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Research article

Approximate controllability for a class of fractional semilinear system with instantaneous and non-instantaneous impulses

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Abstract: This paper is mainly concerned with the existence of mild solutions and approximate controllability for a class of fractional semilinear systems with instantaneous and non-instantaneous impulses. By applying the Kuratowski measure of noncompactness and ρ-set contractive fixed-point theorem, the results for the considered system were obtained. In the end, an example was studied to support the main results.

Keywords: approximate controllability; fractional semilinear system; instantaneous and non-instantaneous impulses; Kuratowski measure of noncompactness

1. Introduction

Many evolutionary processes are characterized by sudden state changes at some points in time, by being affected by short-time perturbations. Compared with the duration of the entire evolutionary process, the duration of these perturbations is negligible. Therefore, if we assume that these disturbances take place in relatively short periods of time, or even instantaneously, in the form of pulses, the processes can be described by impulsive differential equations (IDEs, for short). It is well known that many biological, agricultural, and medical programs and experiments, such as the control of infectious diseases and pests, and the change of human hormone levels under the influence of external factors, involve impulsive effects. Hence, IDEs can be regarded as the relatively accurate description of some specific problems in the real world (see the works [1, 2] and references therein).

On the other hand, some evolution processes, such as intravenous drug injection, periodic fishing, and pest control, cannot be described by instantaneous impulsive systems. To solve this problem, Hernández and O'Regan [3] introduced a new kind of impulses termed non-instantaneous impulses, which start at an arbitrary fixed point and keep active in a finite time interval. Many scholars have conducted extensive research on these two types of IDEs in recent years. For instance, Liu and O′Regan [4] investigated the functional differential equations with instantaneous impulses by using the measure of noncompactness and Mönch fixed-point theorem. Chen and Zhang [5] dealt with the semilinear evolution equations with non-instantaneous impulses by noncompact semigroup. Xu et al. [6] investigated the controllability of non-autonomous and non-instantaneous impulsive systems.

Also, every aspect of a dynamical system cannot be fully understood by considering instantaneous impulse and non-instantaneous impulses separately. In other words, it is necessary to consider the two types of impulses in one system in order to figure out how they affect the system together. For instance, Kumar and Abdal [7] investigated a kind of instantaneous and non-instantaneous impulsive systems using the Sadovskii's fixed-point theorem. Tian and Zhang [8] dealt with the existence of solutions for second-order differential equations with these two kinds of impulses using the variational method. Yao [9] studied the existence and multiplicity of solutions for three-point boundary value problems with instantaneous and non-instantaneous impulses. Kumar and Yadav [10] investigated the approximate controllability of stochastic delay differential systems driven by Poisson jumps with instantaneous and non-instantaneous impulses.

On the other hand, many scholars have already paid close attention to controllability. Li et al. [11] studied impulsive control method. Liu et al. [12] investigated the control design of delayed Boolean control networks. Xu et al. dealt with a class of control networks [6]. Hakkar et al. studied the approximate controllability of delayed fractional stochastic differential systems with mixed noise and impulsive effects [13].

The most effective way to solve this kind of problems is to transform them into fixed-point problems using proper operators in a function space. For instance, the Mönch fixed-point theorem was applied to deal with the controllability of differential equations by Liu [4]. The ρ-set contractive fixed-point theorem was used to investigate the controllability for a kind of fractional non-instantaneous impulsive systems by Meraj and Pandy in [7].

Compared with classical integer derivatives, the fractional derivatives defined by integration have the characteristics of non-local properties and memory properties. Thus, they are widely used to describe more complex phenomena in a variety of fields. It was found that various, especially interdisciplinary, applications can be elegantly modeled with the help of fractional derivatives [14–17]; see also the recent works of [18–20].

For example, Ge and Jhuang [21] dealt with chaos, control, and synchronization of a class of fractional systems. Cheng and Yuan [22] investigated the stability of the equilibria of a kind of equation with fractional diffusion. Jia and Wang [23] studied a fast finite volume method for a classification of fractional equations. Monje et al. [24] introduced the fundamentals and applications of fractional-order systems and controls.

According to the above-mentioned research, we consider the approximate controllability of the following fractional semilinear system with instantaneous and non-instantaneous impulses:

$$\begin{cases} {}^cD^q x(t) = Ax(t) + Bz(t) + f(t, x(t)), t \in \bigcup_{s=0}^{h} (v_s, u_{s+1}] \subset T, \\ t \neq u_s^{j_s+j}, \ \ j = 1, 2, \cdots, (j_{s+1} - j_s), \\ x(t) = \gamma_s \left(t, x\left(u_s^-\right)\right), \ \ t \in \bigcup_{s=1}^{h} (u_s, v_s], \\ x(0) = x_0, \\ \Delta x\left(u_s^{j_s+j}\right) = I_s^{j_s+j}\left(x\left(u_s^{j_s+j^-}\right)\right), \ \ s = 0, 1, \cdots, h, \\ j = 1, 2, \cdots, (j_{s+1} - j_s), \end{cases} \tag{1.1}$$

where $j_0 = 0$, $T = [0, b]$, $b > 0$ is a constant. ${}^cD^q$ is the Caputo fractional derivative of order $q, 0 < q < 1$. A: $D(A) \subset \mathbb{W} \to \mathbb{W}$ is a infinitesimal generator of a C_0-semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$, where $\mathbb{W}$ is a reflexive Banach space, and

$$\begin{aligned} 0 = u_0 = v_0 &< u_0^1 < u_0^2 < \cdots < u_0^{j_1} < u_1 < v_1 \\ &< u_1^{j_1+1} < u_1^{j_1+2} < \cdots < u_1^{j_2} < u_2 < \cdots < v_h \\ &< u_h^{j_h+1} < u_h^{j_h+2} < \cdots < u_h^{j_{h+1}} < u_{h+1} = b. \end{aligned}$$

The state variable $x(\cdot) \in \mathbb{W}$, $z(\cdot) \in L^2(T; \ \mathbb{V})$ is the control variable, where $\mathbb{V}$ is another Banach space, and $B : \mathbb{V} \to \mathbb{W}$ is a bounded linear operator. f: $T \times \mathbb{W} \to \mathbb{W}$ is a given function satisfying some hypotheses that will be specified later, and the functions γ_s: $(u_s, v_s] \times \mathbb{W} \to \mathbb{W}$ represent non-instantaneous impulses. The jump in the state x at time t is defined by

$$\Delta x(t) = x\left(t^+\right) - x\left(t^-\right).$$

As far as we know, no one has conducted research on such class of systems yet. Kumar and Abdal [25] studied (1.1) only in the form of classical integer derivatives. Meraj and Pandey [7] dealt with (1.1) without instantaneous impulses. Compared with previous research, the following distinguishing features are presented in this article. Firstly, compared with [25], (1.1) is in the form of fractional derivatives. In addition, the nonlinear term and the two types of impulses here are no longer required to meet Lipschitz conditions. Secondly, compared with [7], instantaneous impulses are involved in (1.1), and we also weaken the conditions that the impulses need to satisfy. Compared with [26, 27], we not only consider the impact of instantaneous and non-instantaneous impulses at the same time in (1.1), but also discuss the approximate controllability of the system. In addition, (1.1) considers the impact of instantaneous impulses in comparison to [13].

The rest of the article is organized as follows. In Section 2, some fundamental concepts and results are listed. In Section 3, by the ρ-set contractive fixed-point theorem, the existence of mild solutions for (1.1) is discussed. In Section 4, we show that (1.1) is approximately controllable. Finally, in Section 5, a reasonable instance is worked out to support the main results.

2. Preliminaries

In this section, we first present a set of piecewise continuous functions. Next, we define a mild solution of (1.1) and conclude an expression of the resolvent operator. Some related definitions and lemmas are also listed.

Assume that $\mathbb{W}$ is a Banach space with the norm $\|\cdot\|$.

Define $PC(T;\mathbb{W}) = \Big\{x : T \to \mathbb{W} \mid x$ is continuous at $t \neq u_s^{j_s+j}$, $t \neq u_{s+1}$, and $x(u_s^{j_s+j^-})$, $x(u_s^{j_s+j^+})$, $x(u_{s+1}^-)$, $x(u_{s+1}^+)$ exist, with $x(u_s^{j_s+j^-}) = x(u_s^{j_s+j})$ and $x(u_{s+1}^-) = x(u_{s+1})$, for $s = 0, 1, \cdots, h$, $j = 1, 2, \cdots, (j_{s+1} - j_s)\Big\}$.

Obviously, $PC(T;\mathbb{W})$ is a Banach space with the norm

$$\|x\|_{PC} = \sup_{t\in T} \|x(t)\|.$$

Definition 2.1. *([28]) If $f(t) \in C^n[0,\infty)$, then the Caputo fractional derivative of f of order $\alpha (\alpha > 0)$ is defined as follows:*

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integral part of $\alpha > 0$.

Lemma 2.1. *([29]) If f satisfies a uniform Hölder continuity with exponent $\beta \in (0, 1]$, the unique solution of the Cauchy problem*

$$\begin{cases} {}^cD^q x(t) = Ax(t) + f(t), \ t \in T, \\ \quad x(0) = x_0 \in \mathbb{W}, \end{cases}$$

is given by

$$x(t) = U(t)x_0 + \int_0^t (t-s)^{q-1} V(t-s) f(s) ds,$$

where

$$U(t) = \int_0^\infty \zeta_q(\theta)\mathcal{T}(t^q\theta)\, d\theta, V(t) = q\int_0^\infty \theta\, \zeta_q(\theta)\mathcal{T}(t^q\theta)\, d\theta,$$

$$\zeta_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}}\rho_q\left(\theta^{-\frac{1}{q}}\right),$$

$$\rho_q(\theta) = \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q),\ \theta \in (0,\infty),$$

and $\zeta_q(\theta)$ is a probability density function defined on $(0,\infty)$.

Remark 2.1. $\zeta_q(\theta) \geq 0,\ \theta \in (0,\infty)$,

$$\int_0^\infty \zeta_q(\theta) d\theta = 1$$

and

$$\int_0^\infty \theta\zeta_q(\theta) d\theta = \frac{1}{\Gamma(1+q)}.$$

Similar to [7,25], the mild solution of (1.1) can be defined as follows:

Definition 2.2. *For a given $z(\cdot) \in L^2(T;\ \mathbb{V})$, $x(\cdot\,,\ x_0,\ z)$: $T \to \mathbb{W}$ is called a mild solution of (1.1), if $x \in PC(T;\mathbb{W})$ and satisfies*

$$x(t) = \begin{cases} U(t)x_0 + \int_0^t (t-\tau)^{q-1}V(t-\tau)[f(\tau,x(\tau)) + Bz(\tau)]d\tau \\ + \sum_{0<u_0^j<t} U(t-u_0^j)I_0^j(x(u_0^{j^-})),\ \ t \in [0,u_1], \\ \gamma_s(t, x(u_s^-)),\ t \in (u_s, v_s],\ \ s = 1,2,\cdots,h, \\ U(t-v_s)\gamma_s(v_s, x(u_s^-)) \\ + \int_{v_s}^t (t-\tau)^{q-1}V(t-\tau)[f(\tau,x(\tau)) + Bz(\tau)]d\tau \\ + \sum_{v_s<u_s^{j_s+j}<t} U(t-u_s^{j+j_s})I_s^{j+j_s}(x(u_s^{j_s+j^-})), \\ t \in (v_s, u_{s+1}],\ \ s = 1,2,\cdots,h. \end{cases}$$

Definition 2.3. *([30]) Equation (1.1) is said to be approximately controllable on T if, for any given final state $x^b \in \mathbb{W}$ and arbitrary ε, there exists a control $z \in L^2(T,\mathbb{V})$ and a corresponding solution $x(t)$ of (1.1) such that*

$$\|x(b) - x^b\| < \varepsilon.$$

Represent the adjoint of V and B with V^* and B^*, respectively. For arbitrary $\lambda > 0$, define the resolvent operator

$$R\left(\lambda, \Gamma_0^b\right) = \left(\lambda I + \Gamma_0^b\right)^{-1}, \tag{2.1}$$

where

$$\Gamma_0^b = \int_0^b (b-s)^{q-1}V(b-s)BB^*V^*(b-s)ds.$$

Now, introduce some relevant results of the Kuratowski measure of noncompactness $\mathcal{X}$ defined on bounded subsets of the Banach space $\mathbb{W}$. For more detailed information, please see [5,31,32] and references therein.

Lemma 2.2. *([33]) Let $\mathbb{W}$ be a Banach space and W_1 is a bounded subset of $\mathbb{W}$; then there exists a countable set*

$$D = \{x_n\}_{n=1}^{\infty} \subset W_1,$$

such that

$$\mathcal{X}(W_1) \leq 2\mathcal{X}(D).$$

Definition 2.4. *([5]) Let $\mathbb{W}$ be a Banach space and W_1 be a nonempty subset of $\mathbb{W}$. If there exists a constant $\rho \in [0, 1)$ such that*

$$\mathcal{X}(\nu(\Omega)) \leq \rho\mathcal{X}(\Omega)$$

for every bounded set $\Omega \subset W_1$, the continuous map $\nu{:}W_1 \to \mathbb{W}$ is called a ρ-set contractive map.

Theorem 2.1. *([5]) Let $\mathbb{W}$ a Banach space and W_1 be a closed bounded and convex subset. Suppose that $\nu : W_1 \to \mathbb{W}$ is a ρ -set contractive map. Then, ν has at least one fixed point in W_1.*

3. Main results

3.1. Existence of mild solutions

In this section, we discuss the existence of mild solutions of (1.1). For this sake, some hypotheses are listed as follows:
(H1) For $t > 0, \{\mathcal{T}(t)\}$ is compact, and there exists $\mathcal{P} \geq 1$ such that

$$\|\mathcal{T}(t)\| \leq \mathcal{P}, \ \forall\, t \in T.$$

(H2) f: $T \times \mathbb{W} \to \mathbb{W}$ is continuous, and there exists a constant $q_1 \in (0, q)$ and $a(t) \in L^{\frac{1}{q_1}}(T, R^+)$ such that

$$\|f(t, x)\| \leq a(t)\|x\|, \ \forall x \in \mathbb{W}, \ t \in T.$$

(H3) γ_s: $T_s \times \mathbb{W} \to \mathbb{W}$, $T_s = (u_s, v_s]$, $s = 1, 2, \cdots, h$ are continuous, and there are positive constants $\mathcal{G}_{\gamma_s}$ and functions $\mathcal{H}_{\gamma_s}(t) \in L^1(T, R^+)$ such that

$$\|\gamma_s(t, x)\| \leq \mathcal{G}_{\gamma_s}\|x\|, \ \forall t \in T_s, \ \forall x \in \mathbb{W}$$

and

$$\mathcal{X}(\gamma_s(t, D)) \leq \mathcal{H}_{\gamma_s}(t)\mathcal{X}(D)$$

for any bounded $D \subset \mathbb{W}$ and $\forall t \in T_s$.
(H4) $I_s^{j_s+j}$: $\mathbb{W} \to \mathbb{W}$ are continuous for $s = 0, 1, \cdots, h$, and $j = 1, 2, \cdots, (j_{s+1} - j_s)$, and there are positive constants $\mathcal{I}_s^{j_s+j}$ and $\mathcal{H}_s^{j_s+j}$ such that

$$\left\|I_s^{j_s+j}(x)\right\| \leq \mathcal{I}_s^{j_s+j}\|x\|, \ \forall x \in \mathbb{W}$$

and

$$\mathcal{X}(I_s^{j_s+j}(D)) \leq \mathcal{H}_s^{j_s+j}\mathcal{X}(D)$$

for any bounded $D \subset \mathbb{W}$.
(H5) $\lambda R\left(\lambda, \ \Gamma_0^b\right)$ tends to 0 as $\lambda \to 0^+$ in the strong operator topology.

For convenience, denote

$$\mathcal{M} = \left(\frac{1 - q_1}{q - q_1} b^{\frac{1-q_1}{q-q_1}}\right)^{1-q_1},$$
$$\mathcal{F} = \|a(t)\|_{L^{\frac{1}{q_1}}(T,R^+)},$$
$$\varrho = \frac{q - 1}{1 - q_1} \in (0, 1).$$

Now we introduce the following results to get the approximate controllability of (1.1) and its corresponding linear system.

Lemma 3.1. *([30]) The following conditions are equivalent:*
(1) (H5) holds.
(2) System

$$\begin{cases} {}^cD^q x(t) = Ax(t) + Bz(t), \ t \in T, \\ x(0) = x_0, \end{cases}$$

is approximately controllable on T.

Lemma 3.2. *([34]) The operators U and V have the following properties:*
(i) $U(t)$ and $V(t)$ are strongly continuous for $t \geq 0$.
(ii) $U(t)$ and $V(t)$ are linear and bounded operators for arbitrary fixed $t \geq 0$ and they satisfy

$$\|U(t)x\| \leq \mathcal{P}\|x\|, \ \|V(t)x\| \leq \frac{\mathcal{P}}{\Gamma(q)}\|x\|$$

for arbitrary $x \in \mathbb{W}$.
(iii) If $\mathcal{T}(t)(t > 0)$ is a compact semigroup, $U(t)$ and $V(t)$ are compact operators on $\mathbb{W}$ for $t > 0$.

For an arbitrary $\lambda > 0$, we define the following control $z^\lambda(t,\mu,x)$:

$$z^\lambda(t,\mu,x)=\begin{cases} B^*V^*(b-t)R\left(\lambda,\Gamma_0^b\right)\Big[\mu-U(b)x_0- \\ \int_0^b(b-\tau)^{q-1}V(b-\tau)f(\tau,\ x(\tau))d\tau \\ -\sum_{0<u_0^j<b}U\left(b-u_0^j\right)I_0^j\left(x\left(u_0^{j^-}\right)\right)\Big],\ \ t\in[0,u_1], \\ B^*V^*(b-t)R\left(\lambda,\Gamma_0^b\right)\Big[\mu-U(b-v_s)\gamma_s(v_s,\ x(u_s^-)) \\ -\int_{v_s}^b(b-\tau)^{q-1}V(b-\tau)f(\tau,\ x(\tau))d\tau \\ -\sum_{v_s<u_s^{j_s+j}<b}U\left(b-u_s^{j_s+j}\right)I_s^{j_s+j}\left(x\left(u_s^{j_s+j^-}\right)\right)\Big], \\ t\in(v_s,u_{s+1}], \end{cases} \tag{3.1}$$

where $s=1,\cdots,h$, for $t\in T,\ \mu\in\mathbb{W}$ and $x\in PC(T,\mathbb{W})$.

Lemma 3.3. *For arbitrary $\lambda>0$ and $\mu\in\mathbb{W}$, the set $\{z^\lambda(t,\mu,x): x\in B_\delta\}$ is bounded on T, where*

$$B_\delta=\{x\in PC(T;\mathbb{W}):\|x\|_{PC}\le\delta\}.$$

Proof. Notice that, by Lemma 3.2,

$$\begin{aligned}\left\|z^\lambda(t,\mu,x)\right\|=&\Big\|B^*V^*(b-t)R\left(\lambda,\Gamma_0^b\right)\Big[\mu-U(b)x_0 \\ &-\int_0^b(b-\tau)^{q-1}V(b-\tau)f(\tau,\ x(\tau))d\tau \\ &-\sum_{0<u_0^j<b}U\left(b-u_0^j\right)I_0^j\left(x\left(u_0^{j^-}\right)\right)\Big]\Big\| \\ \le&\frac{\|B\|\mathcal{P}}{\lambda\Gamma(q)}\Big[\|\mu\|+\mathcal{P}\|x_0\|+\frac{\mathcal{P}\mathcal{F}\delta}{\Gamma(q)}\mathcal{M}+\sum_{j=1}^{j_1}\mathcal{P}\mathcal{I}_0^j\delta\Big]\end{aligned}$$

for $t\in(0,u_1]$.

Similarly,

$$\begin{aligned}\left\|z^\lambda(t,\mu,x)\right\|=&\Big\|B^*V^*(b-t)R\left(\lambda,\Gamma_0^b\right)\Big[\mu-U(b-v_s)\gamma_s(v_s,\ x(u_s^-)) \\ &-\int_{v_s}^b(b-\tau)^{q-1}V(b-\tau)f(\tau,x(\tau))d\tau \\ &-\sum_{v_s<u_s^{j_s+j}<b}U\left(b-u_s^{j_s+j}\right)I_s^{j_s+j}\left(x\left(u_s^{j_s+j^-}\right)\right)\Big]\Big\| \\ \le&\frac{\|B\|\mathcal{P}}{\lambda\Gamma(q)}\Big[\|\mu\|+\mathcal{P}\mathcal{G}_{\gamma_s}\delta+\frac{\mathcal{P}\delta\mathcal{F}}{\Gamma(q)}\mathcal{M}+\sum_{j=1}^{j_{s+1}-j_s}\mathcal{P}\mathcal{I}_s^{j_s+j}\delta\Big]\end{aligned}$$

for t $\in(v_s,u_{s+1}]$, where $s=1,\cdots,h$.

Now we are in the position to give the existence results for (1.1).

Theorem 3.1. *Suppose that (H1)–(H4) hold. Then, for every $\lambda>0$, there exists at least one solution of (1.1) on T provided that*

$$\mathcal{G}_{\gamma_s}\le 1,$$

$$\begin{aligned}P_0:=&\frac{\mathcal{P}^2\|B\|^2t^q}{\lambda(\Gamma(q))^2q}\Big[\frac{\mathcal{P}\mathcal{F}}{\Gamma(q)}\mathcal{M}+\sum_{j=1}^{j_1}\mathcal{P}\mathcal{I}_0^j\Big] \\ &+\frac{\mathcal{P}\mathcal{F}}{\Gamma(q)}\mathcal{M}+\sum_{0<u_0^j<t}\mathcal{P}\mathcal{I}_0^j\in(0,1),\end{aligned}$$

$$\begin{aligned}P_s:=&\mathcal{P}\mathcal{G}_{\gamma_s}+\frac{\mathcal{P}^2\|B\|^2t^q}{\lambda(\Gamma(q))^2q}\Big[\mathcal{P}\mathcal{G}_{\gamma_s}+\frac{\mathcal{P}\mathcal{F}}{\Gamma(q)}\mathcal{M}+\sum_{j=1}^{j_{s+1}-j_s}\mathcal{P}\mathcal{I}_s^{j_s+j}\Big] \\ &+\frac{\mathcal{P}\mathcal{F}_1}{\Gamma(q)}\mathcal{M}+\sum_{v_s<u_s^{j_s+j}<t}\mathcal{P}\mathcal{I}_s^{j_s+j}\in(0,1)\end{aligned}$$

and

$$\Lambda:=2\Big(\sum_{j=1}^{j_{s+1}-j_s}\mathcal{P}\mathcal{H}_s^{j_s+j}+\mathcal{P}\mathcal{H}_{\gamma_s}(t)\Big)<1,$$

where $s=1,2,\cdots,h$ and $t\in T$.

Proof. First, for an arbitrary $\lambda>0$, define the following operator Υ^λ on $PC(T,\mathbb{W})$.

$$\left(\Upsilon^\lambda x\right)(t)=\begin{cases} U(t)x_0+\int_0^t(t-\tau)^{q-1}V(t-\tau)[f(\tau,x(\tau))+ \\ Bz^\lambda(\tau)]d\tau+\sum_{0<u_0^j<t}\ U(t-u_0^j)I_0^j(x(u_0^{j^-})), \\ t\in[0,u_1], \\ \gamma_s(t,x(u_s^-)),\ \ t\in(u_s,v_s], \\ U(t-v_s)\gamma_s(v_s,x(u_s^-)) \\ +\int_{v_s}^t(t-\tau)^{q-1}V(t-\tau)[f(\tau,x(\tau))+Bz^\lambda(\tau)]d\tau \\ +\sum_{v_s<u_s^{j_s+j}<t}U(t-u_s^{j+j_s})I_s^{j+j_s}(x(u_s^{j_s+j^-})), \\ t\in(v_s,u_{s+1}], \end{cases}$$

where $s=1,2,\cdots,h$, $z^\lambda(t)=z^\lambda(t,\ \mu,\ x)$ is defined as in (3.1).

It is obvious that the existence of fixed points of Υ^λ is equivalent to the existence of mild solutions of (1.1).

For convenience, rewrite Υ^λ as follows:

$$(\Upsilon^\lambda x)(t)=(\Upsilon_1x)(t)+(\Upsilon_2x)(t),\ \ t\in T,$$

where

$$(\Upsilon_1x)(t)=\begin{cases} U(t)x_0+\sum_{0<u_0^j<t}\ U(t-u_0^j)I_0^j(x(u_0^{j^-})), \\ t\in[0,u_1],\ \gamma_s(t,x(u_s^-)),\ t\in(u_s,v_s], \\ U(t-v_s)\gamma_s(v_s,x(u_s^-)) \\ +\sum_{v_s<u_s^{j_s+j}<t}U(t-u_s^{j+j_s})I_s^{j+j_s}(x(u_s^{j_s+j^-})), \\ t\in(v_s,u_{s+1}], \end{cases}$$

$$(\Upsilon_2x)(t)=\begin{cases} \int_0^t(t-\tau)^{q-1}V(t-\tau)[f(\tau,x(\tau))+Bz^\lambda(\tau)]d\tau, \\ t\in[0,u_1], \\ 0,\ \ t\in(u_s,v_s], \\ \int_{v_s}^t(t-\tau)^{q-1}V(t-\tau)[f(\tau,x(\tau))+Bz^\lambda(\tau)]d\tau, \\ t\in(v_s,u_{s+1}], \end{cases}$$

where $s = 1, 2, \cdots, h$.

Next, we prove the existence of fixed points of Υ^λ. The process is divided into four steps.

Step 1. Show that, for an arbitrary $\lambda > 0$, there exists a constant $\delta = \delta(\lambda) > 0$ such that $\Upsilon^\lambda(B_\delta) \subset B_\delta$.

Choose δ satisfying:

$$\delta \geq \max_{1\leq s\leq h}\left[\frac{Q_0}{1-P_0}, \frac{Q_s}{1-P_s}\right], \tag{3.2}$$

where

$$Q_0 = \mathcal{P}\|x_0\| + \frac{\mathcal{P}^2\|B\|^2 t^q}{\lambda(\Gamma(q))^2 q}\Big(\|\mu\| + \mathcal{P}\|x_0\|\Big),$$
$$Q_s = \frac{\mathcal{P}^2\|B\|^2(t-v_s)^q}{\lambda(\Gamma(q))^2 q}\|\mu\|.$$

It is time to claim that $\Upsilon^\lambda(B_\delta) \subset B_\delta$, which is equivalent to show that, for an arbitrary $x \in B_\delta$, $\|\Upsilon^\lambda x(t)\| \leq \delta$ on T.

By (H1)–(H4) and (3.2), one can get that

$$\begin{aligned}\|\Upsilon^\lambda x(t)\| \leq& \mathcal{P}\|x_0\| + \frac{\mathcal{P}^2\|B\|^2 t^q}{\lambda(\Gamma(q))^2 q}\Big[\|\mu\| + \mathcal{P}\|x_0\| + \frac{\mathcal{P}\mathcal{F}\delta}{\Gamma(q)}\mathcal{M}\\ &+ \sum_{j=1}^{j_1}\mathcal{P}\mathcal{I}_0^j\delta\Big] + \frac{\mathcal{P}\delta\mathcal{F}}{\Gamma(q)}\mathcal{M} + \sum_{0<u_0^j<t}\mathcal{P}\mathcal{I}_0^j\delta,\end{aligned} \tag{3.3}$$

for $t \in [0, u_1]$. So,

$$\|\Upsilon^\lambda x(t)\| \leq Q_0 + P_0\delta \leq \delta$$

for $t \in [0, u_1]$.

Similarly,

$$\|(\Upsilon^\lambda x)(t)\| \leq \|\gamma_s(t, x(u_s^-))\| \leq \mathcal{G}_{\gamma_s}\delta \leq \delta, \tag{3.4}$$

for $t \in (u_s, v_s]$, $s = 1, 2, \cdots, h$. Thus,

$$\|\Upsilon^\lambda x(t)\| \leq Q_s + P_s\delta \leq \delta$$

for $t \in [u_s, v_s]$.

In addition,

$$\begin{aligned}\|(\Upsilon^\lambda x)(t)\| \leq& \mathcal{P}\mathcal{G}_{\gamma_s}\delta + \frac{\mathcal{P}^2\|B\|^2 t^q}{\lambda(\Gamma(q))^2 q}\Big[\|\mu\| + \mathcal{P}\mathcal{G}_{\gamma_s}\delta + \frac{\mathcal{P}\delta\mathcal{F}}{\Gamma(q)}\mathcal{M}\\ &+ \sum_{j=1}^{j_{s+1}-j_s}\mathcal{P}\mathcal{I}_s^{j_s+j}\delta\Big] + \frac{\mathcal{P}\delta\mathcal{F}_1}{\Gamma(q)}\mathcal{M} + \sum_{v_s<u_s^{j_s+j}<t}\mathcal{P}\mathcal{I}_s^{j_s+j}\delta,\end{aligned} \tag{3.5}$$

for $t \in (v_s, u_{s+1}]$, $s = 1, 2, \cdots, h$. Thus,

$$\|\Upsilon^\lambda x(t)\| \leq Q_s + P_s\delta \leq \delta$$

for $t \in [v_s, u_{s+1}]$.

Combining (3.3)–(3.5), one can obtain that

$$\|\Upsilon^\lambda\zeta(t)\| \leq Q_s + P_s\delta \leq \delta$$

for $t \in T$. That is, $\Upsilon^\lambda(B_\delta) \subset B_\delta$.

Step 2. Claim that Υ^λ is continuous on B_δ.

We first show that the control $z^\lambda(t, \mu, x)$ is continuous with respect to x on B_δ. Let $\{x_n\}_{n=1}^\infty \subset B_\delta$ be a sequence such that $x_n \to x$ as $n \to \infty$, then

$$\begin{aligned}&\|z^\lambda(t,\mu,x_n) - z^\lambda(t,\mu,x)\|\\ &\leq \frac{\|B\|\mathcal{P}}{\lambda\Gamma(q)}\Big\{\frac{b^{q-1}\mathcal{P}}{\Gamma(q)}\int_0^b\Big\|f(\tau,x_n(\tau)) - f(\tau,x(\tau))\Big\|d\tau\\ &\quad+ \sum_{0<u_0^j<b}\mathcal{P}\Big[I_0^j\big(x_n\big(u_0^{j^-}\big)\big) - I_0^j\big(x\big(u_0^{j^-}\big)\big)\Big]\Big\}\end{aligned} \tag{3.6}$$

for $t \in [0, u_1]$.

$$\begin{aligned}&\|z^\lambda(t,\mu,x_n) - z^\lambda(t,\mu,x)\|\\ &\leq \frac{\|B\|\mathcal{P}}{\lambda\Gamma(q)}\Big\{\mathcal{P}\Big\|\gamma_s(v_s, x(u_s^-)) - \gamma_s(v_s, x_n(u_s^-))\Big\|\\ &\quad+ \frac{(b-v_s)^{q-1}\mathcal{P}}{\Gamma(q)}\int_{v_s}^b\Big\|f(\tau,x_n(\tau)) - f(\tau,x(\tau))\Big\|d\tau\\ &\quad+ \sum_{v_s<u_s^{j_s+j}<b}\mathcal{P}\Big\|I_s^{j_s+j}\big(x_n\big(u_s^{j_s+j^-}\big)\big) - I_s^{j_s+j}\big(x\big(u_s^{j_s+j^-}\big)\big)\Big\|\Big\}\end{aligned} \tag{3.7}$$

for $t \in [v_s, u_{s+1}]$, where $s = 1, 2, \cdots, h$.

By (3.6) and (3.7) together with (H2)–(H4), one can get

$$\|z^\lambda(t,\mu,x_n) - z^\lambda(t,\mu,x)\| \to 0, \quad \text{as } n \to \infty.$$

Therefore, the control $z^\lambda(t, \mu, x)$ is continuous with respect to x on B_δ.

Next, we show that Υ^λ is continuous on B_δ. Let $\{x_n\}_{n=1}^\infty$ be a sequence on B_δ such that $x_n \to x$ as $n \to \infty$.

Notice that, from (H2)–(H4),

$$\begin{aligned}\|(\Upsilon^\lambda x_n)(t) - (\Upsilon^\lambda x)(t)\| \leq& \frac{u_1^q\mathcal{P}}{q\Gamma(q)}\Big[\|B\|\|z^\lambda(t,\mu,x_n(t)) - z^\lambda(t,\mu,x(s))\|\\ &+ \|f(t, x_n(t)) - f(t, x(t))\|\Big]\\ &+ \sum_{0<u_0^j<t}\mathcal{P}\Big\|I_0^j(x_n(u_0^{j^-}) - x(u_0^{j^-}))\Big\|\\ \to& 0,\end{aligned} \tag{3.8}$$

as $n \to \infty$, for $t \in [0,\ u_1]$.

$$\|(\Upsilon^\lambda x_n)(t) - (\Upsilon^\lambda x)(t)\| = \|\gamma_s(t,\ x_n(u_s^-)) - \gamma_s(t,\ x(u_s^-))\| \to 0 \tag{3.9}$$

as $n \to \infty$, for $t \in (u_s,\ v_s]$, where $s = 1, 2, \cdots, h$.

$$\begin{aligned}
&\|(\Upsilon^\lambda x_n)(t) - (\Upsilon^\lambda x)(t)\| \\
&\leq \mathcal{P}\|\gamma_s(t,\ x_n(u_s^-)) - \gamma_s(t,\ x(u_s^-))\| \\
&\quad + \frac{(u_{s+1} - v_s)^q \mathcal{P}}{q\Gamma(q)}\Big[\|B\|\|z^\lambda(t, \mu, x_n(s)) - z^\lambda(t, \mu, x(t))\| \\
&\quad + \|f(t,\ x_n(t)) - f(t,\ x(t))\|\Big] \\
&\quad + \sum_{v_s < u_s^{j_s+j} < t} \mathcal{P}\Big\| I_s^{j_s+j}(x_n(u_s^{j_s+j})) - I_s^{j_s+j}(x(u_s^{j_s+j}))\Big\| \\
&\to 0
\end{aligned} \tag{3.10}$$

as $n \to\ \infty$, for $t \in (v_s,\ u_{s+1}]$, where $s = 1, 2, \cdots, h$.

According to (3.8)–(3.10), one can obtain that

$$\|(\Upsilon^\lambda x_n) - (\Upsilon^\lambda x)\|_{PC} \to 0$$

as $n \to \infty$. Therefore, Υ^λ is continuous on B_δ.

Step 3. Claim that Υ_2 is compact on B_δ.

We first show that, for an arbitrary $t \in T$, the set $\{(\Upsilon_2 x)(t) : x \in B_\delta\}$ is relatively compact in $\mathbb{W}$. For $t = 0$ and $t \in (u_s,\ v_s]$, $(\Upsilon_2 x)(t) = 0$, where $s = 1, 2, \cdots, h$.

Let $t \in (v_s,\ u_{s+1}]$, $s = 0, 1, 2, \cdots, h$ be fixed. For an arbitrary $\varepsilon \in (0, t - v_s)$, define $\Upsilon_2^{\varepsilon,\sigma}$ on B_δ as follows:

$$\begin{aligned}
&(\Upsilon_2^{\varepsilon,\sigma} x)(t) \\
&= q\int_{v_s}^{t-\varepsilon}\int_\sigma^\infty \theta(t-\tau)^{q-1}\zeta_q(\theta)\mathcal{T}((t-\tau)^q(\theta))\Big[f(\tau, x(\tau)) \\
&\quad + Bz^\lambda(\tau)\Big]d\theta d\tau \\
&= \mathcal{T}(\varepsilon^q\sigma) q\int_{v_s}^{t-\varepsilon}\int_\sigma^\infty \theta(t-\tau)^{q-1}\zeta_q(\theta)\mathcal{T}((t-\tau)^q(\theta) \\
&\quad - \varepsilon^q\sigma)\Big[f(\tau, x(\tau)) + Bz^\lambda(\tau)\Big]d\theta d\tau \\
&:= \mathcal{T}(\varepsilon^q\sigma) y(t, \varepsilon).
\end{aligned}$$

Because $\{y(t, \varepsilon)\colon x \in B_\delta\}$ is bounded in $\mathbb{W}$ and $\mathcal{T}(\varepsilon)(\varepsilon > 0)$ is compact, the set $\{(\Upsilon_2^{\varepsilon,\sigma} x)(t)\colon x \in B_\delta\}$ is relatively compact in $\mathbb{W}$.

In addition,

$$\begin{aligned}
&\|(\Upsilon_2 x)(t) - (\Upsilon_2^{\varepsilon,\sigma} x)(t)\| \\
&= q\Big\| \int_{v_s}^t\int_0^\sigma \theta(t-\tau)^{q-1}\zeta_q(\theta)\mathcal{T}((t-\tau)^q(\theta))\Big[f(\tau, x(\tau)) \\
&\quad + Bz^\lambda(\tau)\Big]d\theta d\tau + \int_{v_s}^t\int_\sigma^\infty \theta(t-\tau)^{q-1}\zeta_q(\theta)\mathcal{T}((t-\tau)^q(\theta)) \\
&\quad \Big[f(\tau, x(\tau)) + Bz^\lambda(\tau)\Big]d\theta d\tau \\
&\quad - \int_{v_s}^{t-\varepsilon}\int_\sigma^\infty \theta(t-\tau)^{q-1}\zeta_q(\theta)\mathcal{T}((t-\tau)^q(\theta))\Big[f(\tau, x(\tau)) \\
&\quad + Bz^\lambda(\tau)\Big]d\theta d\tau\Big\| \\
&= q\Big\| \int_{v_s}^t\int_0^\sigma \theta(t-\tau)^{q-1}\zeta_q(\theta)\mathcal{T}((t-\tau)^q(\theta))\Big[f(\tau, x(\tau)) \\
&\quad + Bz^\lambda(\tau)\Big]d\theta d\tau \\
&\quad + \int_{t-\varepsilon}^t\int_\sigma^\infty \theta(t-\tau)^{q-1}\zeta_q(\theta)\mathcal{T}((t-\tau)^q(\theta))\Big[f(\tau, x(\tau)) \\
&\quad + Bz^\lambda(\tau)\Big]d\theta d\tau\Big\| \\
&\leqq (K_1 + K_2),
\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}
K_1 &= \Big\| \int_{v_s}^t\int_0^\sigma \theta(t-\tau)^{q-1}\zeta_q(\theta)\mathcal{T}((t-\tau)^q(\theta))\Big[f(\tau, x(\tau)) \\
&\quad + Bz^\lambda(\tau)\Big]d\theta d\tau\Big\| \\
&\leq \mathcal{P}(\int_0^\sigma \theta\zeta_q(\theta)d\theta)\Big[\Big\|\int_{v_s}^t (t-\tau)^{q-1} f(\tau,\ x(\tau))d\tau\Big\| \\
&\quad + \Big\|\int_{v_s}^t (t-\tau)^{q-1} Bz^\lambda(\tau)d\tau\Big\|\Big] \\
&\leq \mathcal{P}(\int_0^\sigma \theta\zeta_q(\theta)d\theta)\Big[\mathcal{F}\mathcal{M}\delta + \frac{(b-v_s)^q}{q}\|B\|\|z^\lambda(\tau)\|\Big],
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
K_2 &= \Big\| \int_{t-\varepsilon}^t\int_\sigma^\infty \theta(t-\tau)^{q-1}\zeta_q(\theta)\mathcal{T}((t-\tau)^q(\theta))\Big[f(\tau, x(\tau)) \\
&\quad + Bz^\lambda(\tau)\Big]d\theta d\tau\Big\| \\
&\leq \mathcal{P}(\int_\sigma^\infty \theta\zeta_q(\theta)d\theta)\Big[\Big\|\int_{t-\varepsilon}^t (t-\tau)^{q-1} f(\tau,\ x(\tau))d\tau\Big\| \\
&\quad + \Big\|\int_{t-\varepsilon}^t (t-\tau)^{q-1} Bz^\lambda(\tau)d\tau\Big\|\Big] \\
&\leq \mathcal{P}(\int_\sigma^\infty \theta\zeta_q(\theta)d\theta)\Big[\mathcal{F}\delta\Big(\frac{1-q_1}{q-q_1}\varepsilon^{\frac{q-q_1}{1-q_1}}\Big)^{1-q_1} + \frac{\varepsilon^q}{q}\|B\|\|z^\lambda(\tau)\|\Big].
\end{aligned} \tag{3.13}$$

This, together with (3.11) and (3.12), guarantees that

$$\|(\Upsilon_2 x)(t) - (\Upsilon_2^{\varepsilon,\sigma} x)(t)\| \to 0$$

as $\varepsilon \to 0$ and $\delta \to 0$. That is, $\{(\Upsilon_2 x)(t)\colon x \in B_\delta\}$ is relatively compact in $\mathbb{W}$ for an arbitrary $t \in T$.

Next, it is time to claim that $\{(\Upsilon_2 x):\ x \in B_\delta\}$ are equicontinuous. For any $x \in B_\delta$ and $v_s \le t_1 < t_2 \le u_{s+1}$ for $s = 0, 1, 2, \cdots, h$,

$$
\begin{aligned}
&\|(\Upsilon_2 x)(t_2) - (\Upsilon_2 x)(t_1)\| \\
&= \Big\| \int_{v_s}^{t_2} (t_2-\tau)^{q-1} V(t_2-\tau)\Big[f(\tau, x(\tau)) + Bz^\lambda(\tau)\Big] d\tau \\
&\quad - \int_{v_s}^{t_1} (t_1-\tau)^{q-1} V(t_1-\tau)\Big[f(\tau, x(\tau)) + Bz^\lambda(\tau)\Big] d\tau \Big\| \\
&\le \Big\| \int_{t_1}^{t_2} (t_2-\tau)^{q-1} V(t_2-\tau) f(\tau, x(\tau)) d\tau \Big\| \\
&\quad + \Big\| \int_{t_1}^{t_2} (t_2-\tau)^{q-1} V(t_2-\tau) Bz^\lambda(\tau) d\tau \Big\| \\
&\quad + \Big\| \int_{v_s}^{t_1} \Big[(t_2-\tau)^{q-1} - (t_1-\tau)^{q-1}\Big] V(t_2-\tau) f(\tau, x(\tau)) d\tau \Big\| \\
&\quad + \Big\| \int_{v_s}^{t_1} \Big[(t_2-\tau)^{q-1} - (t_1-\tau)^{q-1}\Big] V(t_2-\tau) Bz^\lambda(\tau) d\tau \Big\| \\
&\quad + \Big\| \int_{v_s}^{t_1} (t_1-\tau)^{q-1}\Big[V(t_2-\tau) - V(t_1-\tau)\Big] f(\tau, x(\tau)) d\tau \Big\| \\
&\quad + \Big\| \int_{v_s}^{t_1} (t_1-\tau)^{q-1}\Big[V(t_2-\tau) - V(t_1-\tau)\Big] Bz^\lambda(\tau) d\tau \Big\| \\
&= L_1 + L_2 + L_3 + L_4 + L_5 + L_6.
\end{aligned} \tag{3.14}
$$

Now, we only need to show that L_1–L_6 tend to 0 independently of $x \in B_\delta$ when $(t_2 - t_1) \to 0$.

By Lemma 3.2 and (H1)–(H4), one can obtain that

$$
\begin{aligned}
L_1 \le& \frac{\mathcal{P}\mathcal{F}\delta}{\Gamma(q)} \cdot \left[\frac{1-q_1}{q-q_1}(t_2-t_1)^{\frac{q-q_1}{1-q_1}}\right]^{1-q_1}, \\
L_2 \le& \frac{\|B\|\|z^\lambda(t)\|\mathcal{P}}{\Gamma(q)} \cdot \left[\frac{1-q_1}{q-q_1}(t_2-t_1)^{\frac{q-q_1}{1-q_1}}\right]^{1-q_1}, \\
L_3 \le& \frac{\mathcal{P}\mathcal{F}\delta}{\Gamma(q)} \cdot \frac{(t_2-t_1)^{(\varrho+1)(1-q_1)}}{(1+\varrho)^{1-q_1}}, \\
L_4 \le& \frac{\|B\|\|z^\lambda(t)\|\mathcal{P}}{\Gamma(q)} \cdot \frac{(t_2-t_1)^{(\varrho+1)(1-q_1)}}{(1+\varrho)^{1-q_1}}, \\
L_5 \le& \mathcal{F}\delta\Big[\frac{1-q_1}{q-q_1}(t_1-v_s)^{\frac{q-q_1}{1-q_1}}\Big]^{1-q_1} \sup_{\tau\in(v_s,\ t_1]} \|V(t_2-\tau) \\
&- V(t_1-\tau)\|, \\
L_6 \le& \|B\|\|z^\lambda(t)\|\Big[\frac{1-q_1}{q-q_1}(t_1 \\
&- v_s)^{\frac{q-q_1}{1-q_1}}\Big]^{1-q_1} \sup_{\tau\in(v_s,\ t_1]} \|V(t_2-\tau) - V(t_1-\tau)\|.
\end{aligned} \tag{3.15}
$$

Through calculation,

$$L_i \to 0 \quad \text{as} \quad (t_2 - t_1) \to 0$$

for $i = 1, 2, \cdots, 6$. That is,

$$\|(\Upsilon_2 x)(t_2) - (\Upsilon_2 x)(t_1)\| \to 0$$

as $(t_2 - t_1) \to 0$. So, $\{(\Upsilon_2 x) : x \in B_\delta\}$ are equicontinuous on $[v_s,\ u_{s+1}]$, where $s = 0, 1, 2, \cdots, h$.

It is obvious that $\{(\Upsilon_2 x) : x \in B_\delta\}$ is bounded. Thus, Υ_2 is compact on B_δ by Arzelà-Ascoli theorem.

Step 4. Claim that Υ^λ is a Λ-set contractive map.

First, clearly, Υ_2 is a completely continuous operator on B_δ according to *Steps 2* and *3*.

Next, for an arbitrary bounded set $D \subset B_\delta$, by Lemma 2.2, there exists a countable set

$$D_0 = \{x_n\}_{n=1}^\infty \subset D,$$

such that

$$\mathcal{X}(\Upsilon_1(D)) \le 2\mathcal{X}(\Upsilon_1(D_0)). \tag{3.16}$$

Notice that

$$\|x_n(t) - x_m(t)\| \le \|x_n - x_m\|_{PC}, \quad t \in T,$$

which implies

$$\mathcal{X}(\{x_n(t)\}_{n=1}^\infty) \le \mathcal{X}_{PC}(\{x_n\}_{n=1}^\infty), \quad t \in T, \tag{3.17}$$

where $\mathcal{X}_{PC}(\cdot)$ denotes the Kuratowski measure of noncompactness of a bounded set in $PC(T, \mathbb{W})$.

According to (H2)–(H4),

$$
\begin{aligned}
\mathcal{X}(\{(\Upsilon_1 x_n)(t)\}_{n=1}^\infty) \le& \sum_{j=1}^{j_{s+1}-j_s} \mathcal{P}(\mathcal{X}(\{(I_s^{j_s+j} x_n)(t)\}_{n=1}^\infty)) \\
&+ \mathcal{P}\mathcal{X}(\gamma_s(t, \{x_n(t)\}_{n=1}^\infty)) \\
\le& \sum_{j=1}^{j_{s+1}-j_s} \mathcal{P}\mathcal{H}_s^{j_s+j}\mathcal{X}(\{x_n(t)\}_{n=1}^\infty) \\
&+ \mathcal{P}\mathcal{H}_{\gamma_s}(t)\mathcal{X}(\{x_n(t)\}_{n=1}^\infty) \\
=& \Big(\sum_{j=1}^{j_{s+1}-j_s} \mathcal{P}\mathcal{H}_s^{j_s+j} + \mathcal{P}\mathcal{H}_{\gamma_s}(t)\Big)\mathcal{X}(\{x_n(t)\}_{n=1}^\infty)
\end{aligned} \tag{3.18}
$$

for $t \in [v_s, u_{s+1}]$, $s = 0, 1, 2, \cdots, h$.

Similar to the procedure in *Step 3*, one can obtain that $\Upsilon_1(D_0)$ is equicontinuous on $[v_s, u_{s+1}], s = 0, 1, 2, \ldots, h$. Meanwhile, the boundedness of $\Upsilon_1(D_0)$ is obvious. Therefore (from a well-known result on measures of noncompactness),

$$\mathcal{X}_{PC}\left(\{\Upsilon_1 x_n\}_{n=1}^\infty\right) = \sup_{t\in[v_s,u_{s+1}],\ s=0,1,2,\ldots,h} \mathcal{X}\left(\{(\Upsilon_1 x_n)(t)\}_{n=1}^\infty\right). \tag{3.19}$$

This, together with (3.16)–(3.18) and (H1), guarantees that

$$\begin{aligned}
\mathcal{X}_{PC}(\Upsilon_1(D)) &\le 2\,\mathcal{X}_{PC}(\Upsilon_1(D_0)) \\
&= 2\,\mathcal{X}_{PC}(\{\Upsilon_1 x_n\}_{n=1}^{\infty}) \\
&\le 2\Big(\sum_{j=1}^{j_{s+1}-j_s} \mathcal{PH}_s^{j_s+j} + \mathcal{PH}_{\gamma_s}(t)\Big)\mathcal{X}(\{x_n(t)\}_{n=1}^{\infty}) \\
&\le 2\Big(\sum_{j=1}^{j_{s+1}-j_s} \mathcal{PH}_s^{j_s+j} + \mathcal{PH}_{\gamma_s}(t)\Big)\mathcal{X}_{PC}(\{x_n\}_{n=1}^{\infty}) \\
&= 2\Big(\sum_{j=1}^{j_{s+1}-j_s} \mathcal{PH}_s^{j_s+j} + \mathcal{PH}_{\gamma_s}(t)\Big)\mathcal{X}_{PC}(D_0) \\
&\le 2\Big(\sum_{j=1}^{j_{s+1}-j_s} \mathcal{PH}_s^{j_s+j} + \mathcal{PH}_{\gamma_s}(t)\Big)\mathcal{X}_{PC}(D) \\
&= \Lambda\,\mathcal{X}_{PC}(D),
\end{aligned}$$

where $s = 1, 2, \cdots, h$.

Therefore, Υ^{λ}: $B_\delta \to B_\delta$ is a Λ-set contractive map. By Theorem 2.1, Υ^{λ} has at least one fixed point $x \in B_\delta$, which is a mild solution of (1.1) on T.

Theorem 3.2. *Suppose that (H5) and the hypotheses of Theorem 3.1 hold. Then, (1.1) is approximately controllable on T.*

Proof. By Theorem 3.1, for an arbitrary desired final state $x^b \in \mathbb{W}$, there exists at least one solution denoted by $x^{\lambda}(t)$ of (1.1) corresponding to the following control:

$$z^{\lambda}(t,\ x^b,\ x^{\lambda}) = \begin{cases} B^* V^*(b-t) R\left(\lambda,\ \Gamma_0^b\right)\Big[x^b - U(b)\,x_0 \\ -\int_0^b (b-\tau)^{q-1} V(b-\tau) f(\tau,\ x^{\lambda}(\tau))d\tau \\ -\sum_{0<u_0^j<b} U\left(b-u_0^j\right) I_0^j\left(x^{\lambda}\left(u_0^{j^-}\right)\right)\Big], \\ t \in [0, u_1], \\ B^* V^*(b-t) R\left(\lambda,\ \Gamma_0^b\right)\Big[x^b \\ -U(b-v_s)\gamma_s\left(v_s,\ x^{\lambda}(u_s^-)\right) \\ -\int_{v_s}^b (b-\tau)^{q-1} V(b-\tau) f(\tau,\ x^{\lambda}(\tau))d\tau \\ -\sum_{v_s<u_s^{j_s+j}<b} U\left(b-u_s^{j_s+j}\right) I_s^{j_s+j}\left(x^{\lambda}\left(u_s^{j_s+j^-}\right)\right)\Big], \\ t \in (v_s, u_{s+1}],\ s = 1, \cdots, h. \end{cases}$$

Then, consider

$$\begin{aligned}
x^b - x^{\lambda}(b) =& x^b - U(b-v_s)\gamma_s(v_s, x^{\lambda}(u_s^-)) \\
&- \int_{v_s}^b (b-\tau)^{q-1} V(b-\tau) f(\tau, x^{\lambda}(\tau))d\tau \\
&- \sum_{v_s<u_s^{j_s+j}<b} U(b-u_s^{j+j_s}) I_s^{j+j_s}(x^{\lambda}(u_s^{j_s+j^-})) \\
&- \Gamma_0^b R\left(\lambda, \Gamma_0^b\right)\Big[x^b - U(b-v_s)\gamma_s(v_s, x^{\lambda}(u_s^-)) \\
&- \int_{v_s}^b (b-\tau)^{q-1} V(b-\tau) f(\tau, x^{\lambda}(\tau))d\tau \\
&- \sum_{v_s<u_s^{j_s+j}<b} U(b-u_s^{j+j_s}) I_s^{j+j_s}(x^{\lambda}(u_s^{j_s+j^-}))\Big].
\end{aligned} \tag{3.20}$$

Combining (2.1) and (3.20),

$$\begin{aligned}
x^b - x^{\lambda}(b) =& \lambda R\left(\lambda, \Gamma_0^b\right)\Big[x^b - U(b-v_s)\gamma_s(v_s, x^{\lambda}(u_s^-)) \\
&- \int_{v_s}^b (b-\tau)^{q-1} V(b-\tau) f(\tau, x^{\lambda}(\tau))d\tau \\
&- \sum_{v_s<u_s^{j_s+j}<b} U(b-u_s^{j+j_s}) I_s^{j+j_s}(x^{\lambda}(u_s^{j_s+j^-}))\Big].
\end{aligned}$$

According to (H2), one can obtain that f is bounded. Therefore, there exists a subsequence that is represented in the form of $\left\{f\left(t,\ x^{\lambda}(t)\right)\right\}$ again weakly converging to $f_1(t)$ in $\mathbb{W}$ as $\lambda \to 0^+$.

Now, define

$$\begin{aligned}
\ell =& x^b - U(b-v_s)\gamma_s(v_s, x^{\lambda}(u_s^-)) \\
&- \int_{v_s}^b (b-\tau)^{q-1} V(b-\tau) f_1(\tau)d\tau \\
&- \sum_{v_s<u_s^{j_s+j}<b} U(b-u_s^{j+j_s}) I_s^{j+j_s}(x^{\lambda}(u_s^{j_s+j^-})),
\end{aligned} \tag{3.21}$$

$$\begin{aligned}
\hbar(x^{\lambda}) =& x^b - U(b-v_s)\gamma_s(v_s, x^{\lambda}(u_s^-)) \\
&- \int_{v_s}^b (b-\tau)^{q-1} V(b-\tau) f(\tau, x^{\lambda}(\tau))d\tau \\
&- \sum_{v_s<u_s^{j_s+j}<b} U(b-u_s^{j+j_s}) I_s^{j+j_s}(x^{\lambda}(u_s^{j_s+j^-})).
\end{aligned} \tag{3.22}$$

Similar to [35], it is clear that

$$y(\cdot) \longrightarrow \int_0^{\cdot} (\cdot-\tau)^{q-1} V(\cdot-\tau) y(\tau)d\tau$$

is compact. It follows that

$$\|\hbar(x^{\lambda}) - \ell\| \to 0 \text{ as } \lambda \to 0^+.$$

This, together with (H5), (3.21), and (3.22), implies that

$$\left\|x^b - x^\lambda(b)\right\| \le \left\|\lambda R\left(\lambda, \Gamma_0^b\right)\ell\right\| + \left\|\lambda R\left(\lambda, \Gamma_0^b\right)\right\| \| \hbar(x^\lambda) - \ell\|.$$

Thus,

$$\|x^b - x^\lambda(b)\| \to 0$$

as $\lambda \to 0^+$, which means (1.1) is approximately controllable on T.

To illustrate the effectiveness of the obtained results, now, we work out an example:

Example 3.1. *Consider the following fractional semilinear system with instantaneous and non-instantaneous impulses:*

$$\begin{cases} {}^cD^{\frac{1}{2}}x(t,\iota) \\ = \frac{\partial^2}{\partial\iota^2}x(t,\iota) + z(t,\iota) + \frac{1}{10}\frac{e^{-t}x(t,\iota)}{e^{-t}+e^t} + \frac{1}{3}\int_0^t e^{-(s-t)}x(s,\iota)ds, \\ \qquad \iota \in (0,1),\ t \in (0,\frac{1}{4}] \cup (\frac{3}{4},1], \\ x(t,\iota) = \frac{e^{-(t-\frac{1}{4})}}{5}\frac{x(t,\iota)}{1+x(t,\iota)},\ t \in (\frac{1}{4},\frac{3}{4}],\ \iota \in (0,1), \\ x(t,0) = x(t,1) = 0, \\ x(0,\iota) = x_0(\iota),\ \iota \in [0,1], \\ \Delta x(\frac{1}{6})(\iota) = \int_0^{\frac{1}{6}} \cos\left(\frac{1}{6} - s\right)x(s,\iota)ds,\ \iota \in (0,1), \end{cases} \tag{3.23}$$

where $t \in T = [0,1]$.

Conclusion of example: Equation (3.23) is approximately controllable on T.

Proof. Equation (3.23) can be regarded as a system of the form (1.1), where

$$\begin{aligned} q &= \frac{1}{2},\ \ b = u_2 = 1,\ \ u_0 = v_0 = 0, \\ u_1 &= \frac{1}{4},\ \ v_1 = \frac{3}{4},\ \ u_0^1 = \frac{1}{6},\ j = 1, \\ f(t,x(t)) &= \frac{1}{10}\frac{e^{-t}x(t,\iota)}{e^{-t}+e^t} + \frac{1}{3}\int_0^t e^{-(s-t)}x(s,\iota)ds, \\ \gamma_1(t,x) &= \frac{e^{-(t-\frac{1}{4})}}{5}\frac{x(t,\iota)}{1+x(t,\iota)}, \\ I_0^1(x(\frac{1}{6}^-)) &= \int_0^{\frac{1}{6}} \cos\left(\frac{1}{6} - s\right)x(s,\iota)ds. \end{aligned}$$

Let

$$\mathbb{W} = L^2([0,1])$$

be equipped with the norm defined by

$$\|x\| = \left(\int_0^1 |x(t)|^2 dt\right)^{\frac{1}{2}},\ x \in \mathbb{W}.$$

Define $Ax = x''$, and

$$D(A) = \{x \in \mathbb{W} : x,\ x' \text{ are absolutely continuous and } x'' \in \mathbb{W},\ x(0) = x(1) = 0\}.$$

Thus,

$$Ax = \sum_{n=1}^{\infty} -n^2\langle x, e_n\rangle e_n,\ \ x \in D(A),$$

where

$$e_n(\iota) = \sqrt{\frac{2}{\pi}}\sin(n\iota),\ \ 0 \le \iota \le 1,\ n = 1,2,\cdots.$$

It is well-known that A generates a compact semigroup $\mathcal{T}(t)(t > 0)$ that is given by

$$\mathcal{T}(t)x = \sum_{n=1}^{\infty} e^{-n^2 t}\langle x, e_n\rangle e_n,\ x \in \mathbb{W}.$$

Obviously,

$$\|\mathcal{T}(t)\| \le 1,$$

for arbitrary $t \ge 0$. Put

$$x(t) = x(t,\iota),$$

that is

$$x(t)(\iota) = x(t,\iota),\ \ t \in [0,1],\ \iota \in [0,1].$$

B: $\mathbb{V} \to \mathbb{W}$, which is defined as

$$Bz(t) = z(t,\iota),$$

is a bounded linear operator.

Clearly,

$$\begin{aligned} \|f(t,x(t))\| &\le \frac{1}{10}\left\|\frac{e^{-t}x(t,\iota)}{e^{-t}+e^t}\right\| + \frac{1}{3}\left\|\int_0^t e^{-(s-t)}x(s,\iota)ds\right\| \\ &\le \frac{1}{10}\|x\| + \frac{e^t}{3}\|x\| \qquad (3.24) \\ &= \left(\frac{1}{10} + \frac{e^t}{3}\right)\|x\|, \end{aligned}$$

$$\|\gamma_1(t,x)\| \le \frac{1}{5}\|x\|, \tag{3.25}$$

$$\|I_0^1(x(\frac{1}{6}^-))\| \le \sin\frac{1}{6}\|x\|. \tag{3.26}$$

Combining (3.24)–(3.26), the assumptions (H1)–(H4) hold with

$$a(t) = \frac{1}{10} + \frac{e^t}{3},\ \mathcal{F} = \left(\frac{e^2}{18} + \frac{e}{15} - \frac{101}{900}\right)^{\frac{1}{2}},$$

$$\mathcal{G}_{\gamma_1} = \frac{1}{5},\ \mathcal{I}_0^1 = \sin\frac{1}{6}.$$

Furthermore, similar to [36], the linear system corresponding to (3.23) is approximately controllable on [0,1] which concludes that (H5) also holds. That is, by Theorems 3.1 and 3.2, (3.23) is approximately controllable on T.

4. Conclusions

This paper is mainly concerned with the existence of mild solutions and approximate controllability for a class of fractional semilinear systems with instantaneous and non-instantaneous impulses. The results for the considered system are obtained by applying the Kuratowski measure of noncompactness and the ρ-set contractive fixed-point theorem. The conclusions of this paper are important for fractional systems with instantaneous and non-instantaneous impulses. In the future, the controllability for fractional systems of order $1 < q < 2$ with instantaneous and non-instantaneous impulses can be considered on this basis.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest in this paper.

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