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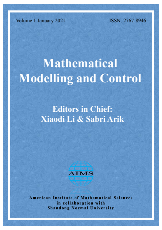

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Research article

Fractional integral approach on nonlinear fractal function and its application

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Abstract: The shape and dimension of the fractal function have been significantly influenced by the scaling factor. This paper investigated the fractional integral of the nonlinear fractal interpolation function corresponding to the iterated function systems employed by Rakotch contraction. We demonstrated how the scaling factors affect the flexibility of fractal functions and their different fractional orders of the Riemann fractional integral using certain numerical examples. The potentiality application of Rakotch contraction of fractal function theory was elucidated based on a comparative analysis of the irregularity relaxation process. Moreover, a reconstitution of epidemic curves from the perspective of a nonlinear fractal interpolation function was presented, and a comparison between the graphs of linear and nonlinear fractal functions was discussed.

Keywords: iterated function system; Rakotch contraction; fractal interpolation function; Riemann-Liouville fractional integral

1. Introduction

Mandelbrot coined the concept of fractal in order to address the issues of irregular natural phenomena that cannot be represented using Euclidean geometry. The fractal theory provides a powerful tool for fitting experimental data to realistic data and complicated curve data in complex modeling [1]. Hutchinson elaborated the concept of fractal through the iterated function system (IFS) and he showed that IFS is one of the significant roles in proving self-similar properties (see more details in [2]). Barnsley has pioneered the concept of constructing a continuous function f from $[t_0, t_N]$ to $\mathbb{R}$ using the framework of IFS theory. This function is the graph of the unique attractor of a special IFS, referred as the fractal interpolation function (FIF) [3]. In recent years [4], FIF has obtained much fascination for its ability to bring together new research disciplines such as approximation theory, interpolation theory, functional analysis, etc. The scaling factor has a significant impact on the fractal characteristics, like box dimension and flexibility of fractal function. Numerous studies on the fractal function have employed constant scaling factors, which produce self-similar properties. Being that there is irregular data with moderately self-similarity, function scaling factors have been proposed in the studies for such a model of function rather than constant scaling factors. Many researchers have examined the concept of IFSs in various manners in order to develop the fractal function theory, and their research has demonstrated that each of those has a unique attractor and these are associated with the graph of continuous mappings that intercalates prescribed data. They investigated the characterization of fractal functions such as their flexibility, fractal dimension, stability, multivariate FIF, and operators (for more information see [5–13]). In [14], the authors investigated the probability distributions of linear FIF amidst different types of random noise. They introduced a technique to approximate the distributions of linear FIF affected by t-distribution noise and also discussed statistical properties, as well as offered numerical approximations for these functions. The authors discussed identification for the parameters of bivariate fractal interpolation surfaces using convex hulls to bounded areas of appropriately chosen data points, achieving a closer fit to the original data [15].

The Banach contraction theorem is the most powerful tool for establishing and studying the convergence of the method of iterative. This traditional theorem that has much significance in both theoretical and real-life applications has been studied in many ways. The Rakotch contraction is the generalization of Banach contraction. Rakotch [16] has proposed the Rakotch fixed point theorem, a novel fixed point theorem that is according with the Rakotch contraction function (sometimes referred to as the nonlinear contraction function). The scaling factor depends on the distance between any two points on a compact domain and the scaling factor is defined as the family of functions, which obeys the condition $0 \leq \alpha(z) < 1$. In [17], the fractal space with the IFS defined on different sorts of the fixed point theorem has been studied by the Rakotch fixed point theorem in place of the Banach contraction theorem. Thereafter, on the basis of the Rakotch fixed point theorem, Song-il Ri generated the new fractal function (known as nonlinear fractal interpolation function (NFIF)) associated with the nonlinear IFS [18]. In this progression, Kim's research team has established different sorts of nonlinear fractal functions such as bivariate functions,and hidden variables, and also analyzed their characterization (see more details in [19, 20]). Thereafter [21], Navascués et al. delineated the $\mathcal{R}$- fractal function as a complete class of attractor functions corresponding to the provided continuous map on a closed subset in $\mathbb{R}$. Moreover, they studied the properties of the $\mathcal{R}$-fractal function with the nonlinear scaling coefficients such as smoothness, fractal operator, etc.

In [22], the box dimension of the graph of the fractal function has been measured through the slope of scaling coefficients. In this development, it is based on the Hölder exponent via calculating the fractal dimension for the graph of the fractal function with both linear and nonlinear scaling factors in [23]. Later on, several researchers estimated the conception fractal dimension for the various sorts of fractal functions like affine, and non-affine (see [24–26]). The relation between the Rakotch contraction factor and the box dimension of the nonlinear fractal function has been proved [27]. The fractal functions of traditional integral have been shown on a compact set in $\mathbb{R}$ but have more complexity and, consequently, they are more appropriate for modeling irregular data or functions in [28].

In [29], research introduced the conception of the fractional integral corresponding to the linear fractal function f and the general connection of the fractional integral order ζ, and

$$\dim_B \Gamma(I^{\zeta} f(t)) = \dim_B \Gamma(f(t)) - \zeta$$

is studied. In numerous papers, the fractional calculus of different kinds of fractal functions and their box dimension have been demonstrated (for more details, see [30–32]). Few applications of different sort of FIFs have been explored in the field of engineering [33], and medical diseases, namely, COVID-19 is discussed and predicated by FIF in [34]. The author in [35] has studied the fractional integral of fractal dimension through the space of all Hölder continuous maps delineated on compact subset in $\mathbb{R}$. The non-affine fractal function associated with fractional integrals has been explored for function order [36]. The fractal operator defined for fractional integrals satisfies linearity and boundedness [37]. Moreover, methods for solving fractal calculus are being investigated [38]. To the best of the authors' knowledge, no research has been carried out related to an analysis of the interaction between fractional calculus and nonlinear fractal interpolation theory. The objective of this study is to extended an approach that is appropriate, such as geometrical approximation. This paper investigates the nonlinear fractal function corresponding to the fractional integral on the closed interval $[t_0, t_N]$ in $\mathbb{R}$. The potentiality application of the Rakotch contraction of the fractal function theory is elucidated based on a comparative analysis of the irregularity relaxation process.

This paper is systematized as follows: Section 2 endows a basic overview. In Section 3, the nonlinear FIF with nonlinear scaling coefficients correlation of the fractional integral on a compact subset $[t_0, t_N]$ in $\mathbb{R}$ is investigated, as well as a discussion about the end point conditions. Section 4 represents the numerical simulation for the prescribed concept of this paper. In Section 5, the application of nonlinear FIF in the time domain of the

COVID-19 data taken for the year 2022 is discussed, and its fractal dimension has been calculated.

2. Preparatory facts

Definition 2.1. *[16, 17] Let X denote the nonempty set. A function f: X → X is said to be ϕ- contraction if for some self-mapping ϕ on* $\mathbb{R}^+ \cup \{0\}$ *such that*

$$\omega(f(t), f(y)) \leq \phi(\omega(t, y)),\ \forall\ t, y \in X,$$

where $0 \leq \phi(z) < 1,\ z > 0$.

Definition 2.2. *[16, 17] Let X denote the complete metric space. A function f: X → X is said to be f is a Rakotch ϕ-contraction if there exists a self-mapping ϕ on*

$$\mathbb{R}^+ \cup \{0\} \quad and \quad 0 \leq \alpha(z)z = \phi(z) < 1,$$

where $z > 0$ *such that*

$$\omega(f(t), f(y)) \leq \alpha(\omega(t, y))\omega(t, y),\ \ \forall\ t, y \in X,$$

where $0 \leq \alpha(z) < 1$*, for each* $z > 0$.

Example 2.1. *If* $X = \mathbb{R}$ *is a complete metric space. If*

$$f_1 : \mathbb{R} \to [c, d], \quad f_2 : \mathbb{R} \to [c, d]$$

defined by

$$f_1 = \frac{z}{1 + z^2} \quad and \quad f_2 = \cos(z), \quad \forall z \in [0, \infty).$$

Then f_r *is a Rakotch* ϕ_r*- contraction where* $r = 1, 2$*, but it is not a Banach contraction on X.*

Remark 2.1. *Each Banach contraction is a Rakotch ϕ-contraction, but the converse need not be true. If*

$$\alpha(z) = \phi(z)z = \alpha z,$$

where α lies between 0 and 1, then the Rakotch ϕ_r*-contraction is a Banach contraction.*

Song-il Ri has developed the generalization of nonlinear fractal functions. He has introduced the concept of the Rakotch contraction in fractal interpolation theory [18]. The construction of a nonlinear fractal function follows as: consider the interpolate points

$$\Delta = \{(t_r, y_r) : r \in N^0\} \subset \mathbb{R}^2,$$

where

$$N^0 = \{0, 1, 2, \ldots, N\},$$

which is a distinct dataset of the closed interval

$$I = [t_0, t_N],$$

and $y_r \in [c, d] \subset \mathbb{R}, r = 0, 1, 2, \ldots N$. Set

$$I_r = [t_{r-1}, t_r] \text{ and } M = I \times [c, d].$$

Let $h_r(t)$: $I \to I_r$ be the contraction mappings and

$$h_r(t_0) = t_{r-1},\ h_r(t_N) = t_r,\ \forall\ r = 1, 2, \ldots, N,$$

such that

$$|h_r(t) - h_r(t^*)| \leq \ell |t - t^*|,$$

for some $\ell \in [0, 1)$. Let G_r: $M \to R$ be the homeomorphism functions and

$$G_r(t_0, y_0) = y_{r-1}, G_r(t_N, y_N) = y_r, \ \ \forall\ r = 1, 2, \ldots, N,$$

such that

$$|G_r(t, y) - G_r(t, y')| \leq \phi(|y - y'|), \ \ \forall y, y' \in [c, d],\ t \in I,$$

where

$$0 \leq \phi(|y - y'|) < 1.$$

Define the IFS as

$$\{X : w_r(t, y) = (h_r(t), G_r(t, y)); r = 1, 2, \ldots, N\},$$

where w_r: $M \to I_r \times [c, d]$ is the N- contraction functions defined on the subset of $\mathbb{R}^2$. Let $\mathcal{H}(X)$ indicate the space of all nonempty closed and bounded subsets of X and it is complete with regard to the Hausdorff distance. A self-mapping W is delineated on $\mathcal{H}(X)$ by

$$W(B^*) = \bigcup_{r=1}^{N} w_r(B^*)$$

for any $B^* \in \mathcal{H}(X)$. As W is the Rakotch contraction map on $\mathcal{H}(X)$, then by the Rakotch fixed point theorem, W admits an unique set $\mathcal{G}_f$ such that

$$\mathcal{G}_f = W(\mathcal{G}_f).$$

The compact set

$$\mathcal{G}_f := \{(t, f(t)) : t \in I\}$$

is the graph of continuous function $f\colon I \to \mathbb{R}$ satisfying $f(t_r) = y_r$ for $r = 0, 1, 2, \ldots, N$. The graph of attractor of the IFS

$$\{X : w_r(t, y) = (h_r(t), G_r(t, y));\ \ r = 1, 2, \ldots, N\}$$

of a function f is associated with the prescribed data points

$$\{(t_r, y_r) \in I \times \mathbb{R} : r = 0, 1, 2, \ldots, N\}$$

is called the NFIF or nonlinear fractal function.

The operator T^* is delineated on the space of all continuous maps on I by

$$T(f(t)) = G_r\left(h_r^{-1}(t), f(h_r^{-1}(t))\right),$$

where $t \in I_r, r = 1, 2, \ldots, N$, which is obeying that the joint-up conditions are $f(t_0) = y_0$ and $f(t_N) = y_N$. The function f being the attractor of T^* is satisfies the following functional equation:

$$\begin{aligned} (T^* f)(t) = f(t) = G_r\left(h_r^{-1}(t), f(h_r^{-1}(t))\right), \\ \text{or} \quad f(h_r(t)) = \wp_r(f(t)) + q_r(t),\ t \in I_r, \end{aligned} \tag{2.1}$$

where $\wp_r, r = 1, 2, \ldots, N$ is the Rakotch scaling factor, and $q_r(t)$ is Lipstchiz's continuous function and obeys the endpoint conditions. Based on $q_r(t)$, numerous authors have developed different kinds of fractal functions (for more details see [7–9]). In this paper, we study the NFIF's both constant and function scale factors [18, 20] delineated by

$$w_r(t, y) = \begin{bmatrix} h_r(t) = b_r t + \beta_r \\ G_r(t, y) \end{bmatrix}, \tag{2.2}$$

where $\forall\ (t, y) \in M$ and the coefficients in (2.2) are real parameters described as follows

$$\begin{cases} b_r = \frac{t_r - t_{r-1}}{t_N - t_0}, \\ \beta_r = \frac{t_N t_{r-1} - t_0 t_r}{t_N - t_0}. \end{cases} \tag{2.3}$$

Throughout this paper, $q_r(t)$ is chosen as linear that is

$$q_r(t) = \alpha_r t + f_r.$$

The following is a fundamental definition of box dimension as:

Definition 2.3. *[5] Let D be any nonempty compact subset of $\mathbb{R}^n$ and let $N_\gamma(D)$ be the minimum number of a set of diameters at most γ, which can cover D. The upper and lower box counting dimensions of D respectively, are as follows:*

$$\underline{\dim}_B D = \underline{\lim}_{\gamma \to 0} \frac{\log N_\gamma(D)}{-\log \gamma},$$
$$\overline{\dim}_B D = \overline{\lim}_{\gamma \to 0} \frac{\log N_\gamma(D)}{-\log \gamma}.$$

If

$$\underline{\dim}_B D = \overline{\dim}_B D.$$

This value is referred to as the box dimension of D:

$$\dim_B D = \lim_{\gamma \to 0} \frac{\log N_\gamma(D)}{-\log \gamma}.$$

The relationship between the scale factor and the fractal dimension of a FIF yields crucial insights into the appearance of fractals. In this context [6], the author defines analytical techniques, such as describing the interrelation between the box dimension of the fractal function and the scaling factors. This exploration elucidates how variations in the scale factors influence the refining of the quality of FIF-generated structure.

Theorem 2.1. *[6] Let*

$$G_r(t, y) = d_r y + q_r(t),$$

where $|d_r| < 1$. If the points $\{(t_r, y_r) \in I \times \mathbb{R} : r = 0, 1, \ldots, N\}$ are noncollinear, then

$$\dim_B \Gamma(f) = \begin{cases} 1 + \frac{\log(\sum_{r=1}^{N} |d_r|)}{\log(N)}, & \text{if } \sum_{r=1}^{N} |d_r| > 1, \\ 1, & \text{otherwise.} \end{cases} \tag{2.4}$$

Suppose

$$d_1 = d_2 = \cdots = d_N = d,$$

then

$$\dim_B \Gamma(f) = 1 + \log_N(|d|).$$

In this sequel, Ri [27] has investigated the box dimension for the NFIF as

$$0 = t_0 \le t_1 \le \cdots \le t_N = 1$$

and

$$(y_0, y_1, \ldots, y_N) \in [a, b]$$

are the given real numbers. Let

$$h_r(t) = b_r t + \beta_r \text{ and } G_r(t, y) = \wp_r(y) + \alpha_r t + f_r,$$

where $\wp_r$: $[a, b] \to [a, b]$ is a Rakotch $\wp_r$- contraction for all $r = 1, 2, \ldots, N$. The set of data points obeying the assumptions of Theorem 2.1 have prescribed the following Theorem 2.2.

Theorem 2.2. *[27] Let*

$$G_r(t, y) = \wp_r(y) + q_r(t)$$

with $|\wp_r(y)| < 1$. *If for all* $\wp_r$ *are self-maps on* $[a, b]$ *and Rakotch* ϕ_r*- contractions, where* $\wp_r$ *is an increasing and continuously differential function on* (t_0, t_N). *If*

$$\sum_{r=1}^{N} \min_{y\in[a,b]} |\wp_r'(y)| > 1, \quad \sum_{r=1}^{N} \max_{y\in[a,b]} |\wp_r'(y)| > 1$$

and the set data points

$$\{(t_r, y_r) \in I \times \mathbb{R} : r = 0, 1, \ldots, N\}$$

are noncollinear, then

$$\begin{aligned} 1 + \log_N\left(\sum_{r=1}^{N} \min_{y\in[a,b]} |\wp_r'(y)|\right) &\le \underline{\dim}_B(\Gamma(f)) \\ &\le \overline{\dim}_B(\Gamma(f)) \qquad (2.5)\\ &\le 1 + \log_N\left(\sum_{r=1}^{N} \max_{y\in[a,b]} |\wp_r'(y)|\right). \end{aligned}$$

3. Riemann-Liouville fractional integral

This section explores the relation between the nonlinear fractal function with various kinds of Rakotch contractions and the fractional integral on I. The Riemann-Liouville fractional integral (RLFI) is delineated as follows:

Definition 3.1. *[37] Let f be a continuous function from I into I, if there is a factor* $0 < \zeta < 1$ *such that*

$$I_{t_0}^{\zeta} f(t) = \frac{1}{\Gamma(\zeta)} \int_{t_0}^{t} (t - u)^{\zeta-1} f(u) du. \qquad (3.1)$$

The above equation is said to be RLFI.

3.1. RLFI for NFIF with constant scaling factor

This section illustrates the nonlinear fractal function of RLFI corresponding to the IFS

$$\{X : (h_r(t), G_r(t, y)); r = 1, 2, 3, \ldots, N\}$$

that is, if f is a nonlinear fractal function on the dataset

$$\Delta = \{(t_r, y_r) : r \in N^0\} \subset \mathbb{R}^2,$$

where

$$N^0 = \{0, 1, 2, \ldots, N\}.$$

Then the RLFI delineated follows as:

$$I_{t_0}^{\zeta} \wp_r(f(t)) = \frac{1}{\Gamma(\zeta)} \int_{t_0}^{t} (t - u)^{\zeta-1} \wp_r(f(u)) du \qquad (3.2)$$

with the initial condition

$$I_{t_0}^{\zeta} \wp_r(f(t_0)) = 0.$$

Theorem 3.1. *Let f be the nonlinear fractal function correlated to the IFS*

$$\{X : (h_r(t), G_r(t, y)); r = 1, 2, 3, \ldots, N\}.$$

Then, $I_{t_0}^{\zeta} f(t)$ *is the nonlinear FIF associated with the new IFS*

$$\{h_r(t), \widehat{G}_r(t, y)\}_{r=1}^{N} \text{ and } \widehat{y}_{0,\zeta} = 0,$$

where

$$\begin{aligned} \widehat{G}_r(t, y) &= b_r^{\zeta} \widehat{\wp}_r(y) + \widehat{q}_r(t), \ \forall \ r = 1, 2, \ldots, N, \\ \widehat{q}_r(t) &= \widehat{y}_{r-1,\zeta} + f_{r,\zeta}(t) + b_r^{\zeta} I_{t_0}^{\zeta} q_r(t), \\ \widehat{y}_{N,\zeta} &= \sum_{i=1}^{N} \left(f_{i,\zeta}(t) + b_i^{\zeta} I_{t_0}^{\zeta} r_i(y_N) + b_i^{\zeta} I_{t_0}^{\zeta} q_i(t_N)\right), \\ f_{r,\zeta} &= \frac{1}{\Gamma(\zeta)} \int_{t_0}^{t_{r-1}} (h_r(t) - u)^{\zeta-1} - (t_{r-1} - u)^{\zeta-1} f(u) du. \end{aligned}$$

Proof. From the functional equation given in (2.1), and thus, for all $t \in [t_0, t_r]$ and $r = 1, 2, \ldots, N$, let $u = h_r(v)$ be employed in the following equation, which is obtained as

$$I_{t_0}^{\zeta} f(h_r(t)) = \frac{1}{\Gamma(\zeta)} \int_{t_0}^{h_r(t)} (h_r(t) - u)^{\zeta - 1} f(u) du,$$
$$I_{t_0}^{\zeta} f(h_r(t)) = \widehat{y}_{r-1,\zeta} + f_{r,\zeta}(t) + b_r^{\zeta} I_{t_0}^{\zeta} \wp_r(f(t)) + b_r^{\zeta} I_{t_0}^{\zeta} q_r(t)$$
$$= \widehat{G}_r(t, \widehat{\wp}_r(y)).$$

Put $t = t_N$ and $h_r(t_N) = t_r$, which implies that

$$\widehat{y}_{N,\zeta} = \widehat{y}_{r-1,\zeta} + f_{r,\zeta}(t) + b_r^{\zeta} I_{t_0}^{\zeta} \wp_r(y_N) + b_r^{\zeta} I_{t_0}^{\zeta} q_r(t_N). \quad (3.3)$$

Since

$$\widehat{y}_{r,\zeta} = \widehat{y}_{0,\zeta} + \sum_{i=1}^{r} \left(\widehat{y}_{i,\zeta} - \widehat{y}_{i-1,\zeta} \right)$$

is applying the Eq (3.3), for each $r = 1, 2, \ldots, N$, and substituting $r = N, \widehat{y}_{r,\zeta}$ is obtained as follows:

$$\widehat{y}_{r,\zeta} = \sum_{i=1}^{N} \left(f_{i,\zeta}(t) + b_i^{\zeta} I_{t_0}^{\zeta} r_i(y_N) + b_i^{\zeta} I_{t_0}^{\zeta} q_i(t_N) \right).$$

The function $I_{t_0}^{\zeta} f(t)$ is a nonlinear FIF, and $\widehat{q}_r(t)$ needs to satisfy the continuity conditions. Note that

$$\widehat{q}_r(t) = \widehat{y}_{r-1,\zeta} + f_{r,\zeta}(t) + b_r^{\zeta} I_{t_0}^{\zeta} q_r(t_n), \quad \widehat{q}_r(t_0) = \widehat{y}_{r-1,\zeta},$$

and

$$\widehat{q}_r(t_N) = \widehat{y}_{r-1,\zeta} + f_{r,\zeta}(t_N) + b_r^{\zeta} I_{t_0}^{\zeta} q_r(t_N).$$

Comparing the above equation with $\widehat{y}_{r,\zeta}$ in Eq (3.3) is

$$\widehat{q}_r(t_N) = \widehat{y}_{r,\zeta} - b_r^{\zeta} I_{t_0}^{\zeta} \wp_r(y_N).$$

This clearly provides that

$$\widehat{G}_r(t_0, \widehat{y}_0) = \widehat{G}_r(t_0, 0) = \widehat{q}_r(t_0) = \widehat{y}_{r-1},$$

and also

$$\widehat{G}_r(t_N, \widehat{y}_N) = I_{t_0}^{\zeta} f(h_r(t_N)) = \widehat{y}_r.$$

Hence, the nonlinear IFS

$$\{X : (h_r(t), \widehat{G}_r(t, y)); r = 1, 2, \ldots, N\}$$

associated the new FIF $I_{t_0}^{\zeta} f$ with initial condition $\widehat{y}_{0,\zeta} = 0$.

Remark 3.1. *In Theorem* 3.1*, the nonlinear FIF with Rakotch contraction factor $\wp_r$ is studied by RLFI. Suppose the Rakotch contraction factor $\wp_r(y)$ in Theorem* 3.1 *is chosen as $\wp_r(y) = d_r y$, then the fractal function with Banach contraction has been discussed through the RLFI and is gathered as follows:*

$$\widehat{G}_r(t, y) = b_r^{\zeta} \widehat{y} + \widehat{q}_r(t), \; \forall \; r = 1, 2, \ldots, N,$$
$$\widehat{q}_r(t) = \widehat{y}_{r-1,\zeta} + f_{r,\zeta}(t) + b_r^{\zeta} I_{t_0}^{\zeta} q_r(t),$$
$$\widehat{y}_N = \frac{\sum_{i=1}^{N} \left(f_{i,\zeta}(t) + b_i^{\zeta} d_i \widehat{y}_N + b_i^{\zeta} I_{t_0}^{\zeta} q_i(t_N) \right)}{(1 - \sum_{i=1}^{N} b_i^{\zeta} d_i)},$$
$$f_{r,\zeta} = \frac{1}{\Gamma(\zeta)} \int_{t_0}^{t_{r-1}} (h_r(t) - u)^{\zeta-1} - (t_{r-1} - u)^{\zeta-1} f(u) du.$$

Let f be a continuous function on I. It is said to be a classical integral on I, and it is denoted as

$$\widehat{f}(t) = \widehat{y}_0 + \int_{t_0}^{t} f(x) dx.$$

This remark has explored the connection between the nonlinear fractal function correlated to the nonlinear IFS

$$\{X : w_r(t, y); r = 1, 2, \ldots, N\}$$

and the classical integral on $[t_0, t_N]$.

Remark 3.2. *(1) In Theorem* 3.1*, the nonlinear FIF with Rakotch contraction factor $\wp_r$ is explored by RLFI. Assume that the fractional order is taken as $\zeta = 1$, then RLFI becomes the classical integral on $[t_0, t_N]$, provided that*

$$\widehat{G}_r(t, y) = b_r \widehat{\wp}_r(y) + \widehat{q}_r(t),$$
$$b_r = \frac{t_r - t_{r-1}}{t_N - t_0} \; \forall \; r = 1, 2, \ldots, N,$$
$$\widehat{q}_r(t) = \widehat{y}_{r-1} + b_r \int_{t_0}^{t} q_r(u) du,$$
$$\widehat{y}_N = \widehat{y}_0 + \sum_{i=1}^{N} b_i \widehat{\wp}_i(y_N) + \sum_{i=1}^{N} b_i \int_{t_0}^{t_N} q_i(u) du.$$

(2) Let $f(t)$ be the nonlinear fractal function correlated to $\{h_r(t), G_r(t, y)\}_{r=1}^{N}$ and

$$\widehat{f}(t) = \widehat{y}_N + \int_{t_N}^{t} f(x) dx.$$

Then, $\widehat{f}(t)$ *is the new nonlinear FIF associated with* $\{h_r(t), \widehat{G}_r(t,y)\}_{r=1}^N$*, where*

$$\widehat{G}_r(t,y) = b_r\widehat{\wp}_r(y) + \widehat{q}_r(t),$$
$$b_r = \frac{t_r - t_{r-1}}{t_N - t_0}, \text{ for all } r = 1,2,\ldots,N,$$
$$\widehat{q}_r(t) = \widehat{y}_r + b_r\int_{t_0}^{t} q_r(u)du,$$
$$\widehat{y}_0 = \widehat{y}_N + \sum_{i=1}^{N} b_i\widehat{\wp}_i(y_0) + \sum_{i=1}^{N} b_i\int_{t_0}^{t_N} q_i(u)du.$$

Definition 3.2. *[37] Let f be a continuous function from I into I, if there is a factor* $0 < \zeta < 1$ *such that*

$$I_{t_N}^{\zeta} f(t) = \frac{1}{\Gamma(\zeta)}\int_{t_N}^{t} (t-u)^{\zeta-1} f(u)du. \tag{3.4}$$

The Eq (3.4) *is called the RLFI with end point condition*

$$I_{t_N}^{\zeta} f(t_N) = 0.$$

Theorem 3.2. *Let f be the NFIF determined by* $\{h_r(t), G_r(t,y)\}_{r=1}^N$*. Then,* $I_{t_N}^{\zeta} f(t)$ *is the nonlinear FIF associated with* $\{h_r(t), \widehat{G}_r(t,y)\}_{r=1}^N$ *and*

$$\widehat{y}_{N,\zeta} = 0,$$

where

$$\widehat{G}_r(t,y) = b_r^{\zeta}\widehat{\wp}_r(y) + \widehat{q}_r(t), \text{ for all } r = 1,2,\ldots,N,$$
$$\widehat{q}_r(t) = \widehat{y}_{r,\zeta} - f_{r,\zeta}(t) + b_r^{\zeta} I_{t_0}^{\zeta} q_r(t),$$
$$\widehat{y}_{0,\zeta} = \sum_{i=1}^{N}\left(f_{i,\zeta}(t) + b_i^{\zeta} I_{t_N}^{\zeta} r_i(y_0) + b_i^{\zeta} I_{t_N}^{\zeta} q_i(t_0)\right),$$
$$f_{r,\zeta}(t) = \frac{1}{\Gamma(\zeta)}\int_{t_r}^{t_N}\left((h_r(t)-u)^{\zeta-1} - (t_r-u)^{\zeta-1}\right) f(u)du.$$

Proof. Let the functional equation be

$$f(h_r(t)) = \wp_r(f(t)) + q_r(t).$$

Thus, for all $t \in [t_0, t_r]$ and $r = 1,2,\ldots,N$, and consider $u = h_r(v)$ in the below mentioned equation, then the obtained result is as follows:

$$I_{t_N}^{\zeta} f(h_r(t)) = \frac{1}{\Gamma(\zeta)}\int_{t_N}^{h_r(t)} (h_r(t)-u)^{\zeta-1} f(u)du,$$
$$I_{t_N}^{\zeta} f(h_r(t)) = \widehat{y}_{r,\zeta} - f_{r,\zeta}(t) - \frac{b_r}{\Gamma(\zeta)}\int_{t}^{t_N} (t-v)^{\zeta-1} f(h_r(v))dv,$$
$$I_{t_N}^{\zeta} f(h_r(t)) = \widehat{y}_{r,\zeta} - f_{r,\zeta}(t) + b_r^{\zeta} I_{t_N}^{\zeta}\wp_r(f(t)) + b_r^{\zeta} I_{t_N}^{\zeta} q_r(t)$$
$$= \widehat{G}_r(t, I_{t_N}^{\zeta}\wp_r(f(t))).$$

Put $t = t_0$ and $h_r(t_0) = t_{r-1}$. This implies that

$$\widehat{y}_{r-1,\zeta} = \widehat{y}_{r,\zeta} - f_{r,\zeta}(t) + b_r^{\zeta} I_{t_N}^{\zeta}\wp_r(f(t_0)) + b_r^{\zeta} I_{t_N}^{\zeta} q_r(t_0). \tag{3.5}$$

Since

$$\widehat{y}_{r-1,\zeta} = \widehat{y}_{r,\zeta} - \sum_{i=r}^{N}\left(\widehat{y}_{i,\zeta} - \widehat{y}_{i-1,\zeta}\right),$$

for each $r = 1,2,\ldots,N$ and substituting $r = 1, \widehat{y}_{r-1,\zeta}$ is obtained as follows:

$$\widehat{y}_{0,\zeta} = \sum_{i=1}^{N}\left(f_{i,\zeta}(t) + b_i^{\zeta} I_{t_N}^{\zeta}\wp_i(y_0) + b_i^{\zeta} I_{t_N}^{\zeta} q_i(t_0)\right).$$

The function $I_{t_N}^{\zeta} f(t)$ is a nonlinear FIF, $\widehat{q}_r(t)$ needs to satisfy the continuity conditions, and $\widehat{q}_r(t)$ has obeyed the joint-up conditions, which are

$$\widehat{q}_r(t_0) = \widehat{y}_{r,\zeta} - f_{r,\zeta}(t_0) + b_r^{\zeta} I_{t_N}^{\zeta} q_r(t_0)$$

and

$$\widehat{q}_r(t_N) = \widehat{y}_{r,\zeta}.$$

It is clearly prescribed that

$$\widehat{G}_r(t_0, \widehat{y}_0) = \widehat{G}_r(t_0, y_0) = \widehat{q}_r(t_0) = \widehat{y}_{r-1},$$

and also

$$\widehat{G}_r(t_N, 0) = \widehat{y}_{r-1,\zeta} + f_{r,\zeta}(t_N) + b_r^{\zeta} I_{t_N}^{\zeta}\wp_r(f(t_N)) + b_r^{\zeta} I_{t_N}^{\zeta} q_r(t_N)$$
$$= I_{t_N}^{\zeta} f(h_r(t_N))$$
$$= \widehat{y}_r.$$

Hence, the nonlinear IFS $\{h_r(t), \widehat{G}_r(t,y)\}_{r=1}^N$ corresponding to the nonlinear FIF is $I_{t_0}^{\zeta} f$ with $\widehat{y}_{N,\zeta} = 0$.

Remark 3.3. *In Theorem* 3.2*, the nonlinear FIF with Rakotch contraction factor* $\wp_r$ *is investigated by the RLFI with the end point condition. If the Rakotch contraction factor* $\wp_r(y)$ *in Theorem* 3.2 *is chosen as* $\wp_r(y) = d_r y$*, then linear FIF with Banach contraction has been discussed through the RLFI with join-up condition and is obtained as follows:*

$$\widehat{G}_r(t,y) = b_r^{\zeta} d_r \widehat{y} + \widehat{q}_r(t), \ \forall\ r = 1,2,\ldots,N,$$

$$\widehat{q_r}(t) = \widehat{y}_{r,\zeta} - f_{r,\zeta}(t) + b_r^{\zeta} I_{t_0}^{\zeta} q_r(t),$$

$$\widehat{y_0} = \frac{\sum_{i=1}^{N}\left(f_{i,\zeta}(t) + b_i^{\zeta} d_i \widehat{y_0} + b_i^{\zeta} I_{t_N}^{\zeta} q_r(t_N)\right)}{(1 - \sum_{i=1}^{N} b_i^{\zeta} d_i)},$$

$$f_{r,\zeta}(t) = \frac{1}{\Gamma(\zeta)} \int_{t_r}^{t_N} \left((h_r(t) - u)^{\zeta-1} - (t_r - u)^{\zeta-1}\right) f(u)du.$$

The following theorem presents the interaction between fractional integral order ζ, the box dimension of nonlinear fractal functions f, and their RLFI on $[t_0, t_N]$.

Theorem 3.3. *Suppose f is the NFIF prescribed by the IFS*

$$\{X : (h_r(t), G_r(t, y)); r = 1, 2, \dots, N\},$$

where

$$h_r(t) = b_r t + \beta_r \text{ and } G_r(t, y) = \wp_r(y) + q_r(t).$$

If

$$\sum_{r=1}^{N} \min_{y\in[c,d]} |\wp_r'(y)| > 1,$$

and

$$1 + \log_N \left(\sum_{r=1}^{N} \min_{y\in[c,d]} |\wp_r'(y)|\right) \le \underline{\dim}_B(\Gamma(f)), \tag{3.6}$$

$$\sum_{r=1}^{N} \max_{y\in[c,d]} |\wp_r'(y)| > 1,$$

$$\overline{\dim}_B(\Gamma(f)) \le 1 + \log_N \left(\sum_{r=1}^{N} \max_{y\in[c,d]} |\wp_r'(y)|\right). \tag{3.7}$$

Then,

$$\dim_B(\Gamma(I_{t_0}^{\zeta} f(t))) = \dim_B(\Gamma f(t)) - \zeta.$$

The following section is insistent on the scaling factor chosen as the function scaling factor. Consider the functional equation form is delineated in [20] as

$$f(h_r(t)) = d_r(t)\wp_r(f(t)) + q_r(t),\ t \in I_r,\ r = 1, 2, \dots, N, \tag{3.8}$$

where

$$\wp_r(t) = \max_{r=1,2,\dots,N} \max_{t\in I} |d_r(t)| t$$

is the Rakotch scaling factor, and

$$|d_r(t)| < 1,\ \forall\, r = 1, 2, \dots, N.$$

Then, f is the nonlinear fractal function associated with the IFS

$$\{X : (h_r(t), G_r(t, y)); r = 1, 2, \dots, N\}.$$

3.2. RLFI for NFIF with variable scaling factor

Theorem 3.4. *Let f be the NFIF determined by the IFS*

$$\{X : (h_r(t), G_r(t, y)); r = 1, 2, \dots, N\}.$$

Then $I_{t_0}^{\zeta} f(t)$ is the nonlinear fractal function associated with

$$\{h_r(t), \widehat{G}_r(t, y)\}_{r=1}^{N} \text{ and } \widehat{y}_{0,\zeta} = 0,$$

where

$$\widehat{G}_r(t, y) = b_r^{\zeta} d_r(t)\widehat{\wp}_r(y) + \widehat{q}_r(t), \text{ for all } r = 1, 2, \dots, N,$$

$$\widehat{q_r}(t) = \widehat{y}_{r-1,\zeta} + f_{r,\zeta}(t) + \frac{b_r^{\zeta}\alpha_r}{\Gamma(\zeta+1)}(t - t_0)^{\zeta} + \frac{b_r^{\zeta} f_r}{\Gamma(\zeta+2)}(t - t_0)^{\zeta+1},$$

$$\widehat{y}_{N,\zeta} = \sum_{i=1}^{N}\left(f_{i,\zeta}(t_N) + b_i^{\zeta} d_i(t_N)\widehat{\wp}_{i,\zeta}(y_N) + \frac{b_i^{\zeta}\alpha_i}{\Gamma(\zeta+1)}(t_N - t_0)^{\zeta} + \frac{b_i^{\zeta} f_i}{\Gamma(\zeta+2)}(t_N - t_0)^{\zeta+1}\right),$$

$$f_{r,\zeta} = \frac{1}{\Gamma(\zeta)} \int_{t_0}^{t_{r-1}} (h_r(t) - u)^{\zeta-1} - (t_{r-1} - u)^{\zeta-1} f(u)du.$$

Proof. The functional Eq (3.8) is employing the Eq (3.1). Consider $u = h_r(v)$, then

$$I_{t_0}^{\zeta} f(h_r(t)) = \widehat{y}_{r-1,\zeta} + f_{r,\zeta}(t) + \frac{1}{\Gamma(\zeta)} \int_{t_{r-1}}^{h_r(t)} (h_r(t) - u)^{\zeta-1} f(u)du + \frac{b_r^{\zeta}\alpha_r}{\Gamma(\zeta+1)}(t - t_0)^{\zeta} + \frac{b_r^{\zeta} f_r}{\Gamma(\zeta+2)}(t - t_0)^{\zeta+1},$$

$$I_{t_0}^{\zeta} f(h_r(t)) = b_r^{\zeta} d_r(t)\widehat{\wp}_{r,\zeta}(f(t)) + \widehat{q_r}(t).$$

Put $t = t_N$ and $h_r(t_N) = t_n$,

$$\widehat{y}_{r,\zeta} = \widehat{y}_{r-1,\zeta} + f_{r,\zeta}(t_N) + b_r^{\zeta} \sum_{k=1}^{\infty} {}^{\zeta}C_k D^k d_k(t_N) I_{t_0}^{\zeta+k} \wp_r(y_N) + \frac{b_r^{\zeta}\alpha_r}{\Gamma(\zeta+1)}(t_N - t_0)^{\zeta} + \frac{b_r^{\zeta} f_r}{\Gamma(\zeta+2)}(t_N - t_0)^{\zeta+1}. \tag{3.9}$$

Since

$$\widehat{y}_{r,\zeta} = \widehat{y}_{0,\zeta} + \sum_{i=1}^{r}\left(\widehat{y}_{i,\zeta} - \widehat{y}_{i-1,\zeta}\right),$$

for each $r = 1, 2, \dots, N$ and substituting $r = N, \widehat{y}_{r,\zeta}$ is obtained as follows:

$$\widehat{y}_{N,\zeta} = \sum_{i=1}^{N}\left(f_{i,\zeta}(t_N) + b_i^{\zeta} d_i(t_N)\widehat{\wp}_{i,\zeta}(y_N) + \frac{b_i^{\zeta}\alpha_i}{\Gamma(\zeta+1)}(t_N - t_0)^{\zeta}\right.$$

$$+\frac{b_i^\zeta f_i}{\Gamma(\zeta+2)}(t_N-t_0)^{\zeta+1}\Bigg).$$

The function $I_{t_0}^\zeta f(t)$ is a nonlinear FIF, and $\widehat{q_r}(t)$ needs to satisfy the continuity conditions. Since $\widehat{q_r}(t)$ obeys the join up conditions

$$\widehat{q_r}(t_0)=\widehat{y}_{r-1,\zeta}$$

and

$$\widehat{q_r}(t_N)=b_r^\zeta\sum_{k=1}^{\infty}{}^\zeta C_k D^k d_k(t_N) I_{t_0}^{\zeta+k}\wp_k(f(t_N))+\widehat{y}_{r-1,\zeta}+f_{r,\zeta}(t_N)$$
$$+\frac{b_r^\zeta\alpha_r}{\Gamma(\zeta+1)}(t_N-t_0)^\zeta+\frac{b_r^\zeta f_r}{\Gamma(\zeta+2)}(t_N-t_0)^{\zeta+1}.$$

Comparing the above equation with $\widehat{y}_{r,\zeta}$ in Eq (3.9) becomes

$$\widehat{q_r}(t_N)=\widehat{y}_{r,\zeta}-b_r^\zeta d_r(t_N) I_{t_0}^\zeta\wp_r(y_N).$$

This clearly describes that

$$\widehat{G}_r(t_0,\widehat{y}_0)=\widehat{G}_r(t_0,0)=\widehat{q_r}(t_0)=\widehat{y}_{r-1}$$

and also

$$\widehat{G}_r(t_N,\widehat{y}_N)=\widehat{y}_r.$$

Hence the nonlinear IFS $\{h_r(t),\widehat{G}_r(t,y)\}_{r=1}^N$ determines a FIF $I_{t_0}^\zeta f$ with initial condition $\widehat{y}_{0,\zeta}=0$.

Remark 3.4. *Suppose the fractional order chosen as $\zeta=1$, then with the Theorem 3.4, consider the function $f(t)$ to be the NFIF corresponding to the IFS*

$$\{X: w_r(t,y); r=1,2,\ldots,N\}.$$

Then, $\widehat{f}(t)$ is a nonlinear FIF correlated to $\{h_r(t),\widehat{G}_r(t,y)\}_{r=1}^N$, where

$$\widehat{G}_r(t,y)=b_r d_r(t)\widehat{\wp}_r(y)+\widehat{q_r}(t),$$
$$\widehat{q_r}(t)=\widehat{y}_{r-1}-b_r\int_{t_0}^{t} d_r'(u)\widehat{\wp}_r(f(u))du+b_r\int_{t_0}^{t} q_r(u)du,$$
$$\widehat{y}_N=\widehat{y}_0+\sum_{i=1}^{N} b_i d_i(u)\widehat{\wp}_i(y_N)-b_i\int_{t_0}^{t} d_i'(u)\widehat{\wp}_i(f(u))du,$$
$$+\sum_{i=1}^{N} b_i\int_{t_0}^{t_N} q_i(u)du.$$

Theorem 3.5. *Let f be the nonlinear FIF determined by $\{h_r(t),G_r(t,y)\}_{r=1}^N$. Then, $I_{t_N}^\zeta f(t)$ is the nonlinear FIF associated with $\{h_r(t),\widehat{G}_r(t,y)\}_{r=1}^N$ and $\widehat{y}_{N,\zeta}=0$, where*

$$\widehat{G}_r(t,y)=b_r^\zeta(t)\widehat{\wp}_r(y)+\widehat{q_r}(t),$$
$$\widehat{q_r}(t)=\widehat{y}_{r,\zeta}-f_{r,\zeta}(t)-b_r^\zeta d_r(t) I_{t_N}^\zeta\wp_r(y)-\frac{b_r^\zeta\alpha_r}{\Gamma(\zeta+1)}(t-t_N)^\zeta$$
$$-\frac{b_r^\zeta f_r}{\Gamma(\zeta+2)}(t-t_N)^{\zeta+1},$$
$$\widehat{y}_{0,\zeta}=\sum_{i=1}^{N}\Bigg(f_{i,\zeta}(t_0)+b_i^\zeta d_i(t_0)\widehat{\wp}_{i,\zeta}(y_0)+\frac{b_i^\zeta\alpha_i}{\Gamma(\zeta+1)}(t_0-t_N)^\zeta$$
$$+\frac{b_i^\zeta f_i}{\Gamma(\zeta+2)}(t_0-t_N)^{\zeta+1}\Bigg),$$
$$f_{r,\zeta}=\frac{1}{\Gamma(\zeta)}\int_{t_r}^{t_N}(h_r(t)-u)^{\zeta-1}(t_r-u)^{\zeta-1} f(u)du.$$

Proof. Once the functional Eq (3.8) is employed in Eq (3.4) and $u=h_r(v)$, the following equation is obtained:

$$I_{t_N}^\zeta f(h_r(t))=\widehat{y}_{r,\zeta}-f_{r,\zeta}(t)+\frac{1}{\Gamma(\zeta)}\int_{t_r}^{h_r(t)}(h_r(t)-u)^{\zeta-1} f(u)du$$
$$=\widehat{y}_{r,\zeta}-f_{r,\zeta}(t)-b_r^\zeta d_r(t)\widehat{\wp}_{r,\zeta}(f(t))$$
$$-\frac{b_r^\zeta\alpha_r}{\Gamma(\zeta+1)}(t_0-t_N)^\zeta-\frac{b_r^\zeta f_r}{\Gamma(\zeta+2)}(t_0-t_N)^{\zeta+1},$$
$$I_{t_N}^\zeta f(h_r(t))=\widehat{G}_r(t,\widehat{\wp}_{r,\zeta}(f(t))).$$

Put $t=t_0$ and $h_r(t_0)=t_{r-1}$, then one is obtained as

$$\widehat{y}_{r-1,\zeta}=\widehat{y}_{r,\zeta}-f_{r,\zeta}(t_0)-b_r^\zeta\sum_{k=1}^{\infty}{}^\zeta C_k D^k d_r(t_0)\widehat{\wp}_{n,\zeta+k}(y_0)$$
$$-\frac{b_r^\zeta\alpha_r}{\Gamma(\zeta+1)}(t_0-t_N)^\zeta-\frac{b_r^\zeta f_r}{\Gamma(\zeta+2)}(t_0-t_N)^{\zeta+1}. \tag{3.10}$$

Since

$$\widehat{y}_{r-1,\zeta}=\widehat{y}_{r,\zeta}-\sum_{i=r}^{N}\left(\widehat{y}_{r,\zeta}-\widehat{y}_{r-1,\zeta}\right),$$

for each $r=1,2,\ldots,N$ and substituting $r=1,\widehat{y}_{r-1,\zeta}$ is obtained as follows:

$$\widehat{y}_{0,\zeta}=\sum_{i=1}^{N}\Big(f_{i,\zeta}(t_0)+b_i^\zeta d_i(t_0)\widehat{\wp}_{i,\zeta}(y_0)$$
$$+\frac{b_i^\zeta\alpha_i}{\Gamma(\zeta+1)}(t_0-t_N)^\zeta+\frac{b_i^\zeta f_i}{\Gamma(\zeta+2)}(t_0-t_N)^{\zeta+1}\Bigg).$$

The function $I_{t_N}^\zeta f(t)$ is be a nonlinear FIF, and $\widehat{q_r}(t)$ needs to satisfy the continuity conditions. Note that

$$\widehat{q_r}(t)=\widehat{y}_{r,\zeta}-f_{r,\zeta}(t)-\frac{b_r^\zeta\alpha_r}{\Gamma(\zeta+1)}(t-t_N)^\zeta-\frac{b_r^\zeta f_r}{\Gamma(\zeta+2)}(t-t_N)^{\zeta+1},$$

and comparing the aforementioned equation with $\widehat{y}_{r-1,\zeta}$ in Eq (3.10) obeys the join up conditions as

$$\widehat{q}_r(t_0) = \widehat{y}_{r-1,\zeta} - b_r^{\zeta} d_r(t_0)\widehat{\wp}_{r,\zeta}(y_0) \text{ and } \widehat{q}_r(t_N) = \widehat{y}_{r,\zeta}.$$

This clearly provides that

$$\widehat{G}_r(t_0, \widehat{y}_0) = \widehat{y}_{r-1}$$

and also

$$\widehat{G}_r(t_N, \widehat{y}_N) = I_{t_N}^{\zeta} f(h_r(t_N)) = \widehat{y}_r.$$

Hence, the nonlinear IFS is

$$\{h_r(t), \widehat{G}_r(t, y)\}_{r=1}^{N}$$

associated with a new FIF $I_{t_N}^{\zeta} f$ with end point condition $\widehat{y}_{N,\zeta} = 0$.

Remark 3.5. *If the fractional integral order $\zeta = 1$ in Theorem 3.5 is prescribed the classical integral function, $f(t)$ is the NFIF associated with the IFS $\{h_r(t), G_r(t, y)\}_{r=1}^{N}$. Then, $\widehat{f}(t)$ is a nonlinear FIF corresponding to the new IFS*

$$\{X : (h_r(t), \widehat{G}_r(t, y)); r = 1, 2, \dots, N\},$$

where

$$\widehat{G}_r(t, y) = b_r d_r(t)\widehat{\wp}_r(y) + \widehat{q}_r(t), b_r = \frac{t_r - t_{r-1}}{t_N - t_0},$$

$$\widehat{q}_r(t) = \widehat{y}_r - b_r \int_{t_N}^{t} d_r'(u)\widehat{\wp}_r(f(u))du + b_r \int_{t_N}^{t} q_r(u)du,$$

$$\widehat{y}_0 = \widehat{y}_N + \sum_{i=1}^{N} b_i d_i(u)\widehat{\wp}_i(y_N) - b_i \int_{t_0}^{t_N} d_i'(u)\widehat{\wp}_i(f(u))du + \sum_{i=1}^{N} b_i \int_{t_0}^{t_N} q_i(u)du.$$

The next section illustrates the shape of a nonlinear fractal function with both constant and variable scaling factors models of each division for the presented fractal functions phenomenon by using the acquired numerical technique. The numerical simulation of each part in classical integral and RLFI associated with the fractal functions is presented.

4. Numerical simulation

Let $N = 7$ be the partition of interval $I = [0, 1]$ and

$$t_r = r/N, \quad y_r = (0, 2, 3, 0.5, 1, 3, 2, 1)$$

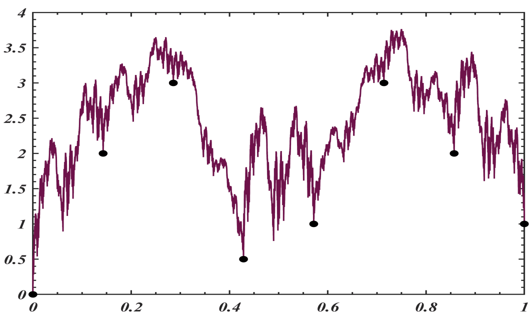
for all $r = 0, 1, 2, \dots, 7$ with various scaling factors

$$d_r = (0.5, 0.3, 0.2, 0.3, 0.25, 0.6, 0.5)$$

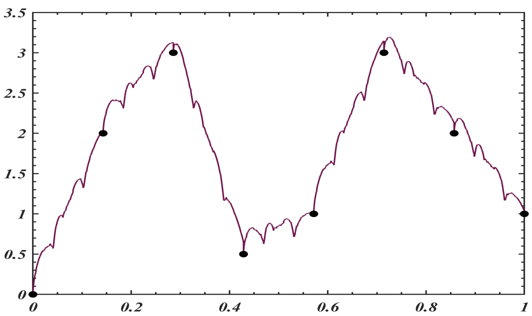
(Banach contraction factors),

$$\wp_r(y) = y/1 + y^2$$

(Rakotch contraction factors). Figure 1a portrayed the graphical approach of the linear fractal function with constant scaling factor, and Figure 1b illuminated the graphical approach of nonlinear fractal function with function scaling factors. Considering the same data points with different scaling factors revealed that the fluctuations observed in Figure 1a surpasses that of Figure 1b, due to the effect exerted by the scale factor.



(a) Linear FIF with $d_r = (0.3, 0.2, 0.3, 0.25, 0.6)$



(b) Nonlinear FIF with $\wp_r(y) = \frac{y}{1+y^2}$

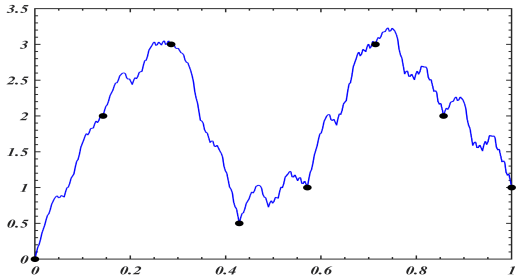
Figure 1. Two types of FIFs with vertical scale vector.

Consequently, the Rakotch scaling factor affords a greater degree of flexibility than the Banach scaling factor. The same data points with different kinds of variable scaling factors

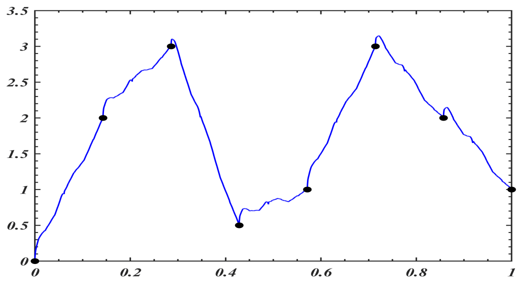
$$d_r(t) = (\log(10t)/5, -e^{t/3}, \log(t+1)/3, t/4, (t+1)/7, 1.3e^{-t}),$$

$\wp_r(y) = d_r(t)y/1 + y^2$

are represented as in Figure 2.



(a) Linear FIF with $d_r(t)$



(b) Nonlinear FIF with $d_r(t)\wp_r(y)$

Figure 2. Two types of FIFs with variable scaling factors are $d_r(t) = (\log(10t)/5, -e^{t/3}, \log(t+1)/3, t/4, (t+1)/7, 1.3e^{-t})$.

Since for all $\wp_r(y)$ are increasing, convex, and satisfy the conditions of the Theorem 2.2, then the box dimension of the nonlinear fractal function follows as:

$$\min_{y\in(0,3)} |\wp_r'(y)| = \frac{1-y^2}{(1+y^2)^2},$$

$$\sum_{r=1}^{6} \min_{y\in(0,3)} |\wp_r'(y)| = \frac{12}{25} + \frac{12}{25} + \frac{12}{25} + \frac{12}{25} + \frac{12}{25} + \frac{12}{25},$$

$$\sum_{r=1}^{6} \min_{y\in(0,3)} |\wp_r'(y)| = 2.88,$$

with

$$\max_{y\in(0,3)} \frac{\wp_r(y)}{y} = \frac{1}{1+y^2},$$

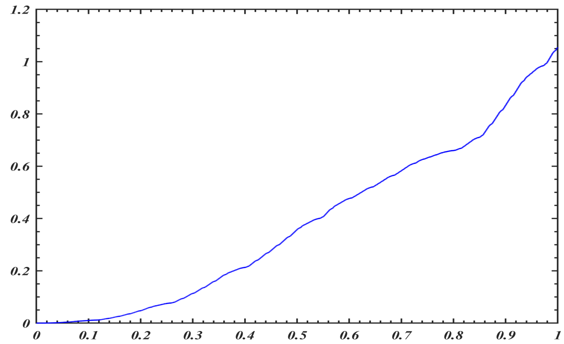
$$\sum_{r=1}^{6} \lim_{y\to 0} \frac{\wp_r(y)}{y} = 1+1+1+1+1+1 = 6.$$

This implies that

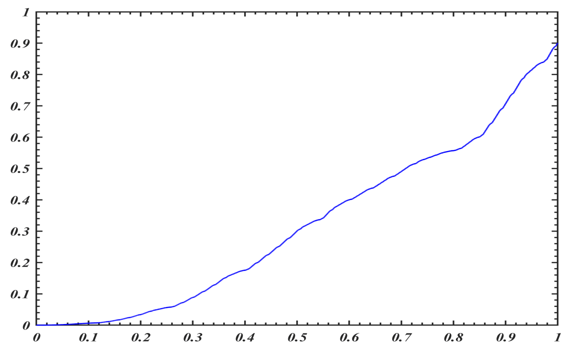
$$1.4594 \le \underline{\dim}_B(\Gamma(f)) \le \overline{\dim}_B(\Gamma(f)) \le 1.7782.$$

However, the fractal dimension of linear FIF is 1.3325. Hence, in the comparison of fractal dimension for linear FIF and NFIF, the NFIF is more suitable to an approach that is precise and appropriate for real-life applications, based on irregular data points.

The visual approaches presented in Figure 3 are linear and nonlinear fractal functions associated with the classical integral for the same data points. The scaling factor and fractional order are interrelated parameters that jointly influence the behavior of a function.



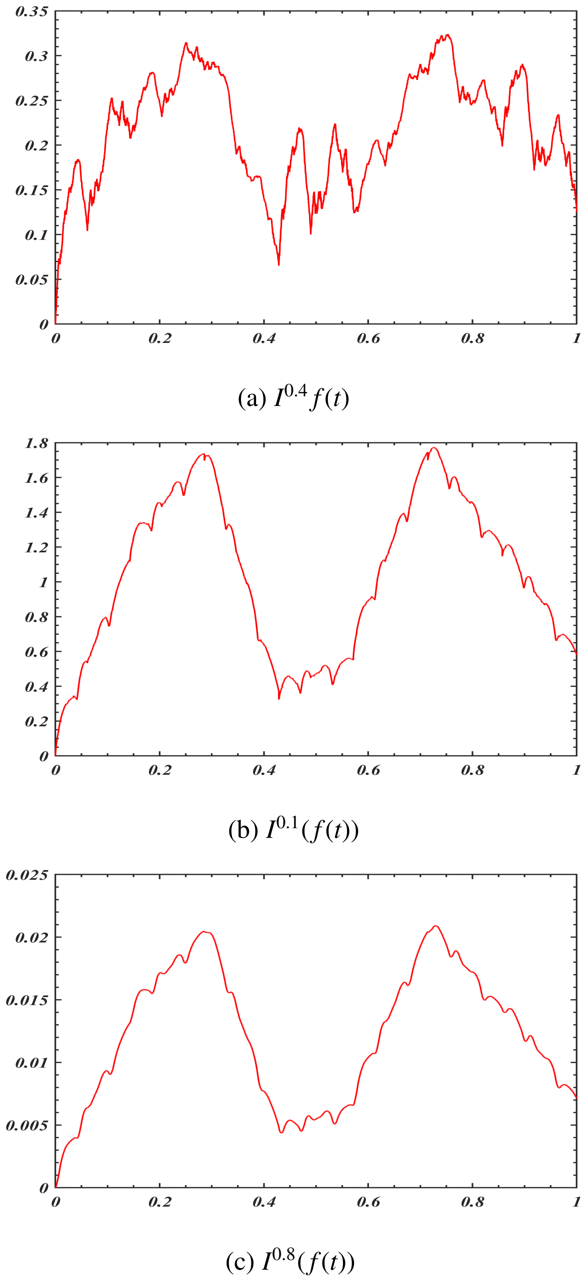
(a) Linear FIF with d_r



(b) Nonlinear FIF with $\wp_r(y) = \frac{y}{1+y^2}$

Figure 3. Two types of FIFs with constant scaling factors of the classical integral.

As the fractional order approaches 0, the scaling factor provides flexibility within the functions range. Conversely, as the fractional order approaches 1, the scaling factor affects flexibility within a different range. Their combined effect alters both the complexity and range of the function, with variations in one parameter impacting the behavior of the other. For instance, the graph of linear and nonlinear fractal functions constituted by the fractional integral with fractional order 0.4, 0.1, and 0.8 are illuminated in Figure 4, respectively.



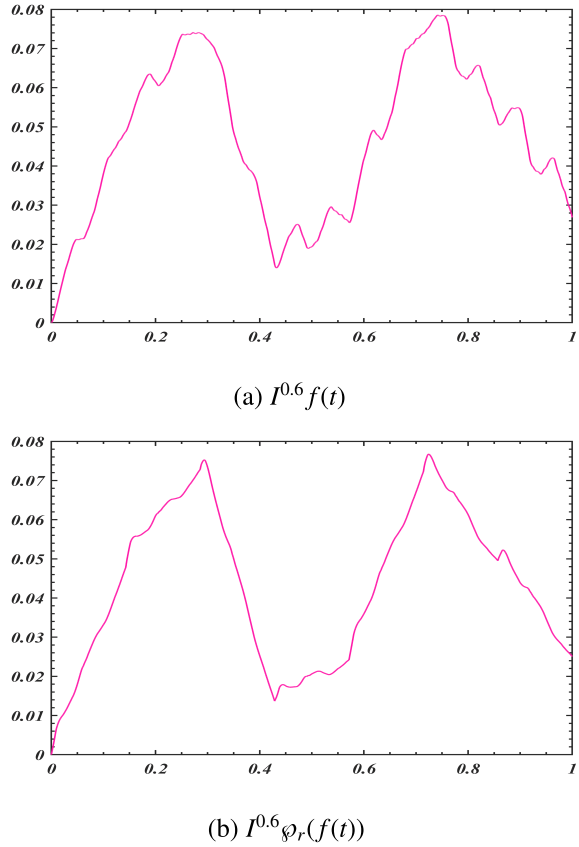
(a) $I^{0.4} f(t)$

(b) $I^{0.1}(f(t))$

(c) $I^{0.8}(f(t))$

Figure 4. Two types of FIFs corresponding to the RLFI with constant scaling factors are $\wp_r(y) = y/1 + y^2$ and $d_r = (0.3, 0.2, 0.3, 0.25, 0.6)$.

Figure 5 demonstrates the graphical approaches of NFIF with various sorts of function scaling factors corresponding to the RLFI with fractional order 0.6, 0.6. Hence, the linear and nonlinear fractal functions approximate the prescribed data points but the linear FIF shows high irregularity with the Banach scaling factor d_r and the nonlinear fractal function provides good smoothness of the prescribed data points with the Rakotch scaling factor $\wp_r(y)$. Therefore, the role of scale factor and fractional integral order plays an important characteristic in the fluctuation of fractal functions associated with their fractional integral function.



(a) $I^{0.6} f(t)$

(b) $I^{0.6}\wp_r(f(t))$

Figure 5. Two types of FIFs corresponding to the RLFI with variable scaling factors are $d_r(t) = (\log(10t)/5, -e^{t/3}, \log(t+1)/3, t/4, (t+1)/7, 1.3e^{-t}), 0.5t$.

5. Application in time series

Consider the set of data points

$$\{(t_r, f(t_r)) \in [t_0, t_N] \times \mathbb{R} : r = 0, 1, 2, \ldots, N\}.$$

The first coordinate t_r is denoted as the time of epidemic for the daily confirmed positive cases occurring from January to December 2022, where t_0=0 and t_r=364 denote the initial and last day of the epidemic, since N=365 is taken as the days-wise over the period 2022 and the second coordinate $f(t_r)$ has been associated with the number of people affected by COVID-19 based on daily confirmed positive cases recorded in the year 2022. The vertical scaling factor d_r was chosen between -0.15≤ d_r ≤0.15. The COVID-19 data has been obtained by the Center for Systems Science and Engineering (CSSE) at Johns Hopkins University (`https:\ourworldindata.org`) from the COVID-19 data repository. It's provided the collection of daily-updated information on COVID-19 for the world. The Johns Hopkins University CSSE established and hosts this data portal.

The most fluctuating number of infected populations in COVID-19 was highest at the end of January, when it has 4.3 × 10^{16}, and the number of the infected people in COVID-19 fluctuates moderately from the first of February to the end of November. The number of infected populations caused by COVID-19 reached its maximum peak at the end of December 2022, which is 8 × 10^{16}, and it is gradually decreasing subsequently, as shown in Figure 6a–c having been employed by three types of fractal interpolation methods in COVID- 19 data. Among these three methods, the linear fractal interpolation method shows more complexities, the nonlinear fractal interpolation method exhibits flexible fluctuations, and the resolution of the map in Figure 6b is lower than that seen in Figure 6a,c. In [39], the authors studied the fractal dimension through the rescaled range analysis technique for COVID-19 data. Consequently, we have employed the same method for both fractal functions to determine their fractal dimensions. This approach reliably approximates time series data based on their fractal dimension. The random walk is generated when the fractal dimension is 1.5. A fractal dimension below 1.5 indicates persistent fluctuations, while a dimension above 1.5 suggests anti-persistent volatility. Later, the dimension of the graph of a function is interpreted as a sampling of the intricacy of fluctuations, representing the geometry of an object in $\mathbb{R}^2$. The flexibleness aids in accurately predicting the total number of people affected by COVID-19 and their fractal dimension is less than 1.5 owing to its variability.

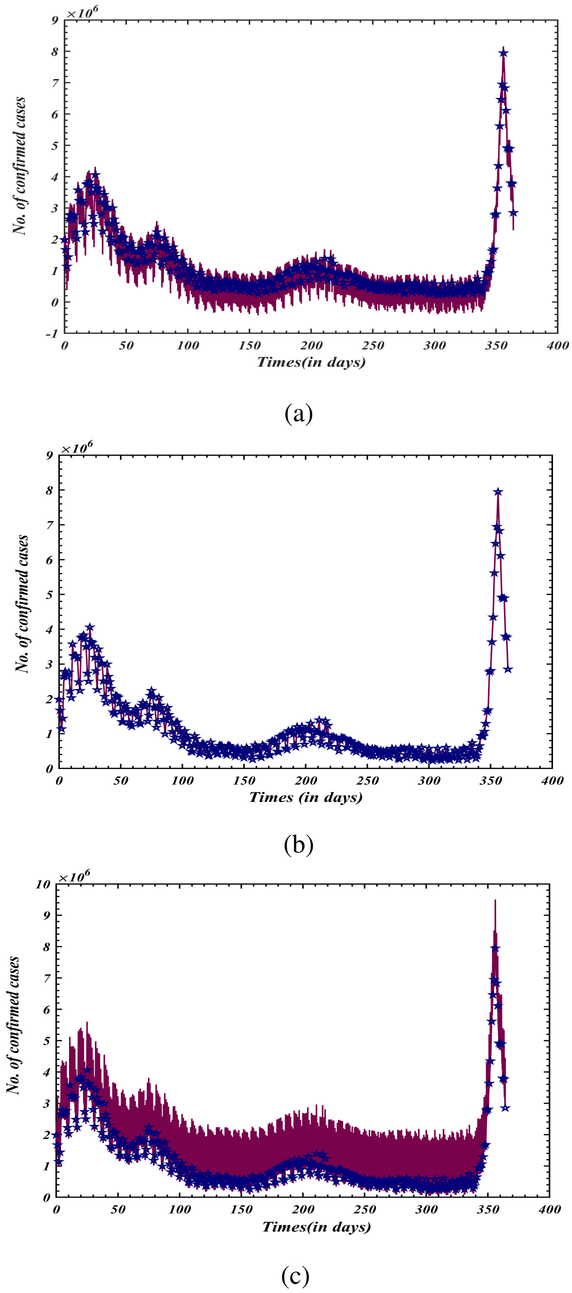
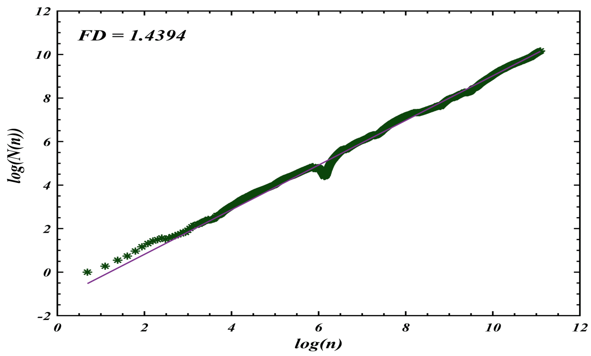
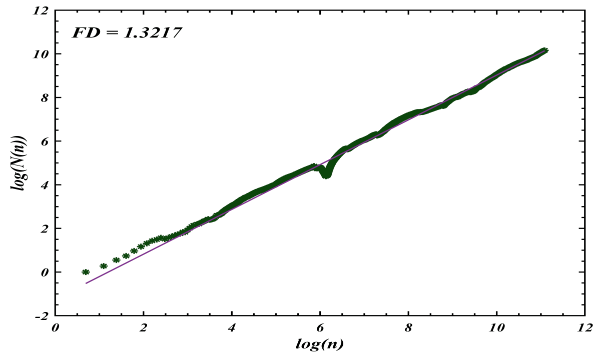


Figure 6. Fractal function associated with the COVID-19 for daily positive cases over the period of 2022. The world data is graphically illustrated based on (a) nonlinear FIF, (b) linear FIF, and (c) both FIF with the constant scaling factor over the period 2022.

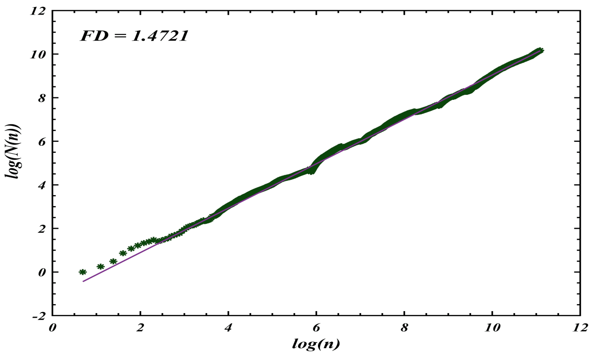
Figure 7a shows that the fractal dimension of the graph derived by the linear fractal interpolation approach is 1.4394 based on the number of people worldwide infected by Covid-19.



(a)



(b)



(c)

Figure 7. COVID-19 graph of log – log plot is depicted for (a) Worldwide affected peoples by COVID-19 for linear FIF, (b) NFIF approaches of the globally affected peoples by COVID-19, and (c) NFIF implies the linear FIF over the period 2022.

By using the same data, the fractal dimension for the graph obtained by the nonlinear FIF approach of worldwide affected populations by COVID-19 is 1.3217, as illustrated in Figure 7b.

With the same data, when using the nonlinear FIF in comparison with the linear FIF approach the obtained the fractal dimension for the graph of worldwide COVID-19-affected populations is 1.4721, as shown in Figure 7c. The distinction between linear and nonlinear FIF approaches can be visualized through the complexity and irregularities of the graph. Moreover, the fractal dimensions of linear and nonlinear FIF are 1.44 and 1.3217, respectively. Thus, it is observed that the dimension values of nonlinear FIF exhibit fewer fluctuations compared to linear FIF. The analysis findings with the Hurst exponent values span from 0.5 to 1, signifying persistence, while the fractal dimension values range from 1 to 1.5. This suggests a persistent degree of fluctuations in the observed COVID-19 cases.

6. Conclusions

This paper has studied that the nonlinear fractal function of the classical integral is again a nonlinear fractal function and meets the end point conditions. The new approaches of RLFI of the nonlinear FIF associated the IFS with Rakotch contraction factor, which is again the nonlinear FIF and its fractal dimension, have been explored. The numerical simulation part has investigated the comparison of the fractal function for both linear and nonlinear scaling factors and we also discussed the graphical description for the nonlinear scale factor replaced by the linear scale factor. Moreover, the application of nonlinear FIF in the time domain of the COVID-19 data taken for the year 2022 is discussed and its fractal dimension has been calculated. The concept of a NFIF method approach for the outbreak curve plays a key role in the analysis and simulation of epidemic models.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest in this paper.

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