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Research article

Global attractivity of a rational difference equation with higher order and its applications

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Abstract: We study in this paper the global attractivity for a higher order rational difference equation. As application, our results not only include and generalize many known ones, but also formulate some new results for several conjectures presented by Camouzis and Ladas, et al.

Keywords: rational difference equation with higher order; global attractivity; globally asymptotically stable; Schwarzian derivative; conjecture

1. Introduction and preliminaries

Difference equations, which come not only from the discretization of differential equations, but also from the modelling of real problems, have many real applications in various disciplines, for example, cybernetics, biology, physics, engineering and other applied fields. Rational difference equations (RDEs) belongs to a kind of typical nonlinear difference equations, whose research history, relatively speaking, is very short. Because the research of many core problems for difference equations may be reduced to the prototype for the corresponding problems of RDEs, the investigations of RDEs have recently become popular, and have rapidly developed. See, for example, [1–17].

The research contents of RDEs are very extensive, and mainly include stability, oscillation, periodicity, bifurcation, chaos, etc. The form of a RDE may look very simple while it may display very complicated dynamical behaviors, such as dichotomy [18, 19], trichotomy [20], homoclinic bifurcation, etc. Recently, there has been a great interest in exploring the local and global stability, the boundedness, and the periodicity of RDEs (see [5, 6, 12, 15]). The investigation of global asymptotical stability of RDEs is more difficult than that of their local asymptotical stability.

In the present paper, we mainly investigate the global attractivity of the following RDE

$$x_{n+1} = \frac{p + qx_n}{A + Bx_n + Cx_{n-k} + Dx_{n-l}}, \quad n = 0, 1, \ldots, \tag{1.1}$$

where the parameters $p, q, A, B, C, D \in [0, \infty)$, k and l are positive integers with $k < l$, and the initial conditions $x_{-l}, \ldots, x_{-1}, x_0 \in (0, \infty)$. To avoid trivial cases, suppose that $p + q > 0$ and $B + C + D > 0$.

The motivation for us to investigate Eq (1.1) comes from the following known work:

Ladas et al. [21] first considered the following special case of Eq (1.1):

$$x_{n+1} = \frac{a + bx_n}{A + x_{n-1}}, \; n = 0, 1, \ldots, \tag{1.2}$$

where the parameters

$$a, b, A \in (0, \infty) \tag{1.3}$$

and the initial values x_{-1}, x_0 are arbitrary positive numbers. The unique positive equilibrium point $\bar{x}$ of Eq (1.2) has been proved to be locally asymptotically stable. For its global dynamics, some results are stated as follows:

Theorem 1.1. *[1, 2, 21] Suppose that (1.3) holds, and one of the following conditions is true:*
(1) $b < A$;
(2) $b \geq A$ and $a \leq Ab$;
(3) $b \geq A$ and $Ab < a < 2A(b + A)$;
(4) $b \geq \sqrt{1 + \sqrt{5}/2}A$, $Ab < a$ and $b^2/A < \bar{x} < 2b$.
Then, $\bar{x}$ is globally asymptotically stable.

Computer simulations show that the local asymptotical stability of the positive equilibrium point $\bar{x}$ of Eq (1.2) implies its global asymptotical stability, which is equivalent to its global attractivity as long as (1.3) holds. However, except for the partial results mentioned in the above Theorem 1.1, this point of view cannot be completely and theoretically proved. So, Ladas presented the following conjecture in [1, 2, 7, 21], respectively.

Conjecture 1.1. *[2, Conjecture 6.1.1] Assume that Eq (1.3) holds. Then, the positive equilibrium point $\bar{x}$ of Eq (1.2) is globally asymptotically stable.*

Those authors [2, 22–24] partly obtained some results for Conjecture 1.1.

From that time on, more and more researchers have paid attention to similar RDEs. Cunningham et al. in [5] investigated the following RDE:

$$x_{n+1} = \frac{\alpha + \beta x_n}{Bx_n + Cx_{n-1}}, \quad n = 0, 1, \ldots, \tag{1.4}$$

where the parameters α, β, B, C are nonnegative real numbers and the initial values x_{-1}, x_0 are arbitrary positive numbers.

The change of variable

$$x_n = \frac{\beta}{B} y_n$$

reduces Eq (1.4) to the following difference equation:

$$y_{n+1} = \frac{y_n + p}{y_n + qx_{n-1}}, \quad n = 0, 1, \ldots, \tag{1.5}$$

where

$$p = \frac{\alpha B}{\beta^2} \text{ and } q = \frac{C}{B}.$$

They obtained the following results:

Theorem 1.2. *[5, Theorem 3] The equilibrium $\bar{y}$ of Eq (1.5) is globally asymptotically stable when*

$$q \leq 4p + 1.$$

Muna and Mohammad [10] studied the following RDE in 2017:

$$x_{n+1} = \frac{\alpha + \beta x_n}{A + Bx_n + Cx_{n-k}}, \quad n = 0, 1, \ldots. \tag{1.6}$$

By analyzing the semicycle, they derived the following results:

Theorem 1.3. *[10, Theorem 3.4] The positive fixed point $\bar{x}$ of Eq (1.7) is globally asymptotically stable.*

In 2021, the first author of this paper and the authors in [15] considered the following difference equation:

$$x_{n+1} = \frac{p + qx_n}{1 + rx_{n-k}}, \quad n = 0, 1, \ldots, \tag{1.7}$$

where $p, q \in [0, \infty), r > 0$, $k \geq 1$ is an integer and initial conditions $x_{-k}, \ldots, x_{-1}, x_0 \in (0, \infty)$.

Furthermore, the authors formulated the conclusion as follows:

Theorem 1.4. *[15, Theorem XY] Assume that $p, q \in [0, \infty)$, $r > 0$ and $k \geq 1$ is a positive integer. Then the unique positive equilibrium $\bar{x}$ of Eq (1.7) is a global attractor of all of its positive solutions.*

There are some other RDEs related to Eq (1.1) that we will not cite one by one here, see [1, 2, 7, 21] and the references cited therein.

All of these problems mentioned above motivate us to investigate in this paper the global attractivity of Eq (1.1), which has more generalized form.

2. Several key lemmas

We present several key auxiliary lemmas that are used to prove our main results in this paper.

Lemma 2.1. *[2, Theorem 2.3.1] Consider the difference equation*

$$x_{n+1} = x_n f(x_n, x_{n-k_1}, \ldots, x_{n-k_r}), \tag{2.1}$$

where $k_1, k_2, \ldots, k_r$ are positive integers. Denote by k the maximum of $k_1, k_2, \ldots, k_r$. Also, assume that the function f satisfies the following hypotheses:

(H1) $f \in C[(0, \infty) \times [0, \infty)^r, (0, \infty)]$ and $g \in C[[0, \infty)^{r+1}, (0, \infty)]$, where

$$g(u_0, u_1, \ldots, u_r) = u_0 f(u_0, u_1, \ldots, u_r)$$

for $u_0 \in (0,\infty)$ *and* $u_1,\ldots,u_r \in [0,\infty)$,

$$g(0,u_1,\ldots,u_r) = \lim_{u_0\to 0^+} g(u_0,u_1,\ldots,u_r);$$

(H2) $f(u_0,u_1,\ldots,u_r)$ *is nonincreasing in* $u_1,\ldots,u_r$*;*

(H3) The equation

$$Ff(x,x,\ldots,x) = 1$$

has a unique positive solution $\overline{x}$*;*

(H4) Either the function $f(u_0,u_1,\ldots,u_r)$ *does not depend on* u_0 *or for every* $x > 0$ *and* $u \geq 0$*,*

$$[f(x,u,\ldots,u) - f(\overline{x},u,\ldots,u)](x-\overline{x}) \leq 0$$

with

$$[f(x,\overline{x},\ldots,\overline{x}) - f(\overline{x},\overline{x},\ldots,\overline{x})](x-\overline{x}) < 0$$

for $x \neq \overline{x}$.

Define a new function F *given by*

$$F(x) = \begin{cases} max_{x\leq y\leq \overline{x}} G(x,y), & for\ \ 0 \leq x \leq \overline{x}, \\ min_{\overline{x}\leq y\leq x} G(x,y), & for\ \ x > \overline{x}, \end{cases} \tag{2.2}$$

where

$$G(x,y) = yf(y,x,\ldots,x)f(\overline{x},\overline{x},\ldots,\overline{x},y)[f(\overline{x},x,\ldots,x)]^{k-1}. \tag{2.3}$$

Then,

(a) $F \in C[(0,\infty),(0,\infty)]$ *and* F *is nonincreasing in* $[0,\infty)$*;*

(b) Assume that the function F *has no periodic points of prime period* 2*. Then,* $\overline{x}$ *is a global attractor of all positive solutions of Eq (2.1).*

Lemma 2.2. *[2, Lemma 1.6.3 (a),(d)] Let*

$$F \in [[0,\infty),(0,\infty)]$$

be a nonincreasing function and let $\overline{x}$ *denote the unique fixed point of* F*, then the following statements are equivalent:*

(a) $\overline{x}$ *is the only fixed point of* F^2 *in* $(0,\infty)$*;*

(b) $\overline{x}$ *is a global attractor of all positive solutions of the difference equation*

$$x_{n+1} = F(x_n), \ \ n = 0,1,\ldots \tag{2.4}$$

with $x_0 \in [0,\infty)$.

Lemma 2.3. *[2, 3] Consider the difference Eq (2.4), where* F *is a decreasing function which maps some interval* I *into itself. Assume that* F *has a negative Schwarzian derivative*

$$\begin{aligned} SF(x) &= \frac{F'''(x)}{F'(x)} - \frac{3}{2}\left(\frac{F''(x)}{F'(x)}\right)^2 \\ &= \left[\frac{F''(x)}{F'(x)}\right]' - \frac{1}{2}\left(\frac{F''(x)}{F'(x)}\right)^2 \\ &< 0 \end{aligned}$$

everywhere on I*, except for point* x*, where*

$$F'(x) = 0.$$

Then, the positive equilibrium $\overline{x}$ *of Eq (2.4) is a global attractor of all positive solutions of Eq (2.4).*

3. Main results

In this section, our main result in this paper and its proof will be given. The main idea for the proof is to invoke three key lemmas in Section 2 to transfer the higher order RDE (1.1) into a first order difference equation more easily dealt with.

Equation (1.1) has a unique nonnegative equilibrium point, denoted as $\bar{x}$, namely,

$$\bar{x} = \frac{q - A + \sqrt{(q-A)^2 + 4p(B+C+D)}}{2(B+C+D)}.$$

Our main result in this paper is the following.

Theorem 3.1. *Consider Eq (1.1). Assume that the parameters* $p, q, A, B, C, D \in [0,\infty)$ *with*

$$p + q > 0 \ \ and \ \ B + C + D > 0,$$

and the paramenters k *and* l *are positive integers with* $k < l$*. Then, the unique nonnegative equilibrium* $\bar{x}$ *of Eq (1.1) is a global attractor of all of its positive solutions when*

$$p = 0 \ \ and \ \ 0 < q \leq A$$

or

$$q(A + B\bar{x} + C\bar{x}) \leq pD.$$

Proof. When $p = 0$ and $q \in (0, A]$, $\bar{x} = 0$. It follows from Eq (1.1) that

$$x_{n+1} < \frac{q}{A} x_n,$$

so x_n eventually monotonically tends to $\bar{x}$. Hence, in the sequel, we only consider the case

$$q(A + B\bar{x} + C\bar{x}) \le pD.$$

For convenience of writing, we introduce the following notations:

$$u = B + C + D, \quad v = pD - qr, \quad r = A + B\bar{x} + C\bar{x},$$
$$s = A + B\bar{x}, \quad t = C + D, \quad e = (A + B\bar{x} + C\bar{x} + D\bar{x})^l.$$

Obviously, $u \in (0, \infty)$, $s, t, e \in [0, \infty)$,

$$v = pD - q(A + B\bar{x} + C\bar{x}) \ge 0.$$

Moreover, it is easy to see that the following inequalities hold:

$$\begin{aligned} &i)\ Ds - tr \le 0; \\ &ii)\ DA - ur \le 0; \\ &iii)\ qs - pt \le 0; \\ &iv)\ qA - pu \le 0. \end{aligned} \tag{3.1}$$

These inequalities can be verified by the following observations:

$$\begin{aligned} i)\ Ds - tr &= Ds - (C + D)(s + C\bar{x}) \\ &= -Cr - CD\bar{x} \le 0; \\ ii)\ DA - ur &= DA - (B + C + D)(A + B\bar{x} + C\bar{x}) \\ &= -(B + C)(r + D\bar{x}) \le 0; \\ iii)\ qs - pt &= q(r - C\bar{x}) - p(C + D) \\ &= -v - C(p + q\bar{x}) \le 0; \\ iv)\ qA - pu &= q(r - B\bar{x} - C\bar{x}) - p(B + C + D) \\ &= -v - (B + C)(p + q\bar{x}) \le 0. \end{aligned}$$

Now, we continue to consider the global attractivity of Eq (1.1). First of all, Eq (1.1) can be written as

$$x_{n+1} = x_n \frac{\frac{p}{x_n} + q}{A + Bx_n + Cx_{n-k} + Dx_{n-l}}. \tag{3.2}$$

Set

$$f(u_0, u_1, \ldots, u_k, \ldots, u_l) = \frac{\frac{p}{u_0} + q}{A + Bu_0 + Cu_k + Du_l}.$$

It is easy to verify that the function f satisfies the conditions (H1)–(H4) of Lemma 2.1. So, the function G defined by Eq (2.3) may be derived as

$$\begin{aligned} G(x, y) &= y \frac{\frac{p}{y} + q}{A + By + tx} \frac{\frac{p}{\bar{x}} + q}{r + Dy} \left(\frac{\frac{p}{\bar{x}} + q}{s + tx}\right)^{l-1} \\ &= \frac{(p + qy)e}{(A + By + tx)(r + Dy)(s + tx)^{l-1}}. \end{aligned}$$

Moreover,

$$\frac{\partial G(x, y)}{\partial y} = \frac{e}{(s + tx)^{l-1}} \frac{(qr - pD)(A + tx) - \delta}{(A + By + tx)^2 (r + Dy)^2}, \tag{3.3}$$

where

$$\delta = prB + 2pBDy + qBDy^2 > 0.$$

The known assumption

$$q(A + B\bar{x} + C\bar{x}) \le pD$$

implies $qr \le pD$. So, from (3.3) we have

$$\frac{\partial G(x, y)}{\partial y} \le 0.$$

Accordingly, the function F defined by (2.2),

$$\begin{aligned} F(x) &= G(x, x) \\ &= \frac{e(p + qx)}{(s + tx)^{l-1}(A + ux)(r + Dx)}. \end{aligned} \tag{3.4}$$

In order to apply Lemma 2.1 (b), according to Lemma 2.2, we must prove that $\bar{x}$ is a global attractor of Eq (2.4). Accordingly, in view of Lemma 2.3, one has to verify that F has a negative Schwarzian derivative.

To do this, notice

$$F'(x) = \frac{e[q - (p + qx)(\frac{u}{A+ux} + \frac{D}{r+Dx} + \frac{(l-1)t}{s+tx})]}{(A + ux)(r + Dx)(s + tx)^{l-1}},$$

where

$$\begin{aligned} q - (p + qx)\left(\frac{D}{r + Dx}\right) &= \frac{qr - Dp}{r + Dx} \\ &= -\frac{v}{r + Dx} \\ &\le 0, \end{aligned}$$

hence, $F'(x) \leq 0$.

Take

$$I = [0, \frac{ep}{Ars^{l-1}}].$$

For any given $x \in I$, one has

$$\begin{aligned} 0 < F(x) \leq F(0) = & \lim_{x \to 0^+} F(x) \\ = & \frac{ep}{Ars^{l-1}}. \end{aligned}$$

So, $F(I) \subset I$. Moreover,

$$\begin{aligned} F''(x) =& \frac{e}{(A+ux)(r+Dx)(s+tx)^{l-1}}[\frac{2u^2(p+qx)}{(A+ux)^2} \\ &+ \frac{2(l-1)tu(p+qx)}{(A+ux)(s+tx)} + \frac{l(l-1)t^2(p+qx)}{(s+tx)^2} \\ &+ \frac{2Du(p+qx)}{(A+ux)(r+Dx)} - \frac{2qu}{A+ux} + \frac{2(l-1)tD(p+qx)}{(s+tx)(r+Dx)} \\ &- \frac{2q(l-1)t}{s+tx} + \frac{2D^2(p+qx)}{(r+Dx)^2} - \frac{2Dq}{r+Dx}] \\ =& \frac{e}{(A+ux)(r+Dx)(s+tx)^{l-1}}[(p+qx)(\frac{2u^2}{(A+ux)^2} \\ &+ \frac{l(l-1)t^2}{(s+tx)^2} + \frac{2(l-1)tu}{(A+ux)(s+tx)}) \\ &+ \frac{2v}{r+Dx}(\frac{u}{A+ux} + \frac{(l-1)t}{s+tx} + \frac{D}{r+Dx})] \\ >& 0. \end{aligned}$$

For convenience of expression, let

$$E =: (p+qx)M + \frac{2v}{r+Dx}N$$

and

$$F =: q - (p+qx)N,$$

where

$$\begin{aligned} M =: & \frac{2u^2}{(A+ux)^2} + \frac{l(l-1)t^2}{(s+tx)^2} + \frac{2(l-1)tu}{(A+ux)(s+tx)} \\ & > 0 \end{aligned}$$

and

$$\begin{aligned} N =: & \frac{u}{A+ux} + \frac{(l-1)t}{s+tx} + \frac{D}{r+Dx} \\ & > 0. \end{aligned}$$

Then,

$$\begin{aligned} M' =& \frac{-4u^3}{(A+ux)^3} + \frac{-2(l-1)tu^2}{(A+ux)^2(s+tx)} + \frac{-2(l-1)t^2u}{(A+ux)(s+tx)^2} \\ &+ \frac{-2l(l-1)t^3}{(s+tx)^3} \\ <& 0, \\ N' =& \frac{-u^2}{(A+ux)^2} + \frac{-(l-1)t^2}{(s+tx)^2} + \frac{-D^2}{(r+Dx)^2} \\ <& 0, \\ SF(x) =& [\frac{F''(x)}{F'(x)}]' - \frac{1}{2}(\frac{F''(x)}{F'(x)})^2 \\ =& (\frac{E}{F})' - \frac{1}{2}(\frac{E}{F})^2 \\ =& \frac{2E'F - E(2F'+E)}{2F^2}. \end{aligned}$$

By Lemma 2.3, it suffices for us to prove

$$2E'F - E(2F'+E) \leq 0.$$

Hence, the remaining work is to determine the sign of the function $2E'F - E(2F'+E)$. Now, calculate it.

$$\begin{aligned} & 2E'F - E(2F'+E) \\ =& 2[qM + (p+qx)M' - \frac{2DvN}{(r+Dx)^2} + \frac{2vN'}{r+Dx}][q - (p+qx)N] \\ & - [(p+qx)M + \frac{2vN}{r+Dx}][(p+qx)M + \frac{2vN}{r+Dx} \\ & - 2qN - 2(p+qx)N'] \\ =& 2q^2M - 2q(p+qx)MN + 2q(p+qx)M' - 2(p+qx)^2M'N \\ & - 2q\frac{2Dv}{(r+Dx)^2}N + \frac{4Dv(p+qx)}{(r+Dx)^2}N^2 + \frac{4vq}{r+Dx}N' \\ & + 2q(p+qx)MN + 2(p+qx)^2MN' - (p+qx)^2M^2 \\ & - \frac{4v(p+qx)}{r+Dx}MN + \frac{4vq}{r+Dx}N^2 - \frac{4v^2}{(r+Dx)^2}N^2 \\ =& (p+qx)^2[2(MN' - M'N) - M^2] \\ & + \frac{2v(p+qx)N}{r+Dx}(\frac{2DN}{r+Dx} - M) + \frac{2vN}{r+Dx}[-\frac{2Dq}{r+Dx} \\ & + 2qN - (p+qx)M] + \frac{4vqN'}{r+Dx} + q[2qM \\ & + (p+qx)M'] + q(p+qx)M' - \frac{4v^2N^2}{(r+Dx)^2}. \end{aligned} \tag{3.5}$$

Next, we will simplify

$$2(MN'-M'N)-M^2, \ \frac{2DN}{r+Dx}-M, \ -\frac{2Dq}{r+Dx}+2qN-(p+qx)M$$

and

$$2qM + (p+qx)M'.$$

Using the inequalities in (3.1) and noticing $l \geq 2$ is a positive integer, we determine their signs.

First, we simplify $2(MN'-M'N)-M^2$. Obviously,

$$
\begin{aligned}
&2(MN'-M'N)-M^2\\
=&-2M'N+2MN'-M^2\\
=&2\Big[\frac{4u^3}{(A+ux)^3}+\frac{2(l-1)tu^2}{(A+ux)^2(s+tx)}+\frac{2(l-1)t^2u}{(A+ux)(s+tx)^2}\\
&+\frac{2l(l-1)t^3}{(s+tx)^3}\Big[\frac{u}{A+ux}+\frac{(l-1)t}{s+tx}+\frac{D}{r+Dx}\Big]\\
&-2\Big[\frac{2u^2}{(A+ux)^2}+\frac{l(l-1)t^2}{(s+tx)^2}+\frac{2(l-1)tu}{(A+ux)(s+tx)}\Big]\\
&\times\Big[\frac{u^2}{(A+ux)^2}+\frac{(l-1)t^2}{(s+tx)^2}+\frac{D^2}{(r+Dx)^2}\Big]\\
&-\Big[\frac{2u^2}{(A+ux)^2}+\frac{l(l-1)t^2}{(s+tx)^2}+\frac{2(l-1)tu}{(A+ux)(s+tx)}\Big]^2\\
=&2\frac{4u^3}{(A+ux)^3}\frac{u}{A+ux}-2\frac{2u^2}{(A+ux)^2}\frac{u^2}{(A+ux)^2}-\frac{4u^4}{(A+ux)^4}\\
&+2\frac{4u^3}{(A+ux)^3}\frac{(l-1)t}{s+tx}-2\frac{2u^2}{(A+ux)^2}\frac{2(l-1)tu}{(s+tx)(A+ux)}\\
&+\frac{8u^3D}{(A+ux)^3(r+Dx)}+2\frac{2(l-1)^2t^2u^2+2(l-1)t^3u^2}{(A+ux)^2(s+tx)^2}\\
&-2\frac{2(l-1)t^2u^2+l(l-1)t^2u^2}{(A+ux)^2(s+tx)^2}-\frac{4(l-1)^2t^2u^2}{(A+ux)^2(s+tx)^2}\\
&+2\frac{2(l-1)t^2u}{(A+ux)(s+tx)^2}\frac{(l-1)t}{s+tx}-4\frac{(l-1)tu}{(A+ux)(s+tx)}\frac{(l-1)t^2}{(s+tx)^2}\\
&+2\frac{2l(l-1)t^3}{(s+tx)^3}\frac{(l-1)t}{s+tx}-2\frac{l(l-1)t^2}{(s+tx)^2}\frac{(l-1)t^2}{(s+tx)^2}\\
&-\frac{l^2(l-1)^2t^4}{(s+tx)^4}+2\frac{2(l-1)tu^2}{(A+ux)^2(s+tx)}\frac{u}{A+ux}\\
&-2\frac{2(l-1)tu}{(A+ux)(s+tx)}\frac{u^2}{(A+ux)^2}+2\frac{2l(l-1)t^3}{(s+tx)^3}\frac{D}{r+Dx}\\
&+2\frac{2l(l-1)t^3}{(s+tx)^3}\frac{u}{A+ux}-2\frac{l(l-1)t^2}{(s+tx)^2}\frac{2(l-1)tu}{(A+ux)(s+tx)}\\
&+2\frac{2(l-1)tu^2}{(A+ux)^2(s+tx)}\frac{D}{r+Dx}+2\frac{2(l-1)t^2u}{(A+ux)(s+tx)^2}\frac{D}{r+Dx}\\
&-2\frac{2u^2}{(A+ux)^2}\frac{l(l-1)t^2}{(s+tx)^2}-\frac{2D^2}{(r+Dx)^2}\Big[\frac{2u^2}{(A+ux)^2}\\
&+\frac{l(l-1)t^2}{(s+tx)^2}+\frac{2(l-1)tu}{(A+ux)(s+tx)}\Big]\\
=&\frac{8u^4}{(A+ux)^4}-\frac{4u^4}{(A+ux)^4}-\frac{4u^4}{(A+ux)^4}\\
&+\frac{8(l-1)tu^3}{(A+ux)^3(s+tx)}-\frac{8(l-1)tu^3}{(A+ux)^3(s+tx)}\\
&+\frac{4(l-1)tu^3}{(A+ux)^3(s+tx)}-\frac{4(l-1)tu^3}{(A+ux)^3(s+tx)}\\
&+\frac{4(l-1)^2t^2u^2+2(l-1)t^2u^2}{(A+ux)^2(s+tx)^2}-\frac{4(l-1)t^2u^2+2l(l-1)t^2u}{(A+ux)^2(s+tx)^2}\\
&-\frac{4(l-1)^2t^2u^2}{(A+ux)^2(s+tx)^2}-\frac{4l(l-1)t^2u^2}{(A+ux)^2(s+tx)^2}+\frac{4(l-1)^2t^3u}{(A+ux)(s+tx)^3}\\
&-\frac{4(l-1)^2t^3u}{(A+ux)(s+tx)^3}+\frac{4l(l-1)^2t^3}{(s+tx)^4}-\frac{2l(l-1)^2t^3}{(s+tx)^4}-\frac{l^2(l-1)^2t^2}{(s+tx)^4}\\
&+\frac{4l(l-1)t^3u}{(s+tx)^3(A+ux)}-\frac{4l(l-1)^2t^3u}{(s+tx)^3(A+ux)}\\
&+\frac{4(l-1)t^2uD}{(s+tx)^2(A+ux)(r+Dx)}+\frac{4(l-1)tu^2D}{(s+tx)(A+ux)^2(r+Dx)}\\
&+\frac{8u^3D}{(A+ux)^3(r+Dx)}+\frac{4l(l-1)t^3D}{(s+tx)^3(r+Dx)}-\frac{2D^2}{(r+Dx)^2}M\\
=&\frac{-6l(l-1)t^2u^2}{(A+ux)^2(s+tx)^2}+\frac{4(l-1)t^2uD}{(A+ux)(s+tx)^2(r+Dx)}\\
&+\frac{4(l-1)tu^2D}{(A+ux)^2(s+tx)(r+Dx)}+\frac{l(2-l)(l-1)^2t^4}{(s+tx)^4}-\frac{2D^2}{(r+Dx)^2}M\\
&+\frac{4l(2-l)(l-1)^3t^3u}{(s+tx)^3(A+ux)}+\frac{8u^3D}{(A+ux)^3(r+Dx)}+\frac{4l(l-1)t^3D}{(s+tx)^3(r+Dx)}\\
=:&\frac{\alpha x+\beta}{(A+ux)^2(s+tx)^2(r+Dx)}+\frac{l(2-l)(l-1)^2t^4}{(s+tx)^4}-\frac{2D^2}{(r+Dx)^2}M\\
&+\frac{4l(2-l)(l-1)^3t^3u}{(s+tx)^3(A+ux)}+\frac{8u^3D}{(A+ux)^3(r+Dx)}+\frac{4l(l-1)t^3D}{(s+tx)^3(r+Dx)},
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha =:&4(l-1)tu^2Ds-4l(l-1)t^2u^2r+4(l-1)t^2uDA-2l(l-1)t^2u^2r\\
=&4(l-1)tu^2(Ds-ltr)+2(l-1)t^2u(2DA-lur)\\
=&4(l-1)tu^2(Ds-(1+l-1)tr)\\
&+2(l-1)t^2u[l(DA-ur)+(2-l)DA]\\
=&4(l-1)tu^2(Ds-tr)-4(l-1)^2t^2u^2r\\
&+2l(l-1)t^2u(DA-ur)+2(2-l)(l-1)t^2uDA\\
\le&0,\\
\beta =:&8(l-1)t^2u^2D-6l(l-1)t^2u^2D\\
=&2(l-1)(4-3l)t^2u^2D\\
\le&0.
\end{aligned}
$$

Obviously, except the last two positive terms

$$\frac{8u^3D}{(A+ux)^3(r+Dx)} \quad \text{and} \quad \frac{4l(l-1)l^3D}{(s+tx)^3(r+Dx)},$$

every term of the final expression in $2(MN'-M'N)-M^2$ is non-positive.

Second, simplify $\frac{2DN}{r+Dx}-M$.

$$
\begin{aligned}
&\frac{2DN}{r+Dx}-M\\
=&\frac{2D}{r+Dx}\Big[\frac{u}{A+ux}+\frac{(l-1)t}{s+tx}+\frac{D}{r+Dx}\Big]\\
&-\frac{2u^2}{(A+ux)^2}-\frac{l(l-1)t^2}{(s+tx)^2}-\frac{2(l-1)tu}{(A+ux)(s+tx)}\\
=&\frac{2uD}{(r+Dx)(A+ux)}-\frac{2(l-1)tu}{(A+ux)(s+tx)}+\frac{2(l-1)tD}{(r+Dx)(s+tx)}\\
&-\frac{l(l-1)t^2}{(s+tx)^2}+\frac{2D^2}{(r+Dx)^2}-\frac{2u^2}{(A+ux)^2}\\
=&\frac{2u}{A+ux}\Big[\frac{D}{r+Dx}-\frac{(l-1)t}{s+tx}\Big]+\frac{(l-1)t}{s+tx}\Big[\frac{2D}{r+Dx}-\frac{lt}{s+tx}\Big]\\
&+2\Big[\frac{D}{r+Dx}+\frac{u}{A+ux}\Big]\Big[\frac{D}{r+Dx}-\frac{u}{A+ux}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{2u}{A+ux}\Big[\frac{D}{r+Dx}-\frac{(1+l-2)t}{s+tx}\Big]+\frac{(l-1)t}{s+tx}\Big[\frac{2D}{r+Dx}\\
&\quad-\frac{(2+l-2)t}{s+tx}\Big]+2\Big[\frac{D}{A+ux}+\frac{u}{A+ux}\Big]\Big[\frac{D}{r+Dx}-\frac{u}{A+ux}\Big]\\
&=\frac{2u}{A+ux}\Big[\frac{D}{r+Dx}-\frac{t}{s+tx}\Big]-\frac{2(l-2)tu}{(A+ux)(s+tx)}\\
&\quad+\frac{(l-1)t}{s+tx}\Big[\frac{2D}{r+Dx}-\frac{2t}{s+tx}\Big]-\frac{(l-1)(l-2)t^2}{(s+tx)^2}\\
&\quad+2\Big[\frac{D}{r+Dx}+\frac{u}{A+ux}\Big]\Big[\frac{D}{r+Dx}-\frac{u}{A+ux}\Big]\\
&=\frac{2(Ds-tr)}{(r+Dx)(s+tx)}\Big[\frac{u}{A+ux}+\frac{(l-1)t}{s+tx}\Big]-\frac{2(l-2)tu}{(A+ux)(s+tx)}\\
&\quad-\frac{(l-1)(l-2)t^2}{(s+tx)^2}+\frac{2(DA-ur)}{(r+Dx)(A+ux)}\Big(\frac{D}{r+Dx}+\frac{u}{A+ux}\Big)\\
&\le 0,
\end{aligned}
$$

because $Ds-tr\le 0,\ DA-ur\le 0$, and $l\ge 2$.

Third,

$$
\begin{aligned}
&-\frac{2Dq}{r+Dx}+2qN-(p+qx)M\\
&=-\frac{2Dq}{r+Dx}+2q\Big[\frac{u}{A+ux}+\frac{(l-1)t}{s+tx}+\frac{D}{r+Dx}\Big]\\
&\quad-(p+qx)\Big[\frac{2u^2}{(A+ux)^2}+\frac{l(l-1)t^2}{(s+tx)^2}+\frac{2(l-1)tu}{(A+ux)(s+tx)}\Big]\\
&=\frac{2qu}{A+ux}-\frac{2u^2(p+qx)}{(A+ux)^2}+\frac{2(l-1)qt}{s+tx}\\
&\quad-\frac{l(l-1)t^2(p+qx)}{(s+tx)^2}-\frac{2(l-1)tu(p+qx)}{(A+ux)(s+tx)}\\
&=\frac{2u}{A+ux}\Big[q-\frac{u(p+qx)}{A+ux}\Big]+\frac{(l-1)t}{s+tx}\Big[2q-\frac{lt(p+qx)}{s+tx}\Big]\\
&\quad-\frac{2(l-1)tu(p+qx)}{(A+ux)(s+tx)}\\
&=\frac{2u(qA-pu)}{(A+ux)^2}+\frac{(l-1)t}{s+tx}\Big[2q-\frac{(2+l-2)t(p+qx)}{s+tx}\Big]\\
&\quad-\frac{2(l-1)tu(p+qx)}{(A+ux)(s+tx)}\\
&=\frac{2u(qA-pu)}{(A+ux)^2}+2\Big[\frac{(l-1)t}{s+tx}\Big]\Big[q-\frac{t(p+qx)}{s+tx}\Big]\\
&\quad-\Big[\frac{(l-1)t}{s+tx}\Big]\Big[\frac{(l-2)t(p+qx)}{s+tx}\Big]-\frac{2(l-1)tu(p+qx)}{(A+ux)(s+tx)}\\
&=\frac{2u(qA-pu)}{(A+ux)^2}+\frac{2(l-1)t(qs-pt)}{(s+tx)^2}\\
&\quad-\frac{(l-1)(l-2)t^2(p+qx)}{(s+tx)^2}-\frac{2(l-1)tu(p+qx)}{(A+ux)(s+tx)}\\
&\le 0,
\end{aligned}
$$

because $qA-pu\le 0,\ qs-pt\le 0,\ l\ge 2$.

Fourth,

$$
\begin{aligned}
&2qM+(p+qx)M'\\
&=2q\Big[\frac{2u^2}{(A+ux)^2}+\frac{l(l-1)t^2}{(s+tx)^2}+\frac{2(l-1)tu}{(A+ux)(s+tx)}\Big]-(p+qx)\\
&\quad\times\Big[\frac{4u^3}{(A+ux)^3}+\frac{2(l-1)tu^2}{(A+ux)^2(s+tx)}+\frac{2(l-1)t^2u}{(A+ux)(s+tx)^2}+\frac{2l(l-1)t^3}{(s+tx)^3}\Big]\\
&=2q\frac{2u^2}{(A+ux)^2}-\frac{4u^3(p+qx)}{(A+ux)^3}+2q\frac{2(l-1)tu}{(A+ux)(s+tx)}-(p+qx)\\
&\quad\times\Big[\frac{2(l-1)tu^2}{(A+ux)^2(s+tx)}+\frac{2(l-1)t^2u}{(A+ux)(s+tx)^2}\Big]+2q\frac{l(l-1)t^2}{(s+tx)^2}\\
&\quad-(p+qx)\frac{2l(l-1)t^3}{(s+tx)^3}\\
&=\frac{4u^2}{(A+ux)^2}\Big[q-\frac{u(p+qx)}{A+ux}\Big]+\frac{2(l-1)tu}{(A+ux)(s+tx)}\Big[2q\\
&\quad-\frac{u(p+qx)}{A+ux}\frac{t(p+qx)}{s+tx}\Big]+\frac{2l(l-1)t^2)}{(s+tx)^2}\Big[q-\frac{t(p+qx)}{s+tx}\Big]\\
&=\frac{4u^2(qA-pu)}{(A+ux)^3}+\frac{2(l-1)tu}{(A+ux)(s+tx)}\Big[q-\frac{u(p+qx)}{A+ux}\\
&\quad+q-\frac{t(p+qx)}{s+tx}\Big]+\frac{2l(l-1)t^2(qs-pt)}{(s+tx)^3}\\
&=\frac{4u^2(qA-pu)}{(A+ux)^3}+\Big[\frac{2(l-1)tu}{(A+ux)(s+tx)}\Big]\Big(\frac{qA-pu}{A+ux}+\frac{qs-pt}{s+tx}\Big)\\
&\quad+\frac{2l(l-1)t^2(qs-pt)}{(s+tx)^3}\\
&=\frac{2(l-1)t(qs-pt)}{(s+tx)^2}\Big[\frac{u}{A+ux}+\frac{lt}{s+tx}\Big]\\
&\quad+\frac{2u(qA-pu)}{(A+ux)^2}\Big[\frac{2u}{A+ux}+\frac{(l-1)t}{s+tx}\Big]\\
&\le 0,
\end{aligned}
$$

because $qA-pu\le 0,\ qs-pt\le 0$, and $l\ge 2$.

Now, we deal with the two positive terms

$$
\frac{8u^3D}{(A+ux)^3(r+Dx)}\quad\text{and}\quad\frac{4l(l-1)l^3D}{(s+tx)^3(r+Dx)}.
$$

By combining with other negative items in (3.5), we have

$$
\begin{aligned}
&\frac{8u^3D(p+qx)^2}{(A+ux)^3(r+Dx)}+\frac{4l(l-1)t^3D(p+qx)^2}{(s+tx)^3(r+Dx)}-\frac{4u^2D^2}{(A+ux)^2}\\
&\times\frac{(p+qx)^2}{(r+Dx)^2}+\frac{2l(l-1)t^2D^2(p+qx)^2}{(s+tx)^2(r+Dx)^2}+\frac{2v}{r+Dx}\frac{u}{A+ux}\\
&\times\frac{2u(qA-pu)}{(A+ux)^2}+\frac{4uv(DA-ur)(p+qx)}{(A+ux)^2(r+Dx)^2}\Big(\frac{D}{r+Dx}+\frac{u}{A+ux}\Big)\\
&+\frac{2v(p+qx)}{r+Dx}\frac{(l-1)^2t^2}{(s+tx)^2}\frac{2(Ds-tr)}{(r+Dx)(s+tx)}+\frac{4u^2q(qA-pu)}{(A+ux)^3}\\
&-\frac{2(l-2)(l-1)^2t^3v(p+qx)}{(s+tx)^3(r+Dx)}-\frac{4(l-1)^2t^2uv(p+qx)}{(A+ux)(s+tx)^2(r+Dx)}\\
&+\frac{4(l-1)^2(qs-pt)t^2v}{(s+tx)^3(r+Dx)}+\frac{2l(l-1)t^2q(qs-pt)}{(s+tx)^3}-\frac{4vq}{r+Dx}\\
&\times\Big[\frac{u^2}{(A+ux)^2}+\frac{(l-1)t^2}{(s+tx)^2}\Big]-\frac{8uv^2D}{(A+ux)(r+Dx)^3}\\
&-q(p+qx)\Big[\frac{4u^3}{(A+ux)^3}+\frac{2l(l-1)t^3}{(s+tx)^3}\Big]\\
&=\frac{4u^3D(p+qx)^2}{(A+ux)^3(r+Dx)}+\frac{4uvD(DA-ur)(p+qx)}{(A+ux)^2(r+Dx)^3}\\
&+\frac{4u^2v(DA-ur)(p+qx)}{(A+ux)^3(r+Dx)^2}-\frac{8uv^2D}{(A+ux)(r+Dx)^3}\\
&-\frac{4u^3q(p+qx)}{(A+ux)^3}-\frac{4u^2vq}{(A+ux)^2(r+Dx)}+\frac{4u^3D(p+qx)^2}{(A+ux)^3(r+Dx)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{4u^2v(qA-pu)}{(A+ux)^3(r+Dx)}+\frac{4u^2q(qA-pu)}{(A+ux)^3}-\frac{4u^2D^2(p+qx)^2}{(A+ux)^2(r+Dx)^2}\\
&+\frac{2l(l-1)t^3D(p+qx)^2}{(s+tx)^3(r+Dx)}+\frac{4(l-1)^2t^2v(qs-pt)}{(s+tx)^3(r+Dx)}\\
&-\frac{4(l-1)vt^2q}{(s+tx)^2(r+Dx)}-\frac{2l(l-1)t^3q(p+qx)}{(s+tx)^3}\\
&-\frac{2(l-1)^2(l-2)vt^3(p+qx)}{(s+tx)^3(r+Dx)}+\frac{2l(l-1)t^3D(p+qx)^2}{(s+tx)^3(r+Dx)}\\
&+\frac{4(l-1)^2vt^2(Ds-tr)(p+qx)}{(s+tx)^3(r+Dx)^2}-\frac{4(l-1)^2uvt^2(p+qx)}{(A+ux)(s+tx)^2(r+Dx)}\\
&+\frac{2l(l-1)t^2q(qs-pt)}{(s+tx)^3}-\frac{2l(l-1)t^2D^2(p+qx)^2}{(s+tx)^2(r+Dx)^2}\\
=:&H+I+J+L,
\end{aligned}
$$

$$
\begin{aligned}
H=&\frac{4u^3D(p+qx)^2}{(A+ux)^3(r+Dx)}+\frac{4uvD(DA-ur)(p+qx)}{(A+ux)^2(r+Dx)^3}\\
&+\frac{4u^2v(DA-ur)(p+qx)}{(A+ux)^3(r+Dx)^2}-\frac{8uv^2D}{(A+ux)(r+Dx)^3}\\
&-\frac{4u^3q(p+qx)}{(A+ux)^3}-\frac{4u^2vq}{(A+ux)^2(r+Dx)},\\
I=&\frac{4u^3D(p+qx)^2}{(A+ux)^3(r+Dx)}+\frac{4u^2v(qA-pu)}{(A+ux)^3(r+Dx)}\\
&+\frac{4u^2q(qA-pu)}{(A+ux)^3}-\frac{4u^2D^2(p+qx)^2}{(A+ux)^2(r+Dx)^2},\\
J=&\frac{2l(l-1)t^3D(p+qx)^2}{(s+tx)^3(r+Dx)}+\frac{4(l-1)^2vt^2(qs-pt)}{(s+tx)^3(r+Dx)}\\
&-\frac{4(l-1)vt^2q}{(s+tx)^2(r+Dx)}-\frac{2l(l-1)t^3q(p+qx)}{(s+tx)^3}\\
&-\frac{2(l-1)^2(l-2)vt^3(p+qx)}{(s+tx)^3(r+Dx)},\\
L=&\frac{2l(l-1)t^3D(p+qx)^2}{(s+tx)^3(r+Dx)}+\frac{4(l-1)^2vt^2(Ds-tr)(p+qx)}{(s+tx)^3(r+Dx)^2}\\
&-\frac{4(l-1)^2t^2uv(p+qx)}{(A+ux)(s+tx)^2(r+Dx)}+\frac{2l(l-1)t^2q(qs-pt)}{(s+tx)^3}\\
&-\frac{2l(l-1)t^2D^2(p+qx)^2}{(s+tx)^2(r+Dx)^2}.
\end{aligned}
$$

It suffices for us to verify $H, I, J, L \leq 0$. Now, invoking the inequalities in (3.1) and noticing $l \geq 2$, we begin to verify them one by one.

$$
\begin{aligned}
H=&\frac{4u^3D(p+qx)^2}{(A+ux)^3(r+Dx)}+\frac{4uvD(DA-ur)(p+qx)}{(A+ux)^2(r+Dx)^3}\\
&+\frac{4u^2v(DA-ur)(p+qx)}{(A+ux)^3(r+Dx)^2}-\frac{8uv^2D}{(A+ux)(r+Dx)^3}\\
&-\frac{4u^3q(p+qx)}{(A+ux)^3}-\frac{4u^2vq}{(A+ux)^2(r+Dx)}\\
=&\frac{4u(p+qx)}{(A+ux)(r+Dx)}\Big[\frac{u^2D(p+qx)}{(A+ux)^2}+v\Big(\frac{D}{r+Dx}-\frac{u}{A+ux}\Big)\\
&\times\Big(\frac{D}{r+Dx}+\frac{u}{A+ux}\Big)\Big]-\frac{4u^3q(p+qx)}{(A+ux)^3}\\
&-\frac{4u^2vq}{(A+ux)^2(r+Dx)}-\frac{8uv^2D}{(A+ux)(r+Dx)^3}\\
=&\frac{4u(p+qx)}{(A+ux)(r+Dx)}\Big[\frac{u^2D(p+qx)}{(A+ux)^2}-\frac{u^2v}{(A+ux)^2}+\frac{vD^2}{(r+Dx)^2}\Big]\\
&-\frac{4u^3q(p+qx)}{(A+ux)^3}-\frac{4u^2vq}{(A+ux)^2(r+Dx)}-\frac{8uv^2D}{(A+ux)(r+Dx)^3}\\
=&\frac{4uvD^2(p+qx)}{(A+ux)(r+Dx)^3}+\frac{4u^3q(p+qx)}{(A+ux)^3}-\frac{4u^3q(p+qx)}{(A+ux)^3}\\
&-\frac{8uv^2D}{(A+ux)(r+Dx)^3}-\frac{4u^2vq}{(A+ux)^2(r+Dx)}\\
=&\frac{4uv}{(A+ux)(r+Dx)}\Big[\frac{D^2(p+qx)}{(r+Dx)^2}-\frac{D(pD-qr)}{(r+Dx)^2}\\
&-\frac{uq}{A+ux}\Big]-\frac{4uv^2D}{(A+ux)(r+Dx)^3}\\
=&\frac{4uv}{(A+ux)(r+Dx)}\Big(\frac{qD}{r+Dx}-\frac{uq}{A+ux}\Big)-\frac{4uv^2D}{(A+ux)(r+Dx)^3}\\
=&\frac{4uvq(DA-ur)}{(A+ux)^2(r+Dx)^2}-\frac{4uv^2D}{(A+ux)(r+Dx)^3}\\
\leq&0;
\end{aligned}
$$

$$
\begin{aligned}
I=&\frac{4u^3D(p+qx)^2}{(A+ux)^3(r+Dx)}+\frac{4u^2v(qA-pu)}{(A+ux)^3(r+Dx)}\\
&+\frac{4u^2q(qA-pu)}{(A+ux)^3}-\frac{4u^2D^2(p+qx)^2}{(A+ux)^2(r+Dx)^2}\\
=&\frac{4u^2}{(A+ux)^3}\Big[\frac{uD(p+qx)^2+v(qA-pu)}{r+Dx}\\
&+q(qA-pu)\Big]-\frac{4u^2D^2(p+qx)^2}{(A+ux)^2(r+Dx)^2}\\
=&\frac{4u^2}{(A+ux)^3}\Big[\frac{uD(p+qx)^2}{r+Dx}+\frac{D(qA-pu)(p+qx)}{r+Dx}\Big]\\
&-\frac{4u^2D^2(p+qx)^2}{(A+ux)^2(r+Dx)^2}\\
=&\frac{4u^2qD(p+qx)}{(A+ux)^2(r+Dx)}-\frac{4u^2D^2(p+qx)^2}{(A+ux)^2(r+Dx)^2}\\
=&\frac{4u^2D(p+qx)}{(A+ux)^2(r+Dx)}\Big[q-\frac{D(p+qx)}{r+Dx}\Big]\\
=&-\frac{4u^2vD(p+qx)}{(A+ux)^2(r+Dx)^2}\leq 0;
\end{aligned}
$$

$$
\begin{aligned}
J=&\frac{2l(l-1)t^3D(p+qx)^2}{(s+tx)^3(r+Dx)}+\frac{4(l-1)^2vt^2(qs-pt)}{(s+tx)^3(r+Dx)}\\
&-\frac{4(l-1)vt^2q}{(s+tx)^2(r+Dx)}-\frac{2l(l-1)t^3q(p+qx)}{(s+tx)^3}\\
&-\frac{2(l-1)^2(l-2)vt^3(p+qx)}{(s+tx)^3(r+Dx)}\\
=&\frac{2l(l-1)t^3(p+qx)}{(s+tx)^3}\Big[\frac{D(p+qx)}{r+Dx}-q\Big]\\
&-\frac{4(l-1)t^2vq}{(s+tx)^2(r+Dx)}+\frac{4(l-1)^2vt^2(qs-pt)}{(s+tx)^3(r+Dx)}\\
&-\frac{2(l-1)^2(l-2)vt^3(p+qx)}{(s+tx)^3(r+Dx)}\\
=&\frac{4(l-1)^2vt^2(qs-pt)}{(s+tx)^3(r+Dx)}-\frac{2(l-1)^2(l-2)t^3v(p+qx)}{(s+tx)^3(r+Dx)}\\
&+\frac{2l(l-1)vt^3(p+qx)}{(s+tx)^3(r+Dx)}-\frac{4(l-1)vt^2q}{(s+tx)^2(r+Dx)}\\
=&\frac{2(l-1)^2vt^2}{(s+tx)^2(r+Dx)}\Big[\frac{2(qs-pt)}{s+tx}-\frac{(l-2)t(p+qx)}{s+tx}\Big]\\
&+\frac{2(l-1)vt^2}{(s+tx)^2(r+Dx)}\Big[\frac{lt(p+qx)}{s+tx}-2q\Big]
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{2(l-1)^2vt^2}{(s+tx)^2(r+Dx)}[2(q-\frac{t(p+qx)}{s+tx}-\frac{(l-2)t(p+qx)}{s+tx}]\\
&\quad+\frac{2(l-1)vt^2}{(s+tx)^2(r+Dx)}[\frac{lt(p+qx)}{s+tx}-2q]\\
&=\frac{2(l-1)^2t^2v}{(s+tx)^2(r+Dx)}[2q-\frac{lt(p+qx)}{s+tx}]\\
&\quad+\frac{2(l-1)t^2v}{(s+tx)^2(r+Dx)}\times[\frac{lt(p+qx)}{s+tx}-2q]\\
&=[\frac{lt(p+qx)}{s+tx}-2q][\frac{2(2-l)(l-1)t^2v}{(s+tx)^2(r+Dx)}]\\
&=[\frac{2(2-l)(l-1)t^2v}{(s+tx)^2(r+Dx)}][\frac{2t(p+qx)}{s+tx}-2q+\frac{(l-2)t(p+qx)}{s+tx}]\\
&=[\frac{2(2-l)(l-1)t^2v}{(s+tx)^2(r+Dx)}][\frac{2(pt-qs)}{s+tx}+\frac{(l-2)t(p+qx)}{s+tx}]\\
&\leq 0;
\end{aligned}
$$

$$
\begin{aligned}
L=&\frac{2l(l-1)t^3D(p+qx)^2}{(s+tx)^3(r+Dx)}+\frac{4(l-1)^2vt^2(Ds-tr)(p+qx)}{(s+tx)^3(r+Dx)^2}\\
&-\frac{4(l-1)^2t^2uv(p+qx)}{(A+ux)(s+tx)^2(r+Dx)}+\frac{2l(l-1)t^2q(qs-pt)}{(s+tx)^3}\\
&-\frac{2l(l-1)t^2D^2(p+qx)^2}{(s+tx)^2(r+Dx)^2}\\
=&\frac{2(l-1)t^2(p+qx)}{(s+tx)^2(r+Dx)}[\frac{ltD(p+qx)}{s+tx}+\frac{2(l-1)v(Ds-tr)}{(s+tx)(r+Dx)}\\
&-\frac{2(l-1)uv}{A+ux}]+\frac{2l(l-1)t^2q(qs-pt)}{(s+tx)^3}\\
&-\frac{2l(l-1)t^2D^2(p+qx)^2}{(s+tx)^2(r+Dx)^2}\\
=&\frac{2(l-1)t^2(p+qx)}{(s+tx)^2(r+Dx)}[\frac{ltD(p+qx)}{s+tx}+\frac{2(l-1)vD}{r+Dx}\\
&-\frac{2(l-1)vt}{s+tx}-\frac{2(l-1)uv}{A+ux}]+\frac{2l(l-1)t^2q(qs-pt)}{(s+tx)^3}\\
&-\frac{2l(l-1)t^2D^2(p+qx)^2}{(s+tx)^2(r+Dx)^2}\\
=&\frac{2(l-1)t^2(p+qx)}{(s+tx)^2(r+Dx)}[\frac{ltD(p+qx)}{s+tx}-\frac{(l+l-2)vt}{s+tx}\\
&+\frac{2(l-1)vD}{r+Dx}-\frac{2(l-1)uv}{A+ux}]\\
&+\frac{2l(l-1)t^2q(qs-pt)}{(s+tx)^3}-\frac{2l(l-1)t^2D^2(p+qx)^2}{(s+tx)^2(r+Dx)^2}\\
=&\frac{2(l-1)t^2(p+qx)}{(s+tx)^2(r+Dx)}[\frac{ltD(p+qx)}{s+tx}-\frac{lvt}{s+tx}\\
&-\frac{(l-2)vt}{s+tx}]+\frac{2l(l-1)t^2q(qs-pt)}{(s+tx)^3}\\
&+\frac{4(l-1)^2vt^2(DA-ur)(p+qx)}{(A+ux)(s+tx)^2(r+Dx)^2}-\frac{2l(l-1)t^2D^2(p+qx)^2}{(s+tx)^2(r+Dx)^2}\\
=&\frac{2(l-1)t^2(p+qx)}{(s+tx)^2(r+Dx)}[\frac{lt}{s+tx}(Dp+Dqx-Dp+qr)\\
&-\frac{(l-2)vt}{s+tx}]+\frac{4(l-1)^2vt^2(DA-ur)(p+qx)}{(A+ux)(s+tx)^2(r+Dx)^2}\\
&+\frac{2l(l-1)t^2q(qs-pt)}{(s+tx)^3}-\frac{2l(l-1)t^2D^2(p+qx)^2}{(s+tx)^2(r+Dx)^2}\\
=&\frac{2l(l-1)t^3q(p+qx)}{(s+tx)^3}+\frac{2l(l-1)t^2q(qs-pt)}{(s+tx)^3}\\
&-\frac{2l(l-1)t^2D^2(p+qx)^2}{(s+tx)^2(r+Dx)^2}\\
&-\frac{2(l-1)(l-2)vt^3(p+qx)}{(s+tx)^3(r+Dx)}+\frac{4(l-1)^2vt^2(DA-ur)(p+qx)}{(A+ux)(s+tx)^2(r+Dx)^2}\\
=&\frac{2l(l-1)t^2q^2}{(s+tx)^2}-\frac{2l(l-1)t^2D^2(p+qx)^2}{(s+tx)^2(r+Dx)^2}\\
&-\frac{2(l-1)(l-2)vt^3}{(s+tx)^3}\times\frac{p+qx}{r+Dx}+\frac{4(l-1)^2vt^2(DA-ur)(p+qx)}{(A+ux)(s+tx)^2(r+Dx)^2}\\
=&\frac{2l(l-1)t^2}{(s+tx)^2}[q+\frac{D(p+qx)}{r+Dx}][q-\frac{D(p+qx)}{r+Dx}]\\
&-\frac{2(l-1)(l-2)}{(s+tx)^3}\times\frac{(p+qx)vt^3}{r+Dx}+\frac{4(l-1)^2vt^2(DA-ur)(p+qx)}{(A+ux)(s+tx)^2(r+Dx)^2}\\
=&\frac{2l(l-1)t^2}{(s+tx)^2}[q+\frac{D(p+qx)}{r+Dx}](\frac{-v-Dqx}{r+Dx})\\
&-\frac{2(l-1)(l-2)}{(s+tx)^3}\times\frac{(p+qx)vt^3}{r+Dx}+\frac{4(l-1)^2vt^2(DA-ur)(p+qx)}{(A+ux)(s+tx)^2(r+Dx)^2}\\
\leq&\,0.
\end{aligned}
$$

Up to here, it is verified that

$$2E'F-E(2F'+E)\leq 0.$$

That is $SF(x)\leq 0$. Thus, according to Lemma 2.3, $\bar{x}$ is a global attractor of all positive solutions of Eq (2.4). In turn, according to Lemma 2.2, $\bar{x}$ is the only fixed point of F^2 in $(0,\infty)$. Then, using Lemma 2.1 (b), it is shown that $\bar{x}$ is the global attractor for all positive solutions of Eq (2.1), hence Eq (3.2) and so Eq (1.1). Thereby, the proof of Theorem 3.1 is complete. □

4. Applications

In this section, we state some applications of Theorem 3.1.

Example 4.1. *Consider the following difference equation, which is studied in [5, Eq #142],*

$$x_{n+1}=\frac{p+x_n}{A+x_n+Dx_{n-2}},\quad n=0,1,\ldots \tag{4.1}$$

with the positive parameters p,A,D and arbitrary nonnegative initial conditions x_{-2},x_{-1},x_0. It has a unique positive equilibrium point

$$\bar{x}=\frac{1-A+\sqrt{(1-A)^2+4p(1+D)}}{2(1+D)}.$$

The authors of [4] said that the positive equilibrium point $\bar{x}$ of Eq (4.1) is locally asymptotically stable when

$$A\geq 1,$$

or

$$0 \le A < 1 \ \ and \ \ D < \frac{4A(A+1)}{(2A-1)^2},$$

or

$$0 \le A < 1, D \ge \frac{4A(A+1)}{(2A-1)^2}, \ \ and \ \ p > p^*$$

and unstable when

$$0 \le A < 1, D \ge \frac{4A(A+1)}{(2A-1)^2}, \ \ and \ \ p < p^*,$$

where

$$p^* = \frac{2A - D + 5AD - 4A^2D - 2D^2 + D^3 - 4AD^3 + 4A^2D^3}{2(9D^2+6D+1)}$$
$$\frac{-(1 + A + 2D + AD - D^2 + 2AD^2)\gamma\sqrt{D}}{2(9D^2+6D+1)},$$
$$\gamma = \sqrt{4A^2D - 4AD + D - 4A^2 - 4A}.$$

Furthermore, the authors of [4] proposed a conjecture as follows:

Conjecture 4.1. *[4, Conjecture 5.142.1] Assume that*

$$p^* < p < \frac{(D-1)(1-A)^2}{4}.$$

Show that every solution of Eq (4.1) converges to the equilibrium point $\bar{x}$.

According to our Theorem 3.1, one sees that, when $A + \bar{x} \le pD$, the positive equilibrium $\bar{x}$ of Eq (4.1) is globally asymptotically stable. Precisely speaking, we derive the following result.

Theorem 4.1. *The positive equilibrium $\bar{x}$ of Eq (4.1) is global asymptotically stable when*

$$p \ge \frac{AD^2 + (A+1)D + 1}{D^2(1+D)}.$$

Proof. It follows from our Theorem 3.1 that, when $A + \bar{x} \le pD$, the positive equilibrium $\bar{x}$ of Eq (4.1) is globally asymptotically stable. Notice that $A + \bar{x} \le pD$ is equivalent to

$$A + \frac{1 - A + \sqrt{(1-A)^2 + 4p(1+D)}}{2(1+D)} \le pD.$$

Namely,

$$\sqrt{(1-A)^2 + 4p(1+D)} \le 2(1+D)pD - [A(1+2D)+1]. \tag{4.2}$$

So, for

$$2(1+D)pD - [A(1+2D)+1] \ge 0,$$

i.e.,

$$p \ge \frac{A(1+2D)+1}{2D(1+D)} = \frac{2AD + A + 1}{2D(1+D)},$$

and Eq (4.2) is equivalent to

$$D^2(1+D)^2p^2 - (1+D)[2AD^2 + (A+1)D + 1]p$$
$$+ A[AD^2 + (A+1)D + 1] \ge 0.$$

The above inequality holds for

$$p \le \frac{A}{1+D}$$

or

$$p \ge \frac{AD^2 + (A+1)D + 1}{D^2(1+D)}.$$

Obviously,

$$\frac{AD^2 + (A+1)D + 1}{D^2(1+D)} \ge \frac{2AD + A + 1}{2D(1+D)}.$$

So, for

$$p \ge \frac{AD^2 + (A+1)D + 1}{D^2(1+D)},$$

then, Eq (4.2) holds. Then, the positive equilibrium $\bar{x}$ of Eq (4.1) is globally asymptotically stable. □

Obviously, our result is different with the corresponding one of [4]. We not only present new results for the global asymptotic stability of Eq (4.1), but also give part results for Conjecture 4.1.

Example 4.2. *Consider the following difference equation [4, Eq #67]:*

$$x_{n+1} = \frac{p + x_n}{A + x_{n-2}}, \ \ n = 0, 1, \dots \tag{4.3}$$

with positive parameters p, A and arbitrary nonnegative initial conditions x_{-2}, x_{-1}, x_0. Equation (4.3) has the unique positive equilibrium

$$\bar{x} = \frac{1 - A + \sqrt{(1-A)^2 + 4p}}{2}.$$

The authors in [4] stated that the positive equilibrium $\bar{x}$ of Eq (4.3) is globally asymptotically stable for $A \ge 1$. The authors of [4] also presented the following conjecture:

Conjecture 4.2. *[4, Conjecture 5.67.1] Assume that either*

$$\frac{1}{2} \leq A < 1$$

or

$$\frac{1}{3} \leq A < \frac{1}{2} \text{ and } p < \frac{A^2(-A^2+3A-1)}{(2A-1)^2}$$

holds. Show that the equilibrium $\bar{x}$ of Eq (4.2) is globally asymptotically stable.

By using our result, namely, Theorem 3.1, it is easy to derive that the equilibrium $\bar{x}$ of Eq (4.2) is globally asymptotically stable for $A \leq p$. Namely, we derive the following result:

Theorem 4.2. *The equilibrium $\bar{x}$ of Eq (4.2) is globally asymptotically stable when $0 < A \leq p$.*

So, our Theorem 4.2 shows that the Conjecture 4.2 is partly correct.

Example 4.3. *Consider the following difference equations, which are numbered as Eqs #517, #521, #523, #545, #547 in [4]:*

$$x_{n+1} = \frac{p + qx_n}{Dx_{n-3}}, \quad n = 0, 1, \ldots, \tag{4.4}$$

$$x_{n+1} = \frac{p}{Bx_n + Dx_{n-3}}, \quad n = 0, 1, \ldots, \tag{4.5}$$

$$x_{n+1} = \frac{p}{A + Bx_n + Dx_{n-3}}, \quad n = 0, 1, \ldots, \tag{4.6}$$

$$x_{n+1} = \frac{p}{Cx_{n-1} + Dx_{n-3}}, \quad n = 0, 1, \ldots, \tag{4.7}$$

$$x_{n+1} = \frac{p}{A + Cx_{n-1} + Dx_{n-3}}, \quad n = 0, 1, \ldots. \tag{4.8}$$

They assumed that the parameters $p, q, A, B, C, D>0$ and initial conditions $x_{-3}, x_{-2}, x_{-1}, x_0$ are arbitrarily nonnegative, and they derived that the solutions of these equations are bounded [4].

However, according to the assumptions for the parameters in [4] and our Theorem 3.1, we can easily derive the following result.

Theorem 4.3. *Every equilibrium point $\bar{x}$ of Eqs (4.4)–(4.8) is globally asymptotically stable.*

So, our result improves and generalizes the corresponding ones in [4].

5. Conclusions and discussion

This paper mainly deals with the global attractivity of all positive solutions of a higher order rational difference equation. As some applications of special cases, our result not only improves many known results, but also partly solves several conjectures presented in some known works.

Nevertheless, it is a pity for us not to derive a complete result for the global asymptotic stability of Eq (1.1). Namely, we only know that the nonnegative fixed point of Eq (1.1) is globally asymptotically stable when

$$q(A + B\bar{x} + C\bar{x}) \leq pD.$$

How about

$$q(A + B\bar{x} + C\bar{x}) > pD?$$

We put forward the following question to interested readers.
Open Problem. Consider the global asymptotic stability of Eq (1.1) when

$$q(A + B\bar{x} + C\bar{x}) > pD.$$

The investigations of RDEs are still interesting for many readers.

The forms of RDEs look very simple, so, it is often mistakenly believed that the investigations of the properties of RDEs are simple. In fact, generally speaking, it is extremely difficult for one to derive a complete result for some characteristics in their entire parameter space because such RDEs generally contain many parameters, such as 8 parameters in Eq (1.1), and some calculations to derive such properties are actually very complex and time consuming. Hence, one often only obtains partial results in the entire parameter space. However, RDEs possess many fascinating properties, such as dichotomy [18, 19], trichotomy [20], bifurcation [25–29], chaos [30, 31], etc. Up to now, there are not yet any effective methods or ways to deal with such problems. One always tries to look for effective methods or ways to study such problems. That is why many readers are still interested in the studies of RDEs nowdays, and it is part of the charm of investigating RDEs.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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