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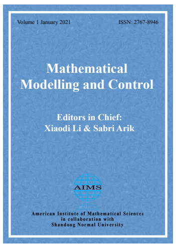

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Research article

Approximation approach for backward stochastic Volterra integral equations

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Abstract: In this paper, we focus on studying a specific type of equations called backward stochastic Volterra integral equations (BSVIEs). Our approach to approximating an unknown function involved using collocation approximation. We used Newton's technique to solve a particular BSVIE by employing block pulse functions (BPFs) and the related stochastic operational matrix of integration. Additionally, we developed considerations for Lipschitz and linear growth, along with linearity conditions, to illustrate error and convergence analysis. We compared the solutions we obtain the values of exact and approximate solutions at selected points with a defined absolute error. The computations were performed using MATLAB R2018a.

Keywords: BSVIEs; BSDEs; BPFs; operational matrix; collocation approximation

1. Introduction

Stochastic differential equations with terminal conditions are referred to as backward stochastic differential equations (BSDEs). These equations have been extensively researched for applications in finance, stochastic games, and optimal control of BSDEs. In 1990, Pardoux and Peng expanded BSDEs to a general nonlinear form [1–3]. Subsequently, the concept of BSDEs was extended to backward stochastic Volterra integral equations (BSVIEs), where both the drift and diffusion coefficients depend on two time moments. The following are general BSDEs with nonlinearities:

$$Y(t) = \psi + \int_t^T g(s, Y(s), Z(s))ds - \int_t^T Z(s)dB(s). \quad (1.1)$$

In contrast, $\{B(t)\}_{t\in[0,T]}$ defines the Wiener process. The terminal condition ψ is an $\mathcal{F}_T$-measurable random variable, and the driver g is a progressively measurable function. The adapted solution of BSDE (1.1) is the pair $(Y(\cdot), Z(\cdot))$ of the adapted processes satisfying (1.1). The second component of the adapted solution $Z(\cdot)$, is known as the martingale integrand.

The study we are conducting takes inspiration from the approach used to estimate the adapted solutions of BSDEs in [4]. We propose to explore the study of backward stochastic Volterra integral equations, building on the latest research [5–9]. BSDEs have been extensively researched and found applications in finance, stochastic games, and optimal control, with the initial research dating back over a dozen years to the work of Pardoux and Peng [10]. Adapted solutions have been studied as existence and uniqueness problems under global Lipschitz conditions, as discussed by Lin [11]. Aman et al. relaxed the global Lipschitz condition on drift [12, 13]. For a comprehensive overview of the theory and applications of BSDE (1.1), including stochastic controls and mathematical finance, readers are referred to the survey work of Karoui et al. [14]. Although there has been relatively limited numerical interest, BSVIEs can be effectively solved using block pulse functions (BPFs) and their stochastic operational matrix of integration. These equations can then be reduced to a linear lower triangular system, which can then be solved by forward substitution, as discussed in [15–22]. The emergence of BSVIEs has resulted in significant development in the area of BSDEs, serving as a natural progression from this field. Let us consider BSVIEs of the form

$$Y(t) =\psi(t) + \int_t^T g(t, s, Y(s), Z(t, s), Z(s, t))ds - \int_t^T Z(t, s)dB(s). \tag{1.2}$$

This equation was originally introduced by Yong [8]. One of the key features of BSVIEs is that they incorporate memories, making them more reflective of real-life situations. The goal is to find the unknown pair $(Y(\cdot), Z(\cdot, \cdot))$, with $Y(\cdot)$ and $Z(t, \cdot)$ being adapted for each $t \in [0, T]$. In the above equation, the free term $\psi(\cdot)$, also referred to as the terminal condition, is permitted to be only a $\mathrm{B}([0,\mathrm{T}]) \otimes \mathcal{F}_{\mathrm{T}}$-measurable stochastic process and may not necessarily be $\mathcal{F}$-adapted. Here, $\mathrm{B}([0,\mathrm{T}])$ represents the Borel σ field of $[0, T]$. The generator or the driver of the BSVIE is a specified map $g(\cdot)$, which can be deterministic or random. The coefficient $g(\cdot)$ depends on both t and s, and $g(\cdot)$ depends not only on $Z(t, s)$ but also on $Z(s, t)$. The drift generally relies on $Z(t, s)$ and $Z(s, t)$. When the driver g is independent of the term $Z(s, t)$, the BSVIE simplifies to:

$$Y(t) =\psi(t) + \int_t^T g(t, s, Y(s), Z(t, s))ds - \int_t^T Z(t, s)dB(s), \tag{1.3}$$

We have the following simple BSVIE (1.4), inspired by the approach used to estimate the adapted solutions of BSDEs in [4]:

$$Y(t) = f(t) + \int_t^T Z(t, s)ds - \int_t^T Z(t, s)dB(s), \tag{1.4}$$

where $t \in [0, T]$. It should be noted that $f(t)$ may not be necessarily adapted. The structure of this work is as follows: In Section 2, we will cover the basic characteristics of BPFs and an approximation of integration operational matrix. In Section 3, we will present the stochastic integration operational matrix. In Section 4, we will solve backward stochastic Volterra integral equations using the stochastic integration operational matrix. Section 5 presents an analysis of the numerical method's errors. In Section 6, we offer numerical results and examples to show the accuracy of the suggested approach.

2. BPFs

This section covers the notations, definitions, known results, and formulas related to BPFs, which are relevant to this paper. These details have been extensively discussed in [23, 24].

An m-set of BPFs is defined over the unit interval $[0, T)$ as: for $0 \leq 1 < m$, and $m \in \{1, 2, \ldots\}$,

$$\phi_i(t) = \begin{cases} 1, & for\ \ (i-1)h \leq t < ih, \\ 0, & otherwise, \end{cases} \tag{2.1}$$

with $t \in [0, T)$, $i = 1, 2, \ldots, m$, and $h = \frac{T}{m}$.

The BPFs have the following properties:

(1) Disjointness: The BPFs are disjointed with each other in the interval $t \in [0, T)$,

$$\Phi_i(t)\Phi_j(t) = \delta_{ij}\Phi_i(t), \tag{2.2}$$

where $i, j = 1, 2, \ldots, m$, and δ_{ij} denotes the Kronecker delta.

(2) Orthogonality: The BPFs are disjointed with each other in the interval $t \in [0, T)$,

$$\int_0^T \Phi_i(t)\Phi_j(t)dt = h\delta_{ij}, \quad i, j = 1, 2, \ldots, m. \tag{2.3}$$

(3) Completeness: If $m \to \infty$, then the BPFs set is complete, i.e., for every $f \in L^2([0, T))$, Parseval's identity holds,

$$\int_0^T f^2(t)dt = \sum_{i=1}^{\infty} f_i^2 \|\phi_i(t)\|^2,$$

where

$$f_i = \frac{1}{h}\int_0^T f(t)\phi_i(t)dt.$$

The set of functions can be described by an m vector,

$$\Phi_m(t) = (\phi_0(t), \phi_1(t), \cdots, \phi_m(t))^T, \quad t \in [0, T).$$

As a result, the following matrix form can be used to represent the connection between BPFs and their integrals.

The above representation and disjointness property follows

$$\Phi(t)\Phi^T(t) = \begin{pmatrix} \phi_1(t) & 0 & \cdots & 0 \\ 0 & \phi_2(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi_m(t) \end{pmatrix}_{m\times m}, \tag{2.4}$$

additionally, we have

$$\Phi(t)^T\Phi(t) = 1$$

and

$$\Phi(t)\Phi^T(t)V = \tilde{V}\Phi(t), \tag{2.5}$$

where V is the m-vector, $\tilde{V}$ is the $m \times m$ matrix,

$$\tilde{V} = diag(V).$$

It can be clearly seen that

$$\Phi^T(t)F\Phi(t) = \tilde{F}^T\Phi(t), \tag{2.6}$$

$\tilde{F}$ is the m-vector whose elements are equivalent to the diagonal entries of matrix F, and F is the $m \times m$ matrix.

2.1. Functions approximation

The expansion of a real bounded function $f(t)$, where $f(t) \in L^2[0, T)$, into a block pulse series is as follows

$$f(t) \simeq \hat{f}_m(t) = \sum_{i=1}^{m} f_i\Phi_i(t), \tag{2.7}$$

where f_i is the block pulse coefficient relative to the ith BPF $\Phi_i(t)$. In the form of a vector we have

$$f(t) \simeq \hat{f}_m(t) = F^T\Phi(t) = \Phi^T(t)F, \tag{2.8}$$

where

$$F = (f_1, f_2, \ldots, f_m)^T.$$

2.2. Integration operational matrix

Computing $\int_0^T \phi_i(s)ds$ follows

$$\int_0^t \phi_i(s)ds = \begin{cases} 0, & 0 \le t < (i-1)h, \\ t-(i-1)h, & (i-1)h \le t < ih, \\ h, & ih \le t < T. \end{cases} \tag{2.9}$$

Note that $t-(i-1)h$ equals to $\frac{h}{2}$ at mid-point of $[(i-1)h, ih)$, thus we can approximate $t-(i-1)h$, for $(i-1)h \le t < ih$, by $\frac{h}{2}$.

From [23], we will have:

$$\int_T^t \Phi(t)dt = \int_T^0 \Phi(t)dt + \int_0^t \Phi(t)dt, \tag{2.10}$$

where the operational integration matrix is provided by

$$P = -\frac{h}{2}\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 2 & 1 & 0 & \cdots & 0 \\ 2 & 2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & 2 & \cdots & 1 \end{pmatrix}_{m\times m} \Phi(t), \tag{2.11}$$

the backward integral of a function's block pulse series can be written as:

$$\begin{aligned} \int_T^t f(t)dt &\simeq \int_T^t F^T\Phi(\tau)d\tau \\ &= -F^TP^T\Phi(t). \end{aligned} \tag{2.12}$$

3. Stochastic integration operational matrix

The Itô integral of each single BPFs $\phi_i(t)$ can be computed as follows:

$$\int_0^t \phi_i(s)dB(s) = \begin{cases} 0, & 0 \le t < (i-1)h, \\ B(t) - B((i-1)h), & (i-1)h \le t < ih, \\ B(ih) - B((i-1)h), & ih \le t < T. \end{cases} \tag{3.1}$$

Now, expressing $\int_0^T \phi_i(s)dB(s)$, in terms of the BPFs follows

$$\begin{aligned} \int_0^T \phi_i(s)dB(s) \simeq& (B(ih/2) - B((i-1)h/2)\phi_i(t) \\ &+ (B(ih) - B((i-1)h)) \sum_{j=i+1}^{m} \phi_j(t). \end{aligned} \tag{3.2}$$

Therefore,

$$\int_T^0 \Phi(s)dB(s) + \int_0^t \Phi(s)dB(s) \simeq -P_S\Phi(t), \tag{3.3}$$

where the stochastic operational integration matrix is provided by

$$P_S = \begin{pmatrix} \gamma_1 & 0 & 0 & \cdots & 0 \\ \rho_2 & \gamma_2 & 0 & \cdots & 0 \\ \rho_3 & \rho_3 & \gamma_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho_4 & \rho_4 & \rho_4 & \cdots & \gamma_m \end{pmatrix}_{m\times m}, \quad (3.4)$$

where

$$\rho_i = B(ih) - B((i-1)h), \quad i = 1, 2, \ldots, m-1$$

and

$$\gamma_j = B(ih/2) - B((i-1)h/2), \quad j = 1, 2, \ldots, m.$$

Therefore, the Itô integral of every function $f(t)$ can be modified as:

$$\begin{aligned} \int_T^t f(s)dBs &\simeq \int_T^t F^T\Phi(\tau)d\tau \\ &= -F^T P_S^T \Phi(t). \end{aligned} \quad (3.5)$$

4. Implementation in stochastic integral equation

Using the block pulse operational matrices, first, we find the collocation approximation to the functions $z_1(t, s)$ and $z_2(t, s)$ for drift and diffusion, respectively, defined by

$$z_1(t, s) = Z(t, s), \quad z_2(t, s) = Z(t, s). \quad (4.1)$$

Plugging Eqs (1.4) and (4.1) as inputs yields

$$Y(t) = f(t) + \int_t^T z_1(t, s)ds - \int_t^T z_2(t, s)dB(s) \quad (4.2)$$

and

$$\begin{cases} z_1(t, s) := Z\Big(t, s, f(t) + \int_t^T z_1(t, s)ds - \int_t^T z_2(t, s)dB(s)\Big), \\ z_2(t, s) := Z\Big(t, s, f(t) + \int_t^T z_1(t, s)ds - \int_t^T z_2(t, s)dB(s)\Big). \end{cases} \quad (4.3)$$

Based on block pulse series, we can approximate the functions $Y(t), f(t), z_1(t, s)$, and $z_2(t, s)$, and we have

$$\begin{cases} Y(t) \simeq Y^T\Phi(t) = \Phi^T(t)Y, \\ f(t) \simeq F^T\Phi(t) = \Phi^T(t)F, \\ z_1(t, s) \simeq \tilde{z}_1(t, s) = \Phi^T(t)Z_1^T\Phi(s) = \Phi^T(t)Z_1\Phi(s), \\ z_2(t, s) \simeq \tilde{z}_2(t, s) = \Phi^T(t)Z_2^T\Phi(s) = \Phi^T(t)Z_2\Phi(s), \end{cases} \quad (4.4)$$

thus, $m \times m$-vectors Z_1, Z_2, and $m \times m$ correspond to the block pulse coefficients of $z_1(t, s)$ and $z_2(t, s)$. When (4.4) is substituted in (4.2), we obtain

$$\begin{aligned} \int_t^T z_1(t, s)ds &\simeq \int_t^T \Phi^T(t)Z_1\Phi(s)ds \\ &= \int_t^T \Phi(t)Z_1\Phi^T(s)ds \\ &= -\Phi^T(t)\tilde{Z}_1 P\Phi(s)ds; \end{aligned} \quad (4.5)$$

in addition, we can express the Itô's integral of (4.2) as follows

$$\begin{aligned} \int_t^T z_2(t, s)ds &\simeq \int_t^T \Phi^T(t)Z_2\Phi(s)dB(s) \\ &= \int_t^T \Phi(t)\Phi^T(s)Z_2 dB(s) \\ &= -\Phi^T(t)\tilde{Z}_2 Ps\Phi(s)dB(s), \end{aligned} \quad (4.6)$$

in this case,

$$\tilde{Z}_1 = diag(Z_1), \quad \tilde{Z}_2 = diag(Z_2).$$

Taking (4.5) and (4.6) and substituting into (4.3) and replacing "≃" with "=", as a result

$$\begin{cases} \Phi(t)Z_1^T\Phi(s) \\ = Z\Big(t, s, f(t) - \Phi^T(t)\tilde{Z}_1 P\Phi(s) + \Phi^T(t)\tilde{Z}_2 Ps\Phi(s)\Big), \\ \Phi(t)Z_2^T\Phi(s) \\ = Z\Big(t, s, f(t) - \Phi^T(t)\tilde{Z}_1 P\Phi(s) + \Phi^T(t)\tilde{Z}_2 Ps\Phi(s)\Big). \end{cases} \quad (4.7)$$

A collocation method based on (4.7), using m nodes

$$t_j, s_j = \frac{j}{m+1}, \quad j = 1, \ldots, m$$

is used for determining the correlation as

$$\begin{cases} \Phi(t_j)Z_1^T\Phi(s_j) \\ = z\Big(t_j, s_j, f(t_j) - \Phi^T(t_j)\tilde{Z}_1 P\Phi(s_j) + \Phi^T(t_j)\tilde{Z}_2 Ps\Phi(t_j)\Big), \\ \Phi(t_j)Z_2^T\Phi(s_j) \\ = Z\Big(t_j, s_j, f(t_j) - \Phi^T(t_j)\tilde{Z}_1 P\Phi(s_j) + \Phi^T(t_j)\tilde{Z}_2 Ps\Phi(t_j)\Big). \end{cases} \quad (4.8)$$

We obtain Z_1 and Z_2 by solving the nonlinear system (4.8). Then, the result $Y(t)$ of (4.2) is approximated as:

$$Y(t) \simeq y_m(t) = f(t) - \Phi^T(t)\tilde{Z}_1 P\Phi(s) + \Phi^T(t)\tilde{Z}_2 Ps\Phi(s). \quad (4.9)$$

5. Error analysis

This section will provide the general estimate for the error and convergence analysis.

Theorem 5.1. *Suppose that*

$$f(t,s) \in [0,1) \times [0,1)$$

and

$$e(t,s) = f(t,s) - \hat{f}_m(t,s), (t,s) \in J = [0,1) \times [0,1),$$

where

$$\hat{f}_m(t,s) = \sum_{i=1}^{m}\sum_{j=1}^{m} f_{ij}\psi_i(t)\phi_j(s)$$

is the block pulse series of $f(t,s)$. Then,

$$\|e(s,t)\| \leq \frac{h}{2\sqrt{3}}(\|f_t'\|_\infty^2 + \|f_s'\|_\infty^2)^{\frac{1}{2}}. \tag{5.1}$$

Proof. The proof is the same as done in [17], so we omit it here. □

Theorem 5.2. *Let*

$$Z : \Delta \times \mathbb{R} \times \Omega \to \mathbb{R}^n$$

be measurable and for some constant $L > 0$,

$$|Z(t,s)| \leq L, \quad (t,s) \in \Delta \tag{5.2}$$

and

$$E\int_0^T \left(\int_t^T |Z(t,s)|ds\right)dt < \infty. \tag{5.3}$$

Then, for any $f(\cdot) \in L_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$, (1.4) admits a unique adapted solution $(Y(\cdot),Z(\cdot,\cdot)) \in H^2_\Delta[0,T]$.

Proof. For a given pair $(y(\cdot),z(\cdot,\cdot)) \in H^2_\Delta[0,T]$, consider the following simple BSVIE:

$$Y(t) = f(t) + \int_t^T \bar{Z}(t,s)ds - \int_t^T Z(t,s)dB(s), \quad t \in [0,T], \tag{5.4}$$

where

$$\bar{Z}(t,s) = Z(t,s), \quad (t,s) \in \Delta.$$

To solve (5.4), we introduce the following family of BSDEs (parameterized by $t \in [0,T]$):

$$\eta(r;t) = f(t) + \int_r^T \bar{Z}(t,s)ds - \int_r^T \zeta(s;t)dB(s), \quad r \in [0,T]. \tag{5.5}$$

It is well-known that the above BSDE admits a unique adapted solution $(\eta(\cdot;t),\zeta(\cdot;t))$ and the following estimate holds:

$$\begin{aligned} &E\left[\sup_{r\in[t,T]}|\eta(r;t)|^p + \left(\int_t^T |\zeta(s;t)|^2 ds\right)^{\frac{p}{2}}\right] \\ &\leq K_p E\left[|f(t)|^p + \left(\int_t^T |\bar{Z}(t,s)|ds\right)^p\right]. \end{aligned} \tag{5.6}$$

Now, let

$$Y(t) = \eta(t;t), \quad Z(t,s) = \zeta(s;t), \quad \forall (t,s) \in \Delta. \tag{5.7}$$

Then, $(Y(\cdot),Z(\cdot,\cdot))$ is an adapted solution to BSVIE (5.4), and

$$\begin{aligned} &E\left[|Y(t)|^p + \left(\int_t^T |Z(t,s)|^2 ds\right)^{\frac{p}{2}}\right] \\ &\leq K_p E\left[|f(t)|^2 + \left(\int_t^T |\bar{Z}(t,s)|ds\right)^p\right]. \end{aligned} \tag{5.8}$$

From this estimate, together with the linearity of (5.4), we see that $(Y(\cdot),Z(\cdot,\cdot))$ is the unique adapted solution to (5.4). Also, we have the following:

$$\begin{aligned} &E\left[|Y(t)|^2 + \int_t^T |Z(t,s)|^2 ds\right] \\ &= E\left|f(t) + \int_t^T \bar{Z}(t,s)ds\right|^2 \\ &\leq 2E\left[|f(t)|^2 + \left(\int_t^T |\bar{Z}(t,s)|ds\right)^2\right]. \end{aligned} \tag{5.9}$$

Then,

$$\begin{aligned} &E\int_0^T \left[|Y(t)|^2 + \int_t^T |Z(t,s)|^2 ds\right]dt \\ &\leq 2E\int_0^T \left[|f(t)|^2 + \left(\int_t^T |\bar{Z}(t,s)|ds\right)^2\right]dt \\ &\leq KE\int_0^T \left[|f(t)|^2\right]dt + KE\int_0^T \left[|y(t)|^2 + \int_t^T |z(t,s)|^2 ds\right]dt. \end{aligned}$$

Thus,

$$(y(\cdot),z(\cdot,\cdot)) \to Y(\cdot),Z(\cdot,\cdot))$$

defines a map from $H^2_\Delta[0,T]$ to itself. We can show that for any

$$(y_i(\cdot),z_i(\cdot,\cdot)) \in H^{2,\beta}_\Delta[0,T],$$

if $Y_i(\cdot), Z_i(\cdot,\cdot))$ is the corresponding adapted solution to BSVIE (1.4), then

$$E\int_0^T\left[e^{\beta t}|Y_1(t)-Y_2(t)|^2+\int_t^T e^{\beta s}|Z_1(t,s)-Z_2(t,s)|^2ds\right]dt$$
$$\leq\frac{K}{\beta}E\int_0^T\left[e^{\beta t}|y_1(t)-y_2(t)|^2+\int_t^T e^{\beta s}|z_1(t,s)-z_2(t,s)|^2ds\right]dt,$$

with $K>0$ being an absolute constant. Hence, the map

$$(y(\cdot),z(\cdot,\cdot))\rightarrow Y(\cdot),Z(\cdot,\cdot))$$

is a contraction on $H^{2,\beta}_{\Delta}[0,T]$ for large enough β >0. We can also obtain an estimate for (Y,Z) that is marginally sharper than the above expression under identical circumstances. □

Theorem 5.3. *Assume that $(C[\Gamma],\|\cdot\|)$ is the Banach space of all continuous functions on*

$$\Gamma=[0,1)\times[0,1)$$

with norm

$$\|Z(t,s)\|=\max_{(t,s)\in\Gamma}|Z(t,s)|. \tag{5.10}$$

Proof. In order to estimate the error of the approximate solution of Eq (1.4), we assume that $Z_{mn}(t,s)$ and $Z(t,s)$ are the approximate and exact solutions of the integral equations, respectively. Let

$$M_1\equiv\sup_{0\leq t,s<1}|(t,s)|<\infty$$

and

$$M_2\equiv\sup_{0\leq t,s<1}|(t,s)|<\infty.$$

Assume the nonlinear terms Z_1 and Z_2 are satisfied in the Lipschitz condition such that

$$\begin{aligned}|Z_1(Z_{mn}(t,s))-Z_1(Z(t,s))|&\leq L_1|Z_{mn}-Z|,\\|Z_2(Z_{mn}(t,s))-Z_2(Z(t,s))|&\leq L_2|Z_{mn}-Z|.\end{aligned}\tag{5.11}$$

Assuming that the error function of the approximation solution $Z_{mn}(t,s)$ to the exact solution $Z(t,s)$ is

$$e_{mn}(t,s)=Z(t,s)-Z_{mn}(t,s),$$

we therefore consider

$$\begin{aligned}\|e_{mn}(t,s)\|=&\|Z_{mn}(t,s)-Z(t,s)\|\\&=\max_{(t,s)\in\Gamma}|Z_{mn}(t,s)-Z(t,s)|\\&\leq\max_{(t,s)\in\Gamma}\int_t^T|Z_{mn}(t,s)-Z(t,s)|ds\\&\quad-\int_t^T|Z_{mn}(t,s)-Z(t,s))|dB(s)\\&\leq(L_1-L_2)\max_{(t,s)\in\Gamma}|Z_{mn}-Z|\\&\leq\beta\max_{(t,s)\in\Gamma}|Z_{mn}-Z|,\end{aligned}\tag{5.12}$$

where

$$\beta=|L_1|-|L_2|,$$

therefore

$$(1-\beta)\|e_{mn}(t,s)\|\leq0.$$

This concludes the proof because if $0<\beta<1$, we have $\|e_{mn}(t,s)\|\rightarrow0$ as $m,n\rightarrow\infty$. □

6. Numerical examples

This section will provide four numerical examples to illustrate the results obtained in Sections 3 and 4. All computations are carried out in MATLAB R2018a, with a precision of 2.22×10^{-16}. By using the definition of absolute error, which is defined as follows, we can compare the values of approximate and exact solutions at some chosen points:

$$\|E\|_\infty=\max_{1\leq i\leq m}|X_i-\bar{X}_i|, \tag{6.1}$$

where X_i and $\bar{X}_i$ are exact and approximate solutions, respectively.

Example 6.1. *(The basic Black-Scholes model) The Black-Scholes model represents a financial market with specific derivative investment instruments, a stochastic mathematical model. Black and Scholes first presented this concept in 1973. A two-dimensional continuous time process*

$$\{(X_0(t),X(t)):0\leq t\leq T\}$$

with the risk-free asset $X_0(t)$ and the risky asset $X(t)$ describes the characteristics of prices in the Black-Scholes model. Assume that the differential equation

$$dX_0(t)=rX_0(t)dt,$$

where $r \geq 0$ is a constant, determines the behavior of $X_0(t)$. As we know, r is an instantaneous interest rate and should not be confused with the one-period rate in discrete-time models. In order for

$$X_0(t) = e^{rt},$$

we put

$$X_0(0) = 1.$$

We believe that the stock price's behavior is predetermined by

$$dX(t) = \lambda X(t)\mathrm{d}t + \mu X(t)dB(t), \quad (6.2)$$

where $\{B(t) : 0 \leq t \leq T\}$ is a standard Brownian motion with

$$B(0) = 0,$$

λ is a constant and $\mu(t)$ as a function. The model is accurate for the range $[0, T]$, where T is the option's maturity.

$$X(t) = \exp\left(\left(\lambda - \frac{1}{2}\mu^2\right)t + \mu B(t)\right)$$

provides the exact solution. In Table 1, the results obtained for

$$t = 0.8, \quad \lambda = -8, \quad \mu = -3 \quad \text{and} \quad n = 100$$

are presented and are shown in Figure 1.

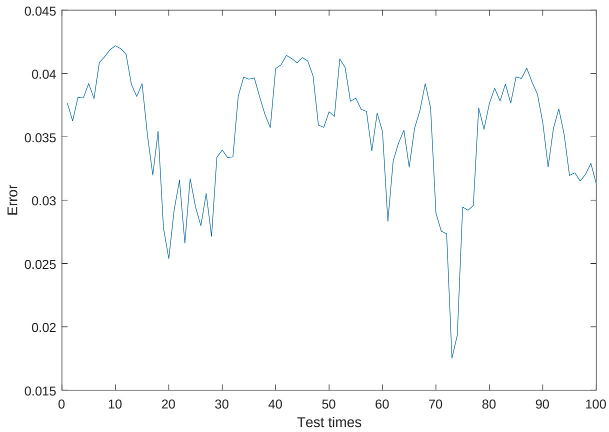

Figure 1. The graph of absolute error function for Example 6.1.

Example 6.2. *Consider the linear stochastic Volterra integral equation*

$$X(t) = X_0 + \int_0^t a^2 \cos(X(s)) \sin^3(X(s))\mathrm{d}s - a\int_0^t \sin^2(X(s))dB(s), \quad s, t \in [0, 1). \quad (6.3)$$

The exact solution is

$$X(t) = \operatorname{arccot}(aB(s) + \cot(X_0)).$$

By taking

$$n = 100, \quad a = 1/4 \quad \text{and} \quad X_0 = 0.1.$$

The numerical results are given in Table 2, and computed errors for

$$n = 100, \quad a = 1/6, \quad X_0 = 0.3 \quad \text{and} \quad t = 0.2$$

are summarized in Figure 2.

Table 1. Mean, standard deviation, and mean confidence interval for error.

t_i	$\bar{X}_E$	S_E	95% confidence interval for mean of E	
			Lower	*Upper*
0.1	1.0024e-03	0.0032e-04	1.0012e-03	1.0028e-03
0.2	1.0045e-03	0.0065e-04	1.0038e-03	1.0082e-03
0.3	1.0134e-03	0.0153e-04	1.0101e-03	1.0167e-03
0.4	1.0218e-02	0.0247e-03	1.0174e-02	1.0282e-02
0.5	1.0354e-02	0.0381e-03	1.0302e-02	1.0455-e02
0.6	1.0487e-02	0.0476e-03	1.0389e-02	1.0646e-02
0.7	1.0594e-01	0.0623e-02	1.0483e-01	1.0708e-01
0.8	1.0854e-01	0.1207e-02	1.0694e-01	1.0956e-01
0.9	1.1053e-01	0.1332e-02	1.0978e-01	1.3121e-01

Table 2. Mean, standard deviation, and mean confidence interval for error.

t_i	$\bar{X}_E$	S_E	95% confidence interval for mean of E	
			Lower	*Upper*
0.1	2.6254e-02	2.5432e-03	1.3526e-02	3.4392e-02
0.2	5.5321e-02	3.9533e-03	3.3449e-02	6.8521e-02
0.3	7.8654e-02	4.8640e-03	6.4678e-02	8.2606e-02
0.4	8.0332e-02	6.2575e-03	7.9340e-02	9.7103e-02
0.5	3.7655e-03	6.0411e-04	2.0574e-03	4.7511e-03
0.6	6.1602e-03	7.3702e-04	5.7631e-03	7.4702e-03
0.7	6.9543e-03	7.5893e-04	6.0422e-03	8.3467e-03
0.8	7.0134e-03	8.3466e-04	6.7640e-03	8.9205e-03
0.9	9.3680e-03	8.6076e-04	7.5013e-03	9.8600e-03

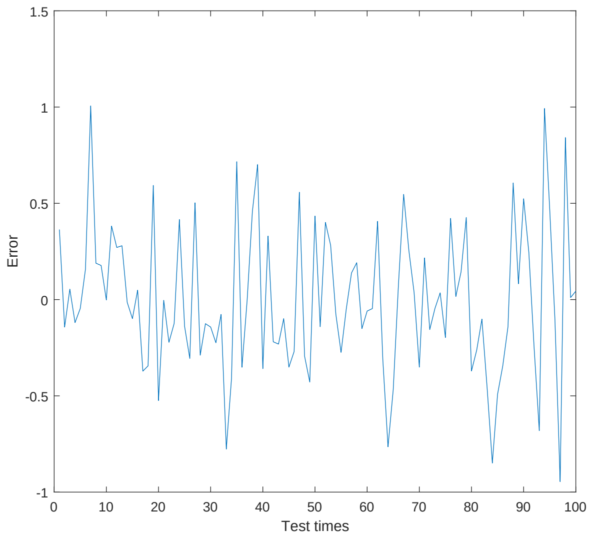

Figure 2. The graph of absolute error function for Example 6.2.

Example 6.3. *Consider the following linear stochastic Volterra integral equation*

$$Y(t) = \psi(t) + \int_0^t u(s)z_1(t,s)ds + \int_0^t \beta(s)z_2(t,s)dB(s), \quad (6.4)$$

where $t, s \in [0, T]$, *and* $\{B(t) : 0 \le t \le T\}$ *is a standard Brownian motion with*

$$B(0) = 0,$$

and $u(t)$ *and* $\beta(t)$ *are two functions. The model is accurate between* $[0, T]$, *where* T *is finite time horizon.* $\psi(t)$ *represents the initial condition of time 0 (or t). A linear stochastic Volterra integral equation equivalent to (1.4), the aforementioned relationship has*

$$z(t,s) = u(s)z_1(t,s),$$
$$z(t,s) = \beta(s)z_2(t,s),$$

the exact solution is

$$Y(t) = \psi(t)\exp\left(\int_0^t (u(s) - \frac{1}{2}\beta^2(s))ds + \int_0^t \beta(s)dB(s)\right).$$

The results obtained for

$$s = 0.3, \quad n = 100, \quad \psi = 0.6, \quad u(s) = \sin(s^2), \quad \beta(s) = \sin(s)$$

and $t \in [0, 1]$ *are summarized in Table 3 and shown in Figure 3.*

Table 3. Mean, standard deviation, and mean confidence interval for error.

t_i, s_i	$\bar{\chi}_E$	S_E	95% confidence interval for mean of E	
			Lower	*Upper*
(0.1,0.1)	5.3831e-01	7.9573e-02	4.8755e-01	5.9607e-01
(0.2,0.2)	2.6743e-01	3.0432e-02	2.1932e-01	3.4303e-01
(0.3,0.3)	7.0415e-02	1.9654e-03	6.4387e-02	8.5674e-02
(0.4,0.4)	3.0964e-02	6.4076e-03	2.5374e-02	4.9501e-02
(0.5,0.5)	1.8703e-02	4.8532e-03	1.3951e-02	2.6433e-02
(0.6,0.6)	4.9411e-03	1.0201e-03	3.2017e-03	6.0178e-03
(0.7,0.7)	1.9332e-03	5.3467e-04	1.4076e-03	2.6944e-03
(0.8,0.8)	6.5401e-04	3.0894e-04	5.1803e-04	7.8503e-04
(0.9,0.9)	2.8063e-04	1.0071e-04	1.9942e-04	3.9710e-04

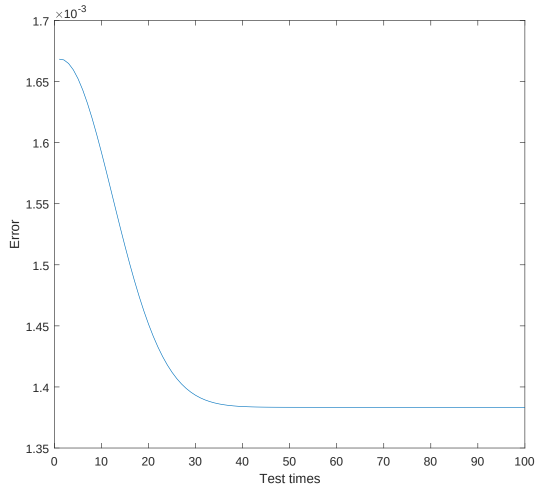

Figure 3. The graph of absolute error function for Example 6.3.

Example 6.4. *Consider the stochastic Itô-Volterra integral equation:*

$$\begin{aligned} Y(t) = &\frac{1}{8} + \frac{1}{64}\int_0^t y(s)(1 - y^2(s))ds \\ &+ \frac{1}{8}\int_0^t (1 - y^2(s))dB(s), \end{aligned} \quad (6.5)$$

with the exact solution,

$$Y(t) = \frac{9e^{0.25B(t)} - 7}{9e^{0.125B(t)} + 7},$$

where $\{B(t) : 0 \le t \le T\}$ *is a normal Brownian motion with*

$$B(0) = 0.$$

The comparison of the absolute error is presented in Table 4.

Table 4. Comparison of the absolute error for Example 6.4.

t_i	$n = 50$	$n = 100$	$n = 150$	$n = 200$
0.1	0.00043	0.00082	0.00102	0.00178
0.3	0.00136	0.00184	0.00263	0.00307
0.5	0.00372	0.00617	0.00928	0.01062
0.7	0.00665	0.00988	0.03021	0.04515
0.8	0.04817	0.05676	0.07389	0.09346
0.9	0.07943	0.09203	0.10535	0.13081

7. Conclusions

This paper concentrates on the simple BSVIEs, where the coefficients only rely on $z(t, s)$. Newton's method solves BSVIEs with BPFs and corresponding stochastic operational matrix. Examples demonstrate estimate analysis and separate convergence of approximating sequences. The concerns identified in the research can be applied to BSVIEs of type II with discretization based on the adapted M-solutions. However, this approach requires a wholly novel methodology and may be the subject of some subsequent studies.

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Conflict of interest

The authors declare no conflicts of interest.

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