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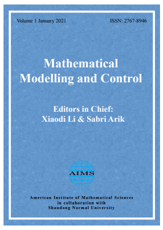

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Research article

On the Ulam stability and existence of L^p-solutions for fractional differential and integro-differential equations with Caputo-Hadamard derivative

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Abstract: In this paper, we investigate the existence and uniqueness of L^p-solutions for nonlinear fractional differential and integro-differential equations with boundary conditions using the Caputo-Hadamard derivative. By employing Hölder's inequality together with the Krasnoselskii fixed-point theorem and the Banach contraction principle, the study establishes sufficient conditions for solving nonlinear problems. The paper delves into preliminary results, the existence and uniqueness of L^p solutions to the boundary value problem, and presents the Ulam-Hyers stability. Furthermore, it investigates the existence, uniqueness, and stability of solutions for fractional integro-differential equations. Through standard fixed-points and rigorous mathematical frameworks, this research contributes to the theoretical foundations of nonlinear fractional differential equations. Also, the Adomian decomposition method ($\mathcal{ADM}$) is used to construct the analytical approximate solutions for the problems. Finally, examples are given that illustrate the effectiveness of the theoretical results.

Keywords: fractional differential equations ($\mathcal{FDE}s$); Caputo-Hadamard ($\mathcal{CH}$) derivative; fixed-point theorems; Ulam-Hyers ($\mathcal{UH}$) stability; Adomian decomposition method ($\mathcal{ADM}$)

1. Introduction

Fractional differential equations ($\mathcal{FDE}s$) have great interest for many mathematicians. This is due to extensive applications of these equations in the mathematical modeling in various fields of both science and engineering such as: control theory, physics, biological phenomena, viscoelasticity, and signal processing (see [1–3]). Furthermore, integro-differential equations are prevalent in various physical phenomena, such as fluid dynamics, biological models, and chemical kinetics. These equations arise due to the complex interactions and behaviors observed in these systems, requiring a combination of differential and integral terms to accurately model their dynamics. For instance, aero-elastic coupling in structures like wings and wind turbine blades leads to integro-differential problems, where control techniques play a crucial role in preventing instabilities. Overall, the presence of integro-differential equations in physical phenomena underscores the need for advanced mathematical tools to understand and predict the behavior of complex systems (see [4–6]). Recently, authors used various fixed-point theorems to prove the existence and uniqueness for the fractional differential equations with initial and boundary conditions. For example, the existence and uniqueness of solutions of differential equations with a mixture of integer and fractional derivatives have been investigated in [7]. The authors in [8] established existence and uniqueness results of solutions for fractional differential equations with integral boundary conditions by means of the Banach contraction mapping principle under sufficient conditions. The existence of solutions of integro-fractional differential equation when $\delta \in (2,3]$ through fixed-point theorem have been studied in [9]. Researchers in [10, 11] study the existence and uniqueness of solutions for certain differential equations by using boundary and initial conditions, along with various techniques based on fixed-point theorems. The existence theory concerning fractional-order three-dimensional differential systems at resonance is presented in [12]; for additional details see these manuscripts [13–15].

On the other hand, the properties of L^p-solutions received a large share of researchers focus. Arshad et al. [16] examined L^p-solutions of fractional integral equations involving the Riemann-Liouville integral operator using a compactness condition. In [17] the author estimated the existence of an integrable solution for the nonlinear fractional differential equations involving two Caputo's fractional derivatives by means of Hölder's inequality together with Banach contraction principle and Schaefer's fixed-point theorem. Also see [18–21]. The Ulam-Hyers ($\mathcal{UH}$) stability analysis has been studied and obtain a great part from the work of audiences [22, 23]. Murad and Ameen in [24] researched the existence and $\mathcal{UH}$ stability of nonlinear fractional differential equations of mixed Caputo-Riemann derivatives. Vu et al. [25] proved the $\mathcal{UH}$ stability for the nonlinear Volterra integro-differential equations. Caputo-Hadamard ($\mathcal{CH}$) fractional differential equations have various applications in modeling complex systems with memory effects in uncertain environments. Some potential applications include: Describing physical systems with memory effects and uncertain parameters, analyzing energy harvesting systems with fractional order properties, modeling biological systems with uncertain dynamics; see [26–28]. A series of research papers investigated the Hadamard derivative and $\mathcal{CH}$ derivative to prove the existence and stability theorems. In [29], existence and uniqueness of solution for Hadamard fractional differential equations on an infinite interval with integral boundary value has been developed. The theoretical analysis of $\mathcal{CH}$ fractional boundary-value problems in L^p-spaces was introduced in [30]. The authors in [31, 32], focus on the existence and Ulam stability of solutions for certain $\mathcal{CH}$ fractional differential equations. The study in [33] highlights the existence of a solution for the boundary value problem of a nonlinear $\mathcal{CH}$ fractional differential equation with integral and anti-periodic conditions. Among the immense number of papers dealing with Caputo-Hadamard and Hadamard fractional differential equations subject to a variety of boundary conditions using fixed-point theory; we refer to [34–36]. Muthaiah et al. [37] discussed existence and of solutions for Hadamard fractional differential equations with integral boundary conditions. In [38] the authors applied the Monch's fixed-point theorem to prove the existence result for the fractional boundary value problems with $\mathcal{CH}$ derivative. Subsequently, many authors discussed the subject of approximation solutions by the Adomian decomposition method ($\mathcal{ADM}$) for various types of $\mathcal{FDE}$, we allude to [39–41]. Abdulahad et al. in [42] proved the existence of L^p-solutions for the following boundary value problem

$$\begin{aligned} &{}^{C}D^{\delta}\phi(t) = V(t, \phi(t)), \quad 0 < \delta < 1, \\ &a\phi(A) + b\phi(T) = c, \quad t \in [A, T]. \end{aligned}$$

Benhamida et al. [43] studied the existence of a solution for the boundary value problem:

$$\begin{aligned} &{}^{CH}D^{\delta}_{1+}\phi(t) = V(t, \phi(t)), \\ &a\phi(1) + b\phi(T) = c, \end{aligned}$$

where ${}^{CH}D^{\delta}_{1+}$ is the $\mathcal{CH}$ derivative, $(0 < \delta \leq 1)$ and a, b, c are constants with

$$a + b \neq 0.$$

Wang et al. [44] employed the existence and uniqueness of positive solutions for the following integral boundary value problem:

$$\begin{cases} D^{\delta}_{0+}\phi(t) + V(t, \phi(t)) = 0, \quad 0 < t < 1, \quad \delta \in (1, 2], \\ \phi(0) = 0, \quad \phi'(1) = \int_0^1 \phi(s)ds, \end{cases}$$

where D^{δ} is the Riemann-Liouville fractional derivative.

In this paper, first we study the following nonlinear fractional differential equation with boundary conditions:

$${}^{\mathcal{CH}}D^{\delta}_{1+}\phi(t) = V(t, \phi(t)), \quad t \in J = [1, e], \tag{1.1}$$

$$\phi(1) = \phi'(1), \qquad \phi(e) = \int_1^e \phi(t)\frac{dt}{t}, \tag{1.2}$$

where ${}^{\mathcal{CH}}D^{\delta}_{1+}$ $\mathcal{CH}$ derivative, with $1 < \delta \leq 2$ and

$$V : [1, e] \times \Re \to \Re$$

is continuous function.

Second, the following fractional integro-differential equations with boundary conditions are investigated:

$${}^{\mathcal{CH}}D^{\delta}_{1+}\phi(t) = V(t, \int_1^t \frac{(\ln(\frac{t}{s}))^{\delta-1}}{\Gamma(\delta)}\Delta(s, \phi(s))\frac{ds}{s}), \tag{1.3}$$

$$\lambda_1\phi(1) + \lambda_2\phi(T) = \lambda_3, \qquad I = [1, T], \tag{1.4}$$

where ${}^{\mathcal{CH}}D^{\delta}_{1+}$ is the $\mathcal{CH}$ derivative and $0 < \delta \leq 1$. Here λ_1–λ_3 are constants

$$\lambda_1 + \lambda_2 \neq 0$$

with

$$V : I \times \Re \to \Re, \ \ \Delta : I \times \Re \to \Re$$

are continuous functions.

This paper is arranged as follows: In Section 2, we present some preliminary results to be used later. In Section 3, the Krasnoselskii's fixed-point theorem and Banach contraction principle are applied to analyze the existence and uniqueness of solutions to the problems (1.1)–(1.4) in L^p-spaces. Moreover, we discuss the $\mathcal{UH}$ and $\mathcal{UH}$-Rassias stability for the problems in Section 4. The $\mathcal{ADM}$ is implemented to find the approximate solutions for the given problems in Section 5. Finally, examples are also given to show the applicability of our results.

2. Preliminaries

Let us give some definitions and lemmas that are basic and needed at various places in this work.

Definition 2.1. *[45] The Hadamard fractional integral of order $\delta \in \Re$ for a continuous function V is defined as*

$$I^{\delta}_{a+}V(t) = \frac{1}{\Gamma(\delta)}\int_{a+}^{t}\left(\ln(\frac{t}{\wp})\right)^{\delta-1}\frac{V(\wp)}{\wp}\,d\wp, \quad \delta > 0$$

provided the integral exists.

Definition 2.2. *[45] The Hadamard derivative of fractional order $\delta \in \Re$ for a continuous function f is defined as*

$$D^{\delta}_{a+}V(t) = \frac{1}{\Gamma(n-\delta)}(t\frac{d}{dt})^{n}\int_{a+}^{t}\left(\ln(\frac{t}{\wp})\right)^{n-\delta-1}\frac{V(\wp)}{\wp}\,d\wp,$$

where

$$n - 1 < \delta < n, \quad n = [\delta] + 1,$$

where $[\delta]$ denotes the integer part of the real number δ.

Definition 2.3. [45] The $\mathcal{CH}$ derivative of fractional order $\delta \in \Re$ for a continuous function V is defined as follows:

$$D^{\delta}_{a+}V(t) = \frac{1}{\Gamma(n-\delta)}\int_{a+}^{t}\left(\ln(\frac{t}{\wp})\right)^{n-\delta-1}\Delta^{n}V(\wp)\frac{d\wp}{\wp}, \tag{2.1}$$

where

$$n - 1 < \delta < n, \quad n = [\delta] + 1, \quad \Delta = (t\frac{d}{dt}),$$

and $[\delta]$ denotes the integer part of the real number δ, and Γ is the gamma function.

Lemma 2.1. *[45] Let*

$$\delta > 0 \quad \text{and} \quad n = [\delta] + 1.$$

If $\phi \in AC^{n}_{\delta}[a, b]$, then the differential equation

$${}^{\mathcal{CH}}D^{\delta}_{a^+}\phi(t) = 0$$

has solutions

$$\phi(t) = \sum_{k=0}^{n-1} c_k(\ln\frac{t}{a})^{k},$$

and the following formula holds:

$$I^{\delta}_{a+}\,{}^{\mathcal{CH}}D^{\delta}_{a+}\phi(t) = \phi(t) + \sum_{k=0}^{n-1} c_k(\ln\frac{t}{a})^{k},$$

where $c_k \in \Re$, $k = 1, 2, \cdots, n-1$.

Definition 2.4. *[46] The Eq (1.1) is $\mathcal{UH}$ stable if there exists a real number $c_f > 0$ such that for each $\varepsilon > 0$ and for each solution $z \in C^1([a, b], \Re)$ of the inequality*

$$|{}^{\mathcal{CH}}D^{\delta}_{1^+}z(t) - V(t, z(t))| \leq \varepsilon, \quad t \in [a, b], \tag{2.2}$$

there exists a solution $\phi \in C^1([a, b], \Re)$ of Eq (1.1) with

$$|z(t) - \phi(t)| \leq c\varepsilon, \quad t \in [a, b].$$

Theorem 2.1. [47] (Krasnoselskii fixed-point theorem)

Let H be a closed, bounded, convex, and nonempty subset of a Banach space V. Let A and B be two operators such that

(1) $Az_1 + Bz_2 \in H$ whenever $z_1, z_2 \in H$;
(2) A is compact and continuous;
(3) B is a contraction mapping.

Then there exists $z \in H$ such that

$$z = Az + Bz.$$

Lemma 2.2. [48] (Bochner integrable)

A measurable function

$$V : [a, b] \times \Re \to \Re$$

is Bochner integrable, if $\|V\|$ is Lebesgue integrable.

Theorem 2.2. [49] (Kolmogorov compactness criterion)

Let $v \subseteq L^p[a,b]$, $1 \leq p < \infty$. If:

(i) v is bounded in $L^p[a,b]$;

(ii) $x_h \to x$ as $h \to 0$ uniformly with respect to $x \in v$, then v is relatively compact in $L^p[a,b]$, where

$$x_h(t) = \frac{1}{h}\int_t^{t+h} x(s)ds.$$

Lemma 2.3. [50] (Hölder's inequality)

Let X be a measurable space, let p and q satisfy

$$1 \leq p < \infty, \quad 1 \leq q < \infty,$$

and

$$\frac{1}{p} + \frac{1}{q} = 1.$$

If $h \in \mathfrak{L}^{\mathrm{p}}(X)$ and $g \in \mathfrak{L}^{\mathrm{q}}(X)$, then (hg) belongs to $L(X)$ and satisfies

$$\int_X |hg|\,dt \leq \left(\int_X |h|^p\,dt\right)^{\frac{1}{p}} \left(\int_X |g|^q\,dt\right)^{\frac{1}{q}}.$$

Lemma 2.4. [36] If

$$0 < \delta < 1, \quad 1 < p < 1/(1-\delta),$$

then

$$\int_1^t (\ln\frac{t}{s})^{p(\delta-1)}\frac{1}{s^p}ds \leq \frac{(\ln t)^{p(\delta-1)+1}}{p(\delta-1)+1}. \tag{2.3}$$

3. Main results

3.1. Existence and uniqueness results for problems (1.1) and (1.2)

This section deals with the existence and uniqueness of a solution for the fractional differential Eq (1.1) with boundary condition (1.2). For measurable functions

$$V : J \times \Re \to \Re$$

define the norm

$$\|V\|_p^p = \int_1^e |V(t)|^p\,dt, \quad (1 \leq p < \infty),$$

where $L^p(J,\Re)$ is the Banach space of all Lebesgue measurable functions. Now, consider the following assumptions:

$(F1)$ There exists a constant $\mu > 0$ such that

$$\left|V(t,\phi(t))\right| \leq \mu\,|\phi(t)|,$$

for each $t \in J$ and for all $\phi \in \Re$.

$(F2)$ $V(t,\phi)$ is continuous and satisfies the Lipschitz condition, there exists a constant $\omega_1 > 0$ such that

$$|V(t,\phi_1(t)) - V(t,\phi_2(t))| \leq \omega_1|\phi_1(t) - \phi_2(t)|,$$

for each $\phi_1, \phi_2 \in \Re$.

For the sake of convenience, we set the notation:

$$\aleph_1 = \Big(\frac{2^{3p}}{(\Gamma(\delta))^p}(\frac{p-1}{p\delta-1})^{p-1} + \frac{2^{3p}}{(\Gamma(\delta+1))^p}(\frac{p-1}{p(\delta+1)-1})^{p-1}\Big)(2^p e - 1),$$
$$\aleph_2 = \Big(\frac{2^{2p}}{(\Gamma(\delta))^p}(\frac{p-1}{p\delta-1})^{p-1} + \frac{2^{2p}}{(\Gamma(\delta+1))^p}(\frac{p-1}{p(\delta+1)-1})^{p-1}\Big)^{\frac{1}{p}}(2^p e - 1)^{\frac{1}{p}},$$
$$\mho_1 = \Big(\aleph_1 + \Big(\frac{2^p}{(\Gamma(\delta))^p}(\frac{p-1}{p\delta-1})^{p-1}\frac{e}{p\delta}\Big)\Big).$$

Lemma 3.1. *For any*

$$\phi(t) \in C(J,\Re), \quad 1 < \delta \leq 2,$$

then the boundary value problems (1.1) and (1.2) have a solution

$$\begin{aligned}\phi(t) =& 2(\ln(t)+1)\int_1^e \frac{(\ln\frac{e}{s} - \delta)(\ln\frac{e}{s})^{\delta-1}}{\Gamma(\delta+1)}V(s,\phi(s))\frac{ds}{s} \\ &+ \int_1^t \frac{(\ln\frac{t}{s})^{\delta-1}}{\Gamma(\delta)}V(s,\phi(s))\frac{ds}{s}.\end{aligned} \tag{3.1}$$

Proof. Applying Lemma 2.1, we can reduce the problems (1.1) and (1.2) to an equivalent integral equation

$$\phi(t) = c_0 + c_1\ln(t) + \int_1^t \frac{(\ln\frac{t}{s})^{\delta-1}}{\Gamma(\delta)}V(s,\phi(s))\frac{ds}{s}$$

to find c_0 and c_1, from the first boundary condition

$$\phi(1) = \phi'(1),$$

we obtain

$$\phi(t) = c_0(\log(t)+1) + \int_1^t \frac{(\log\frac{t}{s})^{\delta-1}}{\Gamma(\delta)}V(s,\phi(s))\frac{ds}{s},$$

by using the condition

$$\phi(e) = \int_1^e \phi(t)\frac{dt}{t},$$

the result is

$$\phi(e) = 2c_0 + \int_1^e \frac{(\log \frac{e}{s})^{\delta-1}}{\Gamma(\delta)} V(s, \phi(s)) \frac{ds}{s}$$

and

$$\int_1^e \phi(t)\frac{dt}{t} = \frac{3}{2}c_0 + \int_1^e \int_1^t \frac{(\log \frac{t}{s})^{\delta-1}}{\Gamma(\delta)} V(s, \phi(s)) \frac{ds}{s}\frac{dt}{t},$$

by using Fubini's theorem, the following is obtain

$$\int_1^e \phi(t)\frac{dt}{t} = \frac{3}{2}c_0 + \int_1^e \frac{(\log \frac{e}{s})^{\delta}}{\delta\Gamma(\delta)} V(s, \phi(s)) \frac{ds}{s}.$$

Hence

$$c_0 = 2\int_1^e \frac{(\log \frac{e}{s} - \delta)(\log \frac{e}{s})^{\delta-1}}{\Gamma(\delta+1)} V(s, \phi(s)) \frac{ds}{s},$$

this implies that

$$\begin{aligned}\phi(t) =& 2(\log(t)+1)\int_1^e \frac{(\log \frac{e}{s} - \delta)(\log \frac{e}{s})^{\delta-1}}{\Gamma(\delta+1)} V(s, \phi(s)) \frac{ds}{s} \\ &+ \int_1^t \frac{(\log \frac{t}{s})^{\delta-1}}{\Gamma(\delta)} V(s, \phi(s)) \frac{ds}{s}.\end{aligned}$$

This completes the poof. □

The first result is based on Banach contraction principle.

Theorem 3.1. *Assume that $(F1)$ and $(F2)$ hold. If*

$$(\mho_1)^{\frac{1}{p}}\ \omega_1 < 1.$$

Then the boundary value problems (1.1) and (1.2) have a unique solution.

Proof. Define the operator $\top$ by

$$\begin{aligned}(\top\phi)(t) =& 2(\ln(t)+1)\int_1^e \frac{(\ln \frac{e}{s} - \delta)(\ln \frac{e}{s})^{\delta-1}}{\Gamma(\delta+1)} V(s, \phi(s)) \frac{ds}{s} \\ &+ \int_1^t \frac{(\ln \frac{t}{s})^{\delta-1}}{\Gamma(\delta)} V(s, \phi(s)) \frac{ds}{s},\end{aligned}$$

we need to prove that the operator $\top$ has a fixed-point on the set

$$\vartheta_u = \{\phi \in L^p(J) : \|\phi\|_p^p \leq u^p, u > 0\}.$$

For $\phi \in \vartheta_u$, we have

$$\begin{aligned}|\top(\phi)(t)|^p \leq& \frac{2^p}{(\Gamma(\delta))^p}\Big(\int_1^t (\ln \frac{t}{s})^{\delta-1}|V(s,\phi(s))|\frac{ds}{s}\Big)^p \\ &+ \frac{2^{3p}}{(\Gamma(\delta+1))^p}(\ln(t)+1)^p\Big(\int_1^e (\ln \frac{e}{s})^{\delta}|V(s,\phi(s))|\frac{ds}{s}\Big)^p \\ &+ \frac{2^{3p}}{(\Gamma(\delta))^p}(\ln(t)+1)^p\Big(\int_1^e (\ln \frac{e}{s})^{\delta-1}|V(s,\phi(s))|\frac{ds}{s}\Big)^p.\end{aligned} \tag{3.2}$$

By Hölder's inequality and Lemma 2.4 we obtain

$$\Big(\int_1^t (\ln \frac{t}{s})^{\delta-1}|V(s,\phi(s))|\frac{ds}{s}\Big)^p \leq \frac{(\ln(t))^{p\delta-1}}{(\frac{p\delta-1}{p-1})^{p-1}}\Big(\int_1^t |V(s,\phi(s))|^p ds\Big). \tag{3.3}$$

Now, by the same way we find that

$$\Big(\int_1^e (\ln \frac{e}{s})^{\delta-1}|V(s,\phi(s))|\frac{ds}{s}\Big)^p \leq \frac{1}{(\frac{p\delta-1}{p-1})^{p-1}}\Big(\int_1^e |V(s,\phi(s))|^p ds\Big) \tag{3.4}$$

and

$$\Big(\int_1^e (\ln \frac{e}{s})^{\delta}|V(s,\phi(s))|\frac{ds}{s}\Big)^p \leq \frac{1}{(\frac{p(\delta+1)-1}{p-1})^{p-1}}\Big(\int_1^e |V(s,\phi(s))|^p ds\Big). \tag{3.5}$$

Thus, Eqs (3.3)–(3.5) are Lebesgue integrable; by using Lemma 2.2, we conclude that $(\ln \frac{e}{s})^{\delta-1}V(s,\phi(s))$, $(\ln \frac{e}{s})^{\delta}V(s,\phi(s))$, and $(\ln \frac{t}{s})^{\delta-1}V(s,\phi(s))$ are Bochner integrable with respect to $s \in [1,t]$; for all $t \in J$, then the Eq (3.2) becomes

$$\begin{aligned}&\int_1^e |\top(\phi)(t)|^p dt \\ &\leq \frac{2^p}{(\Gamma(\delta))^p}\int_1^e \frac{(\ln(t))^{p\delta-1}}{(\frac{p\delta-1}{p-1})^{p-1}}\int_1^t |V(s,\phi(s))|^p ds dt \\ &+\Big(\frac{2^{3p}}{(\Gamma(\delta))^p}\frac{1}{(\frac{p\delta-1}{p-1})^{p-1}} + \frac{2^{3p}}{(\Gamma(\delta+1))^p}\frac{1}{(\frac{p(\delta+1)-1}{p-1})^{p-1}}\Big) \\ &\int_1^e (\ln(t)+1)^p \int_1^e |V(s,\phi(s))|^p ds dt.\end{aligned}$$

Then, by the condition $(F1)$ implies that

$$\begin{aligned}&\int_1^e |\top(\phi)(t)|^p dt \\ &\leq \frac{2^p\,\mu^p}{(\Gamma(\delta))^p}\int_1^e \frac{(\ln(t))^{p\delta-1}}{(\frac{p\delta-1}{p-1})^{p-1}}\int_1^t |\phi(s)|^p ds dt \\ &+\Big(\frac{2^{3p}\,\mu^p}{(\Gamma(\delta))^p}(\frac{p-1}{p\delta-1})^{p-1} + \frac{2^{3p}\,\mu^p}{(\Gamma(\delta+1))^p}(\frac{p-1}{p(\delta+1)-1})^{p-1}\Big) \\ &\int_1^e (\ln(t)+1)^p dt \int_1^e |\phi(s)|^p ds.\end{aligned}$$

Integrate by parts; the following is obtained:

$$\begin{aligned}\|\top\phi\|_p^p \leq& \frac{2^p}{(\Gamma(\delta))^p}(\frac{p-1}{p\delta-1})^{p-1}\mu^p\frac{e}{p\delta}\int_1^e |\phi(t)|^p dt + \Big(\frac{2^{3p}}{(\Gamma(\delta))^p}(\frac{p-1}{p\delta-1})^{p-1} \\ &+ \frac{2^{3p}}{(\Gamma(\delta+1))^p}(\frac{p-1}{p(\delta+1)-1})^{p-1}\Big)\mu^p(2^p e - 1)\int_1^e |\phi(s)|^p ds\end{aligned}$$

and

$$\|\top\phi\|_p^p \le \Big(\aleph_1 + \frac{2^p}{(\Gamma(\delta))^p}(\frac{p-1}{p\delta-1})^{p-1}\frac{e}{p\delta}\Big)\mu^p\ u^p,$$
$$\|\top\phi\|_p \le (\mho_1)^{\frac{1}{p}}\ \mu\ u,$$

which implies that $\top\vartheta_u \subseteq \vartheta_u$.

Hence, $\top(\phi)(t)$ is Lebesgue integrable and $\top$ maps ϑ_u into itself. We have to show that $\top$ is a contraction mapping. Let $\phi_1, \phi_2 \in L^p(J)$, we have

$$\int_1^e |\top(\phi_1(t)) - \top(\phi_2(t))|^p dt \le 2^p \int_1^e \Big(\int_1^t \frac{(\ln\frac{t}{s})^{\delta-1}}{\Gamma(\delta)} \big|V(s,\phi_1(s)) - V(s,\phi_2(s))\big|\frac{ds}{s}\Big)^p dt + 2^{3p}\int_1^e (\ln(t)+1)^p \Big(\int_1^e \frac{(\ln\frac{e}{s})^{\delta-1}}{\Gamma(\delta)}\big|V(s,\phi_1(s)) - V(s,\phi_2(s))\big|\frac{ds}{s}\Big)^p dt + 2^{3p} \int_1^e (\ln(t)+1)^p \Big(\int_1^e \frac{(\ln\frac{e}{s})^{\delta}}{\Gamma(\delta+1)}\big|V(s,\phi_1(s)) - V(s,\phi_2(s))\big|\frac{ds}{s}\Big)^p dt.$$

Using $(F2)$ and Hölder's inequality, one has

$$\int_1^e |\top(\phi_1(t)) - \top(\phi_2(t))|^p dt \le \frac{2^p\omega_1^p}{(\Gamma(\delta))^p}(\frac{p-1}{p\delta-1})^{p-1}\int_1^e (\ln(t))^{p\delta-1} \int_1^t |\phi_1(s)-\phi_2(s)|^p ds dt + \frac{2^{3p}\omega_1^p}{(\Gamma(\delta))^p}(\frac{p-1}{p\delta-1})^{p-1}\int_1^e (\ln(t)+1)^p \int_1^e |\phi_1(s)-\phi_2(s)|^p ds dt + \frac{2^{3p}\omega_1^p}{(\Gamma(\delta+1))^p}(\frac{p-1}{p(\delta+1)-1})^{p-1} \int_1^e (\ln(t)+1)^p \int_1^e |\phi_1(s)-\phi_2(s)|^p ds dt.$$

Integrate by parts, leads to

$$\|\top\phi_1 - \top\phi_2\|_p \le \Big[\frac{2^p}{(\Gamma(\delta))^p}(\frac{p-1}{p\delta-1})^{p-1}\frac{e}{p\delta} + \Big(\frac{2^{3p}}{(\Gamma(\delta))^p}(\frac{p-1}{p\delta-1})^{p-1} + \frac{2^{3p}}{(\Gamma(\delta+1))^p}(\frac{p-1}{p(\delta+1)-1})^{p-1}\Big)(2^pe-1)\Big]^{\frac{1}{p}}\omega_1\|\phi_1-\phi_2\|_p$$

and

$$\|\top\phi_1 - \top\phi_2\|_p \le (\mho_1)^{\frac{1}{p}}\ \omega_1\|\phi_1-\phi_2\|_p.$$

If

$$(\mho_1)^{\frac{1}{p}}\ \omega_1 < 1,$$

then by the contraction mapping principle, the boundary value problems (1.1) and (1.2) have a unique solution. □

The following result is based on Krasnoselskii's fixed-point theorem.

Theorem 3.2. *Assume that* $(F1)$ *and* $(F2)$ *hold. Then the boundary value problems (1.1) and (1.2) have at least one solution.*

Proof. Let us define two operators, χ_1 and χ_2, from Eq (3.1) as

$$(\chi_1\phi)(t) = \int_1^t \frac{(\ln\frac{t}{s})^{\delta-1}}{\Gamma(\delta)}V(s,\phi(s))\frac{ds}{s},$$
$$(\chi_2\phi)(t) = 2(\ln(t)+1)\int_1^e \frac{(\ln\frac{e}{s}-\delta)(\ln\frac{e}{s})^{\delta-1}}{\Gamma(\delta+1)}V(s,\phi(s))\frac{ds}{s}.$$

Consider the set

$$\eth_r = \{\phi \in L^p(J) : \|\phi\|_p^p \le r^p, r > 0\}.$$

For $x, y \in \eth_r$, we have

$$\int_1^e |(\chi_1x)(t) + (\chi_2y)(t)|^p dt \le \frac{2^p}{(\Gamma(\delta))^p}\int_1^e \Big(\int_1^t (\ln\frac{t}{s})^{\delta-1} |V(s,x(s))|\frac{ds}{s}\Big)^p dt + \frac{2^{3p}}{(\Gamma(\delta+1))^p}\int_1^e (\ln(t)+1)^p \Big(\int_1^e ((\ln\frac{e}{s})^{\delta}|V(s,\phi(s))|\frac{ds}{s}\Big)^p dt + \frac{2^{3p}}{(\Gamma(\delta))^p}\int_1^e (\ln(t)+1)^p \Big(\int_1^e (\ln\frac{e}{s})^{\delta-1}|V(s,\phi(s))|\frac{ds}{s}\Big)^p dt. \tag{3.6}$$

By $(F1)$ and Hölders inequality, Eq (3.6) becomes

$$\int_1^e |(\chi_1x)(t) + (\chi_2y)(t)|^p dt \le \frac{2^p\mu^p}{(\Gamma(\delta))^p}\int_1^e \frac{(\ln(t))^{p\delta-1}}{(\frac{p\delta-1}{p-1})^{p-1}} \int_1^t |x(s)|^p ds dt + \frac{2^{3p}\mu^p}{(\Gamma(\delta+1))^p}\Big(\frac{p-1}{p(\delta+1)-1}\Big)^{p-1}\int_1^e (\ln(t)+1)^p\int_1^e |\phi(s)|^p ds dt + \frac{2^{3p}\mu^p}{(\Gamma(\delta))^p}\Big(\frac{p-1}{p\delta-1}\Big)^{p-1}\int_1^e (\ln(t)+1)^p\int_1^e |\phi(s)|^p ds dt.$$

It follows from integration by parts, that

$$\|\chi_1x+\chi_2y\|_p \le \Big[\frac{2^p}{(\Gamma(\delta))^p}(\frac{p-1}{p\delta-1})^{p-1}\frac{e}{p\delta}\|x(t)\|_p^p + \Big(\frac{2^{3p}}{(\Gamma(\delta))^p}(\frac{p-1}{p\delta-1})^{p-1} + \frac{2^{3p}}{(\Gamma(\delta+1))^p}(\frac{p-1}{p(\delta+1)-1})^{p-1}\Big)(2^pe-1)\|\phi(t)\|_p^p\Big]^{\frac{1}{p}}\mu$$

and

$$\|\chi_1x+\chi_2y\|_p \le (\mho_1)^{\frac{1}{p}}\mu r.$$

Hence,

$$\chi_1x+\chi_2y \in \eth_r.$$

Now, to prove that χ_2 is a contraction mapping on $\eth_r$, from $(F2)$ and Hölder inequality, it is easy to see that

$$\|\chi_2\phi_1 - \chi_2\phi_2\|_p^p \leq \Big(\frac{2^{2p}}{(\Gamma(\delta))^p}\Big(\frac{p-1}{p\delta-1}\Big)^{p-1} + \frac{2^{2p}}{(\Gamma(\delta+1))^p}$$
$$\Big(\frac{p-1}{p(\delta+1)-1}\Big)^{p-1}\Big)\omega_1^p \int_1^e (\ln(t)+1)^p dt \int_1^e \left|\phi_1(s)-\phi_2(s)\right|^p ds,$$
$$\|\chi_2\phi_1 - \chi_2\phi_2\|_p \leq \Big(\frac{2^{2p}}{(\Gamma(\delta))^p}(\frac{p-1}{p\delta-1})^{p-1}(2^pe-1)$$
$$+ \frac{2^{2p}}{(\Gamma(\delta+1))^p}(\frac{p-1}{p(\delta+1)-1})^{p-1}(2^pe-1)\Big)^{\frac{1}{p}} \omega_1\|\phi_1-\phi_2\|_p$$

and

$$\|\chi_2\phi_1 - \chi_2\phi_2\|_p \leq \aleph_2\omega_1\|\phi_1-\phi_2\|_p.$$

If $\aleph_2\omega_1 < 1$, then χ_2 is a contraction mapping.

We need to show that χ_1 is compact and continuous, for any $x \in \eth_r$, we have

$$\|\chi_1 x\|_p \leq \frac{1}{\Gamma(\delta)}\Big(\Big(\frac{p-1}{p\delta-1}\Big)^{p-1}\frac{e}{p\delta}\Big)^{\frac{1}{p}}\mu\, r.$$

Hence, χ_1 is uniformly bounded. To show that χ_1 is completely continuous, we apply Theorem 2.2, the Kolmogorov compactness criterion. Let Ω be a bounded subset of $\eth_r$. Then $\chi_1(\Omega)$ is bounded in $L^p(J)$, the condition (i) of Theorem 2.2 is applied. Next we will show that $(\chi_1 x)_h \to \chi_1 x$ in $L^p(J)$ as $h \to 0$, uniformly with respect to $x \in \Omega$. We have the following estimation:

$$\|(\chi_1 x)_h(t) - (\chi_1 x)(t)\|_p^p = \int_1^e \left|(\chi_1 x)_h(s) - (\chi_1 x)(t)\right|^p dt,$$
$$\leq \int_1^e \Big|\frac{1}{h}\int_t^{t+h}(\chi_1 x)(s)ds - (\chi_1 x)(t)\Big|^p dt,$$
$$\|(\chi_1 x)_h(t) - (\chi_1 x)(t)\|_p^p \leq \int_1^e \frac{1}{h}\int_t^{t+h}\Big|I^\delta V(s,x(s)) - I^\delta V(t,x(t))\Big|^p ds dt.$$

Since $V \in L^p(J)$, we get that $I^\delta V \in L^p(J)$

$$\frac{1}{h}\int_t^{t+h}\left|I^\delta V(s,x(s)) - I^\delta V(t,x(t))\right|^p ds \to 0.$$

Hence

$$(\chi_1 x)_h(t) \to (\chi_1 x)(t),$$

uniformly a $h \to 0$.

Then, by Theorem 2.2, we deduce that $\chi_1(\Omega)$ is relatively compact; that is, χ_1 is a compact operator. As a consequence of Krasnoselskiis fixed-point theorem, the boundary value problems (1.1) and (1.2) have at least one solution in $\eth_r$. □

3.2. *Existence of solution for the fractional integro-differential equation*

In this section, we prove the existence and uniqueness of solutions through Krasnoselskiis and Banach fixed-point theorems for the integro-differential equations

$$^{C\mathcal{H}}D_{1+}^{\delta}\phi(t) = V\Big(t, \int_1^t \frac{(\ln(\frac{t}{s}))^{\delta-1}}{\Gamma(\delta)}\Delta(s,\phi(s))\frac{ds}{s}\Big) \qquad (3.7)$$

with the boundary condition

$$\lambda_1\phi(1) + \lambda_2\phi(T) = \lambda_3, \qquad I = [1,T]. \qquad (3.8)$$

For measurable functions

$$V : I \times \Re \to \Re,$$

define the norm

$$\|V\|_p^p = \int_1^T |V(t)|^p\, dt, \quad (1 \leq p < \infty),$$

where $L^p(I,\Re)$ is the Banach space of all Lebesgue measurable functions. In order to achieve the results, the following assumptions are required:

$(P1)$ There exists positive constants η_1 and η_2 such that

$$\left|V(t,\phi(t))\right| \leq \eta_1|\phi(t)|$$

and

$$\left|\Delta(t,\phi(t))\right| \leq \eta_2|\phi(t)|$$

for each $t \in I$ and all $\phi \in \Re$.

$(P2)$ There exists a positive constants $\varrho_1, \varrho_2 > 0$, such that

$$\left|V(t,\phi_1(t)) - V(t,\phi_2(t))\right| \leq \varrho_1|\phi_1(t)-\phi_2(t)|,$$

$$\left|\Delta(t,\phi_1(t)) - \Delta(t,\phi_2(t))\right| \leq \varrho_2|\phi_1(t)-\phi_2(t)|,$$

for each $\phi_1, \phi_2 \in \Re$.

For computational convenience, we set

$$\Lambda_1 = \frac{\lambda_3}{\lambda_1+\lambda_2}, \qquad \Lambda_2 = \frac{\lambda_2}{\lambda_1+\lambda_2},$$
$$\mho_3 = \Big(\frac{(2\varrho_1\varrho_2)^p}{(\Gamma(\delta))^{2p}}(\frac{p-1}{p\delta-1})^{2p-2}\frac{(\ln(T))^{2p\delta}}{2(p\delta)^2}T^2$$
$$+ \frac{(2\Lambda_2\varrho_1\varrho_2)^p}{(\Gamma(\delta))^{2p}p\delta}\frac{(\ln(T))^{2p\delta-1}}{(\frac{p\delta-1}{p-1})^{2p-2}}T(T-1)\Big),$$

$$\begin{aligned}\Im_1 =&\frac{(2\eta_1\eta_2)^p}{2(\Gamma(\delta))^{2p}(p\delta)^2}\frac{T^2(\ln(T))^{2p\delta}}{(\frac{p\delta-1}{p-1})^{2p-2}}\\&+\frac{2^{2p}(\Lambda_2\eta_1\eta_2)^p}{(\Gamma(\delta))^{2p}p\delta}\frac{(\ln(T))^{2p\delta-1}}{(\frac{p\delta-1}{p-1})^{2p-2}}T(T-1),\\\Im_2 =&2^{2p}(T-1)|\Lambda_1|^p,\\\Im_3 =&\Big(\frac{(\Lambda_2\varrho_1\varrho_2)^p}{(\Gamma(\delta))^{2p}p\delta}\frac{(\ln(T))^{2p\delta-1}}{(\frac{p\delta-1}{p-1})^{2p-2}}T(T-1)\Big)^{\frac{1}{p}}.\end{aligned}$$

Lemma 3.2. *Let $\phi \in C(I,\Re)$ and $0 < \delta \leq 1$, then the solution of the boundary value problems (3.7) and (3.8) is given by*

$$\begin{aligned}\phi(t) = &\int_1^t \frac{(\ln(\frac{t}{s}))^{\delta-1}}{\Gamma(\delta)}V(s,\int_1^s \frac{(\ln(\frac{s}{\wp}))^{\delta-1}}{\Gamma(\delta)}\Delta(\wp,\phi(\wp))\frac{d\wp}{\wp})\frac{ds}{s}\\&-\Lambda_2\int_1^T \frac{(\ln(\frac{T}{s}))^{\delta-1}}{\Gamma(\delta)}V(s,\int_1^s \frac{(\ln(\frac{s}{\wp}))^{\delta-1}}{\Gamma(\delta)}\Delta(\wp,\phi(\wp))\frac{d\wp}{\wp})\frac{ds}{s}+\Lambda_1.\end{aligned} \tag{3.9}$$

Proof. By applying Lemma 2.1, we can reduce the problems (3.7) and (3.8) to an integral equation

$$\phi(t) = \int_1^t \frac{(\ln(\frac{t}{s}))^{\delta-1}}{\Gamma(\delta)}V(s,\int_1^s \frac{(\ln(\frac{s}{\wp}))^{\delta-1}}{\Gamma(\delta)}\Delta(\wp,\phi(\wp))\frac{d\wp}{\wp})\frac{ds}{s}+c_1,$$

from the boundary condition (3.8), we obtain

$$c_1 = \Lambda_1-\Lambda_2\int_1^T \frac{(\ln(\frac{T}{s}))^{\delta-1}}{\Gamma(\delta)}V(s,\int_1^s \frac{(\ln(\frac{s}{\wp}))^{\delta-1}}{\Gamma(\delta)}\Delta(\wp,\phi(\wp))\frac{d\wp}{\wp})\frac{ds}{s}.$$

Then the solution is

$$\begin{aligned}\phi(t) = &\int_1^t \frac{(\ln(\frac{t}{s}))^{\delta-1}}{\Gamma(\delta)}V(s,\int_1^s \frac{(\ln(\frac{s}{\wp}))^{\delta-1}}{\Gamma(\delta)}\Delta(\wp,\phi(\wp))\frac{d\wp}{\wp})\frac{ds}{s}\\&-\Lambda_2\int_1^T \frac{(\ln\frac{T}{s})^{\delta-1}}{\Gamma(\delta)}V(s,\int_1^s \frac{(\ln(\frac{s}{\wp}))^{\delta-1}}{\Gamma(\delta)}\Delta(\wp,\phi(\wp))\frac{d\wp}{\wp})\frac{ds}{s}\\&+\Lambda_1.\end{aligned}$$

The proof is completed. □

Our first result is based on krasnoselskii's fixed-point theorem.

Theorem 3.3. *Assume that $(P1)$ and $(P2)$ hold. Then the boundary value problems (3.7) and (3.8) have at least one solution.*

Proof. Let us define two operators, ϖ_1 and ϖ, from Eq (3.9) as

$$\varpi_1(\phi)(t) = \int_1^t \frac{(\ln\frac{t}{s})^{\delta-1}}{\Gamma(\delta)}V(s,\int_1^s \frac{(\ln(\frac{s}{\wp}))^{\delta-1}}{\Gamma(\delta)}\Delta(\wp,\phi(\wp))\frac{d\wp}{\wp})\frac{ds}{s},$$

$$\begin{aligned}\varpi(\phi)(t) =&\Lambda_1-\Lambda_2\int_1^T \frac{(\ln\frac{T}{s})^{\delta-1}}{\Gamma(\delta)}V(s,\int_1^s \frac{(\ln(\frac{s}{\wp}))^{\delta-1}}{\Gamma(\delta)}\\&\Delta(\wp,\phi(\wp))\frac{d\wp}{\wp})\frac{ds}{s}.\end{aligned}$$

Consider the set

$$\delta_r = \{\phi \in I : \|\phi\|_p^p \leq r^p, r > 0\}.$$

For $x,\phi \in \delta_r$, we have

$$\begin{aligned}&\int_1^T |(\varpi_1 x)(t)+(\varpi\phi)(t)|^p dt \leq \frac{2^p}{(\Gamma(\delta))^p}\int_1^T\Big(\int_1^t (\ln\frac{t}{s})^{\delta-1}\\&\Big|V(s,\int_1^s (\frac{\ln(\frac{s}{\wp}))^{\delta-1}}{\Gamma(\delta)}\Delta(\wp,x(\wp))\frac{d\wp}{\wp})\Big|\frac{ds}{s}\Big)^p dt\\&+\frac{2^{2p}\Lambda_2^p}{(\Gamma(\delta))^p}\int_1^T\Big(\int_1^T (\ln\frac{T}{s})^{\delta-1}\Big|V(s,\int_1^s (\frac{\ln(\frac{s}{\wp}))^{\delta-1}}{\Gamma(\delta)}\\&\Delta(\wp,\phi(\wp))\frac{d\wp}{\wp})\Big|\frac{ds}{s}\Big)^p dt+2^{2p}(T-1)|\Lambda_1|^p.\end{aligned}$$

By using $(P1)$ and Hölder's inequality, the below is found:

$$\begin{aligned}&\int_1^T |(\varpi_1 x)(t)+(\varpi\phi)(t)|^p dt \leq \frac{(2\eta_1\eta_2)^p}{(\Gamma(\delta))^{2p}}\int_1^T \frac{(\ln(t))^{p\delta-1}}{(\frac{p\delta-1}{p-1})^{2p-2}}\\&\int_1^t (\ln(s))^{p\delta-1}\int_1^s |x(\wp)|^p d\wp ds dt+2^{2p}(T-1)|\Lambda_1|^p\\&+\frac{2^{2p}(\Lambda_2\eta_1\eta_2)^p}{(\Gamma(\delta))^{2p}}\frac{(\ln(T))^{p\delta-1}}{(\frac{p\delta-1}{p-1})^{2p-2}}\int_1^T\int_1^T (\ln(s))^{p\delta-1}\\&\int_1^s |\phi(\wp)|^p d\wp ds dt.\end{aligned}$$

Integrate by parts, the result are

$$\begin{aligned}\|\varpi_1 x+\varpi\phi\|_p^p \leq&\frac{(2\eta_1\eta_2)^p}{2(\Gamma(\delta))^{2p}(p\delta)^2}\frac{T^2}{(\frac{p\delta-1}{p-1})^{2p-2}}(\ln(T))^{2p\delta}\|x\|_p^p\\&+2^{2p}(T-1)|\Lambda_1|^p\\&+\frac{2^{2p}(\Lambda_2\eta_1\eta_2)^p}{(\Gamma(\delta))^{2p}p\delta}\frac{(\ln(T))^{2p\delta-1}}{(\frac{p\delta-1}{p-1})^{2p-2}}T(T-1)\|\phi\|_p^p\end{aligned}$$

and

$$\|\varpi_1 x+\varpi\phi\|_p \leq (\Im_1\, r^p+\Im_2)^{\frac{1}{p}} \leq r.$$

Hence, $\varpi_1 x+\varpi\phi \in \delta_r$.

Now, to prove that ϖ is a contraction in $L^p(I)$. Letting $\phi_1,\phi_2 \in L^p(I)$, we have

$$\begin{aligned}&\int_1^T |(\varpi\phi_1)(t)-(\varpi\phi_2)(t)|^p dt \leq \frac{\Lambda_2^p}{(\Gamma(\delta))^p}\int_1^T\Big(\int_1^T (\ln(\frac{T}{s}))^{\delta-1}\\&\Big|V(s,\int_1^s \frac{(\ln(\frac{s}{\wp}))^{\delta-1}}{\Gamma(\delta)}\Delta(\wp,\phi_1(\wp))\frac{d\wp}{\wp})\\&-V(s,\int_1^s \frac{(\ln(\frac{s}{\wp}))^{\delta-1}}{\Gamma(\delta)}\Delta(\wp,\phi_2(\wp))\frac{d\wp}{\wp})\Big|\frac{ds}{s}\Big)^p dt.\end{aligned} \tag{3.10}$$

Then from $(P2)$ and Hölder's inequality, Eq (3.10) becomes

$$\|\varpi\phi_1 - \varpi\phi_2\|_p^p \leq \frac{(\Lambda_2\varrho_1\varrho_2)^p}{(\Gamma(\delta))^{2p}p\delta}\frac{(\ln(T))^{2p\delta-1}}{(\frac{p\delta-1}{p-1})^{2p-2}}T(T-1)\,\|\phi_2-\phi_1\|_p^p$$

and

$$\|\varpi\phi_1 - \varpi\phi_2\|_p \leq \mho_3\,\|\phi_1-\phi_2\|_p.$$

If $\mho_3 < 1$, then ϖ is a contraction mapping principle. Moreover, continuity of $x(t)$ implies that the operator $\varpi_1 x$ is continuous

$$\|(\varpi_1 x)(t)\|_p \leq \frac{1}{(\Gamma(\delta))^2}\Big(\frac{(\ln(T))^{2p\delta}}{2(p\delta)^2}\frac{T^2}{(\frac{p\delta-1}{p-1})^{2p-2}}\Big)^{\frac{1}{p}}\eta_1\eta_2 r.$$

Hence, ϖ_1 is uniformly bounded on δ_r.

Next to show that ϖ_1 is completely continuous, we apply Theorem 2.2, the Kolmogorov compactness criterion. Let ζ be a bounded subset of δ_r. Then $\varpi_1(\zeta)$ is bounded in $L^p(I)$ and the condition (i) of Theorem 2.2 is applied. Next, to show that $(\varpi_1 x)_h \to \varpi_1 x$ in $L^p(I)$ as $h \to 0$, uniformly with respect to $x \in \zeta$. Let

$$\Xi_1(s) = \int_1^s \frac{(\ln(\frac{s}{\wp}))^{\delta-1}}{\Gamma(\delta)}\Delta(\wp, x(\wp))\frac{d\wp}{\wp},$$
$$\Xi_2(t) = \int_1^t \frac{(\ln(\frac{t}{s}))^{\delta-1}}{\Gamma(\delta)}\Delta(s, x(s))\frac{ds}{s},$$

and

$$\begin{aligned}
&\int_1^T |\top(\varpi_1 x)_h(t) - \top(\varpi_1 x)(t)|^p dt \\
&\leq \int_1^T |\frac{1}{h}\int_t^{t+h}(\varpi_1 x)(s)ds - (\varpi_1 x)(t)|^p dt, \\
&\int_1^T |\top(\varpi_1 x)_h(t) - \top(\varpi_1 x)(t)|^p dt \\
&\leq \int_1^T \frac{1}{h}\int_t^{t+h}|I^\delta V(s,\Xi_1(s)) - I^\delta V(t,\Xi_2(t))|^p ds dt.
\end{aligned} \tag{3.11}$$

Since $V \in L^p(I)$, we get that $I^\delta V \in L^p(I)$, so we have

$$\frac{1}{h}\int_t^{t+h}|I^\delta V(s,\Xi_1(s)) - I^\delta V(t,\Xi_2(t))|^p ds \to 0.$$

Then by Theorem 2.2, we deduce that $\varpi_1(\zeta)$ is relatively compact; this implies that ϖ_1 is a compact operator. As a consequence of Krasnoselskiis fixed-point theorem the boundary value problems (3.7) and (3.8) have at least one solution. The proof is complete. □

Now, the uniqueness result for the problems (3.7) and (3.8) is based on the Banach contraction principle.

Theorem 3.4. *Suppose that $(P1)$ and $(P2)$ holds. If*

$$(\mho_3)^{\frac{1}{p}} < 1.$$

Then the boundary value problems (3.7) and (3.8) have a unique solution.

Proof. Define the operator

$$\theta : L^p(I) \to L^p(I)$$

as follows:

$$\begin{aligned}
(\theta\phi)(t) = &\int_1^t \frac{(\ln\frac{t}{s})^{\delta-1}}{\Gamma(\delta)}V(s, \int_1^s \frac{(\ln(\frac{s}{\wp}))^{\delta-1}}{\Gamma(\delta)}\Delta(\wp,\phi(\wp))\frac{d\wp}{\wp})\frac{ds}{s} \\
&- \Lambda_2\int_1^T \frac{(\ln\frac{T}{s})^{\delta-1}}{\Gamma(\delta)}V(s, \int_1^s \frac{(\ln(\frac{s}{\wp}))^{\delta-1}}{\Gamma(\delta)}\Delta(\wp,\phi(\wp))\frac{d\wp}{\wp})\frac{ds}{s} + \Lambda_1.
\end{aligned}$$

We need to show that $\theta\sigma_r \subseteq \sigma_r$, where,

$$\sigma_r = \{\phi \in L^p(I) : \|\phi\|_p^p \leq r^p, r > 0\}.$$

For $\phi \in \sigma_r$, we have

$$\begin{aligned}
\int_1^T |(\theta\phi)(t)|^p dt \leq &\frac{2^p}{(\Gamma(\delta))^p}\int_1^T\Big(\int_1^t(\ln\frac{t}{s})^{\delta-1} \\
&\big|V(s,\int_1^s\frac{(\ln(\frac{s}{\wp}))^{\delta-1}}{\Gamma(\delta)}\Delta(\wp,\phi(\wp))\frac{d\wp}{\wp})\big|\frac{ds}{s}\Big)^p dt \\
&+ 2^{2p}\int_1^T|\Lambda_1|^p dt + \frac{2^{2p}\Lambda_2^p}{(\Gamma(\delta))^p}\int_1^T\Big(\int_1^T(\ln(\frac{T}{s}))^{\delta-1} \\
&\big|V(s,\int_1^s\frac{(\ln(\frac{s}{\wp}))^{\delta-1}}{\Gamma(\delta)}\Delta(\wp,\phi(\wp))\frac{d\wp}{\wp}\big|\frac{ds}{s}\Big)^p dt.
\end{aligned}$$

It follows from condition $(P1)$ and Hölder's inequality that

$$\begin{aligned}
\|\theta\phi\|_p^p \leq &\frac{(2\eta_1\eta_2)^p}{2(\Gamma(\delta))^{2p}(p\delta)^2}\frac{T^2}{(\frac{p\delta-1}{p-1})^{2p-2}}(\ln(T))^{2p\delta}\|\phi\|_p^p \\
&+ 2^{2p}(T-1)|\Lambda_1|^p \\
&+ \frac{2^{2p}(\Lambda_2\eta_1\eta_2)^p}{(\Gamma(\delta))^{2p}p\delta}\frac{(\ln(T))^{2p\delta-1}}{(\frac{p\delta-1}{p-1})^{2p-2}}T(T-1)\|\phi\|_p^p.
\end{aligned}$$

Hence, θ maps $L^p(I)$ into itself. Now, to prove that θ is a contraction mapping. Let $\phi_1, \phi_2 \in L^p(I)$, we get

$$\int_1^T |(\theta\phi_1)(t) - (\theta\phi_2)(t)|^p dt \le \frac{2^p}{(\Gamma(\delta))^p} \int_1^T \Big(\int_1^t (\ln(\frac{t}{s}))^{\delta-1} \Big| V(s, \int_1^s \frac{(\ln(\frac{s}{\wp}))^{\delta-1}}{\Gamma(\delta)} \Delta(\wp, \phi_1(\wp)) \frac{d\wp}{\wp}) - V(s, \int_1^s \frac{(\ln(\frac{s}{\wp}))^{\delta-1}}{\Gamma(\delta)} \Delta(\wp, \phi_2(\wp)) \frac{d\wp}{\wp}) \Big| \frac{ds}{s} \Big)^p dt$$
$$+ \frac{2^p \Lambda_2^p}{(\Gamma(\delta))^p} \int_1^T \Big(\int_1^T (\ln(\frac{T}{s}))^{\delta-1} \Big| V(s, \int_1^s \frac{(\ln(\frac{s}{\wp}))^{\delta-1}}{\Gamma(\delta)} \Delta(\wp, \phi_1(\wp)) \frac{d\wp}{\wp}) - V(s, \int_1^s \frac{(\ln(\frac{s}{\wp}))^{\delta-1}}{\Gamma(\delta)} \Delta(\wp, \phi_2(\wp)) \frac{d\wp}{\wp}) \Big| \frac{ds}{s} \Big)^p dt.$$

By repeating the same technique of Theorem 3.3, it immediately follows that

$$\|\theta\phi_1 - \theta\phi_2\|_p \le \Big(\frac{(2\varrho_1\varrho_2)^p}{2(\Gamma(\delta))^{2p}(p\delta)^2} \frac{T^2}{(\frac{p\delta-1}{p-1})^{2p-2}} (\ln(T))^{2p\delta} + \frac{(2\Lambda_2\varrho_1\varrho_2)^p}{(\Gamma(\delta))^{2p} p\delta} \frac{(\ln(T))^{2p\delta-1}}{(\frac{p\delta-1}{p-1})^{2p-2}} T(T-1) \Big)^{\frac{1}{p}} \|\phi_2 - \phi_1\|_p$$

and

$$\|\theta\phi_1 - \theta\phi_2\|_p \le (\mho_3)^{\frac{1}{p}} \|\phi_1 - \phi_2\|_p.$$

If

$$(\mho_3)^{\frac{1}{p}} < 1,$$

then θ is a contraction mapping. Therefore, by using Banach contraction mapping, θ has a unique fixed point, which is a unique solution of the boundary value problems (3.7) and (3.8). □

4. $\mathcal{UH}$ stability

In this section, we will study the analysis of $\mathcal{UH}$ stability of the fractional differential Eq (1.1) with boundary condition (1.2) and for the problems (3.7) and (3.8).

4.1. $\mathcal{UH}$ stability for problems (1.1) and (1.2)

Theorem 4.1. *If the hypothesis $(F2)$ holds with*

$$\omega_1^p \mho_1 < 1.$$

Then the boundary value problems (1.1) and (1.2) are $\mathcal{UH}$ stable.

Proof. For $\epsilon > 0$ and w be a solution that satisfies the following inequality

$$|^{C\mathcal{H}} D_{1^+}^{\delta} w(t) - V(t, w(t))| \le \epsilon, \tag{4.1}$$

there exists a solution $\phi \in L^p(J)$ of the boundary value problems (1.1) and (1.2). Then $\phi(t)$ is given by

$$\phi(t) = 2(\ln(t)+1) \int_1^e \frac{(\ln\frac{e}{s} - \delta)(\ln\frac{e}{s})^{\delta-1}}{\Gamma(\delta+1)} V(s, \phi(s)) \frac{ds}{s} + \int_1^t \frac{(\ln\frac{t}{s})^{\delta-1}}{\Gamma(\delta)} V(s, \phi(s)) \frac{ds}{s}.$$

From the inequality (4.1) and for each $t \in J$, we have

$$|w(t) - 2(\ln(t)+1) \int_1^e \frac{(\ln\frac{e}{s} - \delta)(\ln\frac{e}{s})^{\delta-1}}{\Gamma(\delta+1)} V(s, w(s)) \frac{ds}{s} - \int_1^t \frac{(\ln\frac{t}{s})^{\delta-1}}{\Gamma(\delta)} V(s, w(s)) \frac{ds}{s}|^p \le \Big(\frac{\epsilon(\ln(t))^{\delta}}{\Gamma(\delta+1)} \Big)^p.$$

On the other hand, for each $t \in J$, the below is found

$$|w(t) - \phi(t)|^p \le 2^p \frac{\epsilon^p (\ln(t))^{p\delta}}{(\Gamma(\delta+1))^p} + 2^{2p} \Big(\int_1^t \frac{(\ln\frac{t}{s})^{\delta-1}}{\Gamma(\delta)} |V(s, w(s)) - V(s, \phi(s))| \frac{ds}{s} \Big)^p$$
$$+ 2^{3p}(\ln(t)+1)^p \Big(\int_1^e \frac{(\ln\frac{e}{s})^{\delta}}{\Gamma(\delta+1)} |V(s, w(s)) - V(s, \phi(s))| \frac{ds}{s} \Big)^p$$
$$+ 2^{3p}(\ln(t)+1)^p \Big(\int_1^e \frac{(\ln\frac{e}{s})^{\delta-1}}{\Gamma(\delta)} |V(s, w(s)) - V(s, \phi(s))| \frac{ds}{s} \Big)^p.$$

Thus, by condition $(F2)$ and Hölder inequality which implies that

$$\int_1^e |w(t) - \phi(t)|^p dt \le 2^p \int_1^e \frac{\epsilon^p (\ln(t))^{p\delta}}{(\Gamma(\delta+1))^p} dt + \frac{2^{2p}\omega_1^p}{(\Gamma(\delta))^p} \int_1^e \frac{(\ln(t))^{p\delta-1}}{(\frac{p\delta-1}{p-1})^{p-1}} \int_1^t |w(s) - \phi(s)|^p ds dt$$
$$+ \Big(\frac{2^{3p}\omega_1^p}{(\Gamma(\delta))^p} (\frac{p-1}{p\delta-1})^{p-1} + \frac{2^{3p}\omega_1^p}{(\Gamma(\delta+1))^p} (\frac{p-1}{p(\delta+1)-1})^{p-1} \Big) \int_1^e (\ln(t)+1)^p dt \int_1^e |w(s) - \phi(s)|^p ds.$$

Integrating by parts, we have

$$\|w - \phi\|_p^p \le \frac{2^p\, e\, \epsilon^p}{(\Gamma(\delta+1))^p} + \omega_1^p \mho_1 \|w - \phi\|_p^p.$$

Hence

$$|w - \phi\|_p \le c_c \epsilon,$$

where

$$c_c = \frac{2\, e^{\frac{1}{p}}}{(1 - \omega_1^p \mho_1)^{\frac{1}{p}} \Gamma(\delta+1)},$$

which implies that the boundary value problems (1.1) and (1.2) have $\mathcal{UH}$ stability. □

4.2. $\mathcal{UH}$ stability for problems (3.7) and (3.8)

Theorem 4.2. *If the hypothesis (P2) holds with $\mho_3 < 1$. Then the boundary value problems (3.7) and (3.8) are $\mathcal{UH}$ stable.*

Proof. For $\epsilon > 0$ and each solution $w \in L^p(I)$ of the inequality

$$|^{\mathcal{CH}}D^{\delta}_{1^+}w(t) - V(t, \int_1^t \frac{(\ln(\frac{t}{s}))^{\delta-1}}{\Gamma(\delta)}\Delta(s, w(s))\frac{ds}{s})|^p \le \epsilon^p, \quad (4.2)$$

and there exists a solution $\phi \in L^p(I)$ of the boundary value problems (3.7) and (3.8). Then $\phi(t)$ is given by

$$\begin{aligned}\phi(t) =& \int_1^t \frac{(\ln\frac{t}{s})^{\delta-1}}{\Gamma(\delta)}V(s, \int_1^s \frac{(\ln(\frac{s}{\wp}))^{\delta-1}}{\Gamma(\delta)}\Delta(\wp, \phi(\wp))\frac{d\wp}{\wp})\frac{ds}{s}\\ &- \Lambda_2\int_1^T \frac{(\ln\frac{T}{s})^{\delta-1}}{\Gamma(\delta)}V(s, \int_1^s \frac{(\ln(\frac{s}{\wp}))^{\delta-1}}{\Gamma(\delta)}\Delta(\wp, \phi(\wp))\frac{d\wp}{\wp})\frac{ds}{s}\\ &+ \Lambda_1.\end{aligned}$$

From the inequality (4.2) and for each $t \in I$, we obtain

$$\begin{aligned}|w(t) - &\int_1^t \frac{(\ln(\frac{t}{s}))^{\delta-1}}{\Gamma(\delta)}V(s, \int_1^s \frac{(\ln(\frac{s}{\wp}))^{\delta-1}}{\Gamma(\delta)}\Delta(\wp, w(\wp))\frac{d\wp}{\wp})\frac{ds}{s}\\ + \Lambda_2&\int_1^T \frac{(\ln(\frac{T}{s}))^{\delta-1}}{\Gamma(\delta)}V(s, \int_1^s \frac{(\ln(\frac{s}{\wp}))^{\delta-1}}{\Gamma(\delta)}\Delta(\wp, w(\wp))\frac{d\wp}{\wp})\frac{ds}{s}\\ - \Lambda_1|^p \le& \frac{(\ln(t))^{p\delta}}{(\Gamma(\delta+1))^p}\epsilon^p,\end{aligned}$$

for each $t \in I$, the below is found

$$\begin{aligned}&\int_1^T |w(t) - \phi(t)|^p dt\\ &\le 2^p\epsilon^p \int_1^T \frac{(\ln(t))^{p\delta}}{(\Gamma(\delta+1))^p}dt + \frac{2^{2p}}{(\Gamma(\delta))^p}\\ &\int_1^T \Big(\int_1^t (\ln\frac{t}{s})^{\delta-1}\Big|V(s, \int_1^s \frac{(\ln(\frac{s}{\wp}))^{\delta-1}}{\Gamma(\delta)}\Delta(\wp, w(\wp))\frac{d\wp}{\wp})\\ &- V(s, \int_1^s \frac{(\ln(\frac{s}{\wp}))^{\delta-1}}{\Gamma(\delta)}\Delta(\wp, \phi(\wp))\frac{d\wp}{\wp})\Big|\frac{ds}{s}\Big)^p dt + \frac{2^{2p}|\Lambda_2|^p}{(\Gamma(\delta))^p}\\ &\int_1^T \Big(\int_1^T (\ln(\frac{T}{s}))^{\delta-1}\Big|V(s, \int_1^s \frac{(\ln(\frac{s}{\wp}))^{\delta-1}}{\Gamma(\delta)}\Delta(\wp, w(\wp))\frac{d\wp}{\wp})\\ &- V(s, \int_1^T \frac{(\ln(\frac{T}{s}))^{\delta-1}}{\Gamma(\delta)}\Delta(\wp, \phi(\wp))\frac{d\wp}{\wp})\Big|\frac{ds}{s}\Big)^p dt,\end{aligned}$$

by $(P2)$, for each $t \in I$, we obtain

$$\begin{aligned}&\int_1^T |w(t) - \phi(t)|^p dt\\ &\le 2^p\epsilon^p \int_1^T \frac{(\ln(t))^{p\delta}}{(\Gamma(\delta+1))^p}dt + \frac{2^p(\varrho_1\varrho_2)^p}{(\Gamma(\delta))^p}\\ &\int_1^T \Big(\int_1^t (\ln(\frac{t}{s}))^{\delta-1}\int_1^s \frac{(\ln(\frac{s}{\wp}))^{\delta-1}}{\Gamma(\delta)}\Big|w(\wp) - \phi(\wp)\Big|\frac{d\wp}{\wp}\frac{ds}{s}\Big)^p dt\\ &+ \frac{2^p(\varrho_1\varrho_2)^p\Lambda_2^p}{(\Gamma(\delta))^p}\int_1^T \Big(\int_1^T (\ln(\frac{T}{s}))^{\delta-1}\int_1^s \frac{(\ln(\frac{s}{\wp}))^{\delta-1}}{\Gamma(\delta)}\\ &\Big|w(\wp) - \phi(\wp)\Big|\frac{d\wp}{\wp}\frac{ds}{s}\Big)^p dt.\end{aligned}$$

It follows from Hölder's inequality and integration by parts that

$$\begin{aligned}&\|w - \phi\|_p^p\\ &\le \frac{2^pT(\ln(T))^{p\delta}}{(\Gamma(\delta+1))^p}\epsilon^p + \Big(\frac{(2\varrho_1\varrho_2)^p}{2(\Gamma(\delta))^{2p}(p\delta)^2}\frac{T^2}{(\frac{p\delta-1}{p-1})^{2p-2}}(\ln(T))^{2p\delta}\\ &+ \frac{(2\Lambda_2\varrho_1\varrho_2)^p}{(\Gamma(\delta))^{2p}p\delta}\frac{(\ln(T))^{2p\delta-1}}{(\frac{p\delta-1}{p-1})^{2p-2}}T(T-1)\Big)\|w - \phi\|_p^p\end{aligned}$$

and

$$\|w - \phi\|_p^p \le \frac{2^pT(\ln(T))^{p\delta}}{(1-\mho_3)(\Gamma(\delta+1))^p}\epsilon^p.$$

Hence

$$\|w - \phi\|_p \le c_h\epsilon, \quad t \in I,$$

where

$$c_h = (\frac{2^pT(\ln(T))^{p\delta}}{(1-\mho_3)(\Gamma(\delta+1))^p})^{\frac{1}{p}}.$$

Thus, the solution of (3.7) and (3.8) is $\mathcal{UH}$ stable. □

5. Examples

In this section, some examples are given to illustrate our main results.

Example 5.1. *Consider the fractional boundary differential equation*

$$\begin{aligned}{}^{\mathcal{CH}}D^{\frac{3}{2}}_{1^+}\phi(t) &= \frac{2\cos(t)}{e^{4t}}\frac{|\phi|}{1+|\phi|},\\ \phi(1) = \phi'(1), &\quad \phi(e) = \int_1^e \phi(t)\frac{dt}{t},\end{aligned} \quad (5.1)$$

where

$$\delta = \frac{3}{2}, \qquad V(t, \phi(t)) = \frac{2\cos(t)}{e^{4t}}\frac{|\phi|}{1+|\phi|}$$

and $\mu = 0.0198$*, from condition $(F2)$, we get*

$$\omega_1 = 0.01979196385.$$

To prove the existence of a solution, Theorem 3.2 is applied as follows:

$$\|\chi_1 x+\chi_2 y\|_p \leq\Big[\frac{2^p}{(\Gamma(\frac{3}{2}))^p}\Big(\frac{p-1}{\frac{3p}{2}-1}\Big)^{p-1}\frac{e}{\frac{3p}{2}}+\Big(\frac{2^{3p}}{(\Gamma(\frac{3}{2}))^p}\Big(\frac{p-1}{\frac{3p}{2}-1}\Big)^{p-1} + \frac{2^{3p}}{(\Gamma(\frac{5}{2}))^p}\Big(\frac{p-1}{\frac{5p}{2}-1}\Big)^{p-1}\Big)(2^p e-1)\Big]^{\frac{1}{p}}(0.0198),$$

$$\|\chi_1 x+\chi_2 y\|_p \leq(\mho_1)^{\frac{1}{p}}\mu\, r \leq r.$$

After taken $r = 1$, one can has:

If $p = 2$, then $(\mho_1)^{\frac{1}{p}}$ (0.0198) $r = 0.4489$.

If $p = 3$, then $(\mho_1)^{\frac{1}{p}}$ (0.0198) $r = 0.3485$.

If $p = 4$, then $(\mho_1)^{\frac{1}{p}}$ (0.0198) $r = 0.3137$.

The second step shows that χ_2 is a contraction mapping

$$\|\chi_2\phi_1 - \chi_2\phi_2\|_p \leq \aleph_2\omega_1\|\phi_1-\phi_2\|_p.$$

If $p = 2$, then $(\aleph_2)^{\frac{1}{2}} = 11.0867$, $(\aleph_2)^{\frac{1}{2}}\omega_1 = 0.2194$.

If $p = 3$, then $(\aleph_2)^{\frac{1}{3}} = 8.7781$, $(\aleph_2)^{\frac{1}{3}}\omega_1 = 0.1737$.

If $p = 4$, then $(\aleph_2)^{\frac{1}{4}} = 7.9218$, $(\aleph_2)^{\frac{1}{4}}\omega_1 = 0.1568$.

Hence, χ_2 is a contraction mapping.

The third step shows that χ_1 is compact and continuous, one can has

$$\|\chi_1 x\|_p \leq \frac{1}{\Gamma(3/2)}\Big(\Big(\frac{p-1}{(3/2)p-1}\Big)^{p-1}\frac{e}{(3/2)p}\Big)^{\frac{1}{p}}(0.0198)\, r \leq r.$$

If $p = 2$, then $\|\chi_1 x\|_p < 0.5370 * 0.0198 = 0.0106$.

If $p = 3$, then $\|\chi_1 x\|_p < 0.3115 * 0.0198 = 0.0062$.

If $p = 4$, then $\|\chi_1 x\|_p < 0.2000 * 0.0198 = 0.0040$.

Hence, χ_1 is uniformly bounded and relatively compact. All steps of Theorem 3.2 are satisfied; therefore, we deduce that the problem has at least one solution.

Next, to explain the uniqueness of the solution, and according to Theorem (3.1), the results are:

If $p = 2$, then $(\mho_1)^{\frac{1}{2}} = 22.2253$, $(\mho_1)^{\frac{1}{2}}\omega_1 = 0.43988 < 1$.

If $p = 3$, then $(\mho_1)^{\frac{1}{3}} = 17.5586$, $(\mho_1)^{\frac{1}{3}}\omega_1 = 0.34752 < 1$.

If $p = 4$, then $(\mho_1)^{\frac{1}{4}} = 15.8437$, $(\mho_1)^{\frac{1}{4}}\omega_1 = 0.31357 < 1$.

Then, the problem (5.1) has a unique solution.

Example 5.2. *Consider the following boundary value problem:*

$$\begin{aligned} {}^{CH}D^{\frac{7}{4}}_{1^+}\phi(t) &= \frac{(\ln(t))^3}{19+\sin(3t)}\,\frac{1}{1+|\phi|}, \\ \phi(1) = \phi'(1), \quad \phi(e) &= \int_1^e \phi(t)\frac{dt}{t}, \end{aligned} \tag{5.2}$$

where

$$\delta = \frac{7}{4}$$

and

$$V(t,\phi(t)) = \frac{\ln(t)^3}{19+\sin(3t)}\,\frac{1}{1+|\phi|},$$

by using the condition (F2), one has $\omega_1 = 0.05011255$. *Moreover, from Theorem 3.1, we see that:*

If $p = 2$, *then*

$$(\mho_1)^{\frac{1}{2}} = 18.84098, \quad (\mho_1)^{\frac{1}{2}}\omega_1 = 0.94416 < 1.$$

If $p = 3$, *then*

$$(\mho_1)^{\frac{1}{3}} = 14.77526, \quad (\mho_1)^{\frac{1}{3}}\omega_1 = 0.74042 < 1.$$

If $p = 4$, *then*

$$(\mho_1)^{\frac{1}{4}} = 13.29004, \quad (\mho_1)^{\frac{1}{4}}\omega_1 = 0.6659 < 1.$$

By Theorem 3.1, the problem (5.2) has a unique solution.

Example 5.3. *Consider the boundary value problem*

$$\begin{aligned} {}^{CH}D^{\delta}_{1^+}\phi(t) &= \frac{\sin(t)}{(2+e^t)}\int_1^t \frac{(\ln(\frac{t}{s}))^{\delta-1}}{\Gamma(\delta)}\frac{(\ln(s))^3}{4}\,\phi(s)\frac{ds}{s}, \\ 2\phi(1) + 4\phi(e) &= 1, \end{aligned} \tag{5.3}$$

where

$$\delta = \frac{3}{4}, \quad \lambda_1 = 2, \quad \lambda_2 = 4, \quad \lambda_3 = 1,$$

and using the Lipshitz condition (P2), the out comes are

$$\varrho_1 = 0.17834267 \quad \textit{and} \quad \varrho_2 = 0.25.$$

To estimate the problem has at least one solution, apply Theorem 3.3. For the first step, we have

$$\|\varpi_1 x + \varpi\phi\|_p \leq (\Im_1\, r^p + \Im_2)^{\frac{1}{p}} \leq r.$$

Let

$$r = 1 \quad \text{and} \quad \Im_4 = (\Im_1\, r^p + \Im_2)^{\frac{1}{p}}$$

to get:

If $p = 2$, $\Im_4 = 0.9302 < r$.

If $p = 3$, $\Im_4 = 0.8025 < r$.

If $p = 4$, $\Im_4 = 0.7637 < r$.

Hence,

$$\varpi_1 x + \varpi\phi \in \delta_r.$$

For the second step,

$$\|\varpi\phi_1 - \varpi\phi_2\|_p \leq \mathfrak{I}_3 \|\phi_1 - \phi_2\|_p,$$

to prove that ϖ is a contraction mapping, the results are:

If $p = 2$, $\mathfrak{I}_3 = 0.0699 < 1$.

If $p = 3$, $\mathfrak{I}_3 = 0.0473 < 1$.

If $p = 4$, $\mathfrak{I}_3 = 0.0407 < 1$.

Hence, ϖ is a contraction mapping.

For the third step, show that ϖ_1 is compact, we have

$$\|(\varpi_1 x)(t)\|_p \leq \frac{1}{(\Gamma(\delta))^2}\Big(\frac{(\ln(T))^{2p\delta}}{2(p\delta)^2}\frac{T^2}{(\frac{p\delta-1}{p-1})^{2p-2}}\Big)^{\frac{1}{p}}\eta_1\eta_2 r.$$

If $p = 2$ then, $\|(\varpi_1 x)(t)\|_p < 0.0762$.

If $p = 3$ then, $\|(\varpi_1 x)(t)\|_p < 0.0501$.

If $p = 4$ then, $\|(\varpi_1 x)(t)\|_p < 0.0437$.

Hence, ϖ_1 is uniformly bounded and relatively compact. All conditions of Krasnoselskiis fixed-point theorem are satisfied, then the problem has at least one solution.

Now, To exhibit there is only one solution, the Banach fixed-point Theorem 3.4 is applied as follows

$$\begin{aligned}(\mho_3)^{\frac{1}{p}} =& \Big(\frac{(2\varrho_1\varrho_2)^p}{2(\Gamma(\delta))^{2p}(p\delta)^2}\frac{T^2(\ln(T))^{2p\delta}}{(\frac{p\delta-1}{p-1})^{2p-2}}\\ &+ \frac{(2\Lambda_2\varrho_1\varrho_2)^p}{(\Gamma(\delta))^{2p}p\delta}\frac{(\ln(T))^{2p\delta-1}}{(\frac{p\delta-1}{p-1})^{2p-2}}T(T-1)\Big)^{\frac{1}{p}}.\end{aligned}$$

If $p = 2$ then $(\mho_3)^{\frac{1}{2}} = 0.20659 < 1$.

If $p = 3$ then $(\mho_3)^{\frac{1}{3}} = 0.12266 < 1$.

If $p = 4$ then $(\mho_3)^{\frac{1}{4}} = 0.10042 < 1$.

Then the problem (5.3) has a unique solution.

Example 5.4. *Consider the fractional boundary value problem*

$$\begin{aligned}&{}^{CH}D^{\delta}_{1^+}\phi(t) = \frac{e^{2t-4}}{(3+4t^2)}\int_1^t \frac{(\ln(\frac{t}{s}))^{\delta-1}}{\Gamma(\delta)}\frac{2\sqrt{\ln(s)}}{5}|\phi(s)|\frac{ds}{s},\\ &4\phi(1) + 2\phi(e) = 0.5,\end{aligned} \tag{5.4}$$

where $\delta = 0.6, \lambda_1 = 4, \lambda_2 = 2, \lambda_3 = 0.5$, and by using the condition (P2), we have $\varrho_1 = 0.129198$ and $\varrho_1 = 0.4$, from Theorem 3.4, the results are:

If $p = 2$, then $(\mho_3)^{\frac{1}{2}} = 0.403497 < 1$.

If $p = 3$, then $(\mho_3)^{\frac{1}{3}} = 0.169727 < 1$.

If $p = 4$, then $(\mho_3)^{\frac{1}{4}} = 0.132042 < 1$.

Then by Theorem 3.4, the problem (5.4) has a unique solution.

6. Numerical results

In this section, we deal with $\mathcal{ADM}$ to find the approximate solution of fractional differential and integro-differential equations; some numerical examples are presented to compare between the exact and approximate solutions.

6.1. The $\mathcal{ADM}$

George Adomian established the $\mathcal{ADM}$ in the 1980s. The $\mathcal{ADM}$ has been paid much attention in the recent years in applied mathematics, and in the field of series solutions particular. Moreover, it is a fact that this method is powerful and effective, and it easily solves many types of linear or nonlinear ordinary or partial differential equations, and integral equations; see [51, 52]. This method generates a solution in the form of a series whose terms are determined by a recursive relationship using these Adomian polynomials. A brief outline of the method follows. For every nonlinear differential equation, it can be decomposed into the following form:

$$L(\phi) + R(\phi) + N(\phi) = h, \tag{6.1}$$

where L is the highest order differential operator, $R(\phi)$ is the remainder of the linear part, $N(\phi)$ represents the nonlinear part and h is a given function. In general, the operator L is invertible. If we take L^{-1}(integral operator) on both sides of Eq (6.1), an equivalent expression can be given

$$\phi = -L^{-1}R(\phi) - L^{-1}N(\phi) + L^{-1}h + g, \tag{6.2}$$

here g satisfies $Lg = 0$ and the initial conditions. If L is the second-order derivative, L^{-1} is the two-fold definite integral. For the $\mathcal{ADM}$, the solution u is expressed in terms of a series form

$$\phi = \sum_{k=0}^{\infty}\phi_k.$$

If we have a nonlinear term $N(\phi)$ it is represented by the Adomian polynomials A_k

$$N(\phi) = \sum_{k=0}^{\infty} A_k.$$

A_k depends on $\phi_0, \phi_1, \cdots, \phi_k$ and can be formulated by

$$A_k = \frac{1}{k!}\Big[\frac{d^k}{d\lambda^k} N\Big(\sum_{k=0}^{\infty} \lambda^k \phi_k\Big),\Big]_{\lambda=0}, \qquad k = 0, 1, 2, \cdots.$$

Then Eq (6.2) can be written as

$$\sum_{k=0}^{\infty} \phi_k = -L^{-1} R \sum_{k=0}^{\infty} (\phi_k) - L^{-1} \sum_{k=0}^{\infty} (A_k) + g.$$

Example 6.1. *Consider the boundary value problem*

$$\begin{aligned} {}^{CH}D_{1^+}^{\delta}\phi(t) &= \ln(t) + \phi - \frac{(\ln(t))^{\delta+1}}{\Gamma(\delta+2)} + \frac{2(\delta+1)}{\Gamma(\delta+3)}(\ln(t)+1), \\ \phi(1) = \phi'(1), \quad \phi(e) &= \int_1^e \phi(t)\frac{dt}{t}, \quad 1 < \delta \leq 2. \end{aligned} \tag{6.3}$$

The exact solution is

$$\phi(t) = \frac{(\ln(t))^{\delta+1}}{\Gamma(\delta+2)} - \frac{2(\delta+1)}{\Gamma(\delta+3)}(\ln(t)+1).$$

Applying the inverse operator

$$L^{-1} = {}_1^tI^{\delta}$$

on (6.3), we find that:

$$\phi(t) = {}_1^tI^{\delta}\phi(t) + \phi_0(t).$$

In order to obtain $\phi(t)$, we apply the Adomian iterative scheme

$$\begin{aligned} \phi_{n+1}(t) &= {}_1^t I^{\delta}\phi_n(t), \\ \phi_0(t) &= (k+1)\frac{(\ln(t))^{\delta+1}}{\Gamma(\delta+2)} - \frac{(\ln(t))^{2\delta+1}}{\Gamma(2\delta+2)} \\ &\quad + k\frac{(\ln(t))^{\delta}}{\Gamma(\delta+1)} + c_0 + c_1\ln(t), \\ k &= \frac{2(\delta+1)}{\Gamma(\delta+3)}. \end{aligned}$$

Now, to find, $\phi_1, \phi_2, \phi_3, \cdots$, it follows

$$\begin{aligned} \phi_1(t) =&(k+1)\frac{(\ln(t))^{2\delta+1}}{\Gamma(2\delta+2)} - \frac{(\ln(t))^{3\delta+1}}{\Gamma(3\delta+2)} \\ &+ k\frac{(\ln(t))^{2\delta}}{\Gamma(2\delta+1)} + c_0\frac{(\ln(t))^{\delta}}{\Gamma(\delta+1)} + c_1\frac{(\ln(t))^{\delta+1}}{\Gamma(\delta+2)}, \\ \phi_2(t) =&(k+1)\frac{(\ln(t))^{3\delta+1}}{\Gamma(3\delta+2)} - \frac{(\ln(t))^{4\delta+1}}{\Gamma(4\delta+2)} \\ &+ k\frac{(\ln(t))^{3\delta}}{\Gamma(3\delta+1)} + c_0\frac{(\ln(t))^{2\delta}}{\Gamma(2\delta+1)} + c_1\frac{(\ln(t))^{2\delta+1}}{\Gamma(2\delta+2)}, \\ \phi_3(t) =&(k+1)\frac{(\ln(t))^{4\delta+1}}{\Gamma(4\delta+2)} - \frac{(\ln(t))^{5\delta+1}}{\Gamma(5\delta+2)} \\ &+ k\frac{(\ln(t))^{4\delta}}{\Gamma(4\delta+1)} + c_0\frac{(\ln(t))^{3\delta}}{\Gamma(3\delta+1)} + c_1\frac{(\ln(t))^{3\delta+1}}{\Gamma(3\delta+2)}, \\ &\cdots \end{aligned}$$

The approximate solution of problem (6.3) is:

$$\begin{aligned} \phi(t) =&\phi_0(t) + \phi_1(t) + \phi_2(t) + \phi_3(t) + \cdots, \\ \phi(t) =&(k+1)\sum_{p=0}^{\infty}\frac{(\ln(t))^{(p+1)\delta+1}}{\Gamma((p+1)\delta+2)} - \sum_{p=0}^{\infty}\frac{(\ln(t))^{(p+2)\delta+1}}{\Gamma((p+2)\delta+2)} \\ &+ k\sum_{p=0}^{\infty}\frac{(\ln(t))^{(p+1)\delta}}{\Gamma((p+1)\delta+1)} + c_0\sum_{p=0}^{\infty}\frac{(\ln(t))^{p\delta}}{\Gamma(p\delta+1)} \\ &+ c_1\sum_{p=0}^{\infty}\frac{(\ln(t))^{p\delta+1}}{\Gamma(p\delta+2)}. \end{aligned}$$

Tables 1–3 show the approximate and exact solutions for Example 6.3.

Table 1. Exact and approximate solutions for Example 6.1 where $\delta = 1.2$.

t	Exact	$\mathcal{ADM}$	Error
1	-0.5673	-0.5710	0.0038
1.2718	-0.6857	-0.6911	0.0054
1.5437	-0.7477	-0.7547	0.0070
1.8155	-0.7732	-0.7815	0.0082
2.0873	-0.7746	-0.7835	0.0089
2.3591	-0.7594	-0.7681	0.0087
2.6310	-0.7325	-0.7400	0.0075

Table 2. Exact and approximate solutions for Example 6.1 where $\delta = 1.5$.

t	Exact	$\mathcal{ADM}$	Error
1	-0.4299	0.4303	0.0004
1.2718	-0.5247	-0.5253	0.0006
1.5437	-0.5791	-0.5798	0.0007
1.8155	-0.6036	-0.6044	0.0009
2.0873	-0.6064	-0.6073	0.0009
2.3591	-0.5934	-0.5944	0.0009
2.6310	-0.5688	-0.5696	0.0008

Table 3. Exact and approximate solutions for Example 6.1 where $\delta = 1.9$.

t	Exact	$\mathcal{ADM}$	Error
1	-0.2807	-0.2807	0.00006
1.2718	-0.3451	-0.3452	0.00008
1.5437	-0.3857	-0.3858	0.0001
1.8155	-0.4059	-0.4060	0.0001
2.0873	-0.4096	-0.4100	0.0001
2.3591	-0.4004	-0.4005	0.0001
2.6310	-0.3807	-0.3809	0.0002

One can observe on Figure 1 a decrease of the calculated errors towards zero, which confirms the results of convergences of the approximate solution to the exact solution in Example 6.1 when $t \in [1, e]$ and for various fractional orders $\delta = 1.2, 1.3, \cdots, 1.9$.

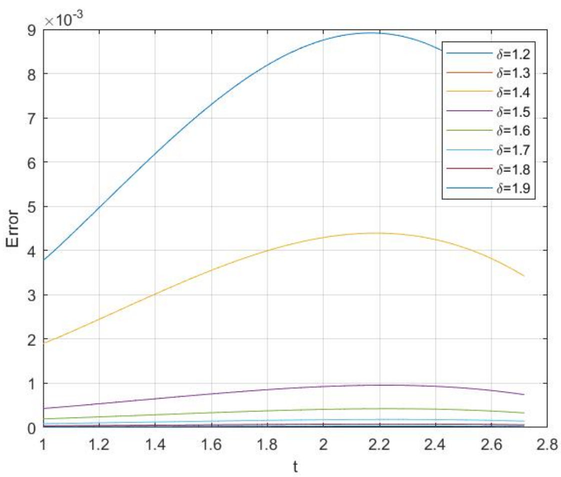

Figure 1. The absolute error between the exact and approximate solutions $\mathcal{ADM}$ for Example 6.1.

Example 6.2. *Consider the boundary value problem*

$$\begin{aligned} {}^{CH}D^{\delta}_{1^+}\phi(t) &= \phi - \ln(t) - 1, \qquad 1 < \delta \leq 2, \\ \phi(1) &= \phi'(1), \quad \phi(e) = 2. \end{aligned} \tag{6.4}$$

The exact solution is

$$\phi(t) = \ln(t) + 1.$$

Now applying the inverse operator

$$L^{-1} = {}^{t}_{1}I^{\delta}$$

on (6.4), one obtains

$$\phi_{n+1}(t) = {}^{t}_{1} I^{\delta}\phi_n(t),$$

where

$$\phi_0(t) = c_0 + c_1 \ln(t) - \frac{(\ln(t))^{\delta}}{\Gamma(\delta+1)} - \frac{(\ln(t))^{\delta+1}}{\Gamma(\delta+2)}.$$

By the same technique, we find

$$\begin{aligned}
\phi_1(t) &= c_0\frac{(\ln(t))^{\delta}}{\Gamma(\delta+1)} + c_1\frac{(\ln(t))^{\delta+1}}{\Gamma(\delta+2)} - \frac{(\ln(t))^{2\delta}}{\Gamma(2\delta+1)} - \frac{(\ln(t))^{2\delta+1}}{\Gamma(2\delta+2)},\\
\phi_2(t) &= c_0\frac{(\ln(t))^{2\delta}}{\Gamma(2\delta+1)} + c_1\frac{(\ln(t))^{2\delta+1}}{\Gamma(2\delta+2)} - \frac{(\ln(t))^{3\delta}}{\Gamma(3\delta+1)} - \frac{(\ln(t))^{3\delta+1}}{\Gamma(3\delta+2)},\\
\phi_3(t) &= c_0\frac{(\ln(t))^{3\delta}}{\Gamma(3\delta+1)} + c_1\frac{(\ln(t))^{3\delta+1}}{\Gamma(3\delta+2)} - \frac{(\ln(t))^{4\delta}}{\Gamma(4\delta+1)} - \frac{(\ln(t))^{4\delta+1}}{\Gamma(4\delta+2)},\\
\phi_4(t) &= c_0\frac{(\ln(t))^{4\delta}}{\Gamma(4\delta+1)} + c_1\frac{(\ln(t))^{4\delta+1}}{\Gamma(4\delta+2)} - \frac{(\ln(t))^{5\delta}}{\Gamma(5\delta+1)} - \frac{(\ln(t))^{5\delta+1}}{\Gamma(5\delta+2)}.
\end{aligned}$$

The solution is the given by

$$\begin{aligned}
\phi(t) =& \phi_0(t) + \phi_1(t) + \phi_2(t) + \phi_3(t) + \cdots,\\
\phi(t) =& c_0\sum_{p=0}^{\infty}\frac{(\ln(t))^{p\delta}}{\Gamma(p\delta+1)} + c_1\sum_{p=0}^{\infty}\frac{(\ln(t))^{p\delta+1}}{\Gamma(p\delta+2)}\\
&- \sum_{p=0}^{\infty}\Big(\frac{(\ln(t))^{(p+1)\delta}}{\Gamma((p+1)\delta+1)}\frac{(\ln(t))^{(p+1)\delta+1}}{\Gamma((p+1)\delta+2))}\Big).
\end{aligned}$$

Tables 4–6, approximate and exact solutions for Eq (6.4).

Table 4. Approximate solution for Example 6.2 when $\delta = 1.2$.

t	Exact	$\mathcal{ADM}$	Error
1	1.000	1.0209	0.0209
1.2	1.1823	1.2098	0.0028
1.4	1.3365	1.3710	0.0033
1.6	1.4700	1.511	0.0414
1.8	1.5878	1.6362	0.0487
2	1.6931	1.7485	0.0554
2.2	1.7885	1.8510	0.0622
2.4	1.8754	1.9443	0.0690
2.6	1.9555	2.0310	0.0754

Table 5. Exact and approximate solutions for Example 6.2 when $\delta = 1.5$.

t	Exact	$\mathcal{ADM}$	Error
1	1.0000	1.0065	0.0065
1.2	1.1823	1.9037	0.0080
1.4	1.3365	1.3462	0.0098
1.6	1.4700	1.4815	0.0114
1.8	1.5878	1.6010	0.0132
2	1.6932	1.7081	0.0150
2.2	1.7885	1.8051	0.0167
2.4	1.8754	1.8939	0.0184
2.6	1.9555	1.9756	0.0201

Table 6. Approximate solution for Example 6.2 when $\delta = 1.9$

t	Exact	$\mathcal{ADM}$	Error
1	1.0000	1.00097	0.00097
1.2	1.1823	1.8349	0.0012
1.4	1.3365	1.3378	0.0014
1.6	1.4700	1.4716	0.0016
1.8	1.5878	1.5896	0.0018
2	1.6932	1.6951	0.0020
2.2	1.7885	1.7906	0.0022
2.4	1.8754	1.8779	0.0024
2.6	1.9555	1.9581	0.0026

The behavior of the absolute errors approaching zero, as displayed in Figure 2, supports the convergence results of the approximate solution toward the exact solution referenced in Example 6.2.

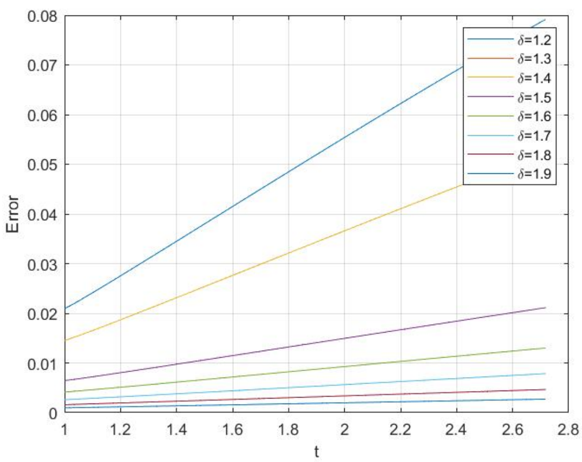

Figure 2. The absolute error between the exact and approximate solutions $\mathcal{ADM}$ for Example 6.2.

Example 6.3. *Consider the fractional integro-differential equation with initial condition*

$$\begin{aligned} {}^{\mathcal{CH}}D^{\delta}_{1^+}\phi(t) =& \phi(t) + \int_1^t \ln(s)\phi(s)\frac{ds}{s} + \frac{2(\ln(t))^{2-\delta}}{\Gamma(3-\delta)} \\ &- (\ln(t))^2 - \frac{(\ln(t))^4}{4}, \quad 0 < \delta \leq 1, \\ \phi(1) =& 0. \end{aligned} \tag{6.5}$$

The exact solution is

$$\phi(t) = (\ln(t))^2.$$

Applying the inverse operator

$$L^{-1} = {}_1^tI^{\delta}$$

on (6.5), we find that

$$\phi = L^{-1}\phi + L^{-1}\int_1^t \ln(s)\phi(s)\frac{ds}{s} + \phi_0(t),$$

where

$$\begin{aligned} \phi_0(t) &= (\ln(t))^2 - \frac{\Gamma(3)(\ln(t))^{\delta+2}}{\Gamma(\delta+3)} - \frac{\Gamma(4)(\ln(t))^{\delta+4}}{\Gamma(\delta+5)} + c_0, \\ \phi_{n+1} &= L^{-1}\phi_n + L^{-1}\int_1^t \ln(s)\phi_n(s)\frac{ds}{s}, \end{aligned}$$

and

$$\begin{aligned} \phi_1(t) =& \frac{c_0 + 2\Gamma(3)}{2\Gamma(\delta+3)}(\ln(t))^{\delta+2} - \frac{\Gamma(4) + 2(\delta+3)}{\Gamma(2\delta+5)}(\ln(t))^{2\delta+4} \\ &- \frac{\Gamma(3)(\ln(t))^{2\delta+2}}{\Gamma(2\delta+3)} + \frac{c_0(\ln(t))^{\delta}}{\Gamma(\delta+1)} + \frac{\Gamma(4)(\ln(t))^{\delta+4}}{\Gamma(\delta+5)} \\ &+ \frac{\Gamma(4)(\delta+5)}{\Gamma(2\delta+7)}(\ln(t))^{2\delta+6}, \\ \phi_2(t) =& \frac{c_0(2\delta+3)+4}{2\Gamma(2\delta+3)}(\ln(t))^{2\delta+2} - \frac{6(\delta+3)}{\Gamma(3\delta+5)}(\ln(t))^{3\delta+4} \\ &- \frac{\Gamma(3)(\ln(t))^{3\delta+2}}{\Gamma(3\delta+3)} + \frac{c_0(\ln(t))^{2\delta}}{\Gamma(2\delta+1)} \\ &+ \frac{\Gamma(4) + (c_0+4)(\delta+3)}{\Gamma(2\delta+5)}(\ln(t))^{2\delta+4} \\ &+ \frac{4\delta^2 + 40\delta + 78}{\Gamma(3\delta+7)}(\ln(t))^{3\delta+6} - \frac{\Gamma(4)(\delta+5)}{\Gamma(2\delta+7)}(\ln(t))^{2\delta+6} \\ &- \frac{\Gamma(4)(\delta+5)(2\delta+7)}{\Gamma(3\delta+9)}(\ln(t))^{3\delta+8}, \end{aligned}$$

The analogous process gives

$$\phi(t) = \phi_0(t) + \phi_1(t) + \phi_2(t) + \phi_3(t) + \cdots .$$

Tables 7–9 show the approximate and exact solutions for Eq (6.5).

Table 7. Approximate solution for Example 6.3 when $\delta = 0.3$.

t	Exact	$\mathcal{ADM}$	Error
1	0	0	0
1.2	0.0332	0.0305	2.7×10^{-3}
1.4	0.1132	0.0969	1.6×10^{-2}
1.6	0.2209	0.1768	4.4×10^{-2}
1.8	0.3454	0.2581	8.7×10^{-2}
2.0	0.4804	0.3332	1.5×10^{-1}
2.2	0.6216	0.3969	2.2×10^{-1}
2.4	0.7664	0.4454	3.2×10^{-1}
2.6	0.9131	0.4755	4.4×10^{-1}

Table 8. Exact and approximate solutions for Example 6.3 where $\delta = 0.5$.

t	Exact	$\mathcal{ADM}$	Error
1	0	0	0
1.2	0.0332	0.0328	4.3×10^{-4}
1.4	0.1132	0.1094	3.7×10^{-3}
1.6	0.2209	0.2087	1.2×10^{-2}
1.8	0.3454	0.3184	2.7×10^{-2}
2.0	0.4804	0.4309	5.0×10^{-2}
2.2	0.6216	0.5411	8.0×10^{-2}
2.4	0.7664	0.6455	1.2×10^{-1}
2.6	0.9131	0.7413	1.7×10^{-1}

Table 9. Exact and approximate solutions for Example 6.3 where $\delta = 0.9$.

t	Exact	$\mathcal{ADM}$	Error
1	0	0	0
1.2	0.0332	0.0332	5.5×10^{-7}
1.4	0.1132	0.1131	1.4×10^{-4}
1.6	0.2209	0.2202	6.6×10^{-4}
1.8	0.3454	0.3436	1.8×10^{-3}
2.0	0.4804	0.4764	4.0×10^{-3}
2.2	0.6216	0.61437	7.2×10^{-3}
2.4	0.7664	0.7543	1.2×10^{-2}
2.6	0.9131	0.8942	1.9×10^{-2}

The Figure 3 shows that the absolute errors between the approximate solution and the exact solution described in Example 6.3 are approaching zero.

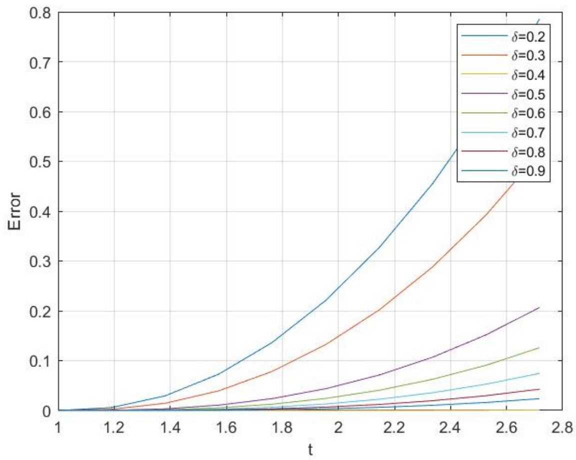

Figure 3. The absolute error between the exact and approximate solutions $\mathcal{ADM}$ for Example 6.3.

This indicates that the $\mathcal{ADM}$ is a powerful and effective technique for obtaining accurate results.

7. Conclusions

In this paper, we investigated the L^p-solutions for nonlinear fractional differential and integro-differential equations with boundary conditions in the sense of the $\mathcal{CH}$ derivative. By means of the Krasnoselskii fixed-point theorem and the Banach contraction principle, we have established sufficient conditions for the existence and uniqueness of solutions for a nonlinear problems. In addition, the $\mathcal{UH}$ stability of the solutions for the indicated problems is studied. We also employed the $\mathcal{ADM}$ to estimate the approximate solutions. Finally, we present examples to demonstrate the consistency of the theoretical findings. In future works, one can extend the given problems to more fractional derivatives such as the Hilfer derivative and Caputo-Fabrizio derivative.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no conflicts of interest in this paper.

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