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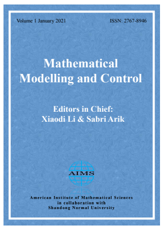

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Research article

Some positive results for exponential-kernel difference operators of Riemann-Liouville type

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Abstract: We established positivity of ∇f obtained from a systematic computation of a composition of sequential fractional differences of the function f that satisfy certain conditions in a negative lower bound setup. First, we considered the different order sequential fractional differences in which we need a complicated condition. Next, we equalled the order of fractional differences and we saw that a simpler condition will be needed. We illustrated our positivity results for an increasing function of the rising type.

Keywords: Riemann-Liouville fractional operators; exponential kernel; positivity analyses; negative lower bound

1. Introduction

Discrete operators and fractional differences/sums are important in many fields of science, including applied science, mathematics, engineering sciences, and physics, as well as some related research fields such as quantum mechanics, number theory, fluid dynamics, mathematical physics, and ordinary/partial differential equation (see, e.g., [1–7]). Recently, Wu et al. [8] studied the inverse problem model and concept of inverse-time fractional chaotic maps with some application involving Riemann-Liouville fractional difference operators; Wu et al. [9] studied Liouville-Caputo fractional difference operators with some more definitions of fractional differences and their applications to fractional maps were compared; Abdeljawad and Baleanu [10] defined Caputo-Fabrizio fractional difference operators and they studied the integration by parts and Euler-Lagrange equations on these operators; and Mohammed et al. [11] examined sharpness results analytically and numerically for those operators involving Atangana-Baleanu fractional differences.

The positivity analysis, which is derived from discrete fractional operators, is one of the significant models in the context of discrete fractional calculus. This analysis is also known as monotonicity analysis, and it represents the positivity of ∇f. The positivity analyses have been frequently utilized to check if a function is increasing or decreasing. Some of the featured applications of positivity analysis are related to Riemann-Liouville fractional difference type [12, 13], Liouville-Caputo fractional difference type [14, 15], Caputo-Fabrizio fractional difference type [16–18], and Atangana-Baleanu fractional difference type [19, 20]. The published articles [21–24] are also important from a sequential aspect to understanding the behavior of monotonicity and positivity analyses in a composition of two discrete fractional operators.

In the present study, we consider the sequential fractional difference operator

$$\left({}^{\mathrm{CF}}_{a+1}\nabla^{\alpha}_{rl}\ {}^{\mathrm{CF}}_{a}\nabla^{\beta}_{rl}f\right)(\tau) \tag{1.1}$$

defined on the pair set

$$\mathscr{D}_1 := \left\{(\alpha,\beta);\ 0<\alpha,\beta<1 \text{ and } 1 \leqq \alpha+\beta<2 \text{ for } \beta\neq\alpha\right\}, \tag{1.2}$$

or, in the case when $\alpha = \beta$, on the set

$$\mathscr{D}_2 := \{(\alpha,\beta);\ 0 < \alpha, \beta < 1 \text{ and } 1 \leqq \alpha + \beta < 2 \text{ for } \beta = \alpha\} \quad (1.3)$$

for $\alpha,\beta \in \mathbb{R}$. We will analyze (1.1) to produce the positivity of $(\nabla f)(\tau)$ under certain conditions. Incidentally, we can say that this article is the extension of the Liouville-Caputo work [17], but here in the sense of Riemann-Liouville operators.

The subsequent sections of the article are organized as follows: In Section 2, we recall the definition of the Caputo-Fabrizio that occurred in (1.1) and we formulated the main lemmas. Moving on to Section 3, we delve into the topics of positivity and monotonicity in analysis for sequential operator Eq (1.1). We end with some concluding remarks about the main results in Section 4.

2. Caputo-Fabrizio and important lemmas

The discrete operator that appeared in (1.1) is the fractional difference operator with exponential kernel or, briefly, is the Caputo-Fabrizio fractional difference operator in the sense of Riemann-Liouville. It is defined by the following summation formula (see [1,4]):

$$\left({}^{\mathrm{CF}}_{\ a}\nabla^{\alpha}_{rl} f\right)(\tau) = B(\alpha)\,\nabla_\tau\left[\sum_{s=a+1}^{\tau} f(s)(1-\alpha)^{\tau-s}\right], \quad (2.1)$$

where $\tau \in \mathbb{N}_{a+1}$, $a \in \mathbb{R}$, $\alpha \in [0,1)$, and $B(\alpha) > 0$ is a normalization constant. It is also worth mentioning that $(\nabla f)(\tau)$ is the ∇ difference operator of f and is given by $f(\tau) - f(\tau-1)$.

Lemma 2.1. *If $(\alpha,\beta) \in \mathscr{D}_1$, then*

$$\sigma_1(i) := \frac{1}{\alpha-\beta}\Big[\beta(1-\beta)^i - \alpha(1-\alpha)^i\Big] \geqq 0 \quad (2.2)$$

is nonnegative, and

$$\sigma_2(i) := \frac{1}{\alpha-\beta}\Big[(1-\beta)^i - (1-\alpha)^i\Big] \quad (2.3)$$

is positive for each $i \in \mathbb{N}_1$.

Proof. By using the mathematical induction process, first for $i = 1$, we have

$$\sigma_1(1) := \frac{1}{\alpha-\beta}\Big[\beta(1-\beta) - \alpha(1-\alpha)\Big] = -1 + \alpha + \beta \geqq 0,$$

since $\alpha + \beta \geqq 1$. Now, we suppose that

$$\sigma_1(\jmath) = \frac{1}{\alpha-\beta}\Big[\beta(1-\beta)^\jmath - \alpha(1-\alpha)^\jmath\Big] \geqq 0, \quad (2.4)$$

for some $\jmath \in \mathbb{N}_1$, then we have to prove that $\sigma_1(\jmath+1) \geqq 0$. Here, two cases arise: the case when $\alpha > \beta$ leads to

$$\sigma_1(\jmath+1) = \underbrace{\frac{1}{\alpha-\beta}}_{>0}\Big[\beta(1-\beta)^{\jmath+1} - \alpha(1-\alpha)^{\jmath+1}\Big],$$

which remains to prove that

$$\Big[\beta(1-\beta)^{\jmath+1} - \alpha(1-\alpha)^{\jmath+1}\Big] \geqq 0.$$

It is clear that

$$\beta(1-\beta)^{\jmath+1} \underset{(2.4)}{\overset{by}{\geqq}} \alpha(1-\alpha)^\jmath(1-\beta) \geqq \alpha(1-\alpha)^{\jmath+1},$$

where the fact $1-\beta > 1-\alpha > 0$ has been used and thus $\sigma_1(\jmath+1) \geqq 0$ for $\alpha > \beta$. By the same process, we can prove that $\sigma_1(\jmath+1) \geqq 0$ for $\alpha < \beta$. Hence, $\sigma_1(i) \geqq 0$ for each $(\alpha,\beta) \in \mathscr{D}_1$ and $i \in \mathbb{N}_1$. Consequently, the first part of the lemma is proved.

To prove the inequality (2.3): for each $i \geqq 1$, we have

$$\frac{1}{\alpha-\beta}\Big[(1-\beta)^i - (1-\alpha)^i\Big] > \underbrace{\frac{1}{\alpha-\beta}}_{>0}\underbrace{\Big[(1-\beta)^i - (1-\beta)^i\Big]}_{=0} = 0,$$

for $\alpha > \beta$ ($\Longrightarrow 1-\beta > 1-\alpha > 0$), and

$$\begin{aligned}&\frac{1}{\alpha-\beta}\Big[(1-\beta)^i - (1-\alpha)^i\Big]\\ &= \frac{1}{\beta-\alpha}\Big[(1-\alpha)^i - (1-\beta)^i\Big]\\ &> \underbrace{\frac{1}{\beta-\alpha}}_{>0}\underbrace{\Big[(1-\beta)^i - (1-\beta)^i\Big]}_{=0}\\ &= 0,\end{aligned}$$

for $\beta > \alpha$ (hence, $1-\alpha > 1-\beta > 0$). Thus, the proof is completed. □

Lemma 2.2. *If $\alpha \in \left[\frac{1}{2}, 1\right)$, then*

$$\sigma_3(i) := (1-\alpha)^{i-1}[\alpha\, i - (1-\alpha)] \quad (2.5)$$

is nonnegative for each $i \in \mathbb{N}_1$.

Proof. Applying induction on (2.5), we have for $i = 1$, $\sigma_3(1) = 2\alpha - 1$, and it is clear that is nonnegative as $\alpha \in \left[\frac{1}{2}, 1\right)$. Suppose that $\sigma_3(\jmath) \geqq 0$; that is,

$$(1-\alpha)^{\jmath-1}\left[\alpha\,\jmath - (1-\alpha)\right] \geqq 0, \tag{2.6}$$

for some $\jmath \in \mathbb{N}_1$, then we have to prove that $\sigma_3(\jmath+1) \geqq 0$. It is clear that

$$\begin{aligned}
&(1-\alpha)^{\jmath}\left[\alpha\,(\jmath+1) - (1-\alpha)\right] \\
&= \underbrace{(1-\alpha)}_{>0}\ \underbrace{(1-\alpha)^{\jmath-1}\left[\alpha\,\jmath - (1-\alpha)\right]}_{\geqq 0 \text{ by claim (2.6)}} + \underbrace{\alpha\,(1-\alpha)^{\jmath}}_{>0} \\
&\geqq 0,
\end{aligned}$$

and this tells us $\sigma_3(\jmath+1) \geqq 0$. Thus, we see that (2.5) is true for each $i \in \mathbb{N}_1$. The proof is done. □

Lemma 2.3. *If f is a function $f\colon \mathbb{N}_a \to \mathbb{R}$, then we have*

$$\begin{aligned}
\nabla\left({}^{\mathrm{CF}}_{\ a}\nabla^{\alpha}_{rl} f\right)(\tau) =& B(\alpha)\Bigg[(\nabla f)(\tau) - \alpha\, f(a+1)(1-\alpha)^{\tau-2-a} \\
&- \alpha \sum_{s=a+2}^{\tau-1} (\nabla f)(s)(1-\alpha)^{\tau-s-1}\Bigg],
\end{aligned}$$

for $\alpha \in (0,1)$ and $\tau \in \mathbb{N}_{a+2}$.

Proof. Considering the definition of (2.1), we have

$$\begin{aligned}
\left({}^{\mathrm{CF}}_{\ a}\nabla^{\alpha}_{rl} f\right)(\tau) =& B(\alpha)\Bigg[\sum_{s=a+1}^{\tau} f(s)(1-\alpha)^{\tau-s} \\
&- \sum_{s=a+1}^{\tau-1} f(s)(1-\alpha)^{\tau-s-1}\Bigg] \\
=& B(\alpha)\left[f(\tau) - \alpha \sum_{s=a+1}^{\tau-1} f(s)(1-\alpha)^{\tau-s-1}\right].
\end{aligned}$$

It follows from this that

$$\begin{aligned}
\nabla\left({}^{\mathrm{CF}}_{\ a}\nabla^{\alpha}_{rl} f\right)(\tau) =& B(\alpha)\Bigg[(\nabla f)(\tau) - \alpha \sum_{s=a+1}^{\tau-1} f(s)(1-\alpha)^{\tau-s-1} \\
&+ \alpha \sum_{s=a+1}^{\tau-2} f(s)(1-\alpha)^{\tau-s-2}\Bigg] \\
=& B(\alpha)\Bigg[(\nabla f)(\tau) - \alpha\, f(a+1)(1-\alpha)^{\tau-2-a} \\
&- \alpha \sum_{s=a+2}^{\tau-1} f(s)(1-\alpha)^{\tau-s-1} \\
&+ \alpha \sum_{s=a+2}^{\tau-1} f(s-1)(1-\alpha)^{\tau-s-1}\Bigg] \\
=& B(\alpha)\Bigg[(\nabla f)(\tau) - \alpha\, f(a+1)(1-\alpha)^{\tau-2-a} \\
&- \alpha \sum_{s=a+2}^{\tau-1} (\nabla f)(s)(1-\alpha)^{\tau-s-1}\Bigg],
\end{aligned} \tag{2.7}$$

for each $\tau \in \mathbb{N}_{a+2}$. This completes the proof. □

3. Positive analyses

In this section, we give some positivity results on the sequential fractional difference (1.1) when there is a negative lower bound in the right side of the inequality in Theorem 3.1. These results are based on the set $\mathscr{D}_1$ as plotted in Figure 1.

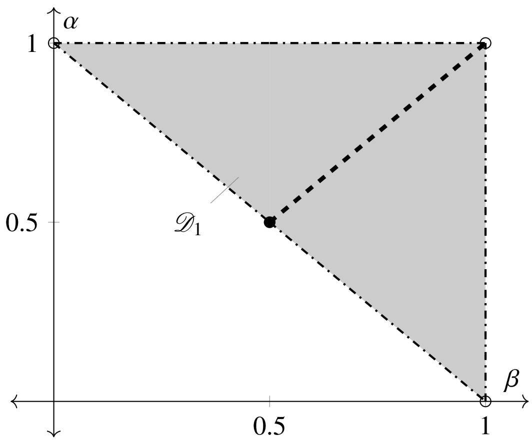

Figure 1. The region of $\mathscr{D}_1$.

Theorem 3.1. *If*

$$\xi \geqq 0 \quad \text{and} \quad \mathbb{N}_a^{\jmath} := \{a, a+1, \ldots, \jmath\}$$

and the function $f\colon \mathbb{N}_a \to \mathbb{R}$ satisfies

(i) $(\nabla f)(a+2) \geqq 0$;

(ii)

$$\left({}^{\mathrm{CF}}_{a+1}\nabla^{\alpha}_{rl}\ {}^{\mathrm{CF}}_{\ a}\nabla^{\beta}_{rl} f\right)(\tau) \geqq -\xi\, B(\alpha)B(\beta)\{f(a+2) - \beta\, f(a+1)\};$$

(iii)

$$\frac{\beta\,(1-\beta)^{\tau-2-a} - \alpha\,(1-\alpha)^{\tau-2-a}}{\alpha-\beta} \geqq \xi;$$

for each $(\alpha,\beta) \in \mathscr{D}_1$ and $\tau \in \mathbb{N}^{\jmath}_{a+3}$, for some $\jmath \in \mathbb{N}_{a+3}$, then, $(\nabla f)(\tau) \geqq 0$ for every $\tau \in \mathbb{N}^{\jmath}_{a+2}$.

Proof. Let

$$\mathrm{h}(\tau) := \left({}^{\mathrm{CF}}_{\ a}\nabla^{\beta}_{rl} f\right)(\tau),$$

then (2.1) enables us to write

$$\begin{aligned}
&\left({}^{\mathrm{CF}}_{a+1}\nabla^{\alpha}_{rl}\ {}^{\mathrm{CF}}_{\ a}\nabla^{\beta}_{rl} f\right)(\tau) = \left({}^{\mathrm{CF}}_{a+1}\nabla^{\alpha}_{rl}\mathrm{h}\right)(\tau)\\
&= B(\alpha)\left[\sum_{s=a+2}^{\tau}\mathrm{h}(s)(1-\alpha)^{\tau-s} - \sum_{s=a+2}^{\tau-1}\mathrm{h}(s)(1-\alpha)^{\tau-1-s}\right] \qquad (3.1)\\
&= B(\alpha)\left[(1-\alpha)^{\tau-a-2}\mathrm{h}(a+2) + \sum_{s=a+3}^{\tau}(\nabla\,\mathrm{h})(s)(1-\alpha)^{\tau-s}\right].
\end{aligned}$$

It follows from Lemma 2.3 that

$$\begin{aligned}
&\left({}^{\mathrm{CF}}_{a+1}\nabla^{\alpha}_{rl}\ {}^{\mathrm{CF}}_{\ a}\nabla^{\beta}_{rl} f\right)(\tau)\\
&= B(\alpha)B(\beta)\Bigg[(1-\alpha)^{\tau-a-2}\Big\{f(a+2)-\beta f(a+1)\Big\}\\
&\quad + (\nabla f)(\tau) + \sum_{s=a+3}^{\tau-1}(\nabla f)(s)(1-\alpha)^{\tau-s}\\
&\quad - \beta f(a+1)\sum_{s=a+3}^{\tau}(1-\beta)^{s-a-2}(1-\alpha)^{\tau-s}\\
&\quad - \beta\sum_{s=a+3}^{\tau}\sum_{r=a+2}^{s-1}(\nabla f)(r)(1-\beta)^{s-r-1}(1-\alpha)^{\tau-s}\Bigg]\\
&:= B(\alpha)B(\beta)\Big[(1-\alpha)^{\tau-a-2}\Big\{f(a+2)-\beta f(a+1)\Big\}\\
&\quad + A_1(\tau) - A_2(\tau)\,f(a+1) - A_3(\tau)\Big], \qquad (3.2)
\end{aligned}$$

where the following is used,

$$\mathrm{h}(a+2) = \left({}^{\mathrm{CF}}_{\ a}\nabla^{\beta}_{rl} f\right)(a+2) = f(a+2) - \beta f(a+1).$$

Compute $A_1(\tau)$, $A_2(\tau)$, and $A_3(\tau)$ to get

$$\begin{aligned}
A_1(\tau) &:= \sum_{s=a+3}^{\tau}(\nabla f)(s)(1-\alpha)^{\tau-s}\\
&= (\nabla f)(\tau) + \sum_{s=a+3}^{\tau-1}(\nabla f)(s)(1-\alpha)^{\tau-s}, \qquad (3.3)
\end{aligned}$$

$$\begin{aligned}
A_2(\tau) &:= \beta\sum_{s=a+3}^{\tau}(1-\beta)^{s-a-2}(1-\alpha)^{\tau-s}\\
&= \beta(1-\beta)(1-\alpha)^{\tau-a-3}\sum_{\kappa=0}^{\tau-a-3}\left(\frac{1-\beta}{1-\alpha}\right)^{\kappa}\\
&= \beta(1-\beta)(1-\alpha)^{\tau-a-3}\cdot\frac{1-\left(\frac{1-\beta}{1-\alpha}\right)^{\tau-a-2}}{1-\frac{1-\beta}{1-\alpha}}\\
&= \beta(1-\beta)\left(\frac{(1-\beta)^{\tau-a-2}-(1-\alpha)^{\tau-a-2}}{\alpha-\beta}\right), \qquad (3.4)
\end{aligned}$$

and

$$\begin{aligned}
A_3(\tau) &:= \beta\sum_{s=a+3}^{\tau}\sum_{r=a+2}^{s-1}(\nabla f)(r)(1-\beta)^{s-r-1}(1-\alpha)^{\tau-s}\\
&= \beta\sum_{r=a+2}^{\tau-1}\left[(\nabla f)(r)\frac{(1-\alpha)^{\tau}}{(1-\beta)^{r+1}}\sum_{s=r+1}^{\tau}\left(\frac{1-\beta}{1-\alpha}\right)^{s}\right]\\
&= \beta\sum_{r=a+2}^{\tau-1}(\nabla f)(r)(1-\alpha)^{\tau-1-r}\cdot\frac{1-\left(\frac{1-\beta}{1-\alpha}\right)^{\tau-r}}{1-\frac{1-\beta}{1-\alpha}}\\
&= \beta\sum_{r=a+2}^{\tau-1}(\nabla f)(r)\left(\frac{(1-\beta)^{\tau-r}-(1-\alpha)^{\tau-r}}{\alpha-\beta}\right)\\
&= \beta\left(\frac{(1-\beta)^{\tau-a-2}-(1-\alpha)^{\tau-a-2}}{\alpha-\beta}\right)(\nabla f)(a+2)\\
&\quad + \beta\sum_{r=a+3}^{\tau-1}(\nabla f)(r)\left(\frac{(1-\beta)^{\tau-r}-(1-\alpha)^{\tau-r}}{\alpha-\beta}\right). \qquad (3.5)
\end{aligned}$$

By making use of (3.3)–(3.5) and (ii) in (3.2), together with the fact that $0 < B(\alpha)B(\beta)$, we obtain

$$\begin{aligned}
(\nabla f)(\tau) &\geqq -(1-\alpha)^{\tau-a-2}\Big\{f(a+2)-\beta f(a+1)\Big\}\\
&\quad - \xi\Big\{f(a+2)-\beta f(a+1)\Big\}\\
&\quad + \beta\left(\frac{(1-\beta)^{\tau-a-2}-(1-\alpha)^{\tau-a-2}}{\alpha-\beta}\right)(\nabla f)(a+2)\\
&\quad + \beta(1-\beta)\left(\frac{(1-\beta)^{\tau-a-2}-(1-\alpha)^{\tau-a-2}}{\alpha-\beta}\right)f(a+1)\\
&\quad - \sum_{s=a+3}^{\tau-1}(\nabla f)(s)(1-\alpha)^{\tau-s}\\
&\quad + \beta\sum_{r=a+3}^{\tau-1}(\nabla f)(r)\left(\frac{(1-\beta)^{\tau-r}-(1-\alpha)^{\tau-r}}{\alpha-\beta}\right)\\
&= \underbrace{\left(\frac{\beta(1-\beta)^{\tau-a-2}-\alpha(1-\alpha)^{\tau-a-2}}{\alpha-\beta}-\xi\right)}_{\geqq 0 \text{ by (iii)}}\\
&\quad \times\underbrace{\Big\{f(a+2)-\beta f(a+1)\Big\}}_{\geqq 0 \text{ by (i)}}\\
&\quad + \sum_{s=a+3}^{\tau-1}(\nabla f)(s)\,\underbrace{\frac{1}{\alpha-\beta}\Big[\beta(1-\beta)^{\tau-s}-\alpha(1-\alpha)^{\tau-s}\Big]}_{\geqq 0 \text{ by (2.2)}},
\end{aligned} \qquad (3.6)$$

for $\tau\in\mathbb{N}^{J}_{a+2}$. Now, we know from (i) that $(\nabla f)(a+2)\geqq 0$, which implies that

$$f(a+2)\geqq f(a+1) > \beta f(a+1).$$

By substituting $\tau = a+3$ into (3.6), we get

$$\begin{aligned}(\nabla f)(a+3) \geqq &\left(\frac{\beta(1-\beta)-\alpha(1-\alpha)}{\alpha-\beta}\right)\\ &\Big\{f(a+2)-\beta f(a+1)\Big\} + \underbrace{\sum_{s=a+3}^{a+2}(\cdot)}_{=0}\\ \geqq &(\beta+\alpha-1)\Big\{f(a+2)-\beta f(a+1)\Big\}\\ \geqq &0.\end{aligned}$$

Repeating this action together with the help of (2.2), we reach that $(\nabla f)(\tau) \geqq 0$ for each $\tau \in \mathbb{N}_{a+2}^{J}$, as desired. □

To confirm the validity of the above theorem, we consider the following example.

Example 3.1. Suppose that f is a function $f\colon \mathbb{N}_0 \to \mathbb{R}$ defined by

$$f(\tau) = \tau^{\overline{\alpha+\beta}}.$$

First of all, for $\alpha = 0.5$ and $\beta = 0.55$, we see that

$$1 < \alpha+\beta < 2 \implies 1 < 1.05 < 2,$$

which verifies that $(\alpha,\beta) \in \mathscr{D}_1$.

Now, by using (3.2) with $\xi = 0.001$, $\tau = a+3$, and $a = 0$, we get

$$\begin{aligned}\left({}^{\mathrm{CF}}_{\ 1}\nabla^{\alpha}_{rl}\ {}^{\mathrm{CF}}_{\ 0}\nabla^{\beta}_{rl} f\right)(2) =& B(\alpha)B(\beta)\Bigg[(1-\alpha)\Big\{f(2)-\beta f(1)\Big\}\\ &+\sum_{s=3}^{3}(\nabla f)(s)(1-\alpha)^{3-s}\\ &-\beta f(1)\sum_{s=3}^{3}(1-\beta)^{s-2}(1-\alpha)^{3-s}\\ &-\beta\sum_{s=3}^{3}\sum_{r=2}^{s-1}(\nabla f)(r)(1-\beta)^{s-r-1}(1-\alpha)^{3-s}\Bigg]\\ =&1.0638\, B(\alpha)B(\beta)\\ \geqq&-0.0015\, B(\alpha)B(\beta).\end{aligned}$$

Hence, the second condition of Theorem 3.1 is satisfied for $\tau = a+3$. Furthermore, the first condition of Theorem 3.1

$$(\nabla f)(2) = 1.0733 \geqq 0$$

holds. The last condition

$$\frac{\beta(1-\beta)-\alpha(1-\alpha)}{\alpha-\beta} = 0.05 \geqq 0.001$$

is satisfied. Hence, $\tau^{\underline{\alpha+\beta}}$ is increasing at $\tau = a+3$, according to Theorem 3.1. Moreover, $\tau^{\underline{\alpha+\beta}}$ is an increasing function on $\mathbb{N}_1$, as its plot has been shown in Figure 2.

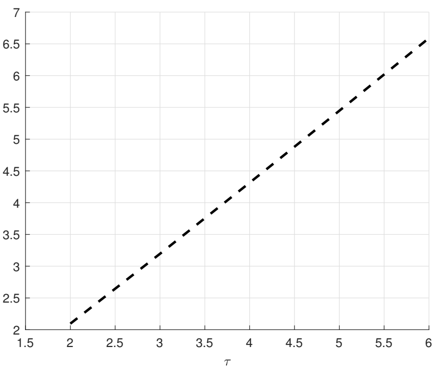

Figure 2. Graph of $\tau^{\overline{\alpha+\beta}}$ for $\alpha = 0.5$ and $\beta = 0.55$.

Our second result is based on the set $\mathscr{D}_2$, which is plotted clearly in Figure 3.

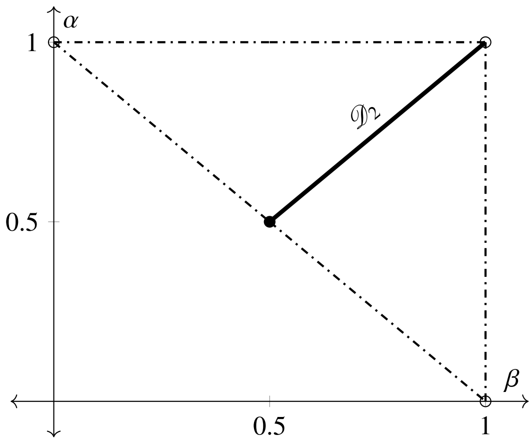

Figure 3. The region of $\mathscr{D}_2$.

Let us start stating and proving our last theorem.

Theorem 3.2. *If* $0 \leqq \xi$ *and* $\alpha \in \left[\frac{1}{2}, 1\right)$ *and the function* $f\colon \mathbb{N}_a \to \mathbb{R}$ *satisfies*

(i) $(\nabla f)(a+2) \geqq 0$;

(ii)

$$\left({}^{\mathrm{CF}}_{a+1}\nabla^{\alpha}_{rl}\ {}^{\mathrm{CF}}_{\ a}\nabla^{\alpha}_{rl} f\right)(\tau) \geqq -\xi\, B^2(\alpha)\Big\{f(a+2)-\alpha f(a+1)\Big\};$$

(iii)

$$\xi \leqq (1-\alpha)^{\tau-a-2}\Big[\alpha(\tau-a-2)-(1-\alpha)\Big];$$

for each $(\alpha,\beta) \in \mathscr{D}_2$ *and* $\tau \in \mathbb{N}_{a+3}^{J}$, *for some* $J \in \mathbb{N}_{a+3}$, *then,* $(\nabla f)(\tau) \geqq 0$ *for every* $\tau \in \mathbb{N}_{a+2}^{J}$.

Proof. We know from (3.1) that when $\beta = \alpha$,

$$\begin{aligned}
&\left({}_{a+1}^{\mathrm{CF}}\nabla_{rl}^{\alpha}\ {}_{a}^{\mathrm{CF}}\nabla_{rl}^{\beta} f\right)(\tau) \\
&= B^2(\alpha)\Bigg[(1-\alpha)^{\tau-a-2}\Big\{f(a+2) - \alpha f(a+1)\Big\} \\
&\quad + \sum_{s=a+3}^{\tau}(\nabla f)(s)(1-\alpha)^{\tau-s} \\
&\quad - \alpha f(a+1)\sum_{s=a+3}^{\tau}(1-\alpha)^{\tau-a-2} \\
&\quad - \alpha\sum_{s=a+3}^{\tau}\sum_{r=a+2}^{s-1}(\nabla f)(r)(1-\alpha)^{\tau-r-1}\Bigg] \\
&:= B^2(\alpha)\Bigg[(1-\alpha)^{\tau-a-2}\Big\{f(a+2) - \alpha f(a+1)\Big\} \\
&\quad + B_1(\tau) - f(a+1)B_2(\tau) - B_3(\tau)\Bigg]. \qquad (3.7)
\end{aligned}$$

Compute $B_1(\tau)$, $B_2(\tau)$, and $B_3(\tau)$ to get

$$\begin{aligned}
B_1(\tau) &:= \sum_{s=a+3}^{\tau}(\nabla f)(s)(1-\alpha)^{\tau-s} \\
&= (\nabla f)(\tau) + \sum_{s=a+3}^{\tau-1}(\nabla f)(s)(1-\alpha)^{\tau-s}, \qquad (3.8) \\
B_2(\tau) &:= \alpha\sum_{s=a+3}^{\tau}(1-\alpha)^{\tau-a-2} \\
&= \alpha(1-\alpha)^{\tau-a-2}\sum_{\kappa=1}^{\tau-a-2}(1) \\
&= \alpha(\tau-a-2)(1-\alpha)^{\tau-a-2} \qquad (3.9)
\end{aligned}$$

and

$$\begin{aligned}
B_3(\tau) &:= \alpha\sum_{s=a+3}^{\tau}\sum_{r=a+2}^{s-1}(\nabla f)(r)(1-\alpha)^{\tau-r-1} \\
&= \alpha\sum_{r=a+2}^{\tau-1}(\nabla f)(r)(1-\alpha)^{\tau-r-1}\sum_{s=r+1}^{\tau}(1) \\
&= \alpha(\tau-a-2)(1-\alpha)^{\tau-a-3}(\nabla f)(a+2) \\
&\quad + \alpha\sum_{r=a+3}^{\tau-1}(\nabla f)(r)(\tau-r)(1-\alpha)^{\tau-r-1}. \qquad (3.10)
\end{aligned}$$

By making use of (3.8)–(3.10) and using (ii) in (3.7), together with the fact that $0 < B(\alpha)B(\beta)$, we get

$$\begin{aligned}
(\nabla f)(\tau) &\geqq -(1-\alpha)^{\tau-a-2}\Big\{f(a+2) - \alpha f(a+1)\Big\} \\
&\quad - \xi\Big\{f(a+2) - \alpha f(a+1)\Big\} \\
&\quad + \alpha(\tau-a-2)(1-\alpha)^{\tau-a-3}(\nabla f)(a+2) \\
&\quad + \alpha(\tau-a-2)(1-\alpha)^{\tau-a-2} f(a+1) \\
&\quad - \sum_{s=a+3}^{\tau-1}(\nabla f)(s)(1-\alpha)^{\tau-s} \\
&\quad + \alpha\sum_{r=a+3}^{\tau-1}(\nabla f)(r)(\tau-r)(1-\alpha)^{\tau-r-1} \\
&= \underbrace{\Big((1-\alpha)^{\tau-a-2}[\alpha(\tau-a-2) - (1-\alpha)] - \xi\Big)}_{\geqq 0 \text{ by (iii)}} \\
&\quad \times \underbrace{\Big\{f(a+2) - \beta f(a+1)\Big\}}_{\geqq 0 \text{ by (i)}} \\
&\quad + \sum_{s=a+3}^{\tau-1}(\nabla f)(s)\underbrace{(1-\alpha)^{\tau-s-1}\Big[\alpha(\tau-s) - (1-\alpha)\Big]}_{\geqq 0 \text{ by (2.5)}}.
\end{aligned} \qquad (3.11)$$

By considering (i) and the last positive inequality (3.11), we can deduce that $(\nabla f)(\tau) \geqq 0$ for each $\tau \in \mathbb{N}_{a+2}^{J}$, as desired. □

We consider the following example in order to see the validity of the above theorem.

Example 3.2. We consider the same function defined in Example 3.1, then for choosing $\alpha = \beta = 0.6$, we have

$$1 < \alpha + \beta < 2 \quad \Longrightarrow \quad 1 < 1.2 < 2,$$

which verifies that $(\alpha,\beta) \in \mathscr{D}_2$. Moreover,

$$(\nabla f)(2) = 0.8713 \geqq 0,$$

and

$$\begin{aligned}
&\left({}_{1}^{\mathrm{CF}}\nabla_{rl}^{\alpha}\ {}_{0}^{\mathrm{CF}}\nabla_{rl}^{\alpha} f\right)(2) \\
&= B^2(\alpha)\Bigg[(1-\alpha)\Big\{f(2) - \alpha f(1)\Big\} + \sum_{s=3}^{3}(\nabla f)(s)(1-\alpha)^{3-s} \\
&\quad - \alpha f(a+1)\sum_{s=3}^{\tau}(1) - \alpha\sum_{s=3}^{3}\sum_{r=2}^{s-1}(\nabla f)(r)(1-\alpha)^{2-r}\Bigg] \\
&= 1.2951\, B^2(\alpha) \\
&\geqq -0.0018\, B^2(\alpha).
\end{aligned}$$

In addition,

$$(1-\alpha)^{\tau-a-2}\big[\alpha(\tau-a-2)-(1-\alpha)\big] = 0.08 \geqq 0.001.$$

Thus, all the conditions of Theorem 3.2 are satisfied. Therefore, the increase of $\tau^{\overline{\alpha+\beta}}$ is proved at $\tau = a+3$. For more clarification, see below Figure 4 as $\tau^{\overline{\alpha+\beta}}$ is increasing on $\mathbb{N}_1$.

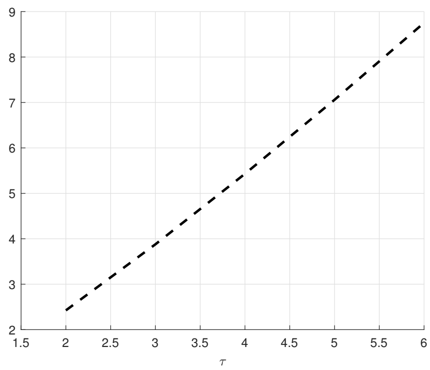

Figure 4. Graph of $\tau^{\overline{\alpha+\beta}}$ for $\alpha = \beta = 0.4$.

4. Conclusions

In this study, we have considered analyzing $\left({}^{\mathrm{CF}}_{a+1}\nabla^{\alpha}_{rl}\ {}^{\mathrm{CF}}_{a}\nabla^{\beta}_{rl} f\right)(\tau)$ on the set $\mathscr{D}_1$ in which $\beta \neq \alpha$, and $\left({}^{\mathrm{CF}}_{a+1}\nabla^{\alpha}_{rl}\ {}^{\mathrm{CF}}_{a}\nabla^{\alpha}_{rl} f\right)(\tau)$ on the set $\mathscr{D}_2$ in which $\beta = \alpha$. The positivity $(\nabla f)(\tau)$ has been examined from analyzing these sequential fractional differences on a finite time set $\mathbb{N}^{J}_{a+2}$ for some $J \in \mathbb{N}_{a+3}$ in both cases when $(\alpha,\beta) \in \mathscr{D}_1$ or $(\alpha,\beta) \in \mathscr{D}_2$. In the first case, we have used a complex condition as appeared in Theorem 3.1 (iii). However, a simpler condition has been applied in the second case as stated in Theorem 3.2 (iii). In the end, we have provided an increasing function to support our claims and results, and it has been shown that the positivity of our main theorems is accurate and can be obtained under the certain conditions.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict interest.

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