$

yields

$$\left(h^*_1, h^*_2, h^*_3\right)^{\mathrm{T}} = |\mathbf{D}|^{-1} \left(|\mathbf{D_1}|, |\mathbf{D_2}|, |\mathbf{D_3}|\right)^{\mathrm{T}}.$$

Based on (4.1), the Hessian matrix $-\mathbf{D}$ is a negative definite. Thus, $Y^*(\mathbf{H})$ has a unique maximum and the unique maximum value point of $Y^*(\mathbf{H})$ is

$$\mathbf{H}^* = \left(h^*_1, h^*_2, h^*_3\right)^{\mathrm{T}}.$$

This completes the proof. □

Proof of (ii). From Theorem 3.1, we only prove that if the following condition holds, then the **OHS** does not exist (i.e., prove (4)):

$$\begin{cases} \Xi_1 \geq 0, \Xi_2 \geq 0, M^{\Theta}_{13} \geq 0, M^{\Theta}_{23} \leq 0, \\ M^{\Theta}_{31} \leq 0, M^{\Theta}_{32} \geq 0, \Theta_i > 0, \ (i = 1, 2, 3), \\ |\mathbf{D}| < 0 \ or \ 4M^{\Theta}_{11} M^{\Theta}_{22} - \left(M^{\Theta}_{12} + M^{\Theta}_{21}\right)^2 < 0. \end{cases} \tag{4.6}$$

$-2M^{\Theta}_{11} < 0$ implies that $-\mathbf{D}$ is not a positive semidefinite. System (4.6) indicates that $-\mathbf{D}$ is not a negative semidefinite. Hence, $-\mathbf{D}$ is indefinite. Thus, $Y^*(\mathbf{H})$ does not exist at the extreme point. Therefore, the **OHE** does not exist. □

5. Examples and numerical simulations

5.1. Example 1

In this section, we will introduce some examples to validate the theoretical results. Let

$$\tau_{ji} = \ln 2, \quad \mu_{ji}(\theta) = \mu_{ji} \mathrm{e}^{\theta}, \quad \gamma_j(\mu, i) = \gamma_j(i),$$
$$\lambda(\mathbb{Z}) = 1, \quad u(t) = 0.3 + 0.05 \frac{\sin t}{t}, \quad h = 0.5.$$

Denote

$$\text{Param(i)} = \begin{pmatrix} r_{11} & g_1 & h_1 & m_1 & \mu_{11} & \mu_{12} & \mu_{13} & \sigma_1 & \gamma_1 \\ r_{22} & g_2 & h_2 & m_2 & \mu_{21} & \mu_{22} & \mu_{23} & \sigma_2 & \gamma_2 \\ r_{33} & g_3 & h_3 & m_3 & \mu_{31} & \mu_{32} & \mu_{33} & \sigma_3 & \gamma_3 \end{pmatrix}$$

and

$$\overline{\text{Param(i)}} = \begin{pmatrix} r_1 & k_1 & a_{11} & a_{12} & a_{13} \\ r_2 & k_2 & a_{21} & a_{22} & a_{23} \\ r_3 & k_3 & a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Let

$$\text{Param(1)} = \begin{pmatrix} 0.1 & 0.1 & 0.1 & 0.3 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.1 & 0.2 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.1 & 0.2 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \end{pmatrix}$$

subject to

$$x_1(\theta) = 0.8e^{\theta}, \quad x_2(\theta) = 0.6e^{\theta}, \quad x_3(\theta) = 0.3e^{\theta}, \quad \theta \in [-\ln 2, 0].$$

Case 1.

$$\overline{\text{Param}(1)} = \begin{pmatrix} 0.1 & 0.2 & 0.2 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.2 & 0.3 & 0.1 & 0.1 & 0.1 \end{pmatrix}.$$

Then,

$$\Xi_1 = -0.0397, \quad \Xi_2 = -0.0247.$$

From Theorem 3.1 (i), all species are extinctive (see Figure 1).

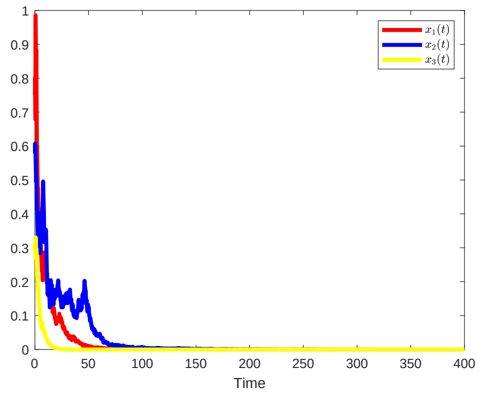

Figure 1. The solution to system (1.2) with Param(1) and $\overline{\text{Param}(1)}$, which represents that all species in **Case 1** are extinctive.

Case 2.

$$\overline{\text{Param}(2)} = \begin{pmatrix} 0.1 & 0.2 & 0.2 & 0.1 & 0.1 \\ 0.3 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.3 & 0.1 \end{pmatrix}.$$

Then, $\Xi_1 = -0.0397$, $\Xi_2 = 0.1753$, $M_{11}^{\Theta_3} = 0.0277$. From Theorem 3.1 (ii), x_1 is extinctive, while x_2 and x_3 are persistent in the mean (see Figure 2) and

$$\overline{\mathbf{x}^{\mathrm{T}}(\infty)} = \left(0, \frac{M_{11}^{\Theta_2}}{M_{11}^{\Theta}}, \frac{M_{11}^{\Theta_3}}{M_{11}^{\Theta}}\right) = (0, 0.8000, 0.3687), \quad a.s. \tag{5.1}$$

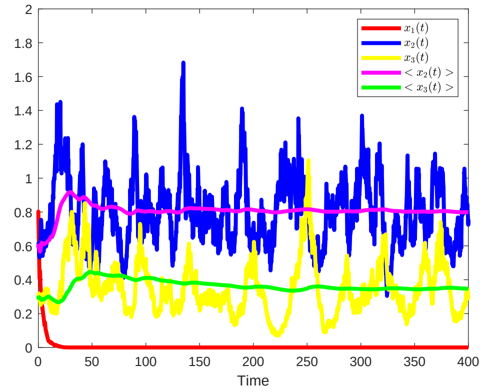

Figure 2. The solution to system (1.2) with Param(1) and $\overline{\text{Param}(2)}$, which represents that both x_2 and x_3 are persistent in the mean in **Case 2**, while x_1 is extinctive.

Case 3.

$$\overline{\text{Param}(3)} = \begin{pmatrix} 0.4 & 0.1 & 0.2 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.4 & 0.1 & 0.1 \end{pmatrix}.$$

Then, $\Xi_1 = 0.2753$, $\Xi_2 = -0.0247$, $M_{22}^{\Theta_3} = 0.0677$. From Theorem 3.1 (iii), x_2 is extinctive, while x_1 and x_3 are persistent in the mean (see Figure 3) and

$$\overline{\mathbf{x}^{\mathrm{T}}(\infty)} = \left(\frac{M_{22}^{\Theta_1}}{M_{22}^{\Theta}}, 0, \frac{M_{22}^{\Theta_3}}{M_{22}^{\Theta}}\right) = (0.7143, 0, 0.6449), \quad a.s. \tag{5.2}$$

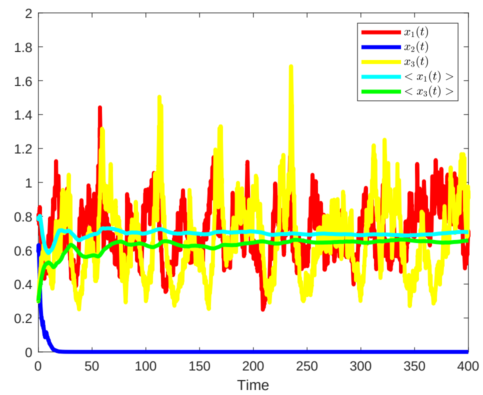

Figure 3. The solution to system (1.2) with Param(1) and $\overline{\text{Param}(3)}$, which represents that both x_1 and x_3 are persistent in the mean in **Case 3**, while x_2 is extinctive.

Case 4.

$$\overline{\text{Param}(4)} = \begin{pmatrix} 0.3 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.2 & 0.1 & 0.1 & 0.2 & 0.2 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.2 \end{pmatrix}.$$

Then, $\Xi_1 = 0.0753$, $\Xi_2 = 0.1753$, $M_{32}^{\Theta} = 0.0150$, $M_{33}^{\Theta_2} = -0.0150$, $M_{22}^{\Theta_3} = -0.0074$. From Theorem 3.1 (iv), x_1 is persistent in the mean, while x_2 and x_3 are extinctive (see Figure 4) and

$$\overline{\mathbf{x}^{\mathrm{T}}(\infty)} = \left(\frac{\Xi_1}{A_{11}}, 0, 0\right) = (1.1687, 0, 0), \quad a.s. \tag{5.3}$$

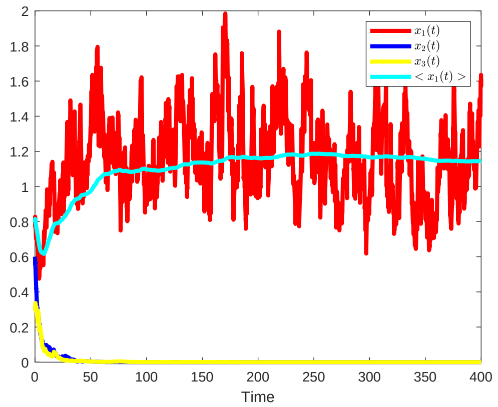

Figure 4. The solution to system (1.2) with Param(1) and $\overline{\text{Param}(4)}$, which represents that x_1 is persistent in the mean in **Case 4**, while x_2 and x_3 are extinctive.

Case 5.

$$\overline{\text{Param}(5)} = \begin{pmatrix} 0.2 & 0.1 & 0.3 & 0.3 & 0.1 \\ 0.2 & 0.1 & 0.1 & 0.2 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.2 \end{pmatrix}.$$

Then, $\Xi_1 = 0.0753$, $\Xi_2 = 0.0753$, $M_{31}^{\Theta} = 0.0150$, $M_{33}^{\Theta_1} = -0.0075$, $M_{11}^{\Theta_3} = -0.0449$. From Theorem 3.1 (v), x_2 is persistent in the mean, while x_1 and x_3 are extinctive (see Figure 5) and

$$\overline{\mathbf{x}^{\mathrm{T}}(\infty)} = \left(0, \frac{\Xi_2}{A_{22}}, 0\right) = (0, 0.3012, 0), \quad a.s. \tag{5.4}$$

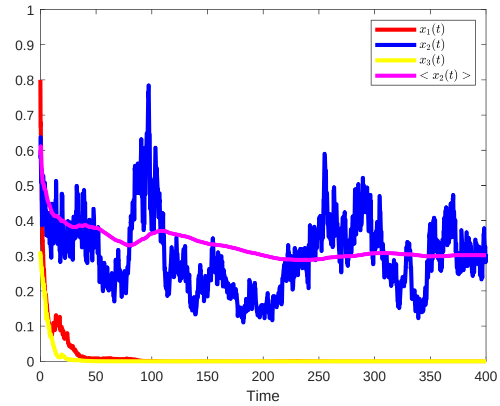

Figure 5. The solution to system (1.2) with Param(1) and $\overline{\text{Param}(5)}$, which represents that x_2 is persistent in the mean in **Case 5**, while x_1 and x_3 are extinctive.

Let

$$\text{Param}(2) = \begin{pmatrix} 0.1 & 0.1 & 0.1 & 0.3 & 0.3 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.1 & 0.2 & 0.2 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.1 & 0.2 & 0.1 & 0.3 & 0.1 & 0.1 & 0.1 \end{pmatrix},$$

subject to $x_1(\theta) = 0.8e^{\theta}$, $x_2(\theta) = 0.6e^{\theta}$, $x_3(\theta) = 0.3e^{\theta}$, $\theta \in [-\ln 2, 0]$.

Case 6.

$$\overline{\text{Param}(6)} = \begin{pmatrix} 0.4 & 0.2 & 0.1 & 0.1 & 0.1 \\ 0.4 & 0.1 & 0.1 & 0.2 & 0.1 \\ 0.1 & 0.3 & 0.1 & 0.1 & 0.1 \end{pmatrix}.$$

Then, $\Xi_1 = 0.2603$, $\Xi_2 = 0.2753$, $M_{31}^{\Theta} = -0.0150$, $M_{32}^{\Theta} = 0.0075$, $\Theta_1 = 0.0068$, $\Theta_2 = 0.0048$, $\Theta_3 = -5.1889 \times 10^{-4}$, $M_{33}^{\Theta_1} = 0.0238$, $M_{33}^{\Theta_2} = 0.0168$. From Theorem 3.1 (vi), x_1 and x_2 are persistent in the mean, while x_3 is extinctive (see Figure 6) and

$$\overline{\mathbf{x}^{\mathrm{T}}(\infty)} = \left(\frac{M_{33}^{\Theta_1}}{M_{33}^{\Theta}}, \frac{M_{33}^{\Theta_2}}{M_{33}^{\Theta}}, 0\right) = (0.7317, 0.5159, 0), \quad a.s. \tag{5.5}$$

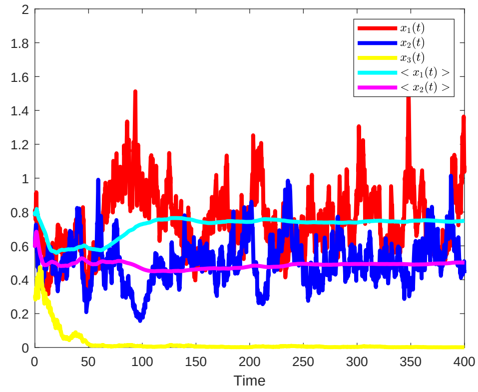

Figure 6. The solution to system (1.2) with Param(2) and $\overline{\text{Param}(6)}$, which represents that both x_1 and x_2 are persistent in the mean in **Case 6**, while x_3 is extinctive.

Case 7.

$$\overline{\text{Param}(7)} = \begin{pmatrix} 0.8 & 0.2 & 0.4 & 0.1 & 0.1 \\ 0.7 & 0.1 & 0.1 & 0.4 & 0.2 \\ 0.2 & 0.3 & 0.4 & 0.1 & 0.5 \end{pmatrix}.$$

Then, $\Xi_1 = 0.6603$, $\Xi_2 = 0.5753$, $M_{13}^{\Theta} = 0.1525$, $M_{23}^{\Theta} = -0.0700$, $M_{31}^{\Theta} = -0.0300$, $M_{32}^{\Theta} = 0.1075$, $\Theta_1 = 0.1463$, $\Theta_2 = 0.1041$, $\Theta_3 = 0.0638$. From Theorem 3.1 (vii), all species are persistent in the mean (see Figure 7) and

$$\overline{\mathbf{x}^{\mathrm{T}}(\infty)} = \left(\frac{\Theta_1}{\Theta}, \frac{\Theta_2}{\Theta}, \frac{\Theta_3}{\Theta}\right) = (0.9144, 0.6505, 0.3989) \quad a.s. \tag{5.6}$$

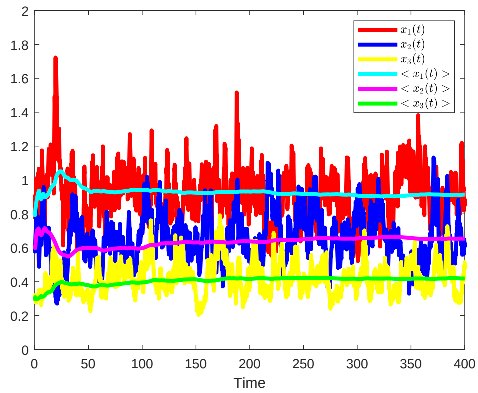

Figure 7. The solution to system (1.2) with Param(2) and $\overline{\text{Param}(7)}$, which represents that all species in **Case 7** are persistent in the mean.

Case 8.

$$\overline{\text{Param}(8)} = \begin{pmatrix} 0.2 & 0.2 & 0.3 & 0.1 & 0.2 \\ 0.3 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.3 & 0.1 & 0.2 & 0.1 \end{pmatrix}.$$

Then, $\Xi_1 = 0.0603$, $\Xi_2 = 0.1753$, $M_{31}^{\Theta} = -0.0150$, $M_{11}^{\Theta_3} = 0.0232$, $\Theta_1 = -0.0109$. From Theorem 3.1 (viii), x_1 is extinctive, while x_2 and x_3 are persistent in the mean (see Figure 8) and

$$\overline{\mathbf{x}^{\mathrm{T}}(\infty)} = \left(0, \frac{M_{11}^{\Theta_2}}{M_{11}^{\Theta}}, \frac{M_{11}^{\Theta_3}}{M_{11}^{\Theta}}\right) = (0, 0.8600, 0.3087), \quad a.s. \tag{5.7}$$

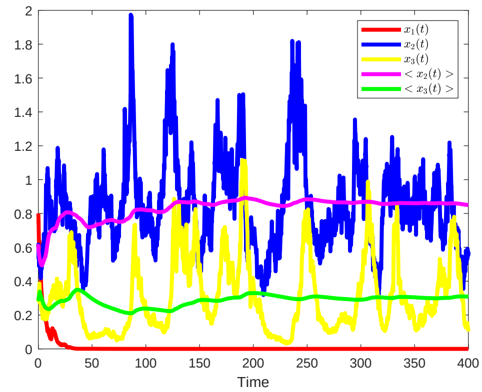

Figure 8. The solution to system (1.2) with Param(2) and $\overline{\text{Param}(8)}$, which represents that x_2 and x_3 are persistent in the mean in **Case 8**, while x_1 is extinctive.

Let

$$\text{Param}(3) = \begin{pmatrix} 0.1 & 0.1 & 0.1 & 0.3 & 0.2 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.1 & 0.2 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.1 & 0.2 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \end{pmatrix}$$

subject to $x_1(\theta) = 0.8e^{\theta}$, $x_2(\theta) = 0.6e^{\theta}$, $x_3(\theta) = 0.3e^{\theta}$, $\theta \in [-\ln 2, 0]$.

Case 9.

$$\overline{\text{Param}(9)} = \begin{pmatrix} 0.4 & 0.2 & 0.1 & 0.1 & 0.1 \\ 0.2 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.3 & 0.3 & 0.1 & 0.1 \end{pmatrix}.$$

Then, $\Xi_1 = 0.2603$, $\Xi_2 = 0.0753$, $M_{32}^{\Theta} = 0.0075$, $M_{22}^{\Theta_3} = 0.0141$, $\Theta_2 = -0.0086$. From Theorem 3.1 (ix), x_1 and x_3 are persistent in the mean, while x_2 are extinctive (see Figure 9) and

$$\overline{\mathbf{x}^{\mathrm{T}}(\infty)} = \left(\frac{M_{22}^{\Theta_1}}{M_{22}^{\Theta}}, 0, \frac{M_{22}^{\Theta_3}}{M_{22}^{\Theta}}\right) = (0.9364, 0, 0.4869) \quad a.s. \quad (5.8)$$

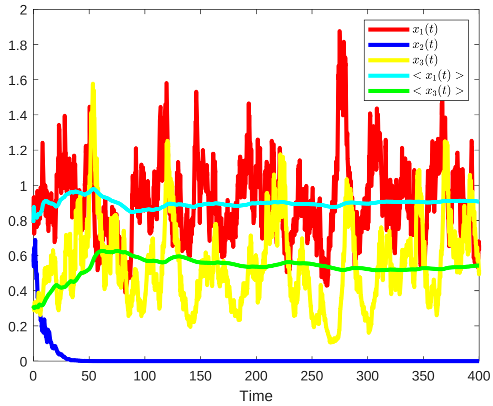

Figure 9. The solution to system (1.2) with $\widetilde{\mathrm{Param}}(3)$ and $\overline{\widetilde{\mathrm{Param}}(9)}$, which represents that x_1 and x_3 are persistent in the mean in **Case 9**, while x_2 is extinctive.

5.2. Example 2

In this example, we will consider the effect of a time delay on the stochastic persistence in the mean and extinction of the species. Let $\tau_{ji} = \ln\omega$, $\mu_{ji}(\theta) = \mu_{ji}e^{\theta}$, $\gamma_j(\mu, i) = \gamma_j(i)$, $\lambda(\mathbb{Z}) = 1$, $u(t) = 0.3 + 0.05\frac{\sin t}{t}$, $h = 0.5$. Denote

$$\widetilde{\mathrm{Param}}(i) = \begin{pmatrix} r_{11} & g_1 & h_1 & m_1 & \mu_{11} & \mu_{12} & \mu_{13} & \sigma_1 & \gamma_1 \\ r_{22} & g_2 & h_2 & m_2 & \mu_{21} & \mu_{22} & \mu_{23} & \sigma_2 & \gamma_2 \\ r_{33} & g_3 & h_3 & m_3 & \mu_{31} & \mu_{32} & \mu_{33} & \sigma_3 & \gamma_3 \end{pmatrix}$$

and

$$\overline{\widetilde{\mathrm{Param}}(i)} = \begin{pmatrix} r_1 & k_1 & a_{11} & a_{12} & a_{13} \\ r_2 & k_2 & a_{21} & a_{22} & a_{23} \\ r_3 & k_3 & a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Let

$$\widetilde{\mathrm{Param}}(1) = \begin{pmatrix} 0.1 & 0.1 & 0.1 & 0.3 & 0.2 & 0.2 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.1 & 0.2 & 0.3 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.1 & 0.2 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \end{pmatrix}$$

and

$$\overline{\widetilde{\mathrm{Param}}(1)} = \begin{pmatrix} 0.4 & 0.1 & 0.45 & 0.2 & 0.1 \\ 0.4 & 0.1 & 0.45 & 0.2 & 0.5 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \end{pmatrix}$$

subject to $x_1(\theta) = 0.8e^{\theta}$, $x_2(\theta) = 0.6e^{\theta}$, $x_3(\theta) = 0.3e^{\theta}$, $\theta \in [-\ln\omega, 0]$.

Case 10. $\omega = 2$. Then, $\Xi_1 = 0.2753$, $\Xi_2 = 0.2753$, $M_{32}^{\Theta} = 0.2125$, $M_{33}^{\Theta_2} = -0.0138$, $M_{22}^{\Theta_3} = -0.0823$. From Theorem 3.1 (iv), x_1 is persistent in the mean, while x_2 and x_3 are extinctive (see Figure 10) and

$$\overline{\mathbf{x}^{\mathrm{T}}(\infty)} = \left(\frac{\Xi_1}{A_{11}}, 0, 0\right) = (0.5006, 0, 0), \quad a.s. \qquad (5.9)$$

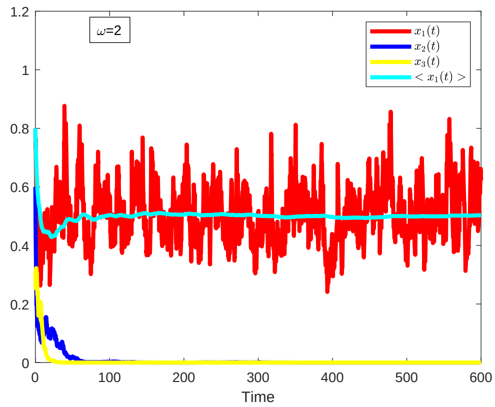

Figure 10. The solution to system (1.2) with $\widetilde{\mathrm{Param}}(1)$, $\overline{\widetilde{\mathrm{Param}}(1)}$ and $\tau_{ji} = \ln 2$, which represents that x_1 is persistent in the mean in **Case 10**, while x_2 and x_3 are extinctive.

Case 11. $\omega = 3$. Then, $\Xi_1 = 0.2753$, $\Xi_2 = 0.2753$, $M_{31}^{\Theta} = 0.1444$, $M_{33}^{\Theta_1} = -0.0184$, $M_{11}^{\Theta_3} = -0.0140$. From Theorem 3.1 (v), x_1 and x_3 are extinctive, while x_2 is persistent in the mean (see Figure 11) and

$$\overline{\mathbf{x}^{\mathrm{T}}(\infty)} = \left(0, \frac{\Xi_2}{A_{22}}, 0\right) = (0, 1.0324, 0), \quad a.s. \qquad (5.10)$$

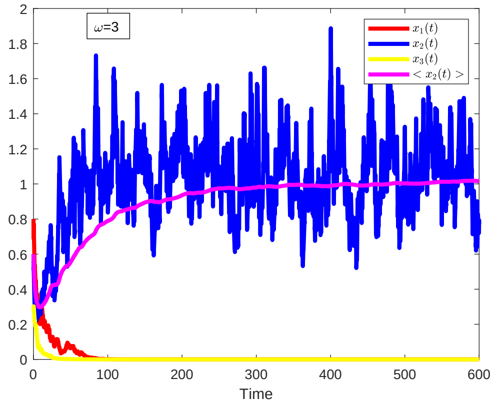

Figure 11. The solution to system (1.2) with $\widetilde{\text{Param}}(1)$, $\overline{\text{Param}(1)}$ and $\tau_{ji} = \ln 3$, which represents that both x_1 and x_3 are extinctive in **Case 11**, while x_2 is persistent in the mean.

From Figures 10 and 11, we observe that time delay can change the survival state of the species.

6. Conclusions and discussion

In this paper, we studied a stochastic, hybrid delay, one-predator-two-prey model with harvesting and jumps in a polluted environment. The main results are presented in Theorems 3.1 and 4.1. Theorem 3.1 provides sufficient conditions for a stochastic persistence in the mean and extinction of each species. Theorem 4.1 obtains sufficient conditions for the existence of an OHS and gives the explicit forms of the OHE and MESY. Our results reveal that a time delay can change the survival state of the species.

Some topics deserve further investigation. For example, it is interesting to study the existence of a stationary distribution for system (1.2) with Markov switching and infinite distributed time delays. On the other hand, one can propose and study some more realistic but complex models with different functional responses and impulsive perturbations. We leave these investigations for future work.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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