Mathematical Modelling and Control

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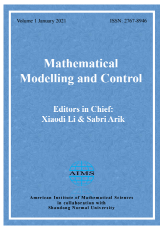

MMC, 5(1): 1–14.
DOI: 10.3934/mmc.2025001
Received: 17 December 2022
Revised: 22 April 2023
Accepted: 20 June 2023
Published: 05 March 2025

Research article

Sign-changing and signed solutions for fractional Laplacian equations with critical or supercritical nonlinearity

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Abstract: In this paper, we investigate the existence of sign-changing and signed solutions for nonlinear elliptic equations driven by nonlocal integro-differential operators with critical or supercritical nonlinearity. By combining an appropriate truncation argument with a constrained minimization method and the Moser iteration method, we obtain a sign-changing solution and a signed solution for them under some suitable assumptions. As a particular case, we drive an existence theorem of sign-changing and signed solutions for the fractional Laplacian equations with critical or supercritical growth.

Keywords: fractional Laplacian equation; Moser iteration method; truncation argument; supercritical growth; variational method

1. Introduction

This paper is devoted to the study of the existence of sign-changing and signed solutions for the following nonlocal elliptic equations:

$$\begin{cases} -\mathcal{L}_K u = \lambda |u|^{p-2}u + f(x,|u|^2)u, & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^n \backslash \Omega, \end{cases} \tag{1.1}$$

where $\mathcal{L}_K$ is the integro-differential operator defined as follows:

$$\mathcal{L}_K u(x) = \frac{1}{2}\int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x))K(y)dy, \ x \in \mathbb{R}^n,$$

here

$$K : \mathbb{R}^n \backslash \{0\} \to (0, +\infty)$$

is a function with the properties that:

(K_1) $mK \in L^1(\mathbb{R}^n)$, where

$$m(x) = \min\{|x|^2, 1\};$$

(K_2) There exist $\gamma > 0$ and $s \in (0,1)$ such that

$$K(x) \geq \gamma |x|^{-(n+2s)}$$

for any $x \in \mathbb{R}^n \setminus \{0\}$.

A typical model for K is given by the singular kernel

$$K(x) = |x|^{-(n+2s)}$$

which coincides with the fractional Laplace operator $-(-\triangle)^s$ of the following fractional Laplacian equations

$$\begin{cases} (-\triangle)^s u = \lambda |u|^{p-2}u + f(x,|u|^2)u, & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^n \backslash \Omega, \end{cases} \tag{1.2}$$

where

$$-(-\triangle)^s u(x) = \frac{1}{2}\int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} dy, \ x \in \mathbb{R}^n.$$

In problems (1.1) and (1.2), the set $\Omega \subset \mathbb{R}^n$ is an open bounded with Lipschitz boundary, $n > 2s$, $s \in (0,1)$, λ is a positive real parameter, $p \geq 2^*$ and

$$2^* := \frac{2n}{n-2s}$$

is the fractional critical Sobolev exponent. The nonlinear term f satisfies the following conditions:

(A_1) $f \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$, there exist $C > 0$ and $2 < q < 2^*$ such that

$$|f(x,t)| \leq C(1 + |t|^{\frac{q-2}{2}}), \quad \forall\ (x,t) \in \bar{\Omega} \times \mathbb{R};$$

(A_2) $\lim\limits_{t \to 0} f(x,t) = 0$ uniformly in $x \in \bar{\Omega}$;

(A_3) $\frac{f(x,t)}{t}$ is increasing in $|t| > 0$ for $a.e.\ x \in \Omega$.

The operator $(-\triangle)^s$ can be seen as the infinitesimal generators of Lévy stable diffusion Processes; see [1] and the references therein. This operator arises in several areas, such as biology, chemistry, physics and finance (see [2–4]). It is easy to see that the integro-differential operator $\mathcal{L}_K$ is a generalization of the fractional Laplace operator $-(-\triangle)^s$ (see [5–7]). Moreover, the interest in non-local integro-differential problems (1.1) goes beyond the mathematical curiosity. They have impressive applications in different fields, such as the thin obstacle problem, portfolio optimization, pricing of financial instruments, phase transitions, stratified materials, statistical mechanics, fluid flow, anomalous diffusion, crystal dislocation, deblurring and denoising of images, and so on, see [8–10]. In the past few years, a great deal of attention has been devoted to nonlocal operators of elliptic type, both for their interesting theoretical structure and in view of concrete applications, see [11, 12] and the references therein. By the minimax method, topological degree theory, or constrained minimization method, many authors obtained the existence results of nontrivial solutions, positive solutions or sign-changing solutions of some nonlinear elliptic equations, see [13–15]. To show their results, the authors always assumed the nonlinearity $f(x,t)$ involves subcritical or critical growth and/or $f(x,t)$ satisfies Ambrosetti-Rabinowitz condition. However, the existence of nontrivial solutions, positive solutions, negative solutions and sign-changing solutions for the nonlocal elliptic problem (1.1) with $p \leq 2^*$ has been investigated by using the variational method, fixed-point index theory, and critical point theorems, see [16–18]. There are only a few results about the existence and multiplicity of solutions for (1.1) with $p > 2^*$. Fortunately, Li et al. [19] investigated the following fractional Schrödinger equation with electromagnetic fields and critical or supercritical nonlinearity:

$$(-\triangle)^s_A u = \lambda |u|^{p-2}u + f(x, |u|^2)u, \quad \text{in } \mathbb{R}^n,$$

where $(-\triangle)^s_A$ is the fractional magnetic operator with

$$n > 2s, \ \ s \in (0,1), \ \ p \geq 2^* = \frac{2n}{n-2s},$$

and λ is a positive real parameter. When the nonlinearity f satisfies the Ambrosetti-Rabinowitz condition, they obtained the existence of a nontrivial solution for the above equation via truncation argument and the mountain pass theorem.

Motivated by the above works, the main purpose of this paper is to study the existence of sign-changing and signed solutions of (1.1) under the conditions (K_1), (K_2) and (A_1)–(A_3). To the best of our knowledge, there are no papers about the existence of sign-changing and signed solutions for (1.1) and (1.2) with supercritical growth.

To state our main result, we define the sets X and X_0 as

$$X = \{u \mid u:\ \mathbb{R}^n \to\ \mathbb{R},\ u\mid_\Omega \in L^2(\Omega) \text{ and } (u(x) - u(y))\sqrt{K(x-y)} \in L^2(\mathbb{R}^{2n} \setminus O)\}$$

and

$$X_0 = \{g \mid g \in X \text{ and } g = 0\ \ a.e. \text{ in } \mathbb{R}^n \setminus \Omega\},$$

where $u|_\Omega$ represents the restriction to Ω of function

$$u(x), O = (\mathbb{R}^n \setminus \Omega) \times (\mathbb{R}^n \setminus \Omega).$$

We note that X and X_0 are non-empty, since

$$C^2_0(\Omega) \subseteq X_0$$

(see [20]). We endows X with the norm defined by

$$\|g\|_X := \|g\|_2 + (\int_Q |g(x) - g(y)|^2 K(x-y)dxdy)^{\frac{1}{2}}, \quad (1.3)$$

where

$$Q = \mathbb{R}^{2n} \setminus O$$

(see [21]). Moreover, we can take the function

$$\|g\| := (\int_{\mathbb{R}^{2n}} |g(x) - g(y)|^2 K(x-y)dxdy)^{\frac{1}{2}} \quad (1.4)$$

as a norm on X_0, which is equivalent to the usual one defined in (1.3) (see [22]). Also, $(X_0, \|\cdot\|)$ is a Hilbert space with a scalar product given by

$$(u,v) := \int_{\mathbb{R}^{2n}} (u(x)-u(y))(v(x)-v(y))K(x-y)dxdy, \forall\, u,v \in X_0. \quad (1.5)$$

Lemma 1.1. *The embedding $X_0 \hookrightarrow L^\nu(\mathbb{R}^n)$ is continuous if $\nu \in [1, 2^*]$ and compact if $\nu \in [1, 2^*)$, where $u \in L^\nu(R^n)$ means $u = 0$ a.e. in $\mathbb{R}^n \backslash \Omega$.*

It is well known that there is the best fractional critical Sobolev constant, such that

$$S^* = \inf_{u \in X_0 \backslash \{0\}} \frac{\int_{\mathbb{R}^{2n}} |u(x) - u(y)|^2 K(x - y) dxdy}{(\int_{\mathbb{R}^n} |u(x)|^{2^*} dx)^{\frac{2}{2^*}}}. \tag{1.6}$$

Observing that the energy functional of (1.1) is given by

$$\begin{aligned}\mathcal{J}(u) =& \frac{1}{2} \int_{\mathbb{R}^{2n}} |u(x) - u(y)|^2 K(x - y) dxdy \\ &- \frac{\lambda}{p} \int_\Omega |u|^p dx - \frac{1}{2} \int_\Omega F(x, |u|^2) dx, \quad u \in X_0.\end{aligned}$$

To the best of our knowledge, the Sobolev embedding theorems no longer hold when $p > 2^*$. On the one hand, it causes the second integral in $\mathcal{J}$ to be divergent, which makes the functional $\mathcal{J}$ cannot be well defined on X_0. On the other hand, it leads to the lack of compactness in studying problem (1.1). Hence, we cannot directly use variational methods to prove the existence of sign-changing and signed solutions.To overcome these difficulties, we use a new method, which came from the papers [19, 23]. The main idea of this method is to reduce the supercritical problem into a subcritical one. In comparison with previous works, this paper has several new features. First, we consider the more general nonlinear term without Ambrosetti-Rabinowitz condition. Second, the nonlinear term involves supcritical growth. Finally, the existence of a sign-changing solution and a signed solution is obtained by combining an appropriate truncation argument with a constrained minimization method and the Moser iteration method. The results in this paper generalize and improve the results in [24–26]]. There have been no previous studies considering the existence of sign-changing and signed solutions for problems (1.1) and (1.2) involving supcritical growth to the best of our knowledge.

The main result of this paper is the following:

Theorem 1.1. *Suppose that (K_1), (K_2), and (A_1)–(A_3) are satisfied. Then there exists $\lambda_* > 0$ for any $\lambda \in (0, \lambda_*]$, problem (1.1) admits a sign-changing solution and a signed solution.*

Remark 1.1. *Comparing with [4, 25, 27], we prove the existence of sign-changing solutions of (1.1) without the Ambrosetti-Rabinowitz condition. The results can be regarded as the complementary work of [4, 25, 27]. Moreover, comparing with [2, 4, 27], we consider the supercritical fractional Laplace equations. Our results are new. Therefore, the results of this paper can enrich the results in the previous papers.*

Theorem 1.2. *Suppose that (A_1)–(A_3) are satisfied. Then there exists $\lambda_{**} > 0$, such that, for any $\lambda \in (0, \lambda_{**}]$, the problem (1.2) admits a sign-changing solution and a signed solution.*

This paper is organized as follows: In Section 2, we will prove the existence of sign-changing and signed solutions for the truncation problem of (1.1). Section 3 is devoted to completing the proof of Theorems 1.1 and 1.2.

2. Preliminaries

In this section, we give a truncation argument in order to overcome the lack of compactness in studying critical and supercritical growth. Let $M > 0$ be a constant. For each $\lambda > 0$ and $M > 0$ fixed, we investigate the existence of sign-changing and signed solutions for the following truncation problem:

$$\begin{cases} -\mathcal{L}_K u = \lambda \varphi(u) u + f(x, |u|^2) u, & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^n \backslash \Omega, \end{cases} \tag{2.1}$$

where

$$\varphi(t) = \begin{cases} |t|^{p-2}, & 0 \leq |t| \leq M, \\ M^{p-q} |t|^{q-2}, & |t| > M. \end{cases}$$

To investigate (2.1), we define the energy functional

$$I_\lambda : X_0 \longrightarrow \mathbb{R}$$

by

$$\begin{aligned}I_\lambda(u) =& \frac{1}{2} \int_{\mathbb{R}^{2n}} |u(x) - u(y)|^2 K(x - y) dxdy \\ &- \frac{1}{2} \int_\Omega F(x, |u|^2) dx - \lambda \int_\Omega \Phi(u) dx, \quad u \in X_0,\end{aligned} \tag{2.2}$$

where

$$\Phi(t) = \int_0^t \varphi(\tau) \tau d\tau.$$

By (A_1) and the standard argument, it is easy to obtain that $I_\lambda \in C^1(X_0, \mathbb{R})$ and

$$\langle I'_\lambda(u), v\rangle = \int_{\mathbb{R}^{2n}} (u(x) - u(y))(v(x) - v(y))K(x - y)dxdy - \int_\Omega f(x, |u|^2)uvdx - \lambda \int_\Omega \varphi(u)uvdx, \quad (2.3)$$

where $u, v \in X_0$.

Let

$$u^+(x) := \max\{u(x), 0\}, \ \ u^-(x) := \min\{u(x), 0\},$$

for any

$$u = u^+ + u^- \in X_0,$$

we have

$$\begin{aligned} \|u\|^2 =& \|u^+\|^2 + \|u^-\|^2 \\ &- \int_{\mathbb{R}^{2n}} (u^+(x)u^-(y) + u^-(x)u^+(y))K(x - y)dxdy \\ &\geq \|u^+\|^2 + \|u^-\|^2, \\ I_\lambda(u) =& I_\lambda(u^+) + I_\lambda(u^-) \\ &- \int_{\mathbb{R}^{2n}} (u^+(x)u^-(y) + u^-(x)u^+(y))K(x - y)dxdy \\ &\geq I_\lambda(u^+) + I_\lambda(u^-) \end{aligned} \quad (2.4)$$

and

$$\langle I'_\lambda(u), u^+\rangle = \langle I'_\lambda(u^+), u^+\rangle - \int_{\mathbb{R}^{2n}} (u^+(x)u^-(y) + u^-(x)u^+(y))K(x - y)dxdy.$$

Obviously, the critical points of I_λ are equivalent to the weak solutions of problem (2.1). Furthermore, if $u \in X_0$ is a solutions of (2.1) and $u^\pm \neq 0$ in Ω, then u is called a sign-changing solution of (2.1). If $u \in X_0$ is a solution of (2.1) and $u > 0$ (or $u < 0$) in Ω, then u is called a signed solution of (2.1).

Next, we consider the minimization problems:

$$m_1 := \inf\{I_\lambda(u) : u \in \mathcal{M}\}, \quad m_2 := \inf\{I_\lambda(u) : u \in \mathcal{N}\}, \quad (2.5)$$

where

$$\mathcal{M} = \{u \in \mathcal{N} : u^\pm \neq 0, \ \langle I'_\lambda(u), u^+\rangle = \langle I'_\lambda(u), u^-\rangle = 0\}$$

and

$$\mathcal{N} = \{u \in X_0 \setminus \{0\} : \langle I'_\lambda(u), u\rangle = 0\}.$$

Theorem 2.1. *Suppose that (K_1), (K_2) and (A_1)–(A_3) are satisfied. Then, for each $\lambda > 0, M > 0$, problem (2.1) admits a sign-changing solution $u_1 \in \mathcal{M}$ and a signed solution $u_2 \in \mathcal{N}$. Furthermore,*

$$I_\lambda(u_1) = \inf_{\mathcal{M}} I_\lambda(u) > 0, \quad I_\lambda(u_2) = \inf_{\mathcal{N}} I_\lambda(u) > 0.$$

In the following, we shall give some properties for $\mathcal{M}$ and $\mathcal{N}$. By (A_1) and (A_2), we easily see that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(x, t^2)| \leq \varepsilon + C_\varepsilon|t|^{q-2}, \ \ |F(x, t^2)| \leq \varepsilon|t|^2 + C_\varepsilon|t|^q \quad (2.6)$$

for all $t \in \mathbb{R}$ and $2 < q < 2^*$. By (A_1)–(A_3), we easily deduce that

$$\begin{aligned} &\frac{1}{2}f(x, t)t - F(x, t) \text{ be increasing in } |t| > 0 \text{ for } a.e.\ x \in \Omega, \\ &f(x, t) \text{ be increasing in } |t| > 0 \text{ for } a.e.\ x \in \Omega, \quad (2.7) \\ &\frac{1}{2}f(x, t)t - F(x, t) > 0, \quad F(x, t) > 0, \quad a.e.\ x \in \Omega, \ t \in \mathbb{R}\backslash\{0\}, \quad (2.8) \end{aligned}$$

and

$$\lim_{|t|\to+\infty} \frac{F(x, t)}{t} = \lim_{|t|\to+\infty} f(x, t) = +\infty, \qquad a.e.\ x \in \Omega. \quad (2.9)$$

First, we show that the sets $\mathcal{M}$ and $\mathcal{N}$ are nonempty in X_0, and then we seek critical points of I_λ by constraint minimizations on $\mathcal{M}$ and $\mathcal{N}$.

Lemma 2.1. *Suppose that (K_1), (K_2) and (A_1)–(A_3) hold.*

(1) If $u \in X_0$ with $u^\pm \neq 0$, then there exists a unique pair $(\alpha_u, \beta_u) \in \mathbb{R}_+ \times \mathbb{R}_+$ such that $\alpha_u u^+ + \beta_u u^- \in \mathcal{M}$.

(2) If $u \in X_0 \setminus \{0\}$, then there exists a unique number $t_u > 0$ such that $t_u u \in \mathcal{N}$ and

$$I_\lambda(t_u u) = \max_{t\geq0} I_\lambda(tu).$$

Proof. (1) For fixed $u \in X_0$ with

$$u^\pm \neq 0,$$

we claim the existence of α_u and β_u.

Set

$$
\begin{aligned}
h_1(\alpha,\beta) =&\langle I'_\lambda(\alpha u^+ + \beta u^-), \alpha u^+\rangle \\
=& \int_{\mathbb{R}^{2n}} (\alpha u^+ + \beta u^-)(x)(\alpha u^+(x) - \alpha u^+(y))K(x-y)dxdy \\
&- \int_{\mathbb{R}^{2n}} (\alpha u^+ + \beta u^-)(y)(\alpha u^+(x) - \alpha u^+(y))K(x-y)dxdy \\
&- \int_\Omega f(x, |\alpha u^+ + \beta u^-|^2)|\alpha u^+|^2 dx - \lambda \int_\Omega \varphi(\alpha u^+)|\alpha u^+|^2 dx \\
=&\alpha^2\|u^+\|^2 - \int_\Omega f(x, |\alpha u^+|^2)|\alpha u^+|^2 dx \\
&- \lambda \int_\Omega \varphi(\alpha u^+)|\alpha u^+|^2 dx \\
&- \alpha\beta \int_{\mathbb{R}^{2n}} (u^-(x)u^+(y) + u^-(y)u^+(x))K(x-y)dxdy,
\end{aligned} \tag{2.10}
$$

$$
\begin{aligned}
h_2(\alpha,\beta) =&\langle I'_\lambda(\alpha u^+ + \beta u^-), \beta u^-\rangle \\
=& \int_{\mathbb{R}^{2n}} (\alpha u^+ + \beta u^-)(x)(\beta u^-(x) - \beta u^-(y))K(x-y)dxdy \\
&- \int_{\mathbb{R}^{2n}} -(\alpha u^+ + \beta u^-)(y)(\beta u^-(x) - \beta u^-(y))K(x-y)dxdy \\
&- \int_\Omega f(x, |\alpha u^+ + \beta u^-|^2)|\beta u^-|^2 dx - \lambda \int_\Omega \varphi(\beta u^-)|\beta u^-|^2 dx \\
=&\beta^2\|u^-\|^2 - \alpha\beta \int_{\mathbb{R}^{2n}} (u^-(x)u^+(y) + u^-(y)u^+(x))K(x-y)dxdy \\
&- \int_\Omega f(x, |\beta u^-|^2)|\beta u^-|^2 dx - \lambda \int_\Omega \varphi(\beta u^-)|\beta u^-|^2 dx.
\end{aligned} \tag{2.11}
$$

By (2.6) and $q \in (2, 2^*)$, we can find that

$$h_1(\alpha,\alpha) > 0, \quad h_2(\alpha,\alpha) > 0$$

for a sufficiently small $\alpha > 0$ and

$$h_1(\beta,\beta) < 0, \quad h_2(\beta,\beta) < 0$$

for a sufficiently large $\beta > 0$. Therefore, there exist $0 < r < R$ such that

$$h_1(r,r) > 0, \ \ h_2(r,r) > 0, \ \ h_1(R,R) < 0, \ \ h_2(R,R) < 0. \tag{2.12}$$

Taking into account (2.10)–(2.12), we deduce

$$h_1(r,\beta) > 0, \ h_1(\beta,R) < 0, \ \forall \beta \in [r,R]$$

and

$$h_2(r,\alpha) > 0, \ h_2(\alpha,R) < 0, \ \forall \alpha \in [r,R].$$

Therefore, there exists some point $(\alpha_u, \ \beta_u)$ with

$$r < \alpha_u, \beta_u < R,$$

such that

$$h_1(\alpha_u,\beta_u) = h_2(\alpha_u,\beta_u) = 0$$

by Miranda's theorem. Thus

$$\alpha_u u^+ + \beta_u u^- \in \mathcal{M}.$$

Next, we prove the uniqueness of the pair (α_u, β_u).

Case 1. $u \in \mathcal{M}$.

Assume $u \in \mathcal{M}$, we have

$$u^+ + u^- = u \in \mathcal{M}.$$

We obtain

$$\langle I'_\lambda(u), u^+\rangle = \langle I'_\lambda(u), u^-\rangle = 0,$$

that is

$$
\begin{aligned}
&\int_{\mathbb{R}^{2n}} (u^-(x)u^+(y) + u^-(y)u^+(x))K(x-y)dxdy \\
&- \int_{\mathbb{R}^{2n}} (u^+(x) - u^+(y))^2 K(x-y)dxdy \\
&= - \int_\Omega f(x, |u^+|^2)|u^+|^2 dx + \lambda \int_\Omega \varphi(u^+)|u^+|^2 dx
\end{aligned} \tag{2.13}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^{2n}} (u^-(x)u^+(y) + u^-(y)u^+(x))K(x-y)dxdy \\
&- \int_{\mathbb{R}^{2n}} (u^-(x) - u^-(y))^2 K(x-y)dxdy \\
&= - \int_\Omega f(x, |u^-|^2)|u^-|^2 dx + \lambda \int_\Omega \varphi(u^-)|u^-|^2 dx.
\end{aligned} \tag{2.14}
$$

Now we prove that there exists a unique pair

$$(\alpha_u, \beta_u) = (1, 1),$$

such that

$$\alpha_u u^+ + \beta_u u^- \in \mathcal{M}.$$

If there exists another pair $(\tilde{\alpha}_u, \tilde{\beta}_u)$ such that

$$\tilde{\alpha}_u u^+ + \tilde{\beta}_u u^- \in \mathcal{M},$$

then we obtain

$$
\begin{aligned}
&\tilde{\alpha}_u\tilde{\beta}_u \int_{\mathbb{R}^{2n}} (u^-(x)u^+(y) + u^-(y)u^+(x))K(x-y)dxdy - \tilde{\alpha}_u^2\|u^+\|^2 \\
&= - \int_\Omega f(x, |\tilde{\alpha}_u u^+|^2)|\tilde{\alpha}_u u^+|^2 dx - \lambda \int_\Omega \varphi(\tilde{\alpha}_u u^+)|\tilde{\alpha}_u u^+|^2 dx
\end{aligned} \tag{2.15}
$$

and

$$\tilde{\alpha}_u\tilde{\beta}_u \int_{\mathbb{R}^{2n}} (u^-(x)u^+(y) + u^-(y)u^+(x))K(x-y)dxdy - \tilde{\beta}_u^2\|u^-\|^2$$
$$= -\int_\Omega f(x, |\tilde{\beta}_u u^-|^2)|\tilde{\beta}_u u^-|^2 dx - \lambda \int_\Omega \varphi(\tilde{\beta}_u u^-)|\tilde{\beta}_u u^-|^2 dx. \tag{2.16}$$

Assume that $0 < \tilde{\alpha}_u \le \tilde{\beta}_u$, by using (2.15), we deduce

$$\tilde{\alpha}_u^2(\|u^+\|^2 - \int_{\mathbb{R}^{2n}} (u^-(x)u^+(y) + u^-(y)u^+(x))K(x-y)dxdy)$$
$$\le \int_\Omega f(x, |\tilde{\alpha}_u u^+|^2)|\tilde{\alpha}_u u^+|^2 dx + \lambda \int_\Omega \varphi(\tilde{\alpha}_u u^+)|\tilde{\alpha}_u u^+|^2 dx.$$

Multiply the above inequality by $\tilde{\alpha}_u^{-2}$, we obtain

$$\|u^+\|^2 - \int_{\mathbb{R}^{2n}} (u^-(x)u^+(y) + u^-(y)u^+(x))K(x-y)dxdy$$
$$\le \int_\Omega f(x, |\tilde{\alpha}_u u^+|^2)|u^+|^2 dx + \lambda \int_\Omega \varphi(\tilde{\alpha}_u u^+)|u^+|^2 dx. \tag{2.17}$$

Putting together (2.17) and (2.13), we have

$$\begin{aligned} &\int_\Omega (f(x, |\tilde{\alpha}_u u^+|^2) - f(x, |u^+|^2))|u^+|^2 dx \\ &+ \lambda \int_\Omega (\varphi(\tilde{\alpha}_u u^+) - \varphi(u^+))|u^+|^2 dx \ge 0. \end{aligned} \tag{2.18}$$

Since $\varphi(t)$ is increasing in $t > 0$, combining (2.7) and (2.18), we obtain

$$1 \le \tilde{\alpha}_u \le \tilde{\beta}_u.$$

Similarly, by (2.16), it results

$$\int_\Omega (f(x, |\tilde{\beta}_u u^-|^2) - f(x, |u^-|^2))|u^-|^2 dx$$
$$+ \lambda \int_\Omega (\varphi(\tilde{\beta}_u u^-) - \varphi(u^-))|u^-|^2 dx \le 0,$$

which implies $\tilde{\beta}_u \le 1$. Then, combining

$$1 \le \tilde{\alpha}_u \le \tilde{\beta}_u,$$

we have

$$\tilde{\alpha}_u = \tilde{\beta}_u = 1.$$

Case 2. $u \notin \mathcal{M}$.

(1) Assume $u \notin \mathcal{M}$, then there exists a pair (α_u, β_u) such that

$$\alpha_u u^+ + \beta_u u^- \in \mathcal{M}.$$

If there exists another pair $(\hat{\alpha}_u, \hat{\beta}_u)$ such that

$$\hat{\alpha}_u u^+ + \hat{\beta}_u u^- \in \mathcal{M}.$$

Set

$$w := \alpha_u u^+ + \beta_u u^-$$

and

$$\hat{w} := \hat{\alpha}_u u^+ + \hat{\beta}_u u^-,$$

we have

$$\frac{\hat{\alpha}_u}{\alpha_u} w^+ + \frac{\hat{\beta}_u}{\beta_u} w^- = \hat{\alpha}_u u^+ + \hat{\beta}_u u^- = \hat{w} \in \mathcal{M}.$$

Since $w \in \mathcal{M}$, we have

$$\alpha_u = \hat{\alpha}_u \quad \text{and} \quad \beta_u = \hat{\beta}_u.$$

So, there exists a unique pair (α_u, β_u) such that

$$\alpha_u u^+ + \beta_u u^- \in \mathcal{M}.$$

(2) For $t > 0$, let

$$h(t) = I_\lambda(tu) = \frac{t^2}{2} \int_{\mathbb{R}^{2n}} |u(x) - u(y)|^2 K(x-y)dxdy$$
$$- \frac{1}{2} \int_\Omega F(x, |tu|^2)dx - \lambda \int_\Omega \Phi(tu)dx.$$

By (2.6) and Lemma 1.1, for $\varepsilon > 0$ sufficiently small we have

$$h(t) \ge \frac{t^2}{4}\|u\|^2 - C_1(C_\varepsilon + \lambda C_0)t^q\|u\|^q,$$

where

$$C_0 = \frac{1}{q} M^{p-q}.$$

Since $q > 2$, we obtain that $h(t) > 0$ for $t > 0$ small. From the Eq (2.9), we easily get that $h(t) \to -\infty$ as $t \to +\infty$. Hence h has a positive maximum at

$$t = t_u > 0.$$

Therefore, $h'(t_u) = 0$ and $t_u u \in \mathcal{N}$. Obviously, $h'(t) = 0$ is equivalent to

$$\|u\|^2 = \int_\Omega f(x, |tu|^2)|u|^2 dx + \lambda \int_\Omega \varphi(tu)|u|^2 dx. \tag{2.19}$$

From (2.7), the right side of (2.19) is increasing for $t > 0$. As a consequence, there exists a unique number $t_u > 0$ such that (2.19) holds. The uniqueness of t_u is proved, and

$$I_\lambda(t_u u) = \max_{t \ge 0} I_\lambda(tu).$$

□

Lemma 2.2. *Suppose that* (K_1), (K_2), *and* (A_1)–(A_3) *hold.*

(1) If

$$\langle I'_\lambda(u), u^\pm\rangle \le 0$$

for fixed $u \in X_0$ *with* $u^\pm \neq 0$, *then there exists a unique pair*

$$(\alpha_u, \beta_u) \in (0,1] \times (0,1],$$

such that

$$\langle I'_\lambda(\alpha_u u^+ + \beta_u u^-), \alpha_u u^+\rangle = \langle I'_\lambda(\alpha_u u^+ + \beta_u u^-), \beta_u u^-\rangle = 0.$$

(2) If

$$\langle I'_\lambda(u), u\rangle \le 0$$

for fixed $u \in X_0 \setminus \{0\}$, *then there exists a unique number* $t_u \in (0,1]$ *such that*

$$\langle I'_\lambda(t_u u), t_u u\rangle = 0.$$

Proof. We only prove Lemma 2.2 (1); the proof of Lemma 2.3 (2) is analogous. □

For fixed $u \in X_0$ with $u^\pm \neq 0$, by Lemma 2.1, we obtain that there exist a unique pair $(\alpha_u,\ \beta_u)$ such that

$$\alpha_u u^+ + \beta_u u^- \in \mathcal{M}.$$

Assume that $\alpha_u \ge \beta_u > 0$. In addition,

$$\begin{aligned}
&\alpha_u^2(\|u^+\|^2 - \int_{\mathbb{R}^{2n}} (u^-(x)u^+(y) + u^-(y)u^+(x))K(x-y)dxdy) \\
&\ge \alpha_u^2\|u^+\|^2 - \alpha_u\beta_u \int_{\mathbb{R}^{2n}} u^-(x)u^+(y)K(x-y)dxdy \\
&- \alpha_u\beta_u \int_{\mathbb{R}^{2n}} u^-(y)u^+(x)K(x-y)dxdy \\
&= \int_\Omega f(x, |\alpha_u u^+|^2)|\alpha_u u^+|^2 dx + \lambda \int_\Omega \varphi(\alpha_u u^+)|\alpha_u u^+|^2 dx.
\end{aligned} \tag{2.20}$$

Since

$$\langle I'_\lambda(u), u^+\rangle \le 0,$$

it holds

$$\begin{aligned}
&\|u^+\|^2 - \int_{\mathbb{R}^{2n}} (u^-(x)u^+(y) + u^-(y)u^+(x))K(x-y)dxdy \\
&\le \int_\Omega f(x, |u^+|^2)|u^+|^2 dx + \lambda \int_\Omega \varphi(u^+)|u^+|^2 dx.
\end{aligned} \tag{2.21}$$

Therefore (2.20) and (2.21) lead to

$$\begin{aligned}
&\int_\Omega (f(x, |\alpha_u u^+|^2) - f(x, |u^+|^2))|u^+|^2 dx \\
&+ \lambda \int_\Omega (\varphi(\alpha_u u^+) - \varphi(u^+))|u^+|^2 dx \le 0.
\end{aligned}$$

By (2.7), we have $\alpha_u \le 1$. Thus, $0 < \beta_u \le \alpha_u \le 1$.

Lemma 2.3. *For fixed* $u \in X_0$ *with* $u^\pm \neq 0$, *then* (α_u, β_u) *obtained in Lemma 2.2 is the unique maximum point of the function*

$$\Theta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R},$$

where

$$\Theta(\alpha, \beta) = I_\lambda(\alpha u^+ + \beta u^-).$$

Proof. By Lemma 2.1, it yields that (α_u, β_u) is the unique critical point of Θ in $\mathbb{R}^+ \times \mathbb{R}^+$. By (2.9), we can see that

$$\Theta(\alpha, \beta) \to -\infty$$

uniformly as

$$|(\alpha, \beta)| \to +\infty,$$

then we can prove that there is no maximum point on the boundary of $(\mathbb{R}^+, \mathbb{R}^+)$. If we suppose that there exists $\bar\beta \ge 0$ such that $(0, \bar\beta)$ is a maximum point of Θ. Since

$$\begin{aligned}
\Theta(\alpha, \bar\beta) =& \frac{1}{2}\int_{\mathbb{R}^{2n}} (\alpha u^+(x) + \bar\beta u^-(x) - \alpha u^+(y) - \bar\beta u^-(y))^2 K(x-y)dxdy \\
&- \frac{1}{2}\int_\Omega F(x, |\alpha u^+ + \bar\beta u^-|^2)dx - \lambda \int_\Omega \Phi(\alpha u^+ + \bar\beta u^-)dx
\end{aligned}$$

is an increasing function of α for α sufficiently small, the pair $(0, \bar\beta)$ cannot be a maximum point of Θ in $\mathbb{R}^+ \times \mathbb{R}^+$. □

Lemma 2.4. *Suppose that* (K_1), (K_2) *and* (A_1)–(A_3) *hold, then*

(1)

$$m_1 = \inf_{\substack{u\in X_0,\\ u^\pm\neq 0}} \max_{\alpha\ge 0, \beta\ge 0} I_\lambda(\alpha u^+ + \beta u^-)$$

and

$$m_2 = \inf_{u\in X_0\setminus\{0\}} \max_{t\ge 0} I_\lambda(tu).$$

(2) $m_1 > 0$ *and* $m_2 > 0$ *can be achieved respectively.*

Proof. (1) By Lemmas 2.1 and 2.3, it is easy to see that

$$m_1 = \inf_{\substack{u\in X_0,\\ u^\pm\neq 0}} \max_{\alpha\ge 0, \beta\ge 0} I_\lambda(\alpha u^+ + \beta u^-)$$

and

$$m_2 = \inf_{u\in X_0\setminus\{0\}} \max_{t\geq 0} I_\lambda(tu).$$

(2) For $u \in \mathcal{M}$, we obtain

$$\langle I'_\lambda(u), u\rangle = 0.$$

By (2.6), for any $\varepsilon > 0$ sufficiently small, we have

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_\Omega F(x,|u|^2)dx - \lambda\int_\Omega \Phi(u)dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{\varepsilon}{2}\int_\Omega |u|^2dx - \frac{C_\varepsilon}{2}\int_\Omega |u|^q dx - \lambda C_0\int_\Omega |u|^q dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{C_2\varepsilon}{2}\|u\|^2 - C_3\|u\|^q \\ &= \frac{1}{2}(1-C_2\varepsilon)\|u\|^2 - C_3\|u\|^q. \end{aligned}$$

Taking

$$\varepsilon = \frac{1}{2C_2},$$

then for sufficiently small $\rho > 0$ where

$$S_\rho := \{u \in X_0 : \|u\| = \rho\},$$

we can know

$$\inf_{u\in S_\rho} I_\lambda(u) > 0.$$

For $u \in \mathcal{M}$, there exists $t > 0$ such that $tu \in S_\rho$. From Lemmas 2.1 and 2.3, we obtain

$$\max_{\alpha\geq 0,\beta\geq 0} I_\lambda(\alpha u^+ + \beta u^-) \geq I_\lambda(tu^+ + tu^-) = I_\lambda(tu) \geq \inf_{u\in S_\rho} I_\lambda(u).$$

Therefore,

$$m_1 := \inf_{\substack{u\in X_0,\\ u^\pm\neq 0}} \max_{\alpha\geq 0,\beta\geq 0} I_\lambda(\alpha u^+ + \beta u^-) \geq \inf_{u\in S_\rho} I_\lambda(u) > 0.$$

Let

$$\{u_n\} \subset \mathcal{M}$$

be such that

$$I_\lambda(u_n) \to m,$$

then we claim that $\{u_n\}$ is bounded. By contradiction, we may suppose that $\|u_n\| \to \infty$ as $n \to \infty$. Let

$$\omega_n = \frac{u_n}{\|u_n\|},$$

then $\omega_n \in X_0$ and $\|\omega_n\| = 1$. Passing to a subsequence if necessary, we may assume that there exists $\omega \in X_0$ such that $\omega_n \rightharpoonup \omega$ in X_0, $\omega_n \to \omega$ in $L^r(\mathbb{R}^n)$, where $2 \leq r < 2^*$, $\omega_n \to \omega$ a.e. in $\mathbb{R}^n$.

If $\omega \neq 0$, then $|\Omega_{\neq}| > 0$, where

$$\Omega_{\neq} = \{x \in \mathbb{R}^n, \omega(x) \neq 0\}.$$

In view of

$$\lim_{n\to\infty}\frac{u_n(x)}{\|u_n\|} = \lim_{n\to\infty}\omega_n(x) = \omega(x) \neq 0,\ x \in \Omega_{\neq}.$$

So

$$|u_n(x)| \to \infty,\ x \in \Omega_{\neq}.$$

Noting that

$$\begin{aligned} m_1 + o(1) &= I(u_n) \\ &= \frac{1}{2}\|u_n\|^2 - \frac{1}{2}\int_\Omega F(x,|u_n|^2)dx - \lambda\int_\Omega \Phi(u_n)dx, \end{aligned}$$

we have

$$0 \leftarrow \frac{m_1+o(1)}{\|u_n\|^2} = \frac{1}{2} - \frac{1}{2}\int_\Omega \frac{F(x,|u_n|^2)}{\|u_n\|^2}dx - \lambda\int_\Omega \frac{\Phi(u_n)}{\|u_n\|^2}dx,$$

consequently,

$$\begin{aligned} 1 &= \int_\Omega \frac{F(x,|u_n|^2)}{|u_n|^2}|\omega_n|^2dx + 2\lambda\int_\Omega \frac{\Phi(u_n)}{\|u_n\|^2}dx + o(1) \\ &\geq \int_{\Omega_{\neq}} \frac{F(x,|u_n|^2)}{|u_n|^2}|\omega_n|^2dx + o(1). \end{aligned}$$

Therefore, by Fatou's lemma and (2.9), we have

$$\begin{aligned} 1 &\geq \liminf_{n\to\infty}\int_{\Omega_{\neq}} \frac{F(x,|u_n|^2)}{|u_n|^2}|\omega_n|^2dx \\ &\geq \int_{\Omega_{\neq}} \liminf_{n\to\infty}\frac{F(x,|u_n|^2)}{|u_n|^2}|\omega_n|^2dx \to +\infty, \end{aligned}$$

which is a contradiction.

If $\omega \equiv 0$, then $\omega_n \to 0$ in $L^r(\mathbb{R}^n)$. So,

$$\int_\Omega F(x,|s\omega_n|^2)dx \to 0 \ \text{ for all } \ s \in \mathbb{R}.$$

So, by Lemma 2.1, we have

$$\begin{aligned} m_1 + 1 &\geq I_\lambda(u_n) \geq I_\lambda(s\omega_n) \\ &= \frac{1}{2}s^2 - \frac{1}{2}\int_\Omega F(x,|s\omega_n|^2)dx - \lambda\int_\Omega \Phi(s\omega_n)dx \\ &\to \frac{1}{2}s^2. \end{aligned}$$

Taking
$$s > \sqrt{2(m_1 + 1)},$$
it is a contradiction. Thus, $\{u_n\}$ is bounded in X_0. By Lemma 1.1, up to a subsequence, we can assume that

$$\begin{aligned} &u_n^{\pm} \rightharpoonup u_1^{\pm} \text{ in } X_0,\\ &u_n^{\pm} \to u_1^{\pm} \text{ in } L^r(\mathbb{R}^n), 2 \le r < 2^*,\\ &u_n^{\pm} \to u_1^{\pm} \text{ a.e. in } \Omega. \end{aligned} \tag{2.22}$$

In addition, $(A_1), (A_2)$, and Lemma 1.1 lead to

$$\begin{aligned} \lim_{n\to\infty}\int_\Omega f(x,|u_n^{\pm}|^2)|u_n^{\pm}|^2dx &= \int_\Omega f(x,|u_1^{\pm}|^2)|u_1^{\pm}|^2dx,\\ \lim_{n\to\infty}\int_\Omega F(x,|u_n^{\pm}|^2)dx &= \int_\Omega F(x,|u_1^{\pm}|^2)dx. \end{aligned} \tag{2.23}$$

Since $u_n \in \mathcal{M}$, then

$$\langle I'_\lambda(u_n), u_n^{\pm}\rangle = 0,$$

that is

$$\begin{aligned} &\|u_n^+\|^2 - \int_{\mathbb{R}^{2n}} (u_n^-(x)u_n^+(y) + u_n^-(y)u_n^+(x))K(x-y)dxdy\\ &= \int_\Omega f(x,|u_n^+|^2)|u_n^+|^2dx + \lambda\int_\Omega \varphi(u_n^+)|u_n^+|^2dx \end{aligned} \tag{2.24}$$

and

$$\begin{aligned} &\|u_n^-\|^2 - \int_{\mathbb{R}^{2n}} (u_n^-(x)u_n^+(y) + u_n^-(y)u_n^+(x))K(x-y)dxdy\\ &= \int_\Omega f(x,|u_n^-|^2)|u_n^-|^2dx + \lambda\int_\Omega \varphi(u_n^-)|u_n^-|^2dx. \end{aligned} \tag{2.25}$$

Thanks to (2.6), (2.24), and (2.25), we have

$$\begin{aligned} \|u_n^{\pm}\|^2 &= \int_\Omega f(x,|u_n^{\pm}|^2)|u_n^{\pm}|^2dx + \lambda\int_\Omega \varphi(u_n^{\pm})|u_n^{\pm}|^2dx\\ &\le \varepsilon\int_\Omega |u_n^{\pm}|^2dx + C_\varepsilon\int_\Omega |u_n^{\pm}|^qdx + \lambda C_0\int_\Omega |u_n^{\pm}|^qdx\\ &\le \varepsilon C_2\|u_n^{\pm}\|^2 + C_4\|u_n^{\pm}\|^q. \end{aligned}$$

Choose
$$\varepsilon = \frac{1}{2C_2}.$$
Thus, there exists a constant $\delta > 0$ such that

$$\|u_n^{\pm}\|^2 \ge \delta.$$

By applying (2.24) and (2.25) again, we deduce that

$$\delta \le \|u_n^{\pm}\|^2 \le \varepsilon\int_\Omega |u_n^{\pm}|^2dx + (C_\varepsilon + \lambda C_0)\int_\Omega |u_n^{\pm}|^qdx.$$

Since $\{u_n\}$ is bounded, by Lemma 1.1, there is

$$C_5 > 0, \quad C_6 > 0,$$

such that
$$\delta \le \varepsilon C_5 + C_6\int_\Omega |u_n^{\pm}|^qdx.$$
Picking
$$\varepsilon = \frac{\delta}{2C_5},$$
we have
$$\int_\Omega |u_n^{\pm}|^qdx \ge \frac{\delta}{2C_6}. \tag{2.26}$$

By (2.22) and (2.26), we have
$$\int_\Omega |u_1^{\pm}|^qdx \ge \frac{\delta}{2C_\varepsilon}.$$
Thus,
$$u_1^{\pm} \ne 0.$$
By Lemma 2.1, there exists

$$\alpha_u \ \ \beta_u > 0$$

such that
$$\bar{u}_1 := \alpha_{u_1}u_1^+ + \beta_{u_1}u_1^- \in \mathcal{M}.$$
Next, we aim to prove that

$$\alpha_{u_1} = \beta_{u_1} = 1.$$

Putting together (2.22), (2.24) and Fatou's lemma, we deduce

$$\begin{aligned} &\|u_1^+\|^2 - \int_{\mathbb{R}^{2n}} (u_1^-(x)u_1^+(y) + u_1^-(y)u_1^+(x))K(x-y)dxdy\\ &\le \int_\Omega f(x,|u_1^+|^2)|u_1^+|^2dx + \lambda\int_\Omega \varphi(u_1^+)|u_1^+|^2dx. \end{aligned} \tag{2.27}$$

By (2.27) and Lemma 2.1, we have

$$\alpha_{u_1} \le 1.$$

In the similar way, we can obtain

$$\beta_{u_1} \le 1.$$

By (2.7), it follows that

$$\begin{aligned}
m_1 \leq & I_\lambda(\bar{u}_1) = I_\lambda(\bar{u}_1) - \frac{1}{4}\langle I'_\lambda(\bar{u}_1), \bar{u}_1\rangle \\
= & \frac{1}{4}\|\bar{u}_1\|^2 + \lambda \int_\Omega (\frac{1}{4}\varphi(\bar{u}_1)|\bar{u}_1|^2 - \Phi(\bar{u}_1))dx \\
& + \frac{1}{2}\int_\Omega (\frac{1}{2}f(x, |\bar{u}_1|^2)|\bar{u}_1|^2 - F(x, |\bar{u}_1|^2))dx \\
= & \frac{1}{4}\|\bar{u}_1\|^2 + \lambda \int_\Omega (\frac{1}{4}\varphi(\alpha_{u_1}u_1^+)|\alpha_{u_1}u_1^+|^2 - \Phi(\alpha_{u_1}u_1^+))dx \\
& + \frac{1}{2}\int_\Omega (\frac{1}{2}f(x, |\alpha_{u_1}u_1^+|^2)|\alpha_{u_1}u_1^+|^2 - F(x, |\alpha_{u_1}u_1^+|^2))dx \\
& + \frac{1}{2}\int_\Omega (\frac{1}{2}f(x, |\beta_{u_1}u_1^-|^2)|\beta_{u_1}u_1^-|^2 - F(x, |\beta_{u_1}u_1^-|^2))dx \\
& + \lambda \int_\Omega (\frac{1}{4}\varphi(\beta_{u_1}u_1^-)|\beta_{u_1}u_1^-|^2 - \Phi(\beta_{u_1}u_1^-))dx \\
\leq & \frac{1}{4}\|\bar{u}_1\|^2 + \frac{1}{2}\int_\Omega (\frac{1}{2}f(x, |u_1^+|^2)|u_1^+|^2 - F(x, |u_1^+|^2))dx \\
& + \frac{1}{2}\int_\Omega (\frac{1}{2}f(x, |u_1^-|^2)|u_1^-|^2 - F(x, |u_1^-|^2))dx \\
& + \lambda \int_\Omega (\frac{1}{4}\varphi(u_1^+)|u_1^+|^2 - \Phi(u_1^+))dx \\
& + \lambda \int_\Omega (\frac{1}{4}\varphi(u_1^-)|u_1^-|^2 - \Phi(u_1^-))dx \\
\leq & \liminf_{n\to\infty}[I_\lambda(u_n) - \frac{1}{4}\langle I'_\lambda(u_n), u_n\rangle] \\
= & m_1.
\end{aligned}$$

Then we have

$$\alpha_{u_1} = \beta_{u_1} = 1.$$

Therefore, we have that

$$u_1 = \bar{u}_1 \in \mathcal{M} \quad \text{and} \quad I(u_1) = m_1.$$

This completes the proof. □

The proof for m_2 is analogous.

Proof. From Lemma 2.4, we get that

$$u_1 \in \mathcal{M} \quad \text{and} \quad I_\lambda(u_1) = m_1 > 0.$$

Similar to the discussion of the last step of Theorem 1.2 in [22], we can obtain that

$$u_1 = u_1^+ + u_1^-$$

is a critical point of I_λ on X_0 and u_1 is a sign-changing solution of (2.1). Similarly, we can obtain $u_2 \in \mathcal{N}$ is a nontrivial solution of (2.1) and

$$I_\lambda(u_2) = m_2 > 0.$$

Thanks to $u_1^\pm \neq 0$, by Lemma 2.2, there exists a unique number $\alpha_{u_1^+} > 0$ such that $\alpha_{u_1^+}u_1^+ \in \mathcal{N}$. Similarly, there is a unique number $\beta_{u_1^-} > 0$ such that $\beta_{u_1^-}u_1^- \in \mathcal{N}$. Therefore, by (2.3), (2.4), and Lemma 2.5, we have

$$\begin{aligned}
0 < 2m_2 &\leq I_\lambda(\alpha_{u_1^+}u_1^+) + I_\lambda(\beta_{u_1^-}u_1^-) \\
&\leq I_\lambda(\alpha_{u_1^+}u_1^+ + \beta_{u_1^-}u_1^-) \leq I_\lambda(u_1^+ + u_1^-) \\
&= m_1,
\end{aligned}$$

that is, $0 < m_2 < m_1$. It follows that $m_2 > 0$ cannot be achieved by a sign-changing function; thus, $u_2 \in \mathcal{N}$ is a signed solution of (2.1). □

3. Proof of main results

In this section, we devote ourselves to completing the proof of Theorems 1.1 and 1.2. From the truncation argument in Section 2, we can see that if the solutions u_1 and u_2 of (2.1) satisfy

$$\|u_i\|_\infty \leq M, \quad i = 1, 2.$$

Then $u_1 \in X_0$ is a sign-changing solution of (1.1), and $u_2 \in X_0$ is a signed solution of (1.1). For convenience, for each $\lambda > 0, M > 0$ fixed, we let

$$\begin{aligned}
g_{\lambda,M}(x,t) &= f(x, |t|^2)t + \lambda\varphi(t)t, \\
G_{\lambda,M}(x,t) &= \int_0^t g_{\lambda,M}(x,\tau)d\tau = \frac{1}{2}F(x,|t|^2) + \lambda\Phi(t).
\end{aligned} \tag{3.1}$$

Lemma 3.1. *Let u_1 and u_2 be a sign-changing solution and a signed solution of problem (2.1), respectively; then there exists a constant $K > 0$ independent of $\lambda, M > 0$ such that*

$$\|u_1\| \leq K \quad \textit{and} \quad \|u_2\| \leq K.$$

Proof. From (2.8), we have

$$f(x, |t|^2)|t|^2 \geq 2F(x, |t|^2) \quad a.e.\ x \in \Omega, \quad t \in \mathbb{R}\backslash\{0\}. \tag{3.2}$$

And as

$$\varphi(t)t^2 \geq q\Phi(t), \quad t \in \mathbb{R}\backslash\{0\}. \tag{3.3}$$

Together with (3.1)–(3.3), we have

$$g_{\lambda,M}(x,t)t \geq \theta G_{\lambda,M}(x,t) \quad a.e.\ x \in \Omega, \quad t \in \mathbb{R}\backslash\{0\}, \tag{3.4}$$

where

$$\theta = \min\{4, q\} > 2.$$

By (3.4), we have

$$\begin{aligned}\theta m_1 &\geq \theta I_\lambda(u_1) - \langle I'_\lambda(u_1), u_1\rangle \\ &= (\frac{\theta}{2} - 1)\|u_1\|^2 + \int_\Omega (g_{\lambda,M}(x,u_1)u_1 - \theta G_{\lambda,M}(x,u_1))dx \\ &\geq (\frac{\theta}{2} - 1)\|u_1\|^2.\end{aligned}$$

So, there exists a constant $K_1 > 0$ independent of $\lambda, M > 0$ such that

$$\|u_1\| \leq K_1.$$

Similarly, we obtain

$$\begin{aligned}\theta m_2 &\geq \theta I_\lambda(u_2) - \langle I'_\lambda(u_2), u_2\rangle \\ &= (\frac{\theta}{2} - 1)\|u_2\|^2 + \int_\Omega (g_{\lambda,M}(x,u_2)u_2 - \theta G_{\lambda,M}(x,u_2))dx \\ &\geq (\frac{\theta}{2} - 1)\|u_2\|^2.\end{aligned}$$

So, there exists a constant $K_2 > 0$ independent of $\lambda, M > 0$ such that

$$\|u_2\| \leq K_2.$$

Taking

$$K = \min\{K_1, K_2\},$$

then

$$\|u_i\| \leq K, \quad i = 1, 2.$$

This completes the proof. □

Lemma 3.2. *Let u_1 and u_2 be a sign-changing solution and a signed solution of problem (2.1), respectively, then there exists a constant $B > 0$ independent on λ and M such that*

$$\|u_i\|_\infty \leq B(1 + \lambda^{\frac{1}{2^*-q}} M^{\frac{p-q}{2^*-q}}), \quad i = 1, 2.$$

We only prove Lemma 3.2 for u_1, the proof for u_2 is analogous.

Proof. For $L > 0$ and $\beta > 1$, set

$$\zeta(t) = t t_L^{2(\beta-1)} \quad \text{and} \quad \Gamma(t) = \int_0^t (\zeta'(\tau))^{\frac{1}{2}} d\tau, \quad \forall\, t \in \mathbb{R},$$

where

$$t_L = \min\{t, L\}.$$

It is easy to obtain that

$$(a-b)[\zeta(a) - \zeta(b)] \geq |\Gamma(a) - \Gamma(b)|^2, \quad \forall a, b \in \mathbb{R} \tag{3.5}$$

and

$$\Gamma(t) \geq \frac{1}{\beta} t t_L^{\beta-1}, \quad \forall t \in \mathbb{R}. \tag{3.6}$$

Recall that

$$u_L = \min\{u_1, L\}.$$

It is easy to see that

$$|u_1 u_L^{2(\beta-1)}| \leq L^{2(\beta-1)} u_1$$

and

$$\zeta(u_1) \in X_0.$$

Choose $\zeta(u_1)$ as a test function in (2.3), combining (3.5) and (3.6), we conclude

$$\begin{aligned}&\frac{1}{\beta^2}\|u_1 u_L^{\beta-1}\|^2 \leq \|\Gamma(u_1)\|^2 \\ &\leq \int_{\mathbb{R}^{2n}} [u_1(x) - u_1(y)][\zeta(u_1(x)) - \zeta(u_1(y))]K(x-y)dxdy \\ &\leq \int_{\mathbb{R}^{2n}} [u_1(x) - u_1(y)]u_1 u_L^{2(\beta-1)}(x)K(x-y)dxdy \\ &\quad - \int_{\mathbb{R}^{2n}} [u_1(x) - u_1(y)]u_1 u_L^{2(\beta-1)}(y)K(x-y)dxdy \\ &= \int_\Omega g_{\lambda,M}(x,u_1)u_1 u_L^{2(\beta-1)}dx,\end{aligned} \tag{3.7}$$

which gives

$$\int_\Omega g_{\lambda,M}(x,u_1)u_1 u_L^{2(\beta-1)}dx \geq 0.$$

By (2.6), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|g_{\lambda,M}(x,t)| \leq \varepsilon|t| + C_\varepsilon(1 + \lambda M^{p-q})|t|^{q-1}. \tag{3.8}$$

Let

$$\omega_L(u_1) = u_1 u_L^{\beta-1},$$

by (3.7) and (3.8) and Hölder's inequality, it holds that

$$\begin{aligned}&\frac{1}{\beta^2}\|\omega_L(u_1)\|^2 \\ &\leq C_\varepsilon(1 + \lambda M^{p-q})(\int_\Omega |u_i(x)|^{2^*}dx)^{\frac{q-2}{2^*}}(\int_\Omega |\omega_L|^{2t}dx)^{\frac{1}{t}} \\ &+ \varepsilon\int_\Omega |\omega_L|^2 dx,\end{aligned} \tag{3.9}$$

where

$$\frac{q-2}{2^*}+\frac{1}{t}=1.$$

It is obvious that $2t \in (2, 2^*)$.

Together with Lemma 1.1 and (1.6), we have

$$S^*|u_1|_{2^*}^2 \leq \|u_1\|^2. \tag{3.10}$$

Therefore, by (3.9) and (3.10), we obtain

$$|\omega_L|_{2^*}^2 \leq C_7\beta^2[|\omega_L|_2^2 + (1+\lambda M^{p-q})|u_1|_{2^*}^{q-2}|\omega_L|_{2t}^2],$$

where $C_7 > 0$. Letting $L \to \infty$,

$$\begin{aligned}|u_1|_{\beta 2^*}^{2\beta} &\leq C_7\beta^2[|u_1|_{2\beta}^{2\beta} + (1+\lambda M^{p-q})|u_1|_{2^*}^{q-2}|u_1|_{2\beta t}^{2\beta}]\\ &\leq C_8\beta^2[1+(1+\lambda M^{r-q})|u_1|_{2^*}^{q-2}]|u_1|_{2\beta t}^{2\beta}.\end{aligned}$$

Thus

$$|u_1|_{\beta 2^*} \leq C_8^{\frac{1}{2\beta}}\beta^{\frac{1}{\beta}}[1+(1+\lambda M^{p-q})|u_1|_{2^*}^{q-2}]^{\frac{1}{2\beta}}|u_1|_{2\beta t}.$$

Let

$$\alpha = \frac{2^*}{2t},$$

then $\alpha > 1$. Taking $\beta = \alpha$, we have

$$|u_1|_{\alpha 2^*} \leq C_8^{\frac{1}{2\alpha}}\alpha^{\frac{1}{\alpha}}[1+(1+\lambda M^{p-q})|u_1|_{2^*}^{q-2}]^{\frac{1}{2\alpha}}|u_1|_{2^*}.$$

Taking $\beta = \alpha^2$, we have

$$|u_1|_{\alpha^2 2^*} \leq C_8^{\frac{1}{2\alpha^2}}\alpha^{\frac{2}{\alpha^2}}[1+(1+\lambda M^{p-q})|u_1|_{2^*}^{q-2}]^{\frac{1}{2\alpha^2}}|u_1|_{2^*\alpha}.$$

Therefore, we have

$$|u_1|_{\alpha^2 2^*} \leq C_8^{\frac{1}{2\alpha}+\frac{1}{2\alpha^2}}\alpha^{\frac{1}{\theta}+\frac{2}{\theta^2}}[1+(1+\lambda M^{p-q})|u_1|_{2^*}^{q-2}]^{\frac{1}{2\alpha}+\frac{1}{2\alpha^2}}|u_1|_{2^*}.$$

Taking $\beta = \alpha^i, i \in \mathbb{N}$, we have

$$|u_1|_{\alpha^i 2^*} \leq C_8^{\sum_{m=1}^{i}\frac{1}{2\alpha^m}}\alpha^{\sum_{m=1}^{i}\frac{m}{\theta^m}}[1+(1+\lambda M^{p-q})|u_1|_{2^*}^{q-2}]^{\sum_{m=1}^{i}\frac{1}{2\alpha^m}}|u_1|_{2^*}.$$

Letting $i \to \infty$, we can know that

$$|u_1|_\infty \leq C_8^{\frac{1}{2(\alpha-1)}}\alpha^{\frac{\alpha}{(\alpha-1)^2}}[1+(1+\lambda M^{p-q})|u_1|_{2^*}^{q-2}]^{\frac{1}{2(\alpha-1)}}|u_1|_{2^*}. \tag{3.11}$$

Finally, by (3.10) and Lemma 3.1, there exists $C_9 > 0$ such that

$$|u_1|_{2^*} \leq C_9.$$

Therefore, it follows from (3.11) and

$$\alpha = \frac{2^*-q+2}{2},$$

there exists a constant $B > 0$ independent on λ and M, such that

$$|u_1|_\infty \leq B(1+\lambda^{\frac{1}{2^*-q}}M^{\frac{p-q}{2^*-q}}).$$

This completes the proof. □

Proof of Theorem 1.1. By Lemma 3.2, there exists a positive constant B independent on λ and M such that

$$\|u_i\|_\infty \leq B(1+\lambda^{\frac{1}{2^*-q}}M^{\frac{p-q}{2^*-q}}), \quad i=1,2.$$

Thus, for large $M > 0$, we can choose small $\lambda_* > 0$ such that

$$\|u_1\|_\infty \leq M \quad \text{and} \quad \|u_2\|_\infty \leq M$$

for all $\lambda \in (0, \lambda_*]$. By Theorem 2.1, problem (1.1) admits a sign-changing solution and a signed solution for $\lambda \in (0, \lambda_*]$.

This completes the proof. □

Proof of Theorem 1.2. We take

$$K(x) = |x|^{-(N+2s)},$$

then it is obvious that $K(x)$ satisfies the conditions $(K_1), (K_2)$ and problem (1.1) turns into problem (1.2). By using [4, Lemma 5], we can obtain that

$$X_0 \subseteq H^s(\mathbb{R}^n).$$

Thus, the assertion of Theorem 1.2 follows from Theorem 1.1. □

4. Conclusions

In this study, we have investigated the existence of sign-changing and signed solutions for nonlinear elliptic equations driven by nonlocal integro-differential operators with critical or supercritical nonlinearity. The main idea of this paper is to reduce the supercritical problem into a subcritical one. In comparison with previous works, this paper has several new features. First, we consider the more general nonlinear term without Ambrosetti-Rabinowitz condition. Second, the nonlinear term involves supcritical growth. Finally, the existence of a sign-changing solution and a signed solution is obtained by combining an appropriate truncation argument with a constrained minimization method and the Moser iteration method. In the future, our work will focus on the existence of normalized solutions to the nonlinear elliptic equations driven by nonlocal integro-differential operators with critical or supercritical nonlinearity.

Use of Generative-AI tools declaration

The authors declares they have not used artificial intelligence (AI) tools in the creation of this article.

Acknowledgments

Supported by the National Natural Science Foundation of China (61803236), Natural Science Foundation of Shandong Province (ZR2018MA022).

Conflict of interest

All authors declare that there are no conflicts of interest in this paper.

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