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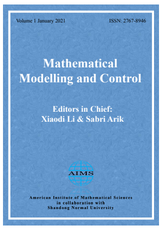

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Research article

Reachable set estimation for 2-D switched nonlinear positive systems with impulsive effects and bounded disturbances described by the Roesser model

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Abstract: The reachable set estimation for two-dimensional (2-D) switched nonlinear positive systems (SNPSs) with bounded disturbances given by the Roesser model is investigated in this paper, in which both the time-varying delays and lagged impulsive effects are taken into account. By applying the average dwell time (ADT) technique, we provide a sufficient condition for the presence of a ball such that any solution of the system converges exponentially within it. An accurate estimate of the convergence rate is provided. We also extend the result to 2-D SNPSs with multi-directional delays, general 2-D switched linear systems, and 2-D SPNSs with heterogeneous delays. Finally, an example is worked out to demonstrate the effectiveness of the main result.

Keywords: 2-D switched nonlinear positive systems; impulsive effects; average dwell time; exogenous disturbances; time-varying delays

1. Introduction

Two-dimensional (2-D) systems are systems which can be used to model real-world engineering systems, and examples such as multi-dimensional digital filtering and circuit analysis [1, 2] can be described by 2-D systems. As special types of 2-D systems, the Roesser model and and the Fornasini-Marchesini (FM) model have been given special attention because of their structures and applications [3–5]. In recent years, stability theory and control synthesis of 2-D systems have been extensively studied [6–9].

Switched positive systems are comprised of a series of positive subsystems, and for any switching signal the states remain nonnegative if the initial conditions are nonnegative. Switched positive systems possess some properties of both switched systems and positive systems, so it is of great interest to study switched positive systems applying the methods which are used to discuss the positive systems, such as, co-positive Lyapunov functions approach (see the researches [10–13]). [14] considered the problem of the existence of common linear co-positive Lyapunov functions for one-dimensional switched positive systems, and the authors introduced multiple linear co-positive Lyapunov functions in [15]. Meanwhile, some practical systems in engineering are described by 2-D switched positive systems, for example, the thermal process with multiple models. The theory of the 2-D switched positive systems have been widely studied in recent years. In [16], the problems of exponential stability for 2-D switched positive systems were considered. The authors investigated the robust observer design for 2-D switched positive systems in [17]. In [18], a necessary and sufficient condition for the asymptotic stability of switched 2-D fractional order positive systems described by the Roesser model is established. Sufficient conditions for the stabilization by state feedback controllers for positive 2-D fractional order sub-systems were reported by [19]. [20] studied the stability problem of uncertain 2-D switched positive systems. Robust stability conditions of 2-D positive systems employing saturation conditions have been reported in [21].

However, the majority of existing research is focused on 2-D switched linear positive systems (SLPSs), and the theory for 2-D switched nonlinear positive systems (SNPSs) is considerably less developed. The methods for studying SLPSs, such as linear copositive functions, are no longer applicable to SNPSs. The pioneering work on the stability analysis of a class of 2-D SNPSs was reported by [22]. Additionally, impulsive phenomena and external disturbances often occur in many real systems of which their states are subject to abrupt changes at certain moments. The research on impulsive systems has emerged in a variety of practical problems, such as in biology and communication networks [23, 24]. Moreover, time-delay phenomena widely exists in practical engineering and it is one of the important reasons for system performance deterioration and instability [25, 26]. Since the activated subsystem is changed at switched and impulsive instants, it is more complicated to makes the system analysis due to the existence of delays for the 2-D SNPSs with impulsive effects. To the best of our knowledge, few studies have attempted to conduct the estimation of reachable sets for the 2-D SNPSs subject to unknown disturbances and delayed impulse effects.

In this paper, we consider the reachable set estimation for 2-D SNPSs given by the Roesser model with unknown exogenous disturbances. Both the systems delay and delayed impulse effects are considered. The contributions of this article are as follows:

First, by applying the multiple max-separable Lyapunov functions approach, we present an explicit sufficient condition for the presence of a ball such that any solution of the system converges exponentially within it for bounded directional delays and delayed impulse effects.

Second, if impulsive matrices and external disturbances are set to zero, then the considered system of this study reduces to existing one in [22]. Therefore, the existing results can be seen as a special case of this article. An accurate estimate of the convergence rate is also provided.

Finally, we also extend the result to 2-D SNPSs with multiple directional delays, general 2-D switched linear systems, and 2-D SNPSs with heterogeneous time-varying delays.

The rest of this article is organized as follows. Some necessary notations, definitions, and problem formulation are presented in Section 2. Our main results and the proofs are provided in Section 3. Section 4 gives an example to justify the efficiency of the obtained results, and the conclusions are stated in Section 5.

2. Preliminaries

$\mathbb{R}$ and $\mathbb{N}$ represent the sets of real and natural numbers, respectively, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $\mathbb{R}^n$ is the set of n-dimensional real vectors, and

$$\mathbb{R}^n_+ := \{\boldsymbol{x} \in \mathbb{R}^n, x_j \geq 0, 1 \leq j \leq n\}.$$

For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, denoted by $\boldsymbol{x} \geq \boldsymbol{y}$ ($\boldsymbol{x} \gg \boldsymbol{y}$, $\boldsymbol{x} \ll \boldsymbol{y}$), if $x_j \geq y_j$ ($x_j > y_j$, $x_j < y_j$) for $1 \leq j \leq n$. Given a positive vector $\boldsymbol{\xi} \gg \boldsymbol{0}$,

$$\|\boldsymbol{x}\|^{\boldsymbol{\xi}}_\infty = \max_{1\leq j\leq n} \frac{|x_j|}{\xi_j}.$$

Denote the weighted l_∞ norm of $\boldsymbol{x} \in \mathbb{R}^n$. Set

$$\|\boldsymbol{x}\|_\infty = \max_{1\leq j\leq n} |x_j|.$$

$\mathbb{R}_{n\times n}$ represents $n \times n$-dimensional real matrices. E_n and O_n denote the identity matrix and zero matrix, respectively.

In this paper, we consider 2-D SPNSs with lagged impulsive effects:

$$\begin{cases} \begin{bmatrix} \boldsymbol{x}^h(k+1,l) \\ \boldsymbol{x}^v(k,l+1) \end{bmatrix} = \boldsymbol{f}_{\sigma(k,l)} \begin{bmatrix} \boldsymbol{x}^h(k,l) \\ \boldsymbol{x}^v(k,l) \end{bmatrix} + \boldsymbol{g}_{\sigma(k,l)} \begin{bmatrix} \boldsymbol{x}^h(k-\tau_h(k),l) \\ \boldsymbol{x}^v(k,l-\tau_v(l)) \end{bmatrix} \\ \qquad +\boldsymbol{\omega}(k,l), \quad k+l \neq \varepsilon_r, \\ \begin{bmatrix} \boldsymbol{x}^h(k,l) \\ \boldsymbol{x}^v(k,l) \end{bmatrix} = \boldsymbol{F}_{\sigma(k,l)} \begin{bmatrix} \boldsymbol{x}^h(k-d_h(k),l) \\ \boldsymbol{x}^v(k,l-d_v(l)) \end{bmatrix}, \quad k+l = \varepsilon_r, \end{cases} \tag{2.1}$$

where $\boldsymbol{x}^h(k,l) \in \mathbb{R}^{n_1}$ and $\boldsymbol{x}^v(k,l) \in \mathbb{R}^{n_2}$ stand for horizontal and vertical state vectors, respectively. $\boldsymbol{x}(k,l) \in \mathbb{R}^n$ represents the whole state with $n = n_1 + n_2$. $\sigma(k,l)$: $\mathbb{N}_0 \times \mathbb{N}_0 \rightarrow M = \{1, 2, 3, \ldots, m\}$ is the switching rule. For any $P \in M$, the vector fields $\boldsymbol{f}_p, \boldsymbol{g}_p$: $\mathbb{R}^n \rightarrow \mathbb{R}^n$ are continuous on $\mathbb{R}^n$. The diagonal matrix

$$\boldsymbol{F}_P = diag\{F_{P_{11}}, F_{P_{11}}, \ldots, F_{P_{nn}}\}$$

is called the impulsive matrix, and we assume $\boldsymbol{F}_{P_{ii}} > 0$ for all $1 \leq i \leq n$. The exogenous disturbances are denoted by $\boldsymbol{\omega}(k,l)$: $\mathbb{N}_0 \times \mathbb{N}_0 \rightarrow \mathbb{R}^n$.

It is assumed in this study that the switching rule $\sigma(k,l)$ relies on ε, that is, if

$$k+l=\widetilde{k}+\widetilde{l}=\varepsilon,$$

then $\sigma(k,l)=\sigma(\widetilde{k},\widetilde{l})$. The switching sequence is stated as follows:

$$(\varepsilon_0,\sigma(\varepsilon_0)),\ (\varepsilon_1,\sigma(\varepsilon_1)),\ldots,(\varepsilon_r,\sigma(\varepsilon_r)),\ldots,$$

where $\varepsilon_r=k_r+l_r$. The $\sigma(\varepsilon_r)$-th subsystem is activated when $k+l\in[\varepsilon_r,\varepsilon_{r+1})$. We suppose system delays $\tau_h(k),\tau_v(l)$ and impulsive delays $d_h(k),d_v(l)$ are all bounded. Therefore, there exist nonnegative real numbers $\widehat{\tau_h},\widehat{\tau_v},\widehat{d_h},\widehat{d_v}$ such that

$$\begin{aligned}&0\le\tau_h(k)\le\widehat{\tau_h},\quad 0\le\tau_v(l)\le\widehat{\tau_v},\\&0\le d_h(k)\le\widehat{d_h},\quad 0\le d_v(l)\le\widehat{d_v},\\&k-d_h(k)\ge-\widehat{\tau_h},\ l-d_v(l)\ge-\widehat{\tau_v}.\end{aligned}$$

Denote

$$\tau_{max}=max(\widehat{\tau_h},\widehat{\tau_v}),\quad d_{max}=max(\widehat{d_h},\widehat{d_v}).$$

The initial conditions are presented as follows:

$$\begin{cases}\boldsymbol{x}^h(k,l)=\boldsymbol{h}(k,l), & -\widehat{\tau_h}\le k\le 0, \quad 0\le l\le\overline{h},\\ \boldsymbol{x}^h(k,l)=\mathbf{0}, & -\widehat{\tau_h}\le k\le 0, \quad l>\overline{h},\\ \boldsymbol{x}^v(k,l)=\boldsymbol{v}(k,l), & -\widehat{\tau_v}\le l\le 0, \quad 0\le k\le\overline{v},\\ \boldsymbol{x}^v(k,l)=\mathbf{0}, & -\widehat{\tau_v}\le l\le 0, \quad k>\overline{v}.\end{cases}\tag{2.2}$$

where $\overline{h}$ and $\overline{v}$ are positive real numbers, and $\boldsymbol{h}(k,l)$, $\boldsymbol{v}(k,l)$ are given positive vectors. Let

$$\hat{\boldsymbol{h}}(r)=\max_{p\in M}\sup_{-\tau_h\le k\le 0}\|h(k,r)\|_\infty^{\boldsymbol{\xi}_{pn_1}}$$

and

$$\hat{\boldsymbol{v}}(s)=\max_{p\in M}\sup_{-\tau_v\le l\le 0}\|\boldsymbol{v}(s,l)\|_\infty^{\boldsymbol{\xi}_{pn_2}},$$

where

$$\boldsymbol{\xi}_{pn_1}=[E_{n_1}\ \ O_{n_1\times n_2}]\ \boldsymbol{\xi}_p,\quad \boldsymbol{\xi}_{pn_2}=[O_{n_2\times n_1}\ \ E_{n_2}]\ \boldsymbol{\xi}_p.$$

Definition 2.1. *The impulsive switched system* (2.1) *is said to be positive if $\boldsymbol{x}^h(k,l)\ge\mathbf{0}$ and $\boldsymbol{x}^v(k,l)\ge\mathbf{0}$ hold for any nonnegative boundary condition $\boldsymbol{h}(k,l)\in\mathbb{R}^{n_1},\boldsymbol{v}(k,l)\in\mathbb{R}^{n_2}$ and any nonnegative disturbance $\boldsymbol{\omega}(k,l)$.*

Definition 2.2. *A vector field $\boldsymbol{f}$: $\mathbb{R}^n\to\mathbb{R}^n$ is called homogeneous of degree one if for any $\boldsymbol{x}\in\mathbb{R}^n$ and $\lambda>0$,*

$$\boldsymbol{f}(\lambda\boldsymbol{x})=\lambda\boldsymbol{f}(\boldsymbol{x}).$$

$\boldsymbol{g}$ is defined to be order-preserving on $\mathbb{R}^n_+$ if $\boldsymbol{g}(\boldsymbol{x})\ge\boldsymbol{g}(\boldsymbol{y})$ for any $\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^n_+$ satisfying $\boldsymbol{x}\ge\boldsymbol{y}$.

Definition 2.3. *For any nonnegative integers i,j and i_0,j_0 with*

$$i+j=\varepsilon\ge\varepsilon_0=i_0+j_0$$

and any switching signal σ, let $N_{\sigma(\varepsilon_0,\varepsilon)}$ denote the number of switching times during the period $[\varepsilon_0,\varepsilon)$. If there exist two constants $N_0>0$ and $\tau_\varepsilon>0$ such that

$$N_{\sigma(\varepsilon_0,\varepsilon)}\le N_0+\frac{\varepsilon-\varepsilon_0}{\tau_\varepsilon},$$

then τ_ε is referred to as the average dwell time (ADT) of the switching signal σ and N_0 is the chatter bound. In this paper, we choose $N_0=0$.

Definition 2.4. *Consider a certain type of ADT switching signals. System* (2.1) *is said to converge exponentially within a ball if there exist constants $a\ge 0$, $b>0$, $0<c<1$, and $0<\gamma<1$ such that*

$$\|x(k,l)\|_\infty^{\boldsymbol{\xi}}\le a+b\Big(\sum_{r=0}^{l}\frac{\hat{h}(r)}{\gamma^{r+1}}+\sum_{s=0}^{k}\frac{\hat{v}(s)}{\gamma^{s+1}}\Big)c^{k+l},$$

where $\boldsymbol{\xi}\gg\mathbf{0}$ is given vector.

Remark 2.1. *It follows from the boundary condition* (2.2) *that*

$$\sum_{r=0}^{l}\frac{\hat{h}(r)}{\gamma^{r+1}}+\sum_{s=0}^{k}\frac{\hat{v}(s)}{\gamma^{s+1}}$$

is bounded by

$$\sum_{r=0}^{\bar{h}}\frac{\hat{h}(r)}{\gamma^{r+1}}+\sum_{s=0}^{\bar{v}}\frac{\hat{v}(s)}{\gamma^{s+1}}.$$

3. Main results

3.1. 2-D SNPSs with delays and lagged impulse

First, two necessary assumptions are proposed on the system (2.1).

Assumption 3.1. *$\boldsymbol{f}_p$ and $\boldsymbol{g}_p$ are order-preserving on $\mathbb{R}^n_+$ and homogeneous of degree one for any $p\in M$.*

Assumption 3.2. *$\omega(k,l) \geq \mathbf{0}$ are external disturbances and satisfy*

$$\|\omega(k,l)\|_\infty \leq \gamma^{k+l}\bar{\omega},$$

where γ and $\bar{\omega}$ are positive constants.

Remark 3.1. *It follows from Assumptions 3.1 and 3.2 that system* (2.1) *is positive for any nonnegative initial condition under arbitrary switching.*

Theorem 3.1. *Let Assumptions 3.1 and 3.2 hold. If for any $p \in M$, there exists a vector $\xi_P \gg \mathbf{0}$ such that*

$$f_p(\xi_p) + g_p(\xi_p) \ll \xi_p,$$

then any solution of system (2.1) *converges exponentially within a ball under suitable ADT switching. The ADT switching signals satisfy*

$$\tau_\varepsilon > -\frac{ln\alpha\beta}{ln\gamma},$$

where

$$\beta = \max_{1\leq i\leq n} \frac{\bar{\xi}_i}{\underline{\xi}_i}$$

with

$$\bar{\xi}_i = \max_{p\in M} \xi_{pi}, \quad \underline{\xi}_i = \min_{p\in M} \xi_{pi}$$

and

$$F = \max_{p\in M, 1\leq i\leq n} F_{pii}, \quad \gamma = \max_{p\in M, 1\leq i\leq n} \gamma_{pi}$$

with γ_{pi} satisfying

$$f_{pi}(\xi_p) + \gamma_{pi}^{-\tau_{\max}} g_{pi}(\xi_p) - \gamma_{pi}\xi_{pi} = 0, \tag{3.1}$$

and

$$\alpha = \begin{cases} \gamma^{-d_{\max}}F, & if\ \gamma^{-d_{\max}}F \geq 1, \\ 1, & if\ \gamma^{-d_{\max}}F < 1. \end{cases}$$

Proof. Let $\|x(k,l)\|_\infty^{\xi_{\sigma(k,l)}}$ be the multiple max-separable Lyapunov function. First, the variable transformation is introduced . Set

$$\begin{bmatrix} x^h(k,l) \\ x^v(k,l) \end{bmatrix} = \begin{bmatrix} \gamma^{k+l} & 0 \\ 0 & \gamma^{k+l} \end{bmatrix} \begin{bmatrix} y^h(k,l) \\ y^v(k,l) \end{bmatrix}, \tag{3.2}$$

then system (2.1) is reduced to

$$\begin{cases} \begin{bmatrix} y^h(k+1,l) \\ y^v(k,l+1) \end{bmatrix} = \gamma^{-1} f_{\sigma(k,l)} \begin{bmatrix} y^h(k,l) \\ y^v(k,l) \end{bmatrix} + g_{\sigma(k,l)} \left(\begin{bmatrix} \gamma^{-\tau_h(k)-1} 0 \\ 0 \gamma^{-\tau_v(l)-1} \end{bmatrix} \begin{bmatrix} y^h(k-\tau_h(k),l) \\ y^v(k,l-\tau_v(l)) \end{bmatrix} \right) \\ \quad + \gamma^{-k-l-1}\omega(k,l), \quad k+l \neq \varepsilon_r, \\ \begin{bmatrix} y^h(k,l) \\ y^v(k,l) \end{bmatrix} = F_{\sigma(k,l)} \left(\begin{bmatrix} \gamma^{-d_h(k)} 0 \\ 0 \gamma^{-d_v(l)} \end{bmatrix} \begin{bmatrix} y^h(k-d_h(k),l) \\ y^v(k,l-d_v(l)) \end{bmatrix} \right), \quad k+l = \varepsilon_r. \end{cases}$$

A set of functions with respect to γ are defined by

$$u_{pi}(\gamma) = f_{pi}(\xi_p) + \gamma^{-\tau_{max}} g_{pi}(\xi_p) - \gamma\xi_{pi}, \tag{3.3}$$

where $\forall p \in M, i = 1,2,3,\ldots,n$, then u_{pi} decreases precisely monotonically for γ and u_{pi} tends to infinity as γ approaches zero. Following from

$$f_p(\xi_p) + g_p(\xi_p) \ll \xi_p,$$

we can get $u_{pi}(1) < 0$. This implies (3.3) has a solution $\gamma_{pi} \in (0,1)$. Let

$$\gamma = \max_{p\in M} \max_{1\leq i\leq n} \gamma_{pi},$$

then $0 < \gamma < 1$ and $u_{pi}(\gamma) \leq 0$. Therefore,

$$f_p(\xi_p) + \gamma^{-\tau_{\max}} g_p(\xi_p) \leq \gamma\xi_p, \quad \forall p \in M. \tag{3.4}$$

When $k+l \in [\varepsilon_0, \varepsilon_1)$, we have $\sigma(k,l) = \sigma(\varepsilon_0)$.

In the following, we demonstrate for any $k+l \in [\varepsilon_0, \varepsilon_1)$

$$\|y(k,l)\|_\infty^{\xi_{\sigma(0,0)}} \leqslant \Phi_0 + [(k+l)-(k_0+l_0)](\gamma^{-1}\frac{\bar{\omega}}{\xi_{\min}}), \tag{3.5}$$

where

$$\xi_{\min} = \min_{p\in M, 1\leq i\leq n} \xi_{pi}$$

and

$$\Phi_0 = \sum_{r=0}^{l} \frac{\hat{h}(r)}{\gamma^{r+1}} + \sum_{s=0}^{k} \frac{\hat{v}(s)}{\gamma^{s+1}}.$$

From (3.2), we have $x(0,0) = y(0,0)$, which implies

$$\|y(0,0)\|_\infty^{\xi_{\sigma(0,0)}} \leqslant \max\{\hat{h}(0), \hat{v}(0)\}.$$

Furthermore, we can get

$$\|y(0,0)\|_\infty^{\xi_{\sigma(0,0)}} \leq \frac{\hat{h}(0)}{\gamma} + \frac{\hat{v}(0)}{\gamma} + [(k_0+l_0)-(k_0+l_0)]\left(\gamma^{-1}\frac{\bar{\omega}}{\xi_{\min}}\right).$$

Therefore, (3.5) is true when $k+l=0$. Assume (3.5) holds for all (k,l) satisfying $k+l \leq u$, where $u \in [\varepsilon_0, \varepsilon_1 - 1)$, $u \in \mathbb{N}$. In the following, we demonstrate that (3.5) is also true for $u+1$. From the definition of l_∞, we have

$$\|y(k,l)\|_\infty^{\xi_{\sigma(0,0)}} \leq \Phi_0 + [(k+l)-(k_0+l_0)](\gamma^{-1}\frac{\bar{\omega}}{\xi_{\min}}), \tag{3.6}$$

where $k + l \leq u$. Since $\boldsymbol{f}_{\sigma(0,0)}$ and $\boldsymbol{g}_{\sigma(0,0)}$ satisfy the Assumption 3.1, from (3.4) and (3.6), we can get

$$\begin{aligned}
&\begin{bmatrix} \boldsymbol{y}^h(k+1,l) \\ \boldsymbol{y}^v(k,l+1) \end{bmatrix} \\
&\leq \gamma^{-1}\boldsymbol{f}_{\sigma(0,0)}\left(\left[\Phi_0 + [(k+l)-(k_0+l_0)]\,\tfrac{\gamma^{-1}\bar{\omega}}{\xi_{\min}}\right]\boldsymbol{\xi}_{\sigma(0,0)}\right) \\
&+\tfrac{\gamma^{-k-l-1}\gamma^{k+l}\bar{\omega}}{\xi_{\min}}\boldsymbol{\xi}_{\sigma(0,0)} + \boldsymbol{g}_{\sigma(0,0)}\left(\left[\Phi_0 + [(k+l)-(k_0+l_0)]\,\tfrac{\gamma^{-1}\bar{\omega}}{\xi_{\min}}\right]\right. \\
&\left. * \begin{bmatrix} \gamma^{-\tau_h(k)-1}, & 0 \\ 0, & \gamma^{-\tau_v(l)-1} \end{bmatrix}\begin{bmatrix} \boldsymbol{\xi}^h_{\sigma(0,0)} \\ \boldsymbol{\xi}^v_{\sigma(0,0)} \end{bmatrix}\right) \\
&\leq \gamma^{-1}\left(\Phi_0 + [(k+l)-(k_0+l_0)]\,\tfrac{\gamma^{-1}\bar{\omega}}{\xi_{\min}}\right)\boldsymbol{f}_{\sigma(0,0)}\left(\boldsymbol{\xi}_{\sigma(0,0)}\right) \\
&+\gamma^{-1}\tfrac{\bar{\omega}}{\xi_{\min}}\boldsymbol{\xi}_{\sigma(0,0)} + \left[\Phi_0 + [(k+l)-(k_0+l_0)]\,\tfrac{\gamma^{-1}\bar{\omega}}{\xi_{\min}}\right] \\
&* \begin{bmatrix} \gamma^{-\tau_{max}-1}, & 0 \\ 0, & \gamma^{-\tau_{max}-1} \end{bmatrix}\boldsymbol{g}_{\sigma(0,0)}\left(\boldsymbol{\xi}_{\sigma(0,0)}\right) \\
&\leq \gamma^{-1}\left[\Phi_0 + [(k+l)-(k_0+l_0)]\,\tfrac{\gamma^{-1}\bar{\omega}}{\xi_{\min}}\right] \\
&* \left[\boldsymbol{f}_{\sigma(0,0)}\left(\boldsymbol{\xi}_{\sigma(0,0)}\right) + \gamma^{-\tau_{\max}}\boldsymbol{g}_{\sigma(0,0)}\left(\boldsymbol{\xi}_{\sigma(0,0)}\right)\right] + \gamma^{-1}\tfrac{\bar{\omega}}{\xi_{\min}}\boldsymbol{\xi}_{\sigma(0,0)} \\
&\leq \left(\Phi_0 + [(k+l)-(k_0+l_0)]\,\tfrac{\gamma^{-1}\bar{\omega}}{\xi_{\min}} + \gamma^{-1}\tfrac{\bar{\omega}}{\xi_{\min}}\right)\boldsymbol{\xi}_{\sigma(0,0)} \\
&= \left(\Phi_0 + [(k+l+1)-(k_0+l_0)]\,\tfrac{\gamma^{-1}\bar{\omega}}{\xi_{\min}}\right)\boldsymbol{\xi}_{\sigma(0,0)},
\end{aligned} \tag{3.7}$$

where $k_0 + l_0 = 0$. Note that (3.7) is true whether or not $k + l - \tau_h(k)$ and $k + l - \tau_v(l)$ are non-negative. It follows from system (2.1) that

$$\boldsymbol{y}^h(k,l+1) = \begin{bmatrix} E_{n_1} & 0_{n_1\times n_2} \end{bmatrix}\begin{bmatrix} \boldsymbol{y}^h(k,l+1) \\ \boldsymbol{y}^v(k-1,l+2) \end{bmatrix}$$

and

$$\boldsymbol{y}^v(k+1,l) = \begin{bmatrix} 0_{n_2\times n_1} & E_{n_2} \end{bmatrix}\begin{bmatrix} \boldsymbol{y}^h(k+2,l-1) \\ \boldsymbol{y}^v(k+1,l) \end{bmatrix}.$$

Then, based on the preceding analysis, it is not difficult to prove

$$\begin{aligned}
&\boldsymbol{y}^h(k,l+1) \\
&\leq \left(\Phi_0 + [(k+l+1)-(k_0+l_0)]\,\tfrac{\gamma^{-1}\bar{\omega}}{\xi_{\min}}\right)\begin{bmatrix} E_{n_1} & 0_{n_1\times n_2} \end{bmatrix}\boldsymbol{\xi}_{\sigma(0,0)}, \\
&\boldsymbol{y}^v(k+1,l) \\
&\leq \left(\Phi_0 + [(k+l+1)-(k_0+l_0)]\,\tfrac{\gamma^{-1}\bar{\omega}}{\xi_{\min}}\right)\begin{bmatrix} 0_{n_2\times n_1} & E_{n_2} \end{bmatrix}\boldsymbol{\xi}_{\sigma(0,0)}.
\end{aligned} \tag{3.8}$$

As $\frac{\gamma^{-1}\bar{\omega}}{\xi_{\min}}$ is non-negative and Φ is nondecreasing in k, l, $\Phi_0 + [(k+l)-(k_0+l_0)]\,\frac{\gamma^{-1}\bar{\omega}}{\xi_{\min}}$ is nondecreasing in k, l. Combining the Eqs (3.7) and (3.8) yields

$$\boldsymbol{y}(k,l) \leq \left(\Phi_0 + [(k+l+1)-(k_0+l_0)]\,\tfrac{\gamma^{-1}\bar{\omega}}{\xi_{\min}}\right)\boldsymbol{\xi}_{\sigma(0,0)},$$

where $k + l = u + 1$. This implies that

$$\|\boldsymbol{y}(k,l)\|_\infty^{\boldsymbol{\xi}_{\sigma(0,0)}} \leq \Phi_0 + [(k+l)-(k_0+l_0)]\,\frac{\gamma^{-1}\bar{\omega}}{\xi_{\min}}, \tag{3.9}$$

where $k + l = u + 1$. Then, when $k + l = \varepsilon_1$, we have

$$\begin{aligned}
\begin{bmatrix} \boldsymbol{y}^h(k,l) \\ \boldsymbol{y}^v(k,l) \end{bmatrix} &= \boldsymbol{F}_{\sigma(k,l)}\left(\begin{bmatrix} \gamma^{-d_h(k)}, 0 \\ 0, \gamma^{-d_v(l)} \end{bmatrix}\begin{bmatrix} \boldsymbol{y}^h(k-d_h(k),l) \\ \boldsymbol{y}^v(k,l-d_v(l)) \end{bmatrix}\right) \\
&\leq \gamma^{-d_{\max}}\boldsymbol{F}_{\sigma(k,l)}\begin{bmatrix} \boldsymbol{y}^h(k-d_h(k),l) \\ \boldsymbol{y}^v(k,l-d_v(l)) \end{bmatrix} \\
&\leq \gamma^{-d_{\max}}F\left(\Phi_0 + [(k+l)-(k_0+l_0)]\,\gamma^{-1}\frac{\bar{\omega}}{\xi_{\min}}\right)\boldsymbol{\xi}_{\sigma(0,0)}.
\end{aligned}$$

Note that

$$\alpha = \begin{cases} \gamma^{-d_{\max}}F, & if\ \gamma^{-d_{\max}}F \geq 1, \\ 1, & if\ \gamma^{-d_{\max}}F < 1, \end{cases}$$

$$\boldsymbol{y}(k,l) \leq \left(\Phi_0 + [(k+l)-(k_0+l_0)]\,\frac{\gamma^{-1}\bar{\omega}}{\xi_{\min}}\right)\boldsymbol{\xi}_{\sigma(0,0)},$$

where $k + l \in [\varepsilon_0, \varepsilon_1)$, which leads to

$$\begin{bmatrix} \boldsymbol{y}^h(k,l) \\ \boldsymbol{y}^v(k,l) \end{bmatrix} \leq \alpha\left[\Phi_0 + [(k+l)-(k_0+l_0)]\,\gamma^{-1}\frac{\bar{\omega}}{\xi_{\min}}\right]\boldsymbol{\xi}_{\sigma(0,0)},$$

where $k + l \in [\varepsilon_0, \varepsilon_1)$. Therefore, we can get

$$\|\boldsymbol{y}(k,l)\|_\infty^{\boldsymbol{\xi}_{\sigma(0,0)}} \leq \alpha\left[\Phi_0 + [(k+l)-(k_0+l_0)]\,\gamma^{-1}\frac{\bar{\omega}}{\xi_{\min}}\right], \tag{3.10}$$

where $k + l \in [\varepsilon_0, \varepsilon_1)$. Denote $\sigma(k_1, l_1) = \sigma(\varepsilon_1)$ as the switching instant, that is, $k + l = \varepsilon_1$. From the definition of l_∞, we can get

$$\begin{aligned}
\|\boldsymbol{y}(k_1,l_1)\|_\infty^{\boldsymbol{\xi}_{\sigma(k_1,l_1)}} &= \max_{1\leq j\leq n}\frac{\boldsymbol{y}_j(k_1,l_1)}{\boldsymbol{\xi}_{\sigma(k_1,l_1)j}} \\
&= \max_{1\leq j\leq n}\frac{\boldsymbol{\xi}_{\sigma(0,0)j}}{\boldsymbol{\xi}_{\sigma(k_1,l_1)j}}\frac{\boldsymbol{y}_j(k_1,l_1)}{\boldsymbol{\xi}_{\sigma(0,0)j}} \\
&\leq \max_{1\leq j\leq n}\frac{\bar{\xi}_j}{\underline{\xi}_j}\frac{\boldsymbol{y}_j(k_1,l_1)}{\boldsymbol{\xi}_{\sigma(0,0)j}} \\
&\leq \beta\|\boldsymbol{y}(k_1,l_1)\|_\infty^{\boldsymbol{\xi}_{\sigma(0,0)}}.
\end{aligned}$$

As a result of (3.10), it is clear that

$$\|\boldsymbol{y}(k_1,l_1)\|_\infty^{\boldsymbol{\xi}_{\sigma(k_1,l_1)}} \leq \beta\alpha\left[\Phi_0 + [(k_1+l_1)-(k_0+l_0)]\,\gamma^{-1}\frac{\bar{\omega}}{\xi_{\min}}\right].$$

Let

$$\Phi_1 = \beta\alpha\left[\Phi_0 + [(k_1 + l_1) - (k_0 + l_0)]\,\gamma^{-1}\frac{\bar{\omega}}{\xi_{\min}}\right].$$

Thus,

$$\boldsymbol{y}(k_1, l_1) \leq \left(\Phi_1 + [(k_1 + l_1) - (k_1 + l_1)]\,\gamma^{-1}\frac{\bar{\omega}}{\xi_{\min}}\right)\boldsymbol{\xi}_{\sigma(k_1,l_1)}. \tag{3.11}$$

Similar to the preceding analysis, the following inequality holds

$$\|\boldsymbol{y}(k,l)\|_\infty^{\boldsymbol{\xi}_{\sigma(k_1,l_1)}} \leq \alpha\left[\Phi_1 + [(k+l) - (k_1 + l_1)]\,\gamma^{-1}\tfrac{\bar{\omega}}{\xi_{\min}}\right],$$

where $k + l \in [\varepsilon_1, \varepsilon_2)$. Furthermore, we have

$$\begin{aligned}&\|\boldsymbol{y}(k,l)\|_\infty^{\boldsymbol{\xi}_{\sigma(k_{m-1},l_{m-1})}}\\ &\leq \alpha\left[\Phi_{m-1} + [(k+l) - (k_{m-1} + l_{m-1})]\,\gamma^{-1}\frac{\bar{\omega}}{\xi_{\min}}\right],\end{aligned} \tag{3.12}$$

where $k + l \in [\varepsilon_{m-1}, \varepsilon_m)$. Let

$$\Phi_m = \beta\alpha\left[\Phi_{m-1} + [(k_m + l_m) - (k_{m-1} + l_{m-1})]\,\gamma^{-1}\frac{\bar{\omega}}{\xi_{\min}}\right].$$

Then, we have

$$\|\boldsymbol{y}(k,l)\|_\infty^{\boldsymbol{\xi}_{\sigma(k_m,l_m)}} \leq \alpha\left[\Phi_m + [(k+l) - (k_m + l_m)]\,\gamma^{-1}\frac{\bar{\omega}}{\xi_{\min}}\right],$$

where $k + l \in [\varepsilon_m, \varepsilon_{m+1})$. According to the definition of Φ_i, combining (3.11) and (3.12) leads to

$$\begin{aligned}&\|\boldsymbol{y}(k,l)\|_\infty^{\boldsymbol{\xi}_{\sigma(k_m,l_m)}}\\
&\leq \alpha\left(\beta\alpha\left[\Phi_{m-1} + [(k_m + l_m) - (k_{m-1} + l_{m-1})]\,\gamma^{-1}\tfrac{\bar{\omega}}{\xi_{\min}}\right]\right)\\
&+\alpha\left([(k+l) - (k_m + l_m)]\,\gamma^{-1}\tfrac{\bar{\omega}}{\xi_{\min}}\right)\\
&= \beta\alpha^2\left[\Phi_{m-1} + (\varepsilon_m - \varepsilon_{m-1})\,\gamma^{-1}\tfrac{\bar{\omega}}{\xi_{\min}}\right] + [(k+l) - \varepsilon_m]\,\gamma^{-1}\tfrac{\alpha\bar{\omega}}{\xi_{\min}}\\
&= \beta\alpha^2\left[\beta\alpha\left(\Phi_{m-2} + (\varepsilon_{m-1} - \varepsilon_{m-2})\,\gamma^{-1}\tfrac{\bar{\omega}}{\xi_{\min}}\right)\right.\\
&\left.+ (\varepsilon_m - \varepsilon_{m-1})\,\gamma^{-1}\tfrac{\bar{\omega}}{\varepsilon_{\min}}\right] + (k + l - \varepsilon_m)\,\gamma^{-1}\tfrac{\alpha\bar{\omega}}{\xi_{\min}}\\
&= \beta^2\alpha^3\Phi_{m-2} + \beta^2\alpha^3\,(\varepsilon_{m-1} - \varepsilon_{m-2})\,\gamma^{-1}\tfrac{\bar{\omega}}{\varepsilon_{\min}}\\
&+\beta\alpha^2\,(\varepsilon_m - \varepsilon_{m-1})\,\gamma^{-1}\tfrac{\bar{\omega}}{\varepsilon_{\min}} + (k + l - \varepsilon_m)\,\gamma^{-1}\tfrac{\alpha\bar{\omega}}{\xi_{\min}}\\
&= \beta^m\alpha^{m+1}\Phi_0 + \beta^m\alpha^{m+1}\,(\varepsilon_1 - \varepsilon_0)\,\gamma^{-1}\tfrac{\bar{\omega}}{\xi_{\min}}\\
&+ \cdots + (k + l - \varepsilon_m)\,\gamma^{-1}\tfrac{\alpha\bar{\omega}}{\xi_{\min}}\\
&\leq \beta^m\alpha^{m+1}\Phi_0 + \alpha(k+l)\left(\beta^m\alpha^m + \beta^{m-1}\alpha^{m-1} + \cdots + 1\right)\gamma^{-1}\tfrac{\bar{\omega}}{\xi_{\min}}\\
&= \beta^m\alpha^m\alpha\Phi_0 + (k+l)\alpha\tfrac{1-\beta^{m+1}\alpha^{m+1}}{1-\alpha\beta}\,\tfrac{\gamma^{-1}\bar{\omega}}{\xi_{\min}},\end{aligned}$$

where $k + l \in [\varepsilon_m, \varepsilon_{m+1})$.

Obviously,

$$m \leq \frac{k+l}{\tau_\varepsilon},$$

where

$$\tau_\varepsilon > -\frac{\ln\alpha\beta}{\ln\gamma}.$$

Hence, we get

$$\|\boldsymbol{y}(k,l)\|_\infty^{\boldsymbol{\xi}_{\sigma(k_m,l_m)}} \leq (\beta\alpha)^{\frac{k+l}{\tau_\varepsilon}}\alpha\Phi_0 + \alpha(k+l)\frac{(\beta\alpha)^{\frac{k+l}{\tau_\varepsilon}}}{\beta\alpha - 1}\beta\alpha\frac{\gamma^{-1}\bar{\omega}}{\xi_{\min}}.$$

We can deduce from (3.2) that

$$\begin{aligned}&\|\boldsymbol{x}(k,l)\|_\infty^{\boldsymbol{\xi}_{\sigma(k_m,l_m)}}\\
&\leq \gamma^{k+l}(\alpha\beta)^{\frac{k+l}{\tau_\varepsilon}}\alpha\Phi_0 + \beta\alpha^2(k+l)\frac{\gamma^{k+l}(\alpha\beta)^{\frac{k+l}{\tau_\varepsilon}}}{\beta\alpha - 1}\frac{\gamma^{-1}\bar{\omega}}{\xi_{\min}}\\
&= \left((\alpha\beta)^{\frac{1}{\tau_\varepsilon}}\gamma\right)^{k+l}\alpha\Phi_0 + \beta\alpha^2(k+l)\frac{\left(\gamma(\alpha\beta)^{\frac{1}{\tau_\varepsilon}}\right)^{k+l}}{\beta\alpha - 1}\frac{\gamma^{-1}\bar{\omega}}{\xi_{\min}}\\
&= \left(e^{\frac{\ln\alpha\beta}{\tau_\varepsilon}+\ln\gamma}\right)^{k+l}\alpha\Phi_0 + \frac{\beta\alpha^2}{\beta\alpha - 1}\frac{\gamma^{-1}\bar{\omega}}{\xi_{\min}}(k+l)\left(e^{\frac{\ln\alpha\beta}{\tau_\varepsilon}+\ln\gamma}\right)^{k+l}.\end{aligned}$$

Denote

$$b = \alpha \quad\text{and}\quad c = e^{\frac{\ln\alpha\beta}{\tau_\varepsilon}+\ln\gamma}.$$

Furthermore, if we let

$$f(x) = xc^x (0 < c < 1),$$

then

$$f_{\max} = f\left(-\frac{1}{\ln c}\right) = -\frac{1}{c^{\frac{1}{\ln c}}\ln c}.$$

Hence,

$$(k+l)\,c^{k+l} \leq f_{\max}, \quad k, l \in N_0.$$

Let

$$a = -\frac{r^{-1}\bar{\omega}\beta\alpha^2}{(\beta\alpha - 1)\xi_{\min}}\frac{1}{c^{\frac{1}{\ln c}}\ln c}$$

and

$$\bar{\boldsymbol{\xi}} = \left[\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_n\right].$$

Then we have

$$\|\boldsymbol{x}(k,l)\|_\infty^{\bar{\boldsymbol{\xi}}} \leq \|\boldsymbol{x}(k,l)\|_\infty^{\boldsymbol{\xi}_{\sigma(k_m,l_m)}} \leqslant a + b\,\Phi_0 c^{k+l}.$$

That is, system (2.1) converges exponentially within a ball. □

Remark 3.2. *Comparing with the main result given in [22], the external disturbances and impulsive effects are considered. If we let* $\boldsymbol{\omega}(k,l) \equiv 0$ *and impulsive matrix* $\boldsymbol{F}_P \equiv 0$ *in Theorem 1, then any solution of system* (2.1) *under the switching signal with ADT*

$$\tau_\varepsilon > -\frac{ln\alpha\beta}{ln\gamma}$$

satisfying

$$\|\boldsymbol{x}(k,l)\|_\infty^{\bar{\xi}} \leq \|\boldsymbol{x}(k,l)\|_\infty^{\xi_{\sigma(k_m,l_m)}} \leqslant b\,\Phi_0 c^{k+l}.$$

That is, Theorem 3.1 in this paper reduces to [22, Theorem 2].

Remark 3.3. *It follows from the proof of Theorem 3.1 that the convergence rate is related to the parameter* γ. *On the other hand,* γ_{pi} *is the unique solution of the Eq* (3.1). *Obviously,* γ_{pi} *is monotonically increasing in* $\widehat{\tau_h}$ *and* $\widehat{\tau_v}$, *and* γ_{pi} *approaches to one as* $max(\widehat{\tau_h}, \widehat{\tau_v})$ *tends to infinity. This implies that system delays have an impact on the convergence rate.*

In the following, we extend the impulse matrix to the nonlinear case.

Corollary 3.1. *If the impulse matrix*

$$\boldsymbol{F}_P = diag\left\{F_{P_{11}}(x), F_{P_{22}}(x), \dots, F_{P_{nn}(x)}\right\}$$

is bounded for any $F_{P_{ii}}(x)$, $i = 1,2,\dots,n$, *then system* (2.1) *converges exponentially within a ball under a class of ADT switching signals.*

Proof. Let

$$F = \sup_{p\in M, 1\leq i\leq n} \sup_x |F_{P_{ii}}(x)|.$$

Then, Corollary 3.1 can be derived from Theorem 3.1. □

3.2. 2-D SNPSs with multi-directional delays

Consider 2-D SNPSs with multiple time-varying delays

$$\begin{cases} \begin{bmatrix} \boldsymbol{x}^h(k+1,l) \\ \boldsymbol{x}^v(k,l+1) \end{bmatrix} = \boldsymbol{f}_{\sigma(k,l)} \begin{bmatrix} \boldsymbol{x}^h(k,l) \\ \boldsymbol{x}^v(k,l) \end{bmatrix} + \sum_{s=1}^N \boldsymbol{g}_{s\sigma(k,l)} \begin{bmatrix} \boldsymbol{x}^h(k-\tau_{hs}(k),l) \\ \boldsymbol{x}^v(k,l-\tau_{vs}(l)) \end{bmatrix} \\ \qquad +\boldsymbol{\omega}(k,l), \quad k+l \neq \varepsilon_r, \\ \begin{bmatrix} \boldsymbol{x}^h(k,l) \\ \boldsymbol{x}^v(k,l) \end{bmatrix} = \sum_{z=1}^Q \boldsymbol{F}^{\,z}_{\sigma(k,l)} \begin{bmatrix} \boldsymbol{x}^h(k-d_{hz}(k),l) \\ \boldsymbol{x}^v(k,l-d_{vz}(l)) \end{bmatrix}, \quad k+l=\varepsilon_r, \end{cases} \tag{3.13}$$

where the delay functions $\tau_{hs}(k)$, $\tau_{vs}(l)$, $d_{hz}(k)$, and $d_{vz}(l)$ satisfy $0 \leq \tau_{hs}(k) \leq \bar{\tau}_{hs}$, $0 \leq \tau_{vs}(l) \leq \bar{\tau}_{vs}$, $0 \leq d_{hz}(k) \leq \bar{d}_{hz}$, $0 \leq d_{vz}(l) \leq \bar{d}_{vz}$,$s \in \{1,2,\dots,N\}$, $z \in \{1,2,\dots,Q\}$.

Now, we give the reachable set estimation for the system (3.13).

Theorem 3.2. *Let Assumptions 3.1 and 3.2 hold and the impulse matrix* $\boldsymbol{F}_P^z$ *be bounded for any* $F^{\,z}_{P_{ii}}(x)$, $i = 1,2\dots,n$. *For any* $p \in M$, *if there exists a vector* $\boldsymbol{\xi}_P \gg 0$ *satisfying*

$$\boldsymbol{f}_p(\boldsymbol{\xi}_p) + \sum_{s=1}^N \boldsymbol{g}_{sp}(\boldsymbol{\xi}_p) \ll \boldsymbol{\xi}_p,$$

then each solution of system (3.13) *converges exponentially within a ball with ADT switching satisfying*

$$\tau_\varepsilon > -\frac{ln\alpha\beta}{ln\gamma},$$

where

$$\alpha = \begin{cases} \sum_{z=1}^Q \gamma^{-d_{z\max}} F_z, & if \sum_{z=1}^Q \gamma^{-d_{z\max}} F_z \geq 1, \\ 1, & if \sum_{z=1}^Q \gamma^{-d_{z\max}} F_z < 1, \end{cases}$$

$$\tau_{s\max} = \max\left(\bar{\tau}_{hs}, \bar{\tau}_{vs}\right), \quad d_{z\max} = \max\left(\bar{d}_{hz}, \bar{d}_{vz}\right),$$

$$F_z = \sup_{p\in M, 1\leq i\leq n} \sup_x |F^{\,z}_{pii}(x)|$$

and

$$\gamma = \max_{p\in M, 0\leq i\leq n} \gamma_{pi}$$

with γ_{pi} *satisfying*

$$f_{pi}(\boldsymbol{\xi}_p) + \sum_{s=1}^N \gamma_{pi}^{-\tau_{s\max}} g_{spi}(\boldsymbol{\xi}_p) - \gamma_{pi}\xi_{pi} = 0.$$

Proof. The same variable transformation as stated in Theorem 3.1 is also used. Then, according to similar analysis to (3.9), one can verify that

$$\|\boldsymbol{y}(k,l)\|_\infty^{\xi_{\sigma(0,0)}} \leq \Phi_0 + [(k+l)-(k_0+l_0)]\,\frac{\gamma^{-1}\bar{\omega}}{\xi_{\min}}, \quad k+l = u+1.$$

As $k+l = \varepsilon_1$, we have

$$\begin{aligned} &\begin{bmatrix} \boldsymbol{y}^h(k,l) \\ \boldsymbol{y}^v(k,l) \end{bmatrix} \\ &= \sum_{z=1}^Q \boldsymbol{F}^{\,z}_{\sigma(k,l)} \left(\begin{bmatrix} \gamma^{-d_{hz}(k)}, 0 \\ 0, \gamma^{-d_{vz}(l)} \end{bmatrix} \begin{bmatrix} \boldsymbol{y}^h\left(k-d_{hz}(k),l\right) \\ \boldsymbol{y}^v\left(k,l-d_{vz}(l)\right) \end{bmatrix} \right) \\ &\leq \sum_{z=1}^Q \gamma^{-d_{z\max}} \boldsymbol{F}^{\,z}_{\sigma(k,l)} \begin{bmatrix} \boldsymbol{y}^h\left(k-d_{hz}(k),l\right) \\ \boldsymbol{y}^v\left(k,l-d_{vz}(l)\right) \end{bmatrix} \\ &\leq \sum_{z=1}^Q \gamma^{-d_{z\max}} F_z \left(\Phi_0 + [(k+l)-(k_0+l_0)]\,\gamma^{-1}\frac{\bar{\omega}}{\xi_{\min}} \right) \xi_{\sigma(0,0)}. \end{aligned}$$

Then, it follows from the definition of α that

$$\|\boldsymbol{y}(k,l)\|_\infty^{\xi_{\sigma(0,0)}} \leq \alpha\left[\Phi_0 + [(k+l)-(k_0+l_0)]\,\gamma^{-1}\frac{\bar{\omega}}{\xi_{\min}}\right],$$

where $k+l \in [\varepsilon_0, \varepsilon_1)$. The rest of the proof can be analyzed applying the same arguments as in the proof of Theorem 3.1. It will be omitted here. □

Theorem 3.2 can be generalized to general 2-D switched linear systems.

$$\begin{cases} \begin{bmatrix} \boldsymbol{x}^h(k+1,l) \\ \boldsymbol{x}^v(k,l+1) \end{bmatrix} = \boldsymbol{A}_{\sigma(k,l)} \begin{bmatrix} \boldsymbol{x}^h(k,l) \\ \boldsymbol{x}^v(k,l) \end{bmatrix} + \boldsymbol{\omega}(k,l) \\ \qquad + \sum_{s=1}^{N} \boldsymbol{B}_{s\sigma(k,l)} \begin{bmatrix} \boldsymbol{x}^h(k-\tau_{hs}(k),l) \\ \boldsymbol{x}^v(k,l_{vs}-\tau(l)) \end{bmatrix}, \\ \qquad k+l \neq \varepsilon_r, \\ \begin{bmatrix} \boldsymbol{x}^h(k,l) \\ \boldsymbol{x}^v(k,l) \end{bmatrix} = \sum_{z=1}^{Q} \boldsymbol{F}^{\,z}_{\sigma(k,l)} \begin{bmatrix} \boldsymbol{x}^h(k-d_{hz}(k),l) \\ \boldsymbol{x}^v(k,l-d_{vz}(l)) \end{bmatrix}, \quad k+l=\varepsilon_r. \end{cases} \tag{3.14}$$

Denote

$$\left|\boldsymbol{A}_p\right| = \left[\left|a_{pij}\right|\right]_{n\times n}, \left|\boldsymbol{B}_{sp}\right| = \left[\left|b^{(s)}_{pij}\right|\right]_{n\times n}.$$

Theorem 3.3. *If for any $p \in M$, there exists a vector $\boldsymbol{\xi}_P \gg 0$ such that*

$$\left(\left|\boldsymbol{A}_p\right| + \sum_{s=1}^{N}\left|\boldsymbol{B}_{sp}\right|\right)\boldsymbol{\xi}_p \ll \boldsymbol{\xi}_p,$$

then any solution of the system (3.14) *converges exponentially within a ball under certain ADT switching. The ADT switching signals satisfy*

$$\tau_\varepsilon > -\frac{ln\alpha\beta}{ln\gamma},$$

where

$$\gamma = \max_{p\in M}\max_{1\leq i\leq n}\gamma_{pi}$$

with γ_{pi}

$$\sum_{j=1}^{n}\left|a_{pij}\right|\xi_{pj} + \sum_{s=1}^{N}(\gamma_{pi}^{-\tau_{s\max}}\sum_{j=1}^{n}\left|b^{(s)}_{pij}\right|\xi_{pj}) - \gamma_{pi}\xi_{pi} = 0.$$

Proof. It is simple to check that

$$\begin{bmatrix} |\boldsymbol{x}^h(k+1,l)| \\ |\boldsymbol{x}^v(k,l+1)| \end{bmatrix} \leq |\boldsymbol{A}_{\sigma(k,l)}| \begin{bmatrix} |\boldsymbol{x}^h(k,l)|\,|\boldsymbol{x}^v(k,l)| \end{bmatrix} + \sum_{s=1}^{N}|\boldsymbol{B}_{s\sigma(k,l)}| \begin{bmatrix} |\boldsymbol{x}^h(k-\tau_{hs}(k),l)| \\ |\boldsymbol{x}^v(k,l-\tau_{vs}(l))| \end{bmatrix} + \boldsymbol{\omega}(k,l).$$

Then, the method to prove Theorem 3.3 is similar to that of Theorem 3.1, and it is omitted. □

3.3. 2-D SPNSs with heterogeneous time-varying delays

Consider 2-D SPNSs with heterogeneous time-varying delays.

$$\begin{cases} x_i^h(k+1,l) = \boldsymbol{f}_{\sigma(k,l)i}\begin{bmatrix} \boldsymbol{x}^h(k,l) \\ \boldsymbol{x}^v(k,l) \end{bmatrix} + \boldsymbol{\omega}_i(k,l) \\ \qquad + \boldsymbol{g}_{\sigma(k,l)i}\begin{bmatrix} \left(x_1^h\left(k-\tau^i_{h1}(k),l\right)\cdots x^h_{n_1}\left(k-\tau^i_{hn_1}(k),l\right)\right)^\top \\ \left(x_1^v\left(k,l-\tau^i_{v1}(l)\right)\cdots x^v_{n_2}\left(k,l-\tau^i_{vn_2}(l)\right)\right)^\top \end{bmatrix}, \\ \qquad k+l\neq\varepsilon_r, \\ x_i^h(k,l) = \boldsymbol{F}_{\sigma(k,l)i}\begin{bmatrix} \left(x_1^h\left(k-d^i_{h_1}(k),l\right)\cdots x^h_{n_1}\left(k-d^i_{hn_1}(k),l\right)\right)^\top \\ \left(x_1^v\left(k,l-d^i_{v1}(l)\right)\cdots x^v_{n_2}\left(k,l-d^i_{vn_2}(l)\right)\right)^\top \end{bmatrix}, \\ \qquad k+l=\varepsilon_r. \end{cases} \tag{3.15}$$

$x_j^h(k,l)$ and $x_j^v(k,l)$ represent the j-th element of the vector functions $\boldsymbol{x}^h(k,l)$ and $\boldsymbol{x}^v(k,l)$, respectively. The delay functions are non-negative and have an upper bound. Denote

$$\tau_{\max} = \max(\tau^i_{h1}(k),\ldots,\tau^i_{hn_1}(k),\tau^i_{v1}(l),\ldots,\tau^i_{vn_2}(l),\ i=1,2,\ldots,n),$$
$$d_{\max} = \max(d^i_{h1}(k),\ldots,d^i_{hn_1}(k),d^i_{v1}(l),\ldots,d^i_{vn_2}(l),\ i=1,2,\ldots,n).$$

Supposing Assumptions 3.1 and 3.2 hold, we can get the following result.

Theorem 3.4. *If for any $p \in M$, there exists a vector $\boldsymbol{\xi}_P \gg 0$ such that*

$$\boldsymbol{f}_p(\boldsymbol{\xi}_p) + \boldsymbol{g}_p(\boldsymbol{\xi}_p) \ll \boldsymbol{\xi}_p,$$

then system (3.15) *converges exponentially within a ball under appropriate ADT switching. Furthermore, the ADT switching signals satisfy*

$$\tau_\varepsilon > -\frac{ln\alpha\beta}{ln\gamma},$$

where α, β, γ are defined in Theorem 3.1.

Proof. Since the heterogeneous time-varying delays are bounded, Theorem 3.4 can be proved by using the same method used in the proof of Theorem 3.1. □

4. Numerical example

Consider the system (3.15) consisting of two subsystems with

$$\boldsymbol{f}_1\left(\begin{bmatrix} \boldsymbol{x}^h(k,l) \\ \boldsymbol{x}^v(k,l) \end{bmatrix}\right) = \begin{bmatrix} 0.14 & 0.16 \\ 0.25 & 0.1 \end{bmatrix}\begin{bmatrix} \boldsymbol{x}^h(k,l) \\ \boldsymbol{x}^v(k,l) \end{bmatrix}$$

$$+\sqrt{(\boldsymbol{x}^h(k,l))^2+(\boldsymbol{x}^v(k,l))^2}\begin{bmatrix}0.01\\0.05\end{bmatrix},$$

$$\boldsymbol{g}_1\left(\begin{bmatrix}\boldsymbol{x}^h(k,l)\\\boldsymbol{x}^v(k,l)\end{bmatrix}\right)=\begin{bmatrix}\frac{0.625\boldsymbol{x}^h(k,l)\boldsymbol{x}^v(k,l)}{\sqrt{(2.3\boldsymbol{x}^h(k,l))^2+(\boldsymbol{x}^v(k,l))^2}}\\\frac{0.5x^h(k,l)\boldsymbol{x}^v(k,l)}{\sqrt{(\boldsymbol{x}^h(k,l))^2+(\boldsymbol{x}^v(k,l))^2}}\end{bmatrix},$$

$$\boldsymbol{f}_2\left(\begin{bmatrix}\boldsymbol{x}^h(k,l)\\\boldsymbol{x}^v(k,l)\end{bmatrix}\right)=\begin{bmatrix}0.3 & 0.23\\0.2 & 0.4\end{bmatrix}\begin{bmatrix}\boldsymbol{x}^h(k,l)\\\boldsymbol{x}^v(k,l)\end{bmatrix}$$

$$+\sqrt{(\boldsymbol{x}^h(k,l))^2+(2\boldsymbol{x}^v(k,l))^2}\begin{bmatrix}0.02\\0.04\end{bmatrix},$$

$$\boldsymbol{g}_2\left(\begin{bmatrix}\boldsymbol{x}^h(k,l)\\\boldsymbol{x}^v(k,l)\end{bmatrix}\right)=\begin{bmatrix}\frac{0.22\boldsymbol{x}^h(k,l)\boldsymbol{x}^v(k,l)}{\sqrt{(2.3\boldsymbol{x}^h(k,l))^2+(\boldsymbol{x}^v(k,l))^2}}\\\frac{0.1x^h(k,l)\boldsymbol{x}^v(k,l)}{\sqrt{(\boldsymbol{x}^h(k,l))^2+(\boldsymbol{x}^v(k,l))^2}}\end{bmatrix},$$

$$\boldsymbol{F}_1=\begin{bmatrix}0.5 & 0\\0 & 1.02\end{bmatrix},\quad \boldsymbol{F}_2=\begin{bmatrix}1.01 & 0\\0 & 0.8\end{bmatrix},$$

$$\boldsymbol{\omega}(k,l)=0.25\begin{bmatrix}|\sin(k)|\\|\cos(l)|\end{bmatrix}.$$

Obviously, the vector fields $\boldsymbol{f}_1$, $\boldsymbol{f}_2$, $\boldsymbol{g}_1$, and $\boldsymbol{g}_2$ are homogeneous of degree one and order preserving. $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, and $\boldsymbol{\omega}(k,l)$ are bounded. It is determined that there exist vectors

$$\boldsymbol{\xi}_1=[1.09,1.09]^T \quad\text{and}\quad \boldsymbol{\xi}_2=[0.8,1.15]^T$$

such that

$$(f_i+g_i)\,\boldsymbol{\xi}_i \ll \boldsymbol{\xi}_i.$$

Let

$$\tau_h(k)=1+3\sin\left(\frac{\pi}{2}k\right),\quad \tau_v(l)=1+3\cos\left(\frac{\pi}{2}l\right)$$

and

$$d_h(k)=1+|\sin\left(\frac{\pi}{2}k\right)|,\quad d_v(l)=1+|\cos\left(\frac{\pi}{2}l\right)|.$$

It follows from Eq (3.1) that

$$\gamma_{11}=0.8821,\ \gamma_{12}=0.9181,\ \gamma_{21}=0.8982,\ \gamma_{22}=0.7595.$$

We pick $\gamma=0.9181$. Then, according to Theorem 1, the SPNS converges exponentially within a ball under ADT switching $\tau_\varepsilon \geq 6.48$. Figure 1 shows the ADT switching signal. Figures 2 and 3 provide the estimates for $x^h(k,l)$ and $x^v(k,l)$ under the switching signal $\tau_\varepsilon=7$, respectively.

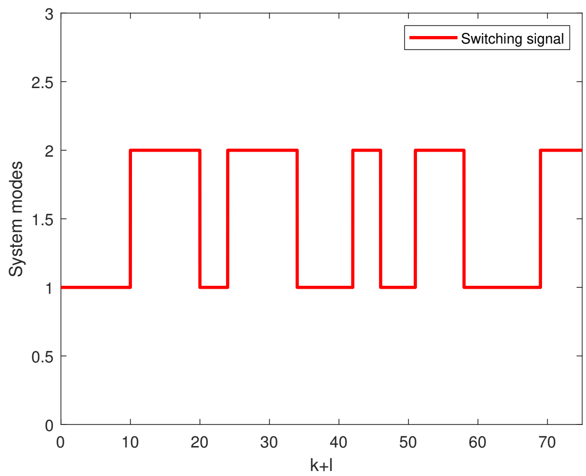

Figure 1. The ADT switching signal.

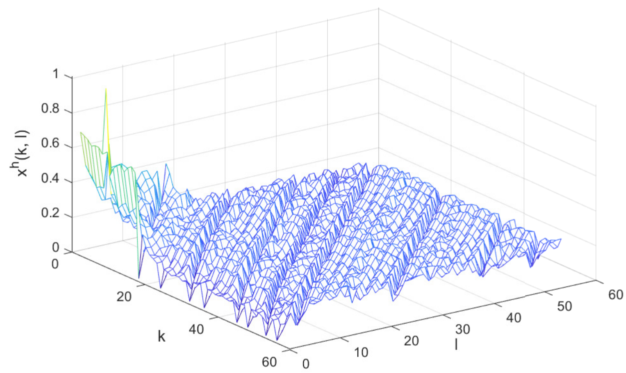

Figure 2. The estimate for $x^h(k,l)$.

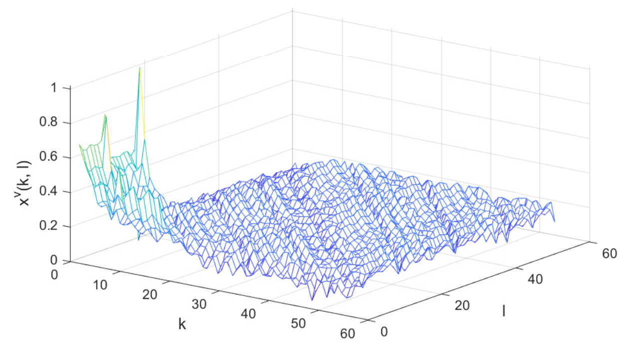

Figure 3. The estimate for $x^v(k,l)$.

5. Conclusions

The reachable set estimation for 2-D SNPSs in the Roesser model with unknown exogenous disturbances are studied. System delays and delayed impulse effects are all considered in the involved systems. For bounded directional delays and delayed impulse effects, an explicit sufficient is presented for the presence of a ball such that any solution of the system converge exponentially within it. The existing result can be seen as a special case of this article. Finally, we also extend the result to 2-D SNPSs with multiple directional delays, general 2-D switched linear systems, and 2-D SNPSs with heterogeneous time-varying delays.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

All authors declare that there are no conflicts of interest in this paper.

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