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Research article

Existence of a unique solution to a fourth-order boundary value problem and elastic beam analysis

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Abstract: We study the existence and uniqueness of solutions to a particular class of two-point boundary value problems involving fourth-order ordinary differential equations. Such problems have exciting applications for modeling the deflections of beams. The primary tools employed in this study include the application of Banach's and Rus's fixed point theorems. Our theoretical results are applied to elastic beam deflections when the beam is subjected to a loading force and both ends are clamped. The existence and uniqueness of solutions to the models are guaranteed for certain classes of linear and nonlinear loading forces.

Keywords: fourth-order boundary value problem; Green's function; fixed point; existence; uniqueness; elastic beam analysis

1. Introduction

In this article, we consider the following nonlinear fourth-order differential equation

$$y'''' + \beta^2 y'' = f(x, y), \quad x \in [0, L], \tag{1.1}$$

together with the boundary condition

$$y(0) = 0, \quad y(L) = 0, \quad y'(0) = 0, \quad y'(L) = 0. \tag{1.2}$$

Here $L, \beta \in \mathbb{R}$, $\beta > 0$, f: $[0, L] \times \mathbb{R} \to \mathbb{R}$ is continuous and $f(x, 0) \neq 0$ for $x \in [0, L]$. The assumption $f(x, 0) \neq 0$ excludes the possibility of the trivial solution. By a solution to (1.1) and (1.2), we mean a function y: $[0, L] \to \mathbb{R}$ such that y is four times differentiable, with a continuous fourth-order derivative on $[0, L]$. We denote this by $y \in C^4([0, L])$, and our y satisfies both (1.1) and (1.2).

This paper aims to establish and compare results on the existence of a unique solution to the boundary value problems (1.1) and (1.2) by applying fixed point theorems. Our main results state that if the function f satisfies the Lipschitz condition and L is not large, then the problem has a unique nontrivial solution. To obtain these results, we first rewrite our problems (1.1) and (1.2) as an equivalent integral equation by constructing the corresponding Green's function. Then, we apply the Banach fixed point theorem on an infinite strip. Next, for the result to apply to a wider class of functions, the Banach fixed point theorem is applied within a closed and bounded set. Finally, we apply Rus's fixed point theorem to increase the length of the interval where the result is valid. To compare the obtained results, we consider examples.

A natural motivation for investigating fourth-order boundary value problems arises in analyzing elastic beam deflections. Consider a slender beam, the ends of which are clamped on the x-axis at $x = 0$ and $x = L$. The beam is subjected to certain forces, such as a compressive force P and a transverse load $h(x)$, which varies along its length. If $y = y(x)$ represents the resultant deflection of the beam at position x, the differential equation

$$y'''' + \beta^2 y'' = h(x), \quad x \in [0, L] \tag{1.3}$$

represents the displacement of the beam in the transverse direction due to buckling with

$$\beta = \sqrt{\frac{P}{EI}}$$

where E is Young's modulus of the slender member and I the moment of inertia of the beam along the direction of its length. For simplicity, assume that $E \cdot I$ and the compressive load P are constants. Trivially, β must be greater than zero; else, if $\beta = 0$, this means that $P = 0$. In this situation, the problem is subjected to the boundary condition (1.2) since the beam has clamped ends at $x = 0$ and $x = L$. If we consider the transverse load on the beam given by $f(x, y)$, which may be nonlinear, then we obtain the fourth-order differential Eq (1.1).

The study of solutions to boundary value problems often involves examining the construction of Green's functions specific to those problems. Consequently, Green's functions hold significance in the theory of boundary value problems. Many researchers have studied fourth-order boundary value problems and their application to elastic beam deflections. Many prominent investigations have centered on determining the solvability of fourth-order boundary value problems and confirming the existence and uniqueness of solutions. Fixed point theorems serve as highly effective and potent tools for establishing the existence or uniqueness of solutions to nonlinear boundary value problems. Numerous authors have studied the existence of solutions for fourth-order boundary value problems using various fixed-point theorems. Among the immense number of papers dealing with the solvability of fourth-order nonlinear differential equations subject to a variety of boundary conditions using fixed point theory, we refer to [1–13] and the references therein for a selection of recent publications in this area.

The problem under consideration is distinct from the aforementioned works. We also point out that our approach of applying Rus's fixed point theorem appears to occupy a unique position within the literature as a strategy to ensure the existence and uniqueness of solutions to fourth-order boundary value problems. The results herein form an advancement over traditional approaches such as applications of Banach's fixed point theorem. This is achieved through the use of two metrics and Rus's fixed point theorem. As we will discover, this enables a greater class of problems to be better understood regarding the existence and uniqueness of solutions. This includes sharpening the Lipschitz constants involved within a global (unbounded) context and closed and bounded domains.

Since our main tools in this paper are fixed point theorems, let us state the Banach and Rus's fixed point theorems for the reader's convenience.

Theorem 1.1. *[14] Let X be a nonempty set, and d be a metric on X such that (X, d) forms a complete metric space. If the mapping $T\colon X \to X$ satisfies*

$$d(Ty, Tz) \le \alpha d(y, z) \text{ for some } \alpha \in (0, 1) \text{ and all } y, z \in X;$$

then there is a unique $y_0 \in X$ such that $Ty_0 = y_0$.

Theorem 1.2. *[15] Let X be a nonempty set, and d and ρ be two metrics on X such that (X, d) forms a complete metric space. If the mapping $T\colon X \to X$ is continuous with respect to d on X and*

(1) There exists $c > 0$ such that

$$d(Ty, Tz) \le c\rho(y, z) \text{ for all } y, z \in X;$$

(2) There exists $\alpha \in (0, 1)$ such that

$$\rho(Ty, Tz) \le \alpha\rho(y, z) \text{ for all } y, z \in X;$$

then there is a unique $y_0 \in X$ such that $Ty_0 = y_0$.

The rest of the paper is organized as follows. In Section 2, we construct the Green's function corresponding to the boundary value problems (1.1) and (1.2) by employing the variation of parameters formula and some additional assumptions. Section 3 is devoted to the estimation of an integral that involves Green's function. In Section 4, we prove our main theorems on the existence of a unique solution to the boundary value problems (1.1) and (1.2). Additionally, we provide a few examples to illustrate the applicability of established results.

2. Construction of the Green's function

The goal of this section is to rewrite the boundary value problems (1.1) and (1.2) as an equivalent integral equation. So, let us consider the linear Eq (1.3) together with the boundary condition (1.2).

Proposition 2.1. *Assume*

$$2-2\cos\beta L-\beta L\sin\beta L\neq 0.$$

If $h\colon [0,L]\to\mathbb{R}$ *is a continuous function, then the boundary value problems* (1.2) *and* (1.3) *have a unique solution, which we can write as*

$$y(x)=\int_0^L G(x,\xi)h(\xi)d\xi,\quad 0\le x\le L, \tag{2.1}$$

where the Green's function is given by

$$G(x,\xi)=\begin{cases}G_1(x,\xi), & 0\le\xi\le x\le L,\\ G_2(x,\xi), & 0\le x\le\xi\le L.\end{cases} \tag{2.2}$$

Here,

$$\begin{aligned}
\mathcal{K}(x,\xi)=&\frac{1}{\beta^3}\left[\beta(x-\xi)-\sin\beta(x-\xi)\right],\\
\mathcal{K}_x(x,\xi)=&\frac{1}{\beta^2}\left[1-\cos\beta(x-\xi)\right],\\
G_1(x,\xi)=&\frac{\mathcal{K}_x(L,\xi)\left[(\beta L-\sin\beta L)(1-\cos\beta x)\right]}{\beta(2-2\cos\beta L-\beta L\sin\beta L)}\\
&+\frac{\mathcal{K}_x(L,\xi)\left[(1-\cos\beta L)(\sin\beta x-\beta x)\right]}{\beta(2-2\cos\beta L-\beta L\sin\beta L)}\\
&+\frac{\mathcal{K}(L,\xi)\left[\sin\beta L(\beta x-\sin\beta x)\right]}{(2-2\cos\beta L-\beta L\sin\beta L)}\\
&+\frac{\mathcal{K}(L,\xi)\left[(1-\cos\beta L)(\cos\beta x-1)\right]}{(2-2\cos\beta L-\beta L\sin\beta L)}+\mathcal{K}(x,\xi),
\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}
G_2(x,\xi)=&\frac{\mathcal{K}_x(L,\xi)\left[(\beta L-\sin\beta L)(1-\cos\beta x)\right]}{\beta(2-2\cos\beta L-\beta L\sin\beta L)}\\
&+\frac{\mathcal{K}_x(L,\xi)\left[(1-\cos\beta L)(\sin\beta x-\beta x)\right]}{\beta(2-2\cos\beta L-\beta L\sin\beta L)}\\
&+\frac{\mathcal{K}(L,\xi)\left[\sin\beta L(\beta x-\sin\beta x)\right]}{(2-2\cos\beta L-\beta L\sin\beta L)}\\
&+\frac{\mathcal{K}(L,\xi)\left[(1-\cos\beta L)(\cos\beta x-1)\right]}{(2-2\cos\beta L-\beta L\sin\beta L)}.
\end{aligned} \tag{2.4}$$

Proof. The general solution of (1.3) is given by

$$y(x)=c_1+c_2x+c_3\cos\beta x+c_4\sin\beta x+\int_0^x\mathcal{K}(x,\xi)h(\xi)d\xi, \tag{2.5}$$

$0\le x\le L$, where c_1–c_4 are arbitrary constants. From (2.5), we have

$$y'(x)=c_2-\beta c_3\sin\beta x+\beta c_4\cos\beta x+\int_0^x\mathcal{K}_x(x,\xi)h(\xi)d\xi, \tag{2.6}$$

where $0\le x\le L$.

Using boundary condition (1.2) in (2.5) and (2.6) and rearranging the terms, we get

$$\begin{aligned}
c_1=&\frac{(\beta L-\sin\beta L)}{\beta(2-2\cos\beta L-\beta L\sin\beta L)}\int_0^L\mathcal{K}_x(L,\xi)h(\xi)d\xi\\
&+\frac{(\cos\beta L-1)}{(2-2\cos\beta L-\beta L\sin\beta L)}\int_0^L\mathcal{K}(L,\xi)h(\xi)d\xi,\\
c_2=&\frac{(\cos\beta L-1)}{(2-2\cos\beta L-\beta L\sin\beta L)}\int_0^L\mathcal{K}_x(L,\xi)h(\xi)d\xi\\
&+\frac{\beta\sin\beta L}{(2-2\cos\beta L-\beta L\sin\beta L)}\int_0^L\mathcal{K}(L,\xi)h(\xi)d\xi,\\
c_3=&-c_1,\\
c_4=&-\frac{1}{\beta}c_2.
\end{aligned}$$

Substituting the constants c_1–c_4 in (2.5) and rearranging the terms, we obtain (2.1). To prove the uniqueness of solutions for the boundary value problems (1.2) and (1.3), we assume that it has multiple solutions. Let u and v be any two solutions of the boundary value problems (1.2) and (1.3). Then, we have

$$\begin{cases}u''''+\beta^2u''=h(x), & x\in[0,L],\\ u(0)=0,\quad u(L)=0,\quad u'(0)=0,\quad u'(L)=0,\end{cases} \tag{2.7}$$

and

$$\begin{cases}v''''+\beta^2v''=h(x), & x\in[0,L],\\ v(0)=0,\quad v(L)=0,\quad v'(0)=0,\quad v'(L)=0.\end{cases} \tag{2.8}$$

Take

$$z(x)=u(x)-v(x)$$

for all $x\in[0,L]$. Then, we obtain the following nonlinear fourth-order differential equation

$$z''''+\beta^2z''=h(x),\quad x\in[0,L], \tag{2.9}$$

together with the boundary condition

$$z(0)=0,\quad z(L)=0,\quad z'(0)=0,\quad z'(L)=0. \tag{2.10}$$

Using the variation of parameters formula, we have

$$z(x)=c_1+c_2x+c_3\cos\beta x+c_4\sin\beta x,\quad 0\le x\le L, \tag{2.11}$$

where c_1–c_4 are arbitrary constants. From (2.11), we have

$$z'(x) = c_2 - \beta c_3 \sin\beta x + \beta c_4 \cos\beta x, \quad 0 \le x \le L. \quad (2.12)$$

Using boundary condition (2.10) in (2.11) and (2.12) and rearranging the terms, we obtain the following homogeneous linear system of four equations in 4 unknowns c_1–c_4:

$$\begin{aligned} &c_1 + c_3 = 0,\\ &c_1 + Lc_2 + c_3\cos\beta L + c_4 \sin\beta L = 0,\\ &c_2 + \beta c_4 = 0,\\ &c_2 - \beta c_3 \sin\beta L + \beta c_4 \cos\beta L = 0, \end{aligned}$$

with the determinant of the coefficient matrix

$$\begin{vmatrix} 1 & 0 & 1 & 0\\ 1 & L & \cos\beta L & \sin\beta L\\ 0 & 1 & 0 & \beta\\ 0 & 1 & -\beta\sin\beta L & \beta\cos\beta L \end{vmatrix} = \beta(2\cos\beta L + \beta L \sin\beta L - 2) \neq 0.$$

Then, the homogeneous linear system has only the trivial solution

$$c_1 = c_2 = c_3 = c_4 = 0$$

implying that

$$z(x) = 0, \quad x \in [0, L].$$

Consequently, we obtain

$$u(x) = v(x), \quad x \in [0, L].$$

Hence, the boundary value problems (1.2) and (1.3) have a unique solution (2.1). To verify that $y \in C^4[0, L]$, one can differentiate (2.1) four times and verify its continuity. □

3. Estimation of the Green's function

In this section, we prove a useful inequality for an integral that involves the Green's function.

Proposition 3.1. *The Green's function in* (2.2) *satisfies*

$$\int_0^L |G(x,\xi)|\,d\xi \le \frac{L^3}{6}k_1 + \frac{L^4}{24}(1 + k_2), \quad (3.1)$$

where

$$k_1 = \sup_{x\in[0,L]} \left| \frac{(\beta L - \sin\beta L)(1-\cos\beta x)}{\beta(2 - 2\cos\beta L - \beta L\sin\beta L)} + \frac{(1-\cos\beta L)(\sin\beta x - \beta x)}{\beta(2 - 2\cos\beta L - \beta L \sin\beta L)} \right| \quad (3.2)$$

and

$$k_2 = \sup_{x\in[0,L]} \left| \frac{\sin\beta L(\beta x - \sin\beta x)}{(2 - 2\cos\beta L - \beta L\sin\beta L)} + \frac{(1-\cos\beta L)(\cos\beta x - 1)}{(2 - 2\cos\beta L - \beta L\sin\beta L)} \right|. \quad (3.3)$$

Proof. For all $x \in [0, L]$, we have

$$\begin{aligned}
&\int_0^L |G(x,\xi)|\,d\xi\\
&= \int_0^x |G(x,\xi)|\,d\xi + \int_x^L |G(x,\xi)|\,d\xi\\
&\le \int_0^x \left| \frac{\mathcal{K}_x(L,\xi)\,[(\beta L - \sin\beta L)(1-\cos\beta x)]}{\beta(2-2\cos\beta L - \beta L\sin\beta L)} \right| d\xi\\
&\quad + \int_0^x \left| \frac{\mathcal{K}_x(L,\xi)\,[(1-\cos\beta L)(\sin\beta x - \beta x)]}{\beta(2-2\cos\beta L - \beta L\sin\beta L)} \right| d\xi\\
&\quad + \int_0^x \left| \frac{\mathcal{K}(L,\xi)\,[\sin\beta L(\beta x - \sin\beta x)]}{(2-2\cos\beta L - \beta L\sin\beta L)} \right| d\xi\\
&\quad + \int_0^x \left| \frac{\mathcal{K}(L,\xi)\,[(1-\cos\beta L)(\cos\beta x - 1)]}{(2-2\cos\beta L - \beta L\sin\beta L)} \right| d\xi\\
&\quad + \int_0^x |\mathcal{K}(x,\xi)|\,d\xi\\
&\quad + \int_x^L \left| \frac{\mathcal{K}_x(L,\xi)\,[(\beta L - \sin\beta L)(1-\cos\beta x)]}{\beta(2-2\cos\beta L - \beta L\sin\beta L)} \right| d\xi\\
&\quad + \int_x^L \left| \frac{\mathcal{K}_x(L,\xi)\,[(1-\cos\beta L)(\sin\beta x - \beta x)]}{\beta(2-2\cos\beta L - \beta L\sin\beta L)} \right| d\xi\\
&\quad + \int_x^L \left| \frac{\mathcal{K}(L,\xi)\,[\sin\beta L(\beta x - \sin\beta x)]}{(2-2\cos\beta L - \beta L\sin\beta L)} \right| d\xi\\
&\quad + \int_x^L \left| \frac{\mathcal{K}(L,\xi)\,[(1-\cos\beta L)(\cos\beta x - 1)]}{(2-2\cos\beta L - \beta L\sin\beta L)} \right| d\xi\\
&\le k_1 \left[\int_0^x \mathcal{K}_x(L,\xi)d\xi + \int_x^L \mathcal{K}_x(L,\xi)d\xi \right]\\
&\quad + k_2 \left[\int_0^x \mathcal{K}(L,\xi)d\xi + \int_x^L \mathcal{K}(L,\xi)d\xi \right] + \int_0^x \mathcal{K}(x,\xi)d\xi\\
&= k_1 \int_0^L \mathcal{K}_x(L,\xi)d\xi + k_2 \int_0^L \mathcal{K}(L,\xi)d\xi + \int_0^x \mathcal{K}(x,\xi)d\xi\\
&\le \frac{L^3}{6}k_1 + \frac{L^4}{24}k_2 + \frac{x^4}{24}\\
&\le \frac{L^3}{6}k_1 + \frac{L^4}{24}(1 + k_2).
\end{aligned}$$

The proof is complete. □

4. Existence of a unique solution

In this section, we will apply fixed point theorems to prove our results on the existence of a unique solution to the boundary value problems (1.1) and (1.2) and compare them. For this, let us define two metrics on the set X of continuous functions defined on $[0, L]$ such that

$$d(y,z) = \sup_{x\in[0,L]} |y(x) - z(x)|$$

and

$$\rho(y,z) = \left(\int_0^L |y(x) - z(x)|^2 \, dx\right)^{\frac{1}{2}}$$

for all $y, z \in X$. It is easy to show that (X, ρ) is a metric space and (X, d) forms a complete metric space.

4.1. Application of Theorem 1.1 on an infinite strip

Theorem 4.1. *Let $f\colon [0, L] \times \mathbb{R} \to \mathbb{R}$ be a continuous function and $f(x, 0) \neq 0$ for $x \in [0, L]$. Assume*

$$2 - 2\cos\beta L - \beta L \sin\beta L \neq 0$$

and f satisfies the Lipschitz condition with respect to its second argument with a Lipschitz constant K. If

$$\frac{L^3}{6}k_1 + \frac{L^4}{24}(1 + k_2) < \frac{1}{K}, \tag{4.1}$$

then there exists a unique non-trivial solution to the boundary value problems (1.1) *and* (1.2).

Proof. It follows from Proposition 2.1 that the boundary value problems (1.1) and (1.2) are equivalent to the integral equation

$$y(x) = \int_0^L G(x,\xi) f(\xi, y(\xi)) d\xi, \quad 0 \le x \le L.$$

Define the mapping $T\colon X \to X$ by

$$(Ty)(x) = \int_0^L G(x,\xi) f(\xi, y(\xi)) d\xi, \quad 0 \le x \le L.$$

Clearly, y is a solution of (1.1) and (1.2) iff y is a fixed point of T. To establish the existence of a unique fixed point of T, we show that the conditions of Theorem 1.1 hold. To see this, let $y, z \in X$, $x \in [0, L]$ and consider

$$\begin{aligned}|(Ty)(x) - (Tz)(x)| &= \left| \int_0^L G(x,\xi) f(\xi, y(\xi)) d\xi - \int_0^L G(x,\xi) f(\xi, z(\xi)) d\xi \right| \\ &\le \int_0^L |G(x,\xi)| \, |f(\xi, y(\xi)) - f(\xi, z(\xi))| \, d\xi \\ &\le K \int_0^L |G(x,\xi)| \, |y(\xi) - z(\xi)| \, d\xi \\ &\le K d(y,z) \int_0^L |G(x,\xi)| \, d\xi \\ &\le K\left(\frac{L^3}{6}k_1 + \frac{L^4}{24}(1 + k_2)\right) d(y,z)\end{aligned}$$

implying that

$$d(Ty, Tz) \le K\left(\frac{L^3}{6}k_1 + \frac{L^4}{24}(1 + k_2)\right) d(y,z)$$

for all $y, z \in X$. Since

$$K\left(\frac{L^3}{6}k_1 + \frac{L^4}{24}(1 + k_2)\right) < 1,$$

the mapping T is a contraction. Hence, by Theorem 1.1, T has a unique fixed point in X. Therefore, the boundary value problems (1.1) and (1.2) have a unique non-trivial solution $y \in X$. The proof is complete. □

4.2. Application of Theorem 1.1 within a closed and bounded set

Consider a closed ball B_N with radius N in X as follows:

$$B_N = \{y \in X : d(y, 0) \le N\}.$$

Since B_N is a closed subspace of X, the pair (B_N, d) forms a complete metric space. Clearly, $T\colon B_N \to X$.

Theorem 4.2. *Let $f\colon [0, L] \times [-N, N] \to \mathbb{R}$ be a continuous function and $f(x, 0) \neq 0$ for $x \in [0, L]$. Assume*

$$2 - 2\cos\beta L - \beta L \sin\beta L \neq 0$$

and f satisfies the Lipschitz condition with respect to its second argument with a Lipschitz constant K. If L satisfies inequality (4.1) *and*

$$\frac{L^3}{6}k_1 + \frac{L^4}{24}(1 + k_2) \le \frac{N}{M}, \tag{4.2}$$

where

$$M = \sup_{(x,y)\in[0,L]\times[-N,N]} |f(x, y)|,$$

then there exists a unique non-trivial solution y to the boundary value problems (1.1) *and* (1.2) *such that*

$$|y(x)| \le N, \quad x \in [0, L].$$

Proof. First, we show that $T\colon B_N \to B_N$. To see this, let $y \in B_N$, $x \in [0, L]$, and consider

$$\begin{aligned}|(Ty)(x)| &\le \int_0^L |G(x,\xi)|\,|f(\xi, y(\xi))|\,d\xi\\ &\le M\int_0^L |G(x,\xi)|\,d\xi\\ &\le M\left(\frac{L^3}{6}k_1 + \frac{L^4}{24}(1+k_2)\right)\end{aligned}$$

implying that

$$d(Ty, 0) \le M\left(\frac{L^3}{6}k_1 + \frac{L^4}{24}(1+k_2)\right) \le N.$$

Thus, $Ty \in B_N$. Therefore, $T\colon B_N \to B_N$. It follows from the proof of Theorem 4.1 that $T\colon B_N \to B_N$ is a contraction. Hence, by Theorem 1.1, T has a unique fixed point in B_N. Therefore, the boundary value problems (1.1) and (1.2) have a unique non-trivial solution $y \in B_N$.

The proof is complete. □

4.3. Application of Theorem 1.2 on an infinite strip

Theorem 4.3. *Let $f\colon [0, L] \times \mathbb{R} \to \mathbb{R}$ be a continuous function and $f(x, 0) \neq 0$ for $x \in [0, L]$. Assume*

$$2 - 2\cos\beta L - \beta L \sin\beta L \neq 0$$

and f satisfies the Lipschitz condition with respect to its second argument with a Lipschitz constant K. If

$$\left(\frac{k_1^2 L^6}{20} + \frac{(5k_2+8)k_1 L^7}{180} + \frac{(5k_2^2+21k_2+5)L^8}{1260}\right)^{\frac{1}{2}} < \frac{1}{K}, \tag{4.3}$$

then there exists a unique non-trivial solution to the boundary value problems (1.1) *and* (1.2).

Proof. To establish the existence of a unique fixed point of T using Theorem 1.2, we have to show that the conditions of Theorem 1.2 hold. For this purpose, let $y, z \in X$, $x \in [0, L]$ and consider

$$\begin{aligned}&|(Ty)(x) - (Tz)(x)|\\ &= \left|\int_0^L G(x,\xi) f(\xi, y(\xi))d\xi - \int_0^L G(x,\xi) f(\xi, z(\xi))d\xi\right|\\ &\le \int_0^L |G(x,\xi)|\,|f(\xi, y(\xi)) - f(\xi, z(\xi))|\,d\xi\\ &\le K\int_0^L |G(x,\xi)|\,|y(\xi) - z(\xi)|\,d\xi\\ &\le K\left(\int_0^L |G(x,\xi)|^2\,d\xi\right)^{\frac{1}{2}} \times \left(\int_0^L |y(\xi) - z(\xi)|^2\,d\xi\right)^{\frac{1}{2}}\\ &\le K \sup_{0\le x\le L}\left(\int_0^L |G(x,\xi)|^2\,d\xi\right)^{\frac{1}{2}} \rho(y,z)\\ &\le c\rho(y,z)\end{aligned}$$

implying that

$$d(Ty, Tz) \le c\rho(y,z)$$

for all $y, z \in X$. Here

$$c = K \sup_{0\le x\le L}\left(\int_0^L |G(x,\xi)|^2\,d\xi\right)^{\frac{1}{2}} > 0.$$

Also,

$$\begin{aligned}\rho(y,z) &= \left(\int_0^L |y(x) - z(x)|^2\,dx\right)^{\frac{1}{2}}\\ &\le \left(\int_0^L \sup_{0\le x\le L} |y(x) - z(x)|^2\,dx\right)^{\frac{1}{2}}\\ &\le \sup_{0\le x\le L} |y(x) - z(x)| \left(\int_0^L dx\right)^{\frac{1}{2}}\\ &= L^{\frac{1}{2}} d(y,z).\end{aligned}$$

Thus, we obtain that

$$d(Ty, Tz) \le c\rho(y,z) \le cL^{\frac{1}{2}} d(y,z)$$

for all $y, z \in X$. Then, for any $\epsilon > 0$, choose

$$\delta = \frac{\epsilon}{cL^{\frac{1}{2}}}$$

such that

$$d(Ty, Tz) < \epsilon$$

whenever

$$d(y,z) < \delta.$$

Therefore, T is continuous with respect to d on X. Consider

$$\begin{aligned}&\left(\int_0^L |(Ty)(x) - (Tz)(x)|^2\,dx\right)^{\frac{1}{2}}\\ &\le \left(\int_0^L \left[K\left(\int_0^L |G(x,\xi)|^2\,d\xi\right)^{\frac{1}{2}} \rho(y,z)\right]^2 dx\right)^{\frac{1}{2}}\end{aligned}$$

$$\leq K\rho(y,z)\left(\int_0^L\left(\int_0^L |G(x,\xi)|^2\,d\xi\right)dx\right)^{\frac{1}{2}}.$$

Now, consider

$$\begin{aligned}
&\int_0^L |G(x,\xi)|^2\,d\xi\\
&= \int_0^x |G(x,\xi)|^2\,d\xi + \int_x^L |G(x,\xi)|^2\,d\xi\\
&\leq \int_0^x \left(\frac{k_1^2(L-\xi)^4}{4} + \frac{k_2^2(L-\xi)^6}{36} + \frac{(x-\xi)^6}{36} + \frac{k_1k_2(L-\xi)^5}{6}\right.\\
&\quad \left. + \frac{k_1(x-\xi)^3(L-\xi)^2}{6} + \frac{k_2(x-\xi)^3(L-\xi)^3}{18}\right)d\xi\\
&\quad + \int_x^L \left(\frac{k_1^2(L-\xi)^4}{4} + \frac{k_2^2(L-\xi)^6}{36} + \frac{k_1k_2(L-\xi)^5}{6}\right)d\xi\\
&\leq \frac{k_1^2L^5}{20} + \frac{(k_2^2+1)L^7}{252} + \frac{k_1k_2L^6}{36}\\
&\quad + \int_0^x \left(\frac{k_1(x-\xi)^3(L-\xi)^2}{6} + \frac{k_2(x-\xi)^3(L-\xi)^3}{18}\right)d\xi\\
&\leq \frac{k_1^2L^5}{20} + \frac{(k_2^2+1)L^7}{252} + \frac{k_1k_2L^6}{36}\\
&\quad + \frac{k_1}{6}\left(-\left[\frac{(x-\xi)^4(L-\xi)^2}{4}\right]_0^x - \int_0^x \frac{2(x-\xi)^4(L-\xi)}{4}d\xi\right)\\
&\quad + \frac{k_2}{18}\left(-\left[\frac{(x-\xi)^4(L-\xi)^3}{4}\right]_0^x - \int_0^x \frac{3(x-\xi)^4(L-\xi)^2}{4}d\xi\right)\\
&\leq \frac{k_1^2L^5}{20} + \frac{(k_2^2+1)L^7}{252} + \frac{k_1k_2L^6}{36}\\
&\quad + \frac{k_1}{6}\left(\frac{L^6}{4} + \left[\frac{(x-\xi)^5(L-\xi)}{10}\right]_0^x + \int_0^x \frac{(x-\xi)^5}{10}d\xi\right)\\
&\quad + \frac{k_2}{18}\left(\frac{L^7}{4} + \left[\frac{3(x-\xi)^5(L-\xi)^2}{20}\right]_0^x\right.\\
&\quad \left. + \int_0^x \frac{3(x-\xi)^5(L-\xi)}{10}d\xi\right)\\
&\leq \frac{k_1^2L^5}{20} + \frac{(k_2^2+1)L^7}{252} + \frac{k_1k_2L^6}{36} + \frac{2k_1L^6}{45}\\
&\quad + \frac{k_2}{18}\left(\frac{L^7}{4} - \left[\frac{(x-\xi)^6(L-\xi)}{20}\right]_0^x - \int_0^x \frac{(x-\xi)^6}{20}d\xi\right)\\
&\leq \frac{k_1^2L^5}{20} + \frac{(k_2^2+1)L^7}{252} + \frac{k_1k_2L^6}{36} + \frac{2k_1L^6}{45} + \frac{k_2}{18}\left(\frac{L^7}{4} + \frac{L^7}{20}\right)\\
&\leq \frac{k_1^2L^5}{20} + \frac{(5k_2+8)k_1L^6}{180} + \frac{(5k_2^2+21k_2+5)L^7}{1260}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left(\int_0^L |(Ty)(x)-(Tz)(x)|^2dx\right)^{1/2}\\
&\leq K\rho(y,z)\left(\int_0^L\left(\frac{k_1^2L^5}{20} + \frac{(5k_2+8)k_1L^6}{180}\right.\right.\\
&\quad \left.\left. + \frac{(5k_2^2+21k_2+5)L^7}{1260}\right)dx\right)^{1/2}\\
&= K\left(\frac{k_1^2L^6}{20} + \frac{(5k_2+8)k_1L^7}{180} + \frac{(5k_2^2+21k_2+5)L^8}{1260}\right)^{\frac{1}{2}}\rho(y,z)
\end{aligned}$$

implying that

$$\rho(Ty,Tz) \leq \alpha\rho(y,z)$$

for all $y, z \in X$. Here,

$$\begin{aligned}
\alpha &= K\left(\frac{k_1^2L^6}{20} + \frac{(5k_2+8)k_1L^7}{180} + \frac{(5k_2^2+21k_2+5)L^8}{1260}\right)^{\frac{1}{2}}\\
&< 1.
\end{aligned}$$

Hence, by Theorem 1.2, T has a unique fixed point in X. Therefore, the boundary value problems (1.1) and (1.2) have a unique non-trivial solution $y \in X$.

The proof is complete. □

5. Examples

In this section, we provide a few examples to illustrate the applicability of results established in the previous section.

Example 1. Consider (1.1) and (1.2) with $\beta = L = 1$ and

$$f(x,y) = \frac{y^2}{y^2+1} + 10x + 1.$$

Clearly, $f\colon [0,1]\times\mathbb{R} \to \mathbb{R}$ is a continuous function, and $f(x,0) \neq 0$ for $x \in [0,1]$. Also,

$$2 - 2\cos 1 - \sin 1 = 0.0779 \neq 0$$

and f satisfies the Lipschitz condition with respect to its second argument with a Lipschitz constant $K = 1$. Further, we obtain

$$\begin{aligned}
k_1 &= \sup_{x\in[0,1]}\left|\frac{(1-\sin 1)(1-\cos x)}{(2-2\cos 1-\sin 1)} + \frac{(1-\cos 1)(\sin x - x)}{(2-2\cos 1-\sin 1)}\right|\\
&\approx 0.1502 \qquad (5.1)
\end{aligned}$$

and

$$\begin{aligned}
k_2 &= \sup_{x\in[0,1]}\left|\frac{\sin 1(x-\sin x)}{(2-2\cos 1-\sin 1)} + \frac{(1-\cos 1)(\cos x - 1)}{(2-2\cos 1-\sin 1)}\right|\\
&\approx 1. \qquad (5.2)
\end{aligned}$$

Clearly,

$$\frac{k_1}{6} + \frac{1+k_2}{24} \approx 0.1084 < 1$$

implying inequality (4.1) holds. Therefore, by Theorem 4.1, the boundary value problems (1.1) and (1.2) have a unique non-trivial solution $y \in X$.

Example 2. Consider (1.1) and (1.2) with $\beta = 1$, $L = 2$, and

$$f(x,y) = \frac{y^2}{y^2+1} + 10x + 1.$$

Clearly, $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $f(x,0) \neq 0$ for $x \in [0,1]$. Also,

$$2 - 2\cos 2 - 2\sin 2 = 1.0137 \neq 0$$

and f satisfies the Lipschitz condition with respect to its second argument with a Lipschitz constant $K = 1$. Further, we obtain

$$\begin{aligned} k_1 &= \sup_{x\in[0,2]} \left| \frac{(2-\sin 2)(1-\cos x)}{(2-2\cos 2 - 2\sin 2)} \right. \\ &\qquad \left. + \frac{(1-\cos 2)(\sin x - x)}{(2-2\cos 2-2\sin 2)} \right| \\ &\approx 0.3182 \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} k_2 &= \sup_{x\in[0,2]} \left| \frac{\sin 2(x-\sin x)}{(2-2\cos 2 - 2\sin 2)} \right. \\ &\qquad \left. + \frac{(1-\cos 2)(\cos x - 1)}{(2-2\cos 2-2\sin 2)} \right| \\ &\approx 1. \end{aligned} \tag{5.4}$$

Since

$$\frac{8k_1}{6} + \frac{16(1+k_2)}{24} \approx 1.7576 > 1,$$

inequality (4.1) does not hold. Hence, Theorem 4.1 is not applicable in this case.

Example 3. Consider (1.1) and (1.2) with $\beta = L = 1$ and

$$f(x,y) = x^2y^2 + 1.$$

Clearly, $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $f(x,0) \neq 0$ for $x \in [0,1]$. Also,

$$2 - 2\cos 1 - \sin 1 = 0.0779 \neq 0.$$

But, f does not satisfy the Lipschitz condition with respect to its second argument. Hence, Theorem 4.1 is not applicable in this case.

Example 4. Consider (1.1) and (1.2) with $\beta = 1$, $L = 0.5$, and

$$f(x,y) = x^2y^2 + 1.$$

Choose $N = 1$. Clearly, $f\colon [0,0.5] \times [-1,1] \to \mathbb{R}$ is a continuous function and $f(x,0) \neq 0$ for $x \in [0,0.5]$. Also,

$$2 - 2\cos(0.5) - (0.5)\sin(0.5) = 0.0051 \neq 0$$

and f satisfies the Lipschitz condition with respect to its second argument with a Lipschitz constant $K = 0.5$. Further, we obtain

$$\begin{aligned} k_1 &= \sup_{x\in[0,0.5]} \left| \frac{(0.5-\sin(0.5))(1-\cos x)}{(2-2\cos(0.5)-(0.5)\sin(0.5))} \right. \\ &\qquad \left. + \frac{(1-\cos(0.5))(\sin x - x)}{(2-2\cos(0.5)-(0.5)\sin(0.5))} \right| \\ &\approx 0.0784, \end{aligned} \tag{5.5}$$

$$\begin{aligned} k_2 &= \sup_{x\in[0,0.5]} \left| \frac{\sin(0.5)(x-\sin x)}{(2-2\cos(0.5)-(0.5)\sin(0.5))} \right. \\ &\qquad \left. + \frac{(1-\cos(0.5))(\cos x - 1)}{(2-2\cos(0.5)-(0.5)\sin(0.5))} \right| \\ &\approx 1 \end{aligned} \tag{5.6}$$

and

$$M = \sup_{(x,y)\in[0,0.5]\times[-1,1]} |f(x,y)| = 1.25.$$

Since

$$\frac{(0.5)^3k_1}{6} + \frac{(0.5)^4(1+k_2)}{24} \approx 0.0068 < 0.8 < 2,$$

where

$$\frac{1}{K} = 2 \quad \text{and} \quad \frac{N}{M} = 0.8,$$

inequalities (4.1) and (4.2) hold. Hence, by Theorem 4.2, the boundary value problems (1.1) and (1.2) have a unique non-trivial solution $y \in B_N$.

Example 5. Consider Example 4. We obtain that

$$\begin{aligned} &\left(\frac{k_1^2L^6}{20} + \frac{(5k_2+8)k_1L^7}{180} + \frac{(5k_2^2+21k_2+5)L^8}{1260} \right)^{\frac{1}{2}} \\ &= 0.0154 < \frac{1}{K}. \end{aligned} \tag{5.7}$$

Then, by Theorem 4.3, the boundary value problems (1.1) and (1.2) have a unique non-trivial solution $y \in X$.

6. Conclusions

In this article, we studied the existence and uniqueness of solutions to a particular class of two-point boundary value problems involving fourth-order ordinary differential equations using Banach's and Rus's fixed point theorems. Such problems have exciting applications for modeling the deflections of beams. The future scope of this research involves studying the existence and uniqueness of solutions of a clamped variable cross-section elastic beam subjected to a loading force or a clamped functionally graded elastic beam subjected to a loading force [16, 17].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors have no competing interests to declare that are relevant to the content of this article.

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