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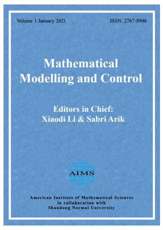

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Research article

A second order quadratic integral inequality associated with regular problems

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Abstract: In this paper, we establish a quadratic integral inequality involving the second order derivative of functions in the following form: for all $f \in D$,

$$\int_a^b r|f''|^2 + p|f'|^2 + q|f|^2 \geq \mu_0 \int_a^b |f|^2.$$

Here r, p, q are real- valued coefficient functions on the compact interval $[a, b]$ with $r(x) > 0$. D is a linear manifold in the Hilbert function space $L^2(a, b)$ such that all integrals of the above inequality are finite and μ_0 is a real number that can be determined in terms of the spectrum of a uniquely determined self adjoint differential operator in $L^2(a, b)$. The inequality is the best possible, i.e., the number μ_0 cannot be increased. f is a complex-valued function in D.

Keywords: regular problems; self-adjoint differential operators

1. Introduction

In this paper, we establish a quadratic integral inequality which involves the second order derivative of functions. The integral inequality is given as below: for all $f \in D$,

$$\int_a^b r|f''|^2 + p|f'|^2 + q|f|^2 \geq \mu_0 \int_a^b |f|^2. \tag{1.1}$$

Here r, p, q are real valued coefficient functions on the compact interval $[a, b]$ with $r(x) > 0$. D is a linear manifold in the Hilbert function space $L^2(a, b)$ such that all integrals of the above inequality are finite and μ_0 is a real number that can be determined in terms of the spectrum of a self adjoint differential operator in $L^2(a, b)$. The inequality is the best possible, i.e., the number μ_0 cannot be increased. f is a complex-valued function in D.

The above inequality is an extension of the following integral inequality:

$$\int_a^b p|f'|^2 + q|f|^2 \geq \mu \int_a^b |f|^2, \qquad f \in D, \tag{1.2}$$

where p and q are given real- valued coefficient functions defined on the interval of integration such that $p(x) > 0$, f is a complex-valued function in a linear manifold D of the Hilbert function space $L^2(a, b)$ where all integrals of (1.2) are finite, and μ is a real number that can be characterized in terms of the spectrum of a uniquely determined self adjoint differential operator in $L^2(a, b)$. The inequality is best possible, i.e., the number μ cannot be increased and all cases of equality are characterized again in terms of the properties of the differential operator in $L^2(a, b)$. Our objective for this paper is to extend the inequality (1.2) to an integral inequality which involves the second -order derivative of functions instead of the first order. A differential operator associated with inequality (1.1) is introduced by minimizing the functional in the calculus of variations.

The inequality (1.2) is established in [1–3] under different conditions. In [2], the authors established the quadratic integral inequality (1.2) for the interval of integration $-\infty < a < b \leq \infty$ where the problem is called regular when $b < \infty$ and singular when $b = \infty$. In fact, we have singular problems associated with inequality (1.2) for either of the following cases:

i) $b = \infty$; or

ii) p^{-1} does not belong to $L(a, b)$.

In [3], the inequality holds for singular problems on a bounded interval $[a, b]$, but in that case the other condition for singular problems holds. In [1], the quadratic integral inequality (1.2) is established for the interval of integration $[a, b)$ by using a new and much improved method compared to that established in [2], where $-\infty < a < b \leq \infty$.

In the present article, singular problems associated with inequality (1.1) occurs for either of the following cases:

i) $b = \infty$; or

ii) r^{-1} does not belong to $L(a, b)$.

In the present paper $b < \infty$ is considered. The positivity condition and absolute continuous property of $r(x)$ on $[a, b]$ together imply that r^{-1} belongs to $L(a, b)$. Thus, the problem is regular. Here, the differential operator associated with inequality (1.1) is introduced by minimizing the functional $\{r|f''|^2 + p|f'|^2 + q|f|^2\}$ in the calculus of variations. Euler-Poisson equation [4] for existence of an extremal for such a problem for $n = 2$ is given by

$$F_y - \frac{d}{dx}(F_{y'}) + \frac{d^2}{dx^2}(F_{y''}) = 0. \tag{1.3}$$

For the functional $\{r|f''|^2 + p|f'|^2 + q|f|^2\}$, the Eq (1.3) yields

$$(ry'')'' - (py')' + qy = 0. \tag{1.4}$$

Now, we define the differential operator associated with inequality (1.1) by

$$M(y) = \lambda y, \tag{1.5}$$

where λ is a parameter and M is the fourth order differential equation such that

$$M(y) = (ry'')'' - (py')' + qy. \tag{1.6}$$

Here in Section 2.1, we state the basic conditions on the coefficients r, p, q that are required for the establishment of our result. In Section 2.2, we first compress the domain of the operator M so that in the contracted domain the operator M is a closed symmetric operator, hence, it has a self-adjoint extension [5], we then derive the domain of the self-adjoint extension. In Section 2.3, we define three linear manifolds $\Delta, \mathcal{D}, D$ to show differences in the domains of the operator. We define the self adjoint differential operator $T_{\alpha,\beta,\gamma,\delta}$ in Section 2.4.

Spectral theory of self adjoint differential operators, the theory of Lebesgue integration and absolute continuity, and also some results from the calculus of variations are different pillars of our results. The knowledge of integral inequalities that depend upon Lebesgue integration and absolute continuity are adopted here from the books "Inequalities" by Hardy et al. [6] and "Priciples of mathematical analysis" by Rudin [7]. The ideas of ordinary quasi- differential expressions and operators and the spectral theory of self adjoint differential operators are found in the books by Akhiezer and Glazman [8], Naimark [5] and Dunford and Schwartz [9]. The concept of the Euler-Poisson equation for minimizing a functional involving the second- order derivative of functions can be found in the book by Elsgolts [4]. There are also references of some other books. Some earlier works on self adjoint differential operators and their associated spectrum are detailed in the papers [2, 9–14]. The quadratic integral inequalities associated with regular and singular problems involving the first-order derivative of functions are detailed [1–3], and an application of a quadratic integral inequality involving the first-order derivative of functions associated with self adjoint differential operator can be found in [15]. Some recent works on self-adjoint differential operators are detailed in [16–18]. Regarding self adjoint differential operators, we are concerned with semi bounded operators; the theory of unbounded linear operators is described in [19].

Some recent works on integral inequalies may be found in [20–25], although the results therein are not directly related to our results.

In Section 3, we prove inequality (1.1) by using a theorem named Theorem 1. We first prove the inequality (1.1) in a subset $\mathcal{D}(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$ of the domain $\mathcal{D}$; we then extend it to the larger set D. For this extension, we establish a lemma, named Lemma 1, and, using this lemma, we extend the inequality (1.1) from $\mathcal{D}(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})$ to the domain $\mathcal{D}$. There is an explicit application of the spectral theory of self adjoint differential operators in our results. Establishing the boundary conditions and the construction of the domain of the self adjoint operator that satisfies these boundary conditions also play significant roles.

2. Basic conditions on the coefficients and construction of the self adjoint differential operator

2.1. Basic conditions on the coefficients r, p, q

In this section, we state the basic conditions on the real-valued coefficient functions r, p, q which are required for our results.

Let the following basic conditions hold for the given real valued coefficient functions r, p and q on a closed and bounded interval $[a, b]$ (for detail explanation, see [5]):

i) r, r' both are absolutely continuous on $[a, b]$ with $r(x) > 0$ on $[a, b]$ and $r''(x) \in L^2(a, b)$;

ii) p is absolutely continuous on [a,b] and $p'(x) \in L^2(a, b)$;

iii) $q \in L(a, b)$.

2.2. Construction of the domain of the self adjoint extension

For a given function y, the self adjointness of the differential expression $(ry'')'' - (py')' + qy$ is ensured from the basic conditions on the coefficients r, p, q assumed in Section 2.1.

The differential operator

$$M[y] = (ry'')'' - (py')' + qy$$

for a given function y, defined in the previous section, is regular on [a,b]. In order to define such an operator, the necessary conditions are that all quasi-derivatives $y^{[k]}$, $k = 0, 1, 2, 3$ should be absolutely continuous on every subinterval $[\alpha,\beta]$ of (a,b) and $M[y] \in L^2(a, b)$. Now, these conditions are clearly true when the the basic conditions on the coefficient functions r, p, q hold.

The quasi derivatives $y^{[k]}$ are defined as follows:

$$y^{[0]} = y, \qquad y^{[1]} = y',$$
$$y^{[2]} = ry'', \quad y^{[3]} = py' - (ry'')'.$$

Let D_0 be the set of all functions $y(x)$ which satisfy the conditions

$$y^{[k]}(a) = y^{[k]}(b), \quad k = 0, 1, 2, 3$$

and M_0 is the restriction of the operator M to D_0, i.e., the operator M_0 has the domain D_0 and is defined by

$$M_0(y) = M(y)$$

for every $y \in D_0$. Now, the operator M_0, being regular, becomes closed symmetric and adjoint to M_0; hence, M_0 has a self adjoint extension [5].

Every self adjoint extension M_u of the operator M_0 is determined by the following linearly independent boundary conditions (for more details, see [5]):

$$\sum_{k=1}^{k=2n} \alpha_{jk} y^{[k-1]}(a) + \sum_{k=1}^{k=2n} \beta_{jk} y^{[k-1]}(b) = 0, \ j = 1, 2, ..., 2n \quad (2.1)$$

and

$$\sum_{v=1}^{v=n} \alpha_{jv}\bar{\alpha}_{k,2n-v+1} - \sum_{v=1}^{v=n} \alpha_{j,2n-v+1}\bar{\alpha}_{kv} = \sum_{v=1}^{v=n} \beta_{jv}\bar{\beta}_{k,2n-v+1} - \sum_{v=1}^{v=n} \beta_{j,2n-v+1}\bar{\beta}_{kv}. \quad (2.2)$$

In our problem, for $n = 2$, the above two Eqs (2.1) and (2.2) take the forms (2.3) and (2.4), respectively:

$$\sum_{k=1}^{k=4} \alpha_{jk} y^{[k-1]}(a) + \sum_{k=1}^{k=2n} \beta_{jk} y^{[k-1]}(b) = 0, \quad j = 1.2, 3, 4 \quad (2.3)$$

and

$$\sum_{v=1}^{v=2} \alpha_{jv}\bar{\alpha}_{k,5-v} - \sum_{v=1}^{v=2} \alpha_{j,5-v}\bar{\alpha}_{kv} = \sum_{v=1}^{v=2} \beta_{jv}\bar{\beta}_{k,5-v} - \sum_{v=1}^{v=2} \beta_{j,5-v}\bar{\beta}_{kv}, \ \ k = 1.2, 3, 4. \quad (2.4)$$

Equation (2.3) gives 4 sets of equations for $j = 1, 2, 3$ and 4, respectively.

For $j = 1$, (2.3) gives

$$\alpha_{11}y(a) + \alpha_{12}y^{[1]}(a) + \alpha_{13}y^{[2]}(a) + \alpha_{14}y^{[3]}(a) + \beta_{11}y(b) + \beta_{12}y^{[1]}(b) + \beta_{13}y^{[2]}(b) + \beta_{14}y^{[3]}(b) = 0. \quad (2.5)$$

For $j = 2$, (2.3) gives

$$\alpha_{21}y(a) + \alpha_{22}y^{[1]}(a) + \alpha_{23}y^{[2]}(a) + \alpha_{24}y^{[3]}(a) + \beta_{21}y(b) + \beta_{22}y^{[1]}(b) + \beta_{23}y^{[2]}(b) + \beta_{24}y^{[3]}(b) = 0. \quad (2.6)$$

For $j = 3$, (2.3) gives

$$\alpha_{31}y(a) + \alpha_{32}y^{[1]}(a) + \alpha_{33}y^{[2]}(a) + \alpha_{34}y^{[3]}(a) + \beta_{31}y(b) + \beta_{32}y^{[1]}(b) + \beta_{33}y^{[2]}(b) + \beta_{34}y^{[3]}(b) = 0. \quad (2.7)$$

For $j = 4$, (2.3) gives

$$\alpha_{41}y(a) + \alpha_{42}y^{[1]}(a) + \alpha_{43}y^{[2]}(a) + \alpha_{44}y^{[3]}(a) + \beta_{41}y(b) + \beta_{42}y^{[1]}(b) + \beta_{43}y^{[2]}(b) + \beta_{44}y^{[3]}(b) = 0. \quad (2.8)$$

Again, for each $j = 1, 2, 3, 4$ from (2.4), we have four different cases for $k = 1, 2, 3, 4$, respectively. There are similar results for $j = 2$, $j = 3$, and $j = 4$.

Thus, Eq (2.4) generates 16 different conditions which are given below: for $j = 1; k = 1$,

$$\alpha_{11}\bar{\alpha}_{14} + \alpha_{12}\bar{\alpha}_{13} - \alpha_{14}\bar{\alpha}_{11} - \alpha_{13}\bar{\alpha}_{12} = \beta_{11}\bar{\beta}_{14} + \beta_{12}\bar{\beta}_{13} - \beta_{14}\bar{\beta}_{11} - \beta_{13}\bar{\beta}_{12}. \quad (2.9)$$

For $j = 1; k = 2$,

$$\alpha_{11}\bar{\alpha}_{24} + \alpha_{12}\bar{\alpha}_{23} - \alpha_{14}\bar{\alpha}_{21} - \alpha_{13}\bar{\alpha}_{22} = \beta_{11}\bar{\beta}_{24} + \beta_{12}\bar{\beta}_{23} - \beta_{14}\bar{\beta}_{21} - \beta_{13}\bar{\beta}_{22}. \quad (2.10)$$

For $j = 1; k = 3$,

$$\alpha_{11}\bar{\alpha}_{34} + \alpha_{12}\bar{\alpha}_{33} - \alpha_{14}\bar{\alpha}_{31} - \alpha_{13}\bar{\alpha}_{32} = \beta_{11}\bar{\beta}_{34} + \beta_{12}\bar{\beta}_{33} - \beta_{14}\bar{\beta}_{31} - \beta_{13}\bar{\beta}_{32}. \quad (2.11)$$

For $j = 1; k = 4$,

$$\alpha_{11}\bar{\alpha}_{44} + \alpha_{12}\bar{\alpha}_{43} - \alpha_{14}\bar{\alpha}_{41} - \alpha_{13}\bar{\alpha}_{42} = \beta_{11}\bar{\beta}_{44} + \beta_{12}\bar{\beta}_{43} - \beta_{14}\bar{\beta}_{41} - \beta_{13}\bar{\beta}_{42}. \quad (2.12)$$

For $j = 2; k = 1$,

$$\alpha_{21}\bar{\alpha}_{14} + \alpha_{22}\bar{\alpha}_{13} - \alpha_{24}\bar{\alpha}_{11} - \alpha_{23}\bar{\alpha}_{12} = \beta_{21}\bar{\beta}_{14} + \beta_{22}\bar{\beta}_{13} - \beta_{24}\bar{\beta}_{11} - \beta_{23}\bar{\beta}_{12}. \quad (2.13)$$

For $j = 2; k = 2$,

$$\alpha_{21}\bar{\alpha}_{24} + \alpha_{22}\bar{\alpha}_{23} - \alpha_{24}\bar{\alpha}_{21} - \alpha_{23}\bar{\alpha}_{22} = \beta_{21}\bar{\beta}_{24} + \beta_{22}\bar{\beta}_{23} - \beta_{24}\bar{\beta}_{21} - \beta_{23}\bar{\beta}_{22}. \quad (2.14)$$

For $j = 2; k = 3$,

$$\alpha_{21}\bar{\alpha}_{34} + \alpha_{22}\bar{\alpha}_{33} - \alpha_{24}\bar{\alpha}_{31} - \alpha_{23}\bar{\alpha}_{32} = \beta_{21}\bar{\beta}_{34} + \beta_{22}\bar{\beta}_{33} - \beta_{24}\bar{\beta}_{31} - \beta_{23}\bar{\beta}_{32}. \quad (2.15)$$

For $j = 2; k = 4$,

$$\alpha_{21}\bar{\alpha}_{44} + \alpha_{22}\bar{\alpha}_{43} - \alpha_{24}\bar{\alpha}_{41} - \alpha_{23}\bar{\alpha}_{42} = \beta_{21}\bar{\beta}_{44} + \beta_{22}\bar{\beta}_{43} - \beta_{24}\bar{\beta}_{41} - \beta_{23}\bar{\beta}_{42}. \quad (2.16)$$

For $j = 3; k = 1$,

$$\alpha_{31}\bar{\alpha}_{14} + \alpha_{32}\bar{\alpha}_{13} - \alpha_{34}\bar{\alpha}_{11} - \alpha_{33}\bar{\alpha}_{12} = \beta_{31}\bar{\beta}_{14} + \beta_{32}\bar{\beta}_{13} - \beta_{34}\bar{\beta}_{11} - \beta_{33}\bar{\beta}_{12}. \quad (2.17)$$

For $j = 3; k = 2$,

$$\alpha_{31}\bar{\alpha}_{24} + \alpha_{32}\bar{\alpha}_{23} - \alpha_{34}\bar{\alpha}_{21} - \alpha_{33}\bar{\alpha}_{22} = \beta_{31}\bar{\beta}_{24} + \beta_{32}\bar{\beta}_{23} - \beta_{34}\bar{\beta}_{21} - \beta_{33}\bar{\beta}_{22}. \quad (2.18)$$

For $j = 3; k = 3$,

$$\alpha_{31}\bar{\alpha}_{34} + \alpha_{32}\bar{\alpha}_{33} - \alpha_{34}\bar{\alpha}_{31} - \alpha_{33}\bar{\alpha}_{32} = \beta_{31}\bar{\beta}_{34} + \beta_{32}\bar{\beta}_{33} - \beta_{34}\bar{\beta}_{31} - \beta_{33}\bar{\beta}_{32}. \quad (2.19)$$

For $j = 3; k = 4$,

$$\alpha_{31}\bar{\alpha}_{44} + \alpha_{32}\bar{\alpha}_{43} - \alpha_{34}\bar{\alpha}_{41} - \alpha_{33}\bar{\alpha}_{42} = \beta_{31}\bar{\beta}_{44} + \beta_{32}\bar{\beta}_{43} - \beta_{34}\bar{\beta}_{41} - \beta_{33}\bar{\beta}_{42}. \quad (2.20)$$

For $j = 4; k = 1$,

$$\alpha_{41}\bar{\alpha}_{14} + \alpha_{42}\bar{\alpha}_{13} - \alpha_{44}\bar{\alpha}_{11} - \alpha_{43}\bar{\alpha}_{12} = \beta_{41}\bar{\beta}_{14} + \beta_{42}\bar{\beta}_{13} - \beta_{44}\bar{\beta}_{11} - \beta_{43}\bar{\beta}_{12}. \quad (2.21)$$

For $j = 4; k = 2$,

$$\alpha_{41}\bar{\alpha}_{24} + \alpha_{42}\bar{\alpha}_{23} - \alpha_{44}\bar{\alpha}_{21} - \alpha_{43}\bar{\alpha}_{22} = \beta_{41}\bar{\beta}_{24} + \beta_{42}\bar{\beta}_{23} - \beta_{44}\bar{\beta}_{21} - \beta_{43}\bar{\beta}_{22}. \quad (2.22)$$

For $j = 4; k = 3$,

$$\alpha_{41}\bar{\alpha}_{34} + \alpha_{42}\bar{\alpha}_{33} - \alpha_{44}\bar{\alpha}_{31} - \alpha_{43}\bar{\alpha}_{32} = \beta_{41}\bar{\beta}_{34} + \beta_{42}\bar{\beta}_{33} - \beta_{44}\bar{\beta}_{31} - \beta_{43}\bar{\beta}_{32}. \quad (2.23)$$

For $j = 4; k = 4$,

$$\alpha_{41}\bar{\alpha}_{44} + \alpha_{42}\bar{\alpha}_{43} - \alpha_{44}\bar{\alpha}_{41} - \alpha_{43}\bar{\alpha}_{42} = \beta_{41}\bar{\beta}_{44} + \beta_{42}\bar{\beta}_{43} - \beta_{44}\bar{\beta}_{41} - \beta_{43}\bar{\beta}_{42}. \quad (2.24)$$

Now, without any loss of generality, we assume that α_{ij} is not purely imaginary for all $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4$.

Clarly, (2.9), (2.14), (2.19) and (2.24) are unconditionally true. Thus, (2.10) is true if

$$\beta_{11} = \beta_{12} = \beta_{13} = \beta_{14} = \alpha_{12} = \alpha_{13} = \alpha_{21} = \alpha_{24} = 0. \quad (2.25)$$

Hence, (2.5) gives

$$\alpha_{11}y^{[0]}(a) + \alpha_{14}y^{[3]}(a) = 0. \quad (2.26)$$

With the conditions of (2.25), if we include $\alpha_{31} = \alpha_{34} = \alpha_{41} = \alpha_{44} = 0$, then (2.11), (2.12), (2.17), (2.21) and (2.13) become true. Combining all conditions we have

$$\beta_{11} = \beta_{12} = \beta_{13} = \beta_{14} = \alpha_{12} = \alpha_{13} = \alpha_{21} = \alpha_{24} = \alpha_{31} = \alpha_{34} = \alpha_{41} = \alpha_{44} = 0. \quad (2.27)$$

Now, with (2.27), we include the following conditions to make (2.15), (2.16), (2.18) and (2.22) valid.

$$\beta_{21} = \beta_{22} = \beta_{23} = \beta_{24} = \alpha_{32} = \alpha_{33} = \alpha_{42} = \alpha_{43} = 0.$$

Then from (2.6), we get

$$\alpha_{22}y^{[1]}(a) + \alpha_{23}y^{[2]}(a) = 0. \quad (2.28)$$

Now, with all of the previous conditions, we take

$$\beta_{32} = \beta_{41} = \beta_{33} = \beta_{44} = 0 \quad (2.29)$$

to make (2.20) and (2.23) valid.

We get the following from (2.7):

$$\beta_{31}y^{[0]}(b) + \beta_{34}y^{[3]}(b) = 0. \quad (2.30)$$

We get the following from (2.8):

$$\beta_{42}y^{[1]}(b) + \beta_{43}y^{[2]}(b) = 0. \quad (2.31)$$

Hence, we obtain a set of 4 different conditions, such as, (2.26), (2.28), (2.30) and (2.31), which provide conditions for the self adjoint extension and act as separated boundary conditions. For $n = 1$, we can see the construction of the domain $\mathcal{D}(\alpha,\beta)$ for self adjoint extension in [1]. We can also get some other type of boundary condition, i.e., periodic boundary conditions. We can obtain it by applying suitable choices of $\alpha_{ij}, \beta_{ij}, i, j = 1, 2, 3, 4$ in a similar way. Considering separated boundary conditions for the self-adjoint extension of the domain $\mathcal{D}$, we construct the domain $\mathcal{D}(\alpha,\beta,\gamma,\delta)$ and define the self adjoint operator $T(\alpha,\beta,\gamma,\delta)$ in $L^2(a,b)$ with this domain as given in Section 2.3 (ii) of this paper.

2.3. *Construction of the linear manifolds $\Delta, \mathcal{D}, D$*

In this section we define the following linear manifolds of the Hilbert function space $L^2(a,b)$:

i) $\Delta = \Delta(r, p, q) = \{f \in \Delta$ if f, f, f'', f''' all are absolutely continuous on [a,b] and $M[f] \in L^2(a,b)\}$.

Here, $\Delta \subset L^2(a,b)$ and $f \in \Delta$ implies that all quasi-derivatives $f, f^{[1]}, f^{[2]}, f^{[3]}$ are absolutely continuous on [a,b] under the given basic conditions on the coefficients r, p, q.

ii) $\mathcal{D} = (\mathcal{D}(\alpha,\beta,\gamma,\delta) \subset \Delta) = \{f \in \mathcal{D} : f \in \Delta; f'(a)cos(\alpha) + r(a)f''(a)sin(\alpha) = f'(b)cos(\beta) + r(b)f''(b)sin(\beta) = f(a)cos(\gamma) + f^{[3]}(a)sin(\gamma) = f(b)cos(\delta) + f^{[3]}(b)sin(\delta) = 0; \alpha,\beta,\gamma,\delta \in [0,\pi)\}$,
where

$$f^{[3]} = pf' - (rf'')'.$$

This domain obtains the separated boundary conditions, such as, (2.26), (2.28), (2.30) and (2.31) given in this paper in Section 2.2.

iii)

$$D = D(r, p, q) = \{f \in D : f \in AC[a,b]; f', f'' \in L^2(a,b)\}.$$

Here $AC[a,b]$ means absolute continuity on $[a,b]$. We note that $f \in D$ implies that $r^{\frac{1}{2}}f'', |p|^{\frac{1}{2}}f', |q|^{\frac{1}{2}}f \in L^2(a,b)$

For detailed explanation, see [2,5].

2.4. Construction of the associated self adjoint differential operator

Below, we define the self-adjoint differential operator in the domain $\mathcal{D}(\alpha,\beta,\gamma,\delta)$.

For each $\alpha,\beta,\gamma,\delta \in [0,\pi)$, we define an operator such that

$$T(\alpha,\beta,\gamma,\delta) : \mathcal{D}(\alpha,\beta,\gamma,\delta) \to L^2(a,b)$$

by

$$T(\alpha,\beta,\gamma,\delta)(f) = M[f], \quad f \in \mathcal{D} = \mathcal{D}(\alpha,\beta,\gamma,\delta).$$

We see that the differential operator $T(\alpha,\beta,\gamma,\delta)$ defined in this way is self adjoint in $L^2(a,b)$ [5].

3. The inequality $\int_a^b r|f''|^2 + p|f'|^2 + q|f|^2 \geq \mu_0 \int_a^b |f|^2$

Below, we state Theorem 3.1 and Lemma 3.1, and we establish inequality (3.1) by applying Theorem 3.1 and Lemma 3.1.

Theorem 3.1. *The coefficients r, p, q satisfy the basic conditions given in Section 2.1; also let the linear manifold D of $L^2(a,b)$ be defined as in Section 2.3; then for any complex-valued function f in D we have the following inequality:*

$$\int_a^b r|f''|^2 + p|f'|^2 + q|f|^2 \geq \mu_0 \int_a^b |f|^2, \quad (f \in D), \tag{3.1}$$

where μ_0 is a real number defined by the smallest eigen value of the self adjoint differential operator $T(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})$, which is bounded below in $L^2(a,b)$. The inequality is the best possible, i.e., the number μ_0 cannot be increased.

Lemma 3.1. *We suppose that the coefficients r, p, q satisfy the basic conditions given in Section 2.1. Then for a given function f in D and $\epsilon > 0$, there exists a function g in $\mathcal{D}(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})$ for which*

$$|\int_a^b r|f''|^2 - \int_a^b r|g''|^2| < \epsilon,$$
$$|\int_a^b p|f'|^2 - \int_a^b p|g'|^2| < \epsilon,$$
$$|\int_a^b q|f|^2 - \int_a^b q|g|^2| < \epsilon,$$
$$|\int_a^b |f|^2 - \int_a^b |g|^2| < \epsilon.$$

3.1. Proof of the inequality (1.1) in $\mathcal{D}(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})$

In this section, we establish inequality (3.1) for the domain $\mathcal{D}(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})$.

Now,

$$\begin{aligned}
&\int_a^b r|f''|^2 + p|f'|^2 + q|f|^2 \\
&= [rf''\overline{f'}]_a^b - \int_a^b (rf'')'\overline{f'} + \int_a^b pf'\overline{f'} + \int_a^b q|f|^2 \\
&= [rf''\overline{f'}]_a^b + \int_a^b \{(-rf'')' + pf'\}\overline{f'} + \int_a^b q|f|^2 \\
&= [rf''\overline{f'}]_a^b + [((-rf'')' + pf')\overline{f}]_a^b + \int_a^b ((rf'')' - pf')'\overline{f} + \int_a^b qf\overline{f} \\
&= [rf''\overline{f'}]_a^b + [\{(-rf'')' + pf'\}\overline{f}]_a^b + \int_a^b \overline{f}M[f].
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_a^b \{r|f''|^2 + p|f'|^2 + q|f|^2\} \\
&= [rf''\overline{f'}]_a^b + [\{(-rf'')' + pf'\}\overline{f}]_a^b + \int_a^b \overline{f}M[f] \\
&= [rf''\overline{f'}]_a^b + [\{(-rf'')' + pf'\}\overline{f}]_a^b + \int_a^b \overline{f}\lambda f,
\end{aligned} \tag{3.2}$$

where we have

$$M[f] = (rf'')'' - (pf')' + qf$$

from (1.6) and the differential operator $M[y] = \lambda y$ from (1.5).

When $\alpha = \beta = \gamma = \delta = \frac{\pi}{2}$, from the construction of $\mathcal{D}(\alpha,\beta,\gamma,\delta)$ in Section 2.3, we have

$$r(a)f''(a) = r(b)f''(b) = f^{[3]}(a) = f^{[3]}(b) = 0,$$

where

$$f^{[3]} = pf' - (rf'')'.$$

Hence, for $\alpha = \beta = \gamma = \delta = \frac{\pi}{2}$, from (3.2), it follows that

$$\int_a^b \{r|f''|^2 + p|f'|^2 + q|f|^2\} = \lambda \int_a^b |f|^2.$$

In Section 2.4, the differential operator

$$T(\alpha,\beta,\gamma,\delta) = M(f)$$

is defined in a domain in such a way that the operator becomes a self adjoint operator in $L^2(a,b)$. Hence, it has a discrete set of eigen- values, which are all real numbers and have a discrete simple spectrum. Again, the operator $T(\alpha,\beta,\gamma,\delta)$ is bounded below in $L^2(a,b)$ for $r(x) > 0$ for all $(\alpha,\beta,\gamma,\delta)$, even when q is Lebesgue-integrable; q need not be bounded below in $L^2(a,b)$ [5].

Hence, if μ_0 is the smallest eigenvalue of the operator $T(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})$ we obtain the following inequality,

$$\int_a^b \{r|f''|^2 + p|f'|^2 + q|f|^2\} = \lambda \int_a^b |f|^2 \geq \mu_0 \int_a^b |f|^2. \quad (3.3)$$

3.2. Proof of Lemma 3.1

Let $f \in D$; then, for a given positive number η, we can choose a continuously differentiable function ϕ with the property that [5]

$$\phi(a) = \phi'(a) = \phi''(a) = \phi(b) = \phi'(b) = \phi''(b) = 0$$

and

$$\int_a^b |f'' - \phi'|^2 < \eta.$$

The existence of such a function ϕ is ensured by the fact that $f \in D$; also, the set of continuously differentiable functions that vanish with their quasi-derivatives at the end points is dense in D of $L^2(a,b)$.

Now we define a function $g(x)$ by

$$g(x) = f(a) + (x-a)f'(a) + \int_a^x \phi'(t)dt. \quad (3.4)$$

Then

$$g'(x) = f'(a) + \phi(x), g''(x) = \phi'(x)$$

and

$$g(a) = f(a), g'(a) = f'(a) + \phi(a),$$

i.e., $g'(a) = f'(a)$ as $\phi(a) = 0$.

So,

$$\int_a^b |f'' - \phi'|^2 < \eta$$

implies that

$$\int_a^b |f'' - g''|^2 < \eta. \quad (3.5)$$

Now with the function g being continuously differentiable on [a,b], and under the basic conditions on the coefficients r, p, q assumed in this paper, we have $g \in \Delta$. Again, we have that $g''(a) = \phi'(a) = 0$. Similarly $g''(b) = \phi'(b) = 0$ and $\phi''(a) = g'''(a) = \phi''(b) = g'''(b) = 0$.

Now, by construction of g, it follows that

$$g'(a)cos(\frac{\pi}{2}) + r(a)g''(a)sin(\frac{\pi}{2}) = g'(b)cos(\frac{\pi}{2}) + r(b)g''(b)sin(\frac{\pi}{2}) = 0$$

and

$$g(a)cos(\frac{\pi}{2}) + g^{[3]}(a)sin(\frac{\pi}{2}) = g(b)cos(\frac{\pi}{2}) + g^{[3]}(b)sin(\frac{\pi}{2}) = 0,$$

where

$$g^{[3]} = pg' - (rg'')'.$$

This implies that $g \in \mathcal{D}(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})$.

Now, we first assume that f is a real valued function. Since $r(x)$ is absolutely continuous on $[a,b]$, there exists a real number

$$R = max\{r(x) : x \in [a,b]\}.$$

Let $\epsilon > 0$. Using the Cauchy-Schwarz integral inequality and the result given by (3.5), we have

$$\begin{aligned} \left|\int_a^b rf''^2 - \int_a^b rg''^2\right| &\leq \int_a^b r|f''^2 - g''^2| \\ &= \int_a^b r|f'' + g''||f'' - g''| \\ &\leq R\{\int_a^b |f'' + g''|^2\}^{\frac{1}{2}}\{\int_a^b |f'' - g''|^2\}^{\frac{1}{2}} \\ &< R\|f'' + g''\|(\eta)^{\frac{1}{2}} \\ &< \epsilon. \end{aligned}$$

Again,

$$\begin{aligned} \int_a^x (f'' - g'') &= (f' - g')(x) - (f' - g')(a) \\ &= f'(x) - g'(x) - f'(a) + g'(a) \\ &= f'(x) - g'(x), \end{aligned}$$

as $g'(a) = f'(a)$. Using the result given by (3.5) and Cauchy-Schwarz integral inequality in the above equation, we have

$$\begin{aligned} |f'(x) - g'(x)| &\leq \int_a^x |f'' - g''| \\ &\leq (x-a)^{\frac{1}{2}}\{\int_a^x |f'' - g''|^2\}^{\frac{1}{2}} \end{aligned}$$

$$\leq (b-a)^{\frac{1}{2}}\{\int_a^b |f''-g''|^2\}^{\frac{1}{2}}$$
$$< (b-a)^{\frac{1}{2}}(\eta)^{\frac{1}{2}}. \tag{3.6}$$

So,

$$\int_a^b |f'(x)-g'(x)|^2 < (b-a)^2\eta. \tag{3.7}$$

Again, since $p(x)$ is absolutely continuous on $[a,b]$, there exists a real number

$$P = max\{|p(x)| : x \in [a,b]\}.$$

Using the Cauchy-Schwarz inequality and inequality (3.7), we have

$$\begin{aligned} |\int_a^b pf'^2 - \int_a^b pg'^2| &\leq \int_a^b |p||f'^2-g'^2| \\ &= \int_a^b |p||f'+g'||f'-g'| \\ &\leq P\{\int_a^b |f'+g'|^2\}^{\frac{1}{2}}\{\int_a^b |f'-g'|^2\}^{\frac{1}{2}} \\ &< P\|f'+g'\|(b-a)(\eta)^{\frac{1}{2}} \\ &< \epsilon. \end{aligned} \tag{3.8}$$

Proceeding in the same way we have

$$\begin{aligned} \int_a^x (f'-g') &= (f-g)(x)-(f-g)(a) \\ &= f(x)-g(x)-f(a)+g(a) \\ &= f(x)-g(x), \text{ as } g(a)=f(a). \end{aligned} \tag{3.9}$$

Using inequality (3.7) and Cauchy-Schwarz integral inequality again, we have

$$\begin{aligned} |f(x)-g(x)| &\leq \int_a^x |f'-g'| \\ &\leq (x-a)^{\frac{1}{2}}\{\int_a^x |f'-g'|^2\}^{\frac{1}{2}} \\ &\leq (b-a)^{\frac{1}{2}}\{\int_a^b |f'-g'|^2\}^{\frac{1}{2}} \\ &< (b-a)^{\frac{3}{2}}(\eta)^{\frac{1}{2}}. \end{aligned} \tag{3.10}$$

So,

$$\int_a^b |f(x)-g(x)|^2 < (b-a)^4\eta. \tag{3.11}$$

Using the Cauchy-Schwarz inequality and inequality (3.11), we have

$$\begin{aligned} |\int_a^b f^2 - \int_a^b g^2| &\leq \int_a^b |f^2-g^2| \\ &= \int_a^b |f+g||f-g| \\ &\leq \{\int_a^b |f+g|^2\}^{\frac{1}{2}}\{\int_a^b |f-g|^2\}^{\frac{1}{2}} \\ &< \|f+g\|(b-a)^2(\eta)^{\frac{1}{2}} \\ &< \epsilon. \end{aligned} \tag{3.12}$$

Again, inequality (3.10) yields

$$|f(x)-g(x)|^2 < (b-a)^3\eta, \tag{3.13}$$

$$\begin{aligned} \int_a^b |q||f(x)-g(x)|^2 &< (b-a)^3\eta \int_a^b |q| \\ &= Q(b-a)^3\eta, \end{aligned} \tag{3.14}$$

where

$$Q = \int_a^b |q| < \infty,$$

as $q \in L(a,b)$. Now, using (3.14), we obtain

$$\begin{aligned} |\int_a^b qf^2 - \int_a^b qg^2| &= |\int_a^b q(f^2-g^2)| \\ &\leq \{\int_a^b |q||f+g||f-g|\} \\ &= \int_a^b |q|^{\frac{1}{2}}|f+g||q|^{\frac{1}{2}}|f-g| \\ &\leq \{\int_a^b |q||f+g|^2\}^{\frac{1}{2}}\{\int_a^b |q||f-g|^2\}^{\frac{1}{2}} \\ &< Q^{\frac{1}{2}}(b-a)^{3/2}(\eta)^{\frac{1}{2}}\{\||q|^{\frac{1}{2}}(f+g)\|\} \\ &< \epsilon. \end{aligned}$$

This completes the proof of the lemma.

Now, if f is a complex valued function, we can write $f - f_1 + if_2$, where f_1 and f_2 are real valued functions, and we have $g_1, g_2 \epsilon \mathcal{D}(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$ such that

$$|\int_a^b r|f_1''|^2 - r|g_1''|^2| \leq \epsilon/2$$

and

$$|\int_a^b r|f_2''|^2 - r|g_2''|^2| \leq \epsilon/2.$$

Let

$$g = g_1 + ig_2.$$

Here, $g\epsilon\mathcal{D}(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})$ given that $g_1, g_2\epsilon\mathcal{D}(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})$. Therefore,

$$\begin{aligned}|\int_a^b r|f''|^2 - r|g''|^2| &\leq |\int_a^b r|f_1''|^2 - r|g_1''|^2| + |\int_a^b r|f_2''|^2 - r|g_2''|^2| \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon.\end{aligned}$$

Similarly, the other results can be proved.

3.3. Extension of the inequality from $\mathcal{D}(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})$ to D

Now, we have seen that the inequality (3.1) holds for the domain $\mathcal{D}(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})$. In this section, we will extend the inequality from the domain $\mathcal{D}(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})$ to D with the help of Lemma 3.1. The domain D is defined as in Section 2.3.

Suppose that, if possible, the inequality (3.1) does not hold for a function $f \in D$; then, there is a real number $\delta > 0$ such that

$$\int_a^b \{r|f''|^2 + p|f'|^2 + (q - \mu_0)|f|^2\} = -\delta.$$

Now, according to the above lemma, we choose $\epsilon < min\{\frac{\delta}{4}, |\mu_0|\frac{\delta}{4}\}$; again for $f \in D$, we have $g \in \mathcal{D}(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})$, which satisfies the results of Lemma 3.1. We have that

$$\begin{aligned}&\int_a^b \{r|g''|^2 + p|g'|^2 + \{q - \mu_0\}|g|^2\} \qquad (3.15)\\ &= \int_a^b \{r|g''|^2 + p|g'|^2 + \{q - \mu_0\}|g|^2\} + \delta - \delta \\ &= \int_a^b \{r|g''|^2 + p|g'|^2 + \{q - \mu_0\}|g|^2\} \\ &\quad - \int_a^b \{r|f''|^2 + p|f'|^2 + \{q - \mu_0\}|f|^2\} - \delta \\ &\leq |\int_a^b r|f''|^2 - \int_a^b r|g''|^2| + |\int_a^b p|f|'^2 - \int_a^b p|g'|^2| \\ &\quad + |\int_a^b q|f|^2 - \int_a^b q|g|^2| + |\mu_0|| \int_a^b |f|^2 - \int_a^b |g|^2\ | - \delta \\ &< 4\epsilon - \delta \leq \delta - \delta = 0.\end{aligned}$$

But, this contradicts the fact that inequality (1.1) holds in $\mathcal{D}(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})$.

Hence the inequality holds for all $f \in D$. We now show the case of equality. We now consider a function f where $f = c\Psi_0$ when c is any non-zero complex number and Ψ_0 is an eigenfunction of the operator $T(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})$ corresponding to the eigen- value μ_0. Then,

$$M[c\Psi_0] = \mu_0 c\Psi_0$$

and we get the following from inequality (1.1):

$$\begin{aligned}\int_a^b \{r|c\Psi_0''|^2 + p|c\Psi_0'|^2 + q|c\Psi_0|^2\} &= \int_a^b M[c\Psi_0]\overline{c\Psi_0} \\ &= \int_a^b \mu_0 c\Psi_0\overline{c\Psi_0} \\ &= \mu_0 \int_a^b |c\Psi_0|^2.\end{aligned}$$

The above shows that the equality in (1.1) is obtained for μ_0. So, μ_0 is the best possible number in sense of equality in (1.1), i.e., the number μ_0 can not be increased.

4. Conclusions

The inequality (1.1) is an extension of the inequality which involves the second- order derivative of the functions, instead of the first -order derivative of the functions considered in inequality (1.2). The integral inequality obtained in this paper is quite interesting, as well as important, as it provides some applications for the determination of domains of self adjoint operators associated with the differential expression obtained via minimization of a quadratic functional involving the second-order derivative. In inequality (1.1), $\mu_0 \int_a^b |f|^2$ becomes a true infimum of the integral $\int_a^b r|f''|^2 + p|f'|^2 + q|f|^2$ because inequality (1.1) yields an equality for the real number μ_0. So, μ_0 is the best possible number in sense of equality in (1.1) as it cannot be increased. The number μ_0 has great importance in the spectral theory of the self adjoint operator associated with the differential expression for minimizing the functional of inequality (1.1). The inequality (1.1) can be applied in the field of operations research for optimization problems, as well as in numerical analysis for finding errors or some other important characteristics. The result obtained in this paper may have important applications in various fields of mathematics, as well as in different branches of science especially in the branch of physics and mathematical sciences.

This paper deals with separated boundary conditions at the end points a and b. We can have different sets of boundary conditions, as in Section 2.2 by applying some other choices of α_{ij} and β_{ij}, which may lead us to periodic boundary conditions of the mixed symmetric boundary condition. We have established our results only for regular cases and the uniqueness of the parameter μ_0 is not discussed in the paper. We also have not discussed the cases for $b = \infty$. These are the major limitations of our paper. The present article may be extended further by considering those limitations in the future.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authos do not have any conflicts of interest with other works.

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