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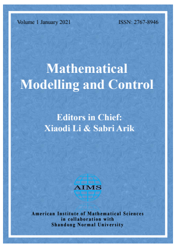


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***Research article***

# Existence of nonoscillatory solutions for higher order nonlinear mixed neutral differential equations

**Hui Li***, **Nana Jin and Yu Zhang**

School of Mathematical Sciences, University of Ji'nan, Ji'nan 250022, China

* **Correspondence:** Email: sms_lih@ujn.edu.cn.

**Abstract:** In this paper, the existence of nonoscillatory solutions for a class of higher-order nonlinear differential equations is investigated. Notably, the equations are of mixed neutral type with a forcing term, which distinguished the equations in this paper from the existing ones and made the qualitative analysis of the solution more complicated. By means of the Schauder-Tychonoff fixed point theorem and inequality techniques, some new sufficient conditions for the existence of nonoscillatory solutions were established. The results in this paper improved and generalized some known results in the existing works. Finally, an example was given to illustrate the effectiveness of the proposed method.

**Keywords:** nonoscillatory solutions; neutral; mixed differential equations; nonlinear analysis

## 1. Introduction

The study of the oscillatory and non-oscillatory properties of solutions is of great significance in mathematics, physics, and engineering. Oscillatory solutions can describe many natural and social phenomena, such as the oscilltion of mechanical systems, the propagation of electromagnetic waves, and oscillations in circuits. Nonoscillatory solutions typically represent the steady-state behavior of a system, such as the DC steady-state in a circuit, the equilibrium state of chemical reactions, etc. In control theory, the oscillation of a system can be used to determine its stability. Nonoscillatory solutions usually correspond to the asymptotic stable state of the system, which is crucial for understanding the long-term behavior of the system.

In recent years, the qualitative properties of differential equations [1–3]. Specifically, the work [1] investigated the global dynamics of the Lotka-Volterra systems with anti-symmetric interactions. Authors in [2] cinsidered a class of survival red blood cells model with time-varying delays and impulsive effects. The authors in [3] discussed some basic properties of solutions to fractional hybrid $q$-difference equations. Additionally, [4] studied the feedback control for uncertain nonlinear systems, that is, the global stabilization via adaptive event-triggered output feedback. As one of the fundamental properties of equations, the oscillation and non oscillation have received increasing attention from scholars (half-linear equations in [5], quasilinear equations in [6, 7], and nonlinear in [8, 9]). Differential equations with forcing terms are used as powerful tools to describe many physical and practical problems, such as classical oscillator in chaotic phenomena, periodic orbit extraction, nonlinear mechanical oscillators, and prediction of diseases [10–12]. The oscillation and other complex behaviours of various differential equations have been widely investigated (see the works [13–15]). Specifically, Oscillation and nonoscillation of solutions of a second-order nonlinear ordinary differential equation was discussed in [13]. The authors in [14] considered a class of fractional partial differential equations with damping term subject to Robin and Dirichlet boundary value conditions. In [15], complicated behaviors of a delay differential equation are explored through the Euler discretization method. Mixed neutral differential equations find numerous applications in natural sciences and technology (see [16, 17]), but they have specific properties that make their study difficult in aspects of ideas and techniques. These difficulties explain the relatively small number of results about this kind of differential equation, especially the higher order mixed neutral delay differential equations with forcing terms. Some related papers can be found in [18–20] and the references cited therein.

Specifically, in 2005, Zhang et al. [18] investigated the existence of nonoscillatory solutions for the first order neutral delay differential equations with variable coefficients

$$[x(t) + P(t)x(t-\tau)]' + Q_1(t)x(t-\sigma_1) - Q_2(t)x(t-\sigma_2) = 0,$$

where $t \geq t_0$. Some sufficient conditions were obtained by means of contraction mapping principle. In 2007, Zhou [19] studied the existence of nonoscillatory solutions for the second order nonlinear neutral delay differential equations

$$[r(t)(x(t) + P(t)x(t-\tau))']' + \sum_{i=1}^{m} Q_i(t)f_i(x(t-\sigma_i)) = 0,$$

where $t \geq t_0$. Some new sufficient conditions for the existence of nonoscillatory solutions for the above equations were obtained by means of the Krasnoselskiis fixed point theorem. In 2015, Candan [20] discussed the existence of nonoscillatory solutions for higher order delay differential equations with the forcing term

$$\left[r(t)x^{(n)}(t)\right]^{(m)} + f(t, x(t)) = g(t), \quad t \geq t_0.$$

By the method of Schauder's fixed point theorem, the author derived some new sufficient conditions that are complements and extensions of the previous papers. All papers above were concerned with the existence of nonoscillatory solutions for neutral differential equations or higher order equations with delay. The only paper that considered the existence of nonoscillatory solutions for mixed neutral delay differential equations is [21]. Candan considered a class of first order mixed neutral delay differential equations

$$\frac{d}{dt}[x(t) + P_1(t)x(t-\tau_1) + P_2(t)x(t+\tau_2)] + Q_1(t)x(t-\sigma_1) - Q_2(t)x(t+\sigma_2) = 0.$$

The author obtained some new sufficient conditions for the existence of nonoscillatory solutions by means of Banach contraction mapping principle.

However, there was no related result for the higher order mixed neutral delay differential equations. In order to make up for this, in this paper, we consider the existence of nonoscillatory solutions for the following higher order nonlinear mixed neutral delay differential equations

$$\left[a(t)(z(t))^{(n)}\right]^{(m)} + q_1(t)f(t, x(t)) - q_2(t)g(t, x(t)) = h(t), \quad (1.1)$$

where

$$z(t) = x(t) + p_1(t)x(t-\tau_1) - p_2(t)x(t+\tau_2), \quad t \in [t_0, \infty).$$

We will assume that the following conditions hold throughout this paper.

$(H_1)$ For any $t \in [t_0, \infty)$,

$$\int_t^\infty \int_s^\infty \frac{s^{n-1}u^{m-1}}{a(s)} h(u)duds < \infty, \quad (1.2)$$

$$\int_t^\infty \int_s^\infty \frac{s^{n-1}u^{m-1}}{a(s)} q_1(u)duds < \infty \quad (1.3)$$

and

$$\int_t^\infty \int_s^\infty \frac{s^{n-1}u^{m-1}}{a(s)} q_2(u)duds < \infty. \quad (1.4)$$

$(H_2)$ $f$ and $g$ are continuous and

$$0 < \frac{f(t,u)}{u} \leq k_1, \quad 0 < \frac{g(t,u)}{u} \leq k_2$$

for any $u \neq 0$, where $k_1$ and $k_2$ are two positive constants.

$(H_3)$ $a(t) \in C([t_0, \infty), (0, \infty))$, $h(t) \in C([t_0, \infty), \mathbb{R})$, $p_i(t), q_i(t) \in C([t_0, \infty), [0, \infty))$, $\tau_i > 0$, and $0 < p_i(t) \leq p_i$, $p_i$ are positive constants with

$$p_1 + p_2 < 1, \quad i = 1, 2.$$

In fact, higher order delay differential equations with forcing terms have practical applications in many scientific and engineering fields. For example, in control theory, time-delay differential equations are commonly used to describe the dynamic behavior of control systems with time delays. The forcing term can represent the external input signal or control action, and these equations are used to analyze and design the aircraft auto drive system and robot control system. In neuroscience, time-delay differential equations can be used to simulate the propagation of action potentials in neurons, and the forcing term can represent external stimuli or interactions in neural networks. In mechanical systems, time-delay differential equations can describe vibration systems with friction and damping, and forcing terms can represent external forces or disturbances.

In these applications, time delay and forcing terms are both very important factors as they can significantly affect the dynamic behavior and stability of the system. For example, time delay can cause oscillations or instability in the system, while forcing terms can cause changes in the system's response. Therefore, studying high-order delay differential equations with forcing terms is essential for understanding and predicting the behavior of these systems.

The main work of this paper can be described as follows: First, we investigate a new class of higher order nonlinear mixed neutral delay differential equations with forcing terms, which is a direct generalization of previous papers. Second, the sufficient conditions for the existence of nonoscillatory solutions are weaker than the ones in the references. Specifically, the nonlinear terms do not need to be monotonic and are not required to satisfy the Lipschitz condition.

The following are the contributions of this paper:

(1) A typical class of differential equations are investigated. Specifically, the nonlinear differential equations considered are not only of high order but also with mixed neutral delay and forcing terms.

(2) A weaker sufficient condition for the existence of nonoscillatory solution has been obtained by Schauder-Tychonoff fixed point theorem and inequality techniques.

This paper is structured as follows: In Section 2, we introduce some necessary notations, lemmas and definitions. Section 3 is fully dedicated to addressing the main results of the paper. In the last section, an example is given to illustrate our results.

## 2. Preliminaries

In this section, we will present some necessary knowledge of the definition, the notations, and Schauder-Tychonoff fixed point theorem.

**Lemma 2.1.** *[22] (Schauder-Tychonoff fixed point theorem) Let $X$ be a locally convex space, $K \subset X$ be nonempty and convex, $S \subset K$, and $S$ be compact. Given a continuous map $F\colon K \to S$, then there exists $\tilde{x} \in S$ such that*

$$F(\tilde{x}) = \tilde{x}.$$

Let $\Omega$ be the set of continuous and bounded functions on $[t_0, \infty)$ with supremum norm. Thus, $\Omega$ is a complete metric space.

As usual, a solution is called oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is said to be nonoscillatory.

## 3. Main results

We are now in a position to state and prove our main results according to the Schauder-Tychonoff fixed point theorem.

**Theorem 3.1.** *Suppose that conditions $(H_1)$–$(H_3)$ hold. Then, there exists a bounded nonoscillatory solution for Eq (1.1).*

*Proof.* Denote the subset $X_1$ of $\Omega$ and the map $F_1\colon X_1 \to \Omega$ by

$$X_1 = \{x \in \Omega \mid M_1 \le x(t) \le M_2,\ t \ge t_0\}$$

and

$$(F_1x)(t) = \begin{cases} \alpha + \dfrac{(-1)^{n+m}}{(n-1)!(m-1)!}\displaystyle\int_t^\infty \int_s^\infty \dfrac{(s-t)^{n-1}(u-s)^{m-1}}{a(s)} \\ \times(h(u) - q_1(u)f(u, x(u)) + q_2(u)g(u, x(u))duds \\ -p_1(t)x(t-\tau_1) + p_2(t)x(t+\tau_2), \quad t \ge t_1, \\ (F_1x)(t_1), \quad t_0 \le t \le t_1, \end{cases}$$

where $t_1$ is sufficiently large, $M_1$, and $M_2$ are positive constants, and

$$\alpha \in (M_1 + p_1M_2, M_2 - p_2M_2).$$

Meanwhile, it is clear that the existence of a nonoscillatory solution for Eq (1.1) is equivalent to the fixed point of $F_1$ in $X_1$. According to Lemma 2.1, we need to separate our proofs into the following four steps.

Step i: $F_1$ is continuous.

For any $\{x_n\} \in X_1, n = 1, 2, \ldots, x \in X_1$ with $x_n \to x$, $n \to \infty$, since $f$ and $g$ are continuous, it suggests that when $n \to \infty$,

$$f(t, x_n(t)) \to f(t, x(t)) \text{ and } g(t, x_n(t)) \to g(t, x(t)). \quad (3.1)$$

From conditions $(H_1)$–$(H_3)$, Eq (3.1), and Lebesgue dominated convergence theorem, we obtain that for any $t \in [t_0, \infty)$,

$$
\begin{aligned}
&|(F_1x_n)(t) - (F_1x)(t)| \\
&\le \frac{1}{(n-1)!(m-1)!}\int_t^\infty \int_s^\infty \frac{(s-t)^{n-1}(u-s)^{m-1}}{a(s)} \\
&\quad \times \Big(q_1(u)\,|f(u, x_n(u)) - f(u, x(u))| \\
&\quad + q_2(u)\,|g(u, x_n(u)) - g(u, x(u))|\,\Big)duds \\
&\quad + p_1(t)|x_n(t-\tau_1) - x(t-\tau_1)| \\
&\quad + p_2(t)|x_n(t+\tau_2) - x(t+\tau_2)| \to 0
\end{aligned}
$$

as $x_n \to x$, that is,

$$\|F_1x_n - F_1x\| \to 0$$

as $x_n \to x$.

Thus, $F_1$ is continuous.

Step ii: $F_1X_1 \subset X_1$.

We will consider the following two cases: $n+m$ is an even number and $n+m$ is an odd number.

i) $n+m$ is an even number.

From condition $(H_1)$, we know that for the sufficiently large $t_2$, we get

$$
\begin{aligned}
&\int_t^\infty \int_s^\infty \frac{s^{n-1}u^{m-1}}{a(s)}\Big(h(u) + k_2M_2q_2(u)\Big)duds \qquad (3.2)\\
&\le (n-1)!(m-1)!(M_2 - p_2M_2 - \alpha)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_t^\infty \int_s^\infty \frac{s^{n-1}u^{m-1}}{a(s)}q_1(u)duds \qquad (3.3)\\
&\le \frac{(n-1)!(m-1)!(\alpha - p_1M_2 - M_1)}{k_2M_2},
\end{aligned}
$$

where $t > t_2$. On the one hand, for any $x \in X_1$, from condition $(H_2)$ and inequality (3.2), we have that for any $t \in [t_0, \infty)$,

$$
\begin{aligned}
(F_1x)(t) &\le \alpha + \frac{1}{(n-1)!(m-1)!}\int_t^\infty \int_s^\infty \frac{(s-t)^{n-1}(u-s)^{m-1}}{a(s)} \\
&\quad \times (h(u) + q_2(u)g(u, x(u)))duds + p_2(t)x(t+\tau_2) \\
&\le \alpha + \frac{1}{(n-1)!(m-1)!}\int_t^\infty \int_s^\infty \frac{s^{n-1}u^{m-1}}{a(s)} \\
&\quad \times (h(u) + q_2(u)\frac{g(u, x(u))}{x(u)}x(u))duds + p_2M_2 \\
&\le \alpha + \frac{1}{(n-1)!(m-1)!}\int_t^\infty \int_s^\infty \frac{s^{n-1}u^{m-1}}{a(s)} \\
&\quad \times (h(u) + k_2M_2q_2(u))duds + p_2M_2 \\
&\le \alpha + (M_2 - p_2M_2 - \alpha) + p_2M_2 \\
&= M_2.
\end{aligned}
$$

On the other hand, for any $x \in X_1$, from condition $(H_2)$ and inequality (3.3), we get that for any $t \in [t_0, \infty)$,

$$
\begin{aligned}
(F_1x)(t) &\ge \alpha - \frac{1}{(n-1)!(m-1)!}\int_t^\infty \int_s^\infty \frac{(s-t)^{n-1}(u-s)^{m-1}}{a(s)} \\
&\quad \times q_1(u)f(u, x(u))duds - p_1(t)x(t-\tau_1) \\
&\ge \alpha - \frac{1}{(n-1)!(m-1)!}\int_t^\infty \int_s^\infty \frac{s^{n-1}u^{m-1}}{a(s)}q_1(u) \\
&\quad \times \frac{f(u, x(u))}{x(u)}x(u)duds - p_1M_2 \\
&\ge \alpha - \frac{k_1M_2}{(n-1)!(m-1)!}\int_t^\infty \int_s^\infty \frac{s^{n-1}u^{m-1}}{a(s)}q_1(u)duds \\
&\quad - p_1M_2 \\
&\ge \alpha - (\alpha - p_1M_2 - M_1) - p_1M_2 \\
&= M_1.
\end{aligned}
$$

ii) $n+m$ is an odd number.

Similar to Eqs (3.2) and (3.3), for the sufficiently large $t_3$, we have

$$
\begin{aligned}
&\int_t^\infty \int_s^\infty \frac{s^{n-1}u^{m-1}}{a(s)}q_1(u)duds \qquad (3.4)\\
&\le \frac{(n-1)!(m-1)!(M_2 - p_2M_2 - \alpha)}{k_1M_2}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_t^\infty \int_s^\infty \frac{s^{n-1}u^{m-1}}{a(s)}(h(u) + k_2M_2q_2(u))duds \qquad (3.5)\\
&\le (n-1)!(m-1)!(\alpha - p_1M_2 - M_1),
\end{aligned}
$$

where $t > t_3$.

On the one hand, for any $x \in X_1$, from condition $(H_2)$ and inequality (3.4), we obtain that for any $t \in [t_0, \infty)$,

$$\begin{aligned}(F_1x)(t) &\leq \alpha + \frac{1}{(n-1)!(m-1)!}\int_t^{\infty}\int_s^{\infty}\frac{(s-t)^{n-1}(u-s)^{m-1}}{a(s)}\\ &\quad\times(q_1(u)f(u,x(u)))duds + p_2(t)x(t+\tau_2)\\ &\leq \alpha + \frac{1}{(n-1)!(m-1)!}\int_t^{\infty}\int_s^{\infty}\frac{s^{n-1}u^{m-1}}{a(s)}\\ &\quad\times\left(q_1(u)\frac{f(u,x(u))}{x(u)}x(u)\right)duds + p_2M_2\\ &\leq \alpha + \frac{k_1M_2}{(n-1)!(m-1)!}\int_t^{\infty}\int_s^{\infty}\frac{s^{n-1}u^{m-1}}{a(s)}\\ &\quad\times q_1(u)duds + p_2M_2\\ &\leq \alpha + (M_2 - p_2M_2 - \alpha) + p_2M_2\\ &= M_2.\end{aligned}$$

On the other hand, for any $x \in X_1$, from condition $(H_2)$ and inequality (3.5), we have that for any $t \in [t_0, \infty)$,

$$\begin{aligned}(F_1x)(t) &\geq \alpha - \frac{1}{(n-1)!(m-1)!}\int_t^{\infty}\int_s^{\infty}\frac{(s-t)^{n-1}(u-s)^{m-1}}{a(s)}\\ &\quad\times(h(u) + q_2(u)g(u,x(u)))duds - p_1(t)x(t-\tau_1)\\ &\geq \alpha - \frac{1}{(n-1)!(m-1)!}\int_t^{\infty}\int_s^{\infty}\frac{s^{n-1}u^{m-1}}{a(s)}\\ &\quad\times\left(h(u) + q_2(u)\frac{g(u,x(u))}{x(u)}x(u)\right)duds - p_1M_2\\ &\geq \alpha - \frac{1}{(n-1)!(m-1)!}\int_t^{\infty}\int_s^{\infty}\frac{s^{n-1}u^{m-1}}{a(s)}\\ &\quad\times(h(u) + k_2M_2q_2(u))duds - p_1M_2\\ &\geq \alpha - (\alpha - p_1M_2 - M_1) - p_1M_2\\ &= M_1.\end{aligned}$$

The above equalities and Step i suggest that

$$F_1X_1 \subset X_1.$$

Step iii: $F_1x$ is uniformly bounded.

Since $F_1X_1 \subset X_1$ for any $x \in X_1$, it holds that

$$\|F_1x\| \leq M_2,$$

which means that $F_1x$ is uniformly bounded.

Step iv: $F_1x$ is equicontinuous.

For any $\varepsilon > 0$ and $t_4, t_5 \in [t_0, \infty)$, there exists $\delta > 0$ such that when

$$|t_4 - t_5| < \delta,$$

we derive

$$\begin{aligned}&|(F_1x)(t_4) - (F_1x)(t_5)|\\ &\leq \Bigg|\frac{1}{(n-1)!(m-1)!}\int_{t_4}^{t_5}\int_s^{\infty}\frac{s^{n-1}u^{m-1}}{a(s)}\\ &\quad\times(h(u) - q_1(u)f(u,x(u)) + q_2(u)g(u,x(u)))duds\\ &\quad - p_1(t_4)x(t_4-\tau_1) + p_1(t_5)x(t_5-\tau_1)\\ &\quad + p_2(t_4)x(t_4+\tau_2) - p_2(t_5)x(t_5+\tau_2)\Bigg|\\ &\leq \frac{1}{(n-1)!(m-1)!}\Bigg(\left|\int_{t_4}^{t_5}\int_s^{\infty}\frac{s^{n-1}u^{m-1}}{a(s)}h(u)duds\right|\\ &\quad + k_1M_2\left|\int_{t_4}^{t_5}\int_s^{\infty}\frac{s^{n-1}u^{m-1}}{a(s)}q_1(u)duds\right|\\ &\quad + k_2M_2\left|\int_{t_4}^{t_5}\int_s^{\infty}\frac{s^{n-1}u^{m-1}}{a(s)}q_2(u)duds\right|\Bigg)\\ &\quad + |p_1(t_4)x(t_4-\tau_1) - p_1(t_5)x(t_5-\tau_1)|\\ &\quad + |p_2(t_4)x(t_4+\tau_2) - p_2(t_5)x(t_5+\tau_2)|.\end{aligned}$$

From conditions $(H_1)$–$(H_3)$, we know that

$$|(F_1x)(t_4) - (F_1x)(t_5)| < \varepsilon$$

when

$$|t_4 - t_5| < \delta,$$

which means that $F_1x$ is equicontinuous. Therefore, $F_1$ is completely continuous on $X_1$. By Lemma 2.1, there exists $\tilde{x} \in X_1$ such that

$$F_1(\tilde{x}) = \tilde{x},$$

that is, there exists a bounded nonoscillatory solution for Eq (1.1). The proof is completed. □

In what follows, we consider a special case of Eq (1.1). Let

$$p_1(t) = p_2(t) = q_2(t) = 0 \quad \text{and} \quad q_1(t) = 1$$

in Eq (1.1). Then, Eq (1.1) can be reduced to the following form

$$\left[a(t)(x(t))^{(n)}\right]^{(m)} + f(t, x(t)) = h(t). \tag{3.6}$$

Therefore, we can easily derive another result according to Theorem 3.1.

**Corollary 3.1.** *Suppose that the following conditions hold.*

$(H_4)$ *For any* $t \in [t_0, \infty)$,

$$\int_t^{\infty}\int_s^{\infty}\frac{s^{n-1}u^{m-1}}{a(s)}h(u)duds < \infty$$

*and*

$$\int_t^\infty \int_s^\infty \frac{s^{n-1}u^{m-1}}{a(s)}duds < \infty.$$

($H_5$) *f is continuous and*

$$0 < \frac{f(t,u)}{u} \le k_1$$

*for any $u \neq 0$, where $k_1$ is a positive constant.*

($H_6$) *$a(t) \in C([t_0,\infty),(0,\infty))$ and $h(t) \in C([t_0,\infty),\mathbb{R})$.*

*Then, there exists a bounded nonoscillatory solution for Eq* (3.6).

**Remark 3.1.** *From Corollary 3.1, we will find that our result in this paper is a direct generalization of [20]. We can also present some other results when set different parameters for the coefficients.*

## 4. Example

An example will be presented in this section to illustrate our main results.

**Example 4.1.** Consider the following fifth-order mixed neutral delay differential equation

$$\left(e^t(x(t)+x(t-1)-x(t+2))''\right)''' + e^{-t}x(t) - e^{-2t}x(t) = e^{-t}, \tag{4.1}$$

where $t \ge 0$.

Comparing with Eq (1.1), we have

$$a(t) = e^t,\ \ p_1(t) = p_2(t) \equiv 1,\ \ q_1(t) = e^{-t},$$
$$q_2(t) = e^{-2t},\ \ h(t) = e^{-t},\ \ n = 2,\ \ m = 3,$$
$$\tau_1 = 1,\ \ \tau_2 = 2,\ \ f(t,u) = g(t,u) = u.$$

Then,

$$\frac{f(t,u)}{u} = \frac{g(t,u)}{u} = 1.$$

It is obvious that conditions ($H_2$) and ($H_3$) are satisfied. Next, we will verify the condition ($H_1$). Since

$$h(u) = q_1(u)$$

in this example, we only need to verify condition (1.2) or (1.3). Then,

$$\int_t^\infty \int_s^\infty \frac{s^{n-1}u^{m-1}}{a(s)}h(u)duds = \int_t^\infty \frac{s}{e^s}\int_s^\infty \frac{u^2}{e^u}duds$$
$$= \left(\frac{1}{2}t^3 + \frac{7}{4}t^2 + \frac{11}{4}t + \frac{11}{8}\right)e^{-2t}$$
$$< \infty$$

and

$$\int_t^\infty \int_s^\infty \frac{s^{n-1}u^{m-1}}{a(s)}q_2(u)duds = \int_t^\infty \frac{s}{e^s}\int_s^\infty \frac{u^2}{e^{2u}}duds$$
$$= \left(\frac{1}{6}t^3 + \frac{1}{3}t^2 + \frac{11}{36}t + \frac{11}{108}\right)e^{-3t}$$
$$< \infty,$$

for any $t \ge 0$. All conditions of Theorem 3.1 are satisfied. Therefore, there exists a nonoscillatory solution for Eq (4.1).

## 5. Conclusions

We considered the existence of nonoscillatory solutions for a class of higher order nonlinear mixed neutral delay differential equations. Not only the equations but also the results obtained are completely new, which are necessary supplements to the known results. The nonlinear terms $f$ and $g$ are sublinear in this paper. The superlinear case could discussed in future research.

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## Conflict of interest

The authors declare that there are no conflicts of interest.

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