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Research article

Finite-time lag synchronization for two-layer complex networks with impulsive effects

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Abstract: This paper mainly considered the finite-time lag synchronization for two-layer complex networks with impulsive effects. Different types of controllers were designed to achieve the lag synchronization of two-layer complex networks. Several sufficient conditions on lag synchronization in the sense of finite time were derived. The time for synchronization was also estimated. It is important to note that synchronization time was influenced by the initial value, as well as the impulses and impulse sequence. This implied that different impulse effects result in varying synchronization times. Additionally, desynchronizing impulses can extend the synchronization time, whereas synchronizing impulses have the opposite effect. Finally, a numerical example was presented to showcase the practicality and validity of the proposed theoretical criteria.

Keywords: two-layer complex networks; finite-time lag synchronization; desynchronizing impulses; synchronizing impulses

1. Introduction

In the past two decades, complex networks have gained significant attention for their prevalence in various real-world applications. Numerous accomplishments [1–6] have been made in this field. In fact, the research on isolated and single networks has been very extensive, and multiple types of interactive networks have been neglected. However, in realistic complex networks, the nodes in networks can participate in a variety of interactions, and this multiple interaction can have not only a simple additive effect on its dynamics and the network structure. For example, people in a society constitute their social network of relationships through their friendships, family relationships, and work-related acquaintances [7]. In a two-layer neural network, both electrical and chemical synapses can transmit neuronal information [8]. Even protein transcriptional regulation, metabolic synthesis, and signaling in cells require multiple layers of interactions and regulations [9]. Obviously, the different interactions in the above cases cannot be simply superimposed, so the multilayer network model is more suitable for dealing with such problems. In 2010, Mucha introduced the concept of multilayer networks for the first time [10], sparking scholarly interest in the study of this interconnected network paradigm. Therefore, this paper carries out research under the framework of the multilayer network model.

The study of complex networks mainly focuses on their topology and dynamics. As a dynamical phenomenon observed in networks, synchronization has received extensive attention and continuous research. According to different practical meanings, complete synchronization, generalized synchronization, phase synchronization, and lag synchronization have been studied [11–17]. Lag synchronization, among the types mentioned earlier, refers to the synchronization of one node's state with another node's state at a specific time in the past. This phenomenon is commonly observed in systems such as laser and electronic networks. It has been proved that it is an applicable strategy from the perspective of engineering applications of secure communication and concurrent image processing [18]. For example, in video calling networks, the sounds and images received at time t are emitted by the originator at time $t - \tau$. Moreover, the synchronization of multilayer networks has rapidly attracted people's attention. Many types of synchronization, such as complete, intralayer, interlayer, and cluster synchronization, have been defined and studied [19–24], which make the research of network science more concrete and realistic. Interlayer synchronization means that the corresponding nodes between layers are synchronized, and some results have been achieved. In [25], the relationship between the network structure of the two-layer network and its interlayer synchronization was explored. The essential requirements for interlayer synchronization to exist and be achieved were analytically derived in [26]. However, there are few results on lag synchronization in the interlayer for multilayer networks.

In all kinds of synchronization behavior research, it is always hoped to realize synchronization as soon as possible. The synchronization rate is an important index used to measure synchronization performance. In most current research on closed-loop systems, the fastest achievable synchronization rate is exponential. This limitation arises due to the necessity of maintaining Lipschitz continuity within the closed-loop system. As a result, synchronization typically falls under the category of infinite-time synchronization, where convergence occurs over an extended period. From a practical standpoint, the lifespan of a man-machine system is finite. Therefore, in engineering applications aimed at enhancing economic benefits and work efficiency, achieving synchronization in finite time is highly desirable. This finite-time synchronization goal is crucial for optimizing system performance within practical constraints and time lines. Based on the above scenario, people are concerned about the finite time synchronization. Finite-time synchronization ensures the fastest synchronization convergence time, but it also effectively suppresses disturbances and exhibits robustness in the face of uncertainties. Therefore, synchronization of finite time has been extensively studied for multilayer networks [27–33]. For example, the relationship between topological structure, multiple weights, internal coupling mode, coupling strength, and cross-layer was established in [27]. The event-triggered intermittent control approach was discussed for achieving finite-time interlayer synchronization in multilayer networks [28]. By designing two different controllers and constructing the Lyapunov function, the bound of the synchronization time of finite time lag synchronization was estimated in [31].

In addition, it is worth noting that during the network synchronization process, the transmission of the signal will inevitably have a sudden discrete change at specific moments, leading to impulsive phenomena [34]. These impulses within the target network can have different effects: Desynchronizing impulses may disrupt synchronization, synchronizing impulses may enhance synchronization, and inactive impulses may have no impact on the existing synchronization state. Understanding and managing these impulses is crucial in ensuring the stability and efficiency of network synchronization processes [35–37]. In recent years, various different control methods have been developed for finite-time synchronization [38–41], such as feedback control, adaptive control, impulsive control, sliding mode controller, and so on. The primary objective when designing a controller for finite-time synchronization is to achieve synchronization within a specific time period. However, it can be challenging to meet additional criteria, such as a straightforward and practical controller structure and the avoidance of chattering phenomena. In many cases, considering all these conditions simultaneously in finite-time synchronization controller design is difficult. Designers often face trade-offs and prioritize the achievement of finite-time synchronization over other considerations. Based on the above considerations, finite time controllers are designed in this paper to realize the finite-time lag synchronization of two-layer complex networks.

This paper aims to investigate finite-time lag synchronization in two-layer complex networks with impulsive effects, distinguishing between synchronizing and desynchronizing impulses. Two types of Lyapunov functions and controllers are proposed, resulting in several synchronization conditions. The key contributions are outlined as follows: First, the focus on lag synchronization in two-layer networks is a novel contribution in the field, given the early stages of research in this area. Second, considering finite-time lag synchronization in two-layer networks, the paper addresses both synchronizing and desynchronizing impulses distinctly. Third, the paper successfully tackles and resolves the challenge of bounded controller implementation, enhancing practicality and convenience in real-world applications.

2. Preliminaries

Let $\mathbb{Z}_+$ denote the set of positive integers. $\mathbb{R}$ and $\mathbb{R}_+$ denote the set of real numbers and the set of all positive real numbers. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ represent n-dimensional and $n\times m$-dimensional real spaces equipped with the Euclidean norm $|\cdot|$. $\lambda_{min}(A)$ and $\lambda_{max}(A)$ denote the minimum and maximum eigenvalues of matrix A, respectively. $A > 0$ $(A < 0)$ means that the matrix A is a symmetric and positive (negative) definite matrix. The notation A^{-1}, A^T represents the inverse and the transpose of A. I is the identity matrix with appropriate dimensions. For any interval $Q \subseteq \mathbb{R}$ and any set $\Omega \subseteq \mathbb{R}^k$, $1 \le k \le n$, let $C(Q,\Omega)$ $=\{v$: $Q \to \Omega$ be a continuous function$\}$ and $PC(Q,\Omega)$=$\{v$: $Q \to \Omega$ be bounded and continuous everywhere except at finite number of points t, at which $v(t^-)$, $v(t^+)$ exist and $v(t^+) = v(t)\}$. $\mathcal{K}$=$\{b(\cdot) \in C(\mathbb{R}_+,\mathbb{R}_+) \mid b(0) = 0$, $b(\delta) > 0$ for $\delta > 0$, and b is strictly increasing in $\delta\}$,

$$\mathcal{K}_\infty = \{b(\cdot) \in \mathcal{K} \mid b(m) \to \infty \text{ as } m \to \infty\}.$$

The right-upper Dini derivative of $f(t)$ is defined as

$$D^+ f(t) = \lim_{h\to 0^+} \sup \frac{1}{h}[f(t+h) - f(t)].$$

Notation $\star$ represents the symmetric part in a matrix.

Consider the following two-layer complex networks, which is consisted of N nodes in each layer:

$$\begin{cases} \dot{x}_i(t) = f(x_i(t)) + c_1 \sum\limits_{j=1}^{N} a_{ij} H x_j(t) + c_2\Gamma(y_i(t) - x_i(t)), \\ \dot{y}_i(t) = f(y_i(t)) + c_1 \sum\limits_{j=1}^{N} a_{ij} H y_j(t) + c_2\Gamma(x_i(t) - y_i(t)), \end{cases} \tag{2.1}$$

where $x_i = (x_{i1},\cdots,x_{in})^T \in \mathbb{R}^n$ and $y_i = (y_{i1},\cdots,y_{in})^T \in \mathbb{R}^n$ are the state variables of the i $(i = 1,\cdots,N)$th node of the x-layer and y-layer, respectively. Function

$$f(\cdot) = (f_1(\cdot),\cdots,f_n(\cdot))^T$$

is the self-dynamics of each node. The constants c_1 and c_2 are intralayer coupling strength and inter-layer coupling strength.

$$H = diag\{h_1, h_2,\cdots,h_n\} > 0$$

and

$$\Gamma = diag\{\gamma_1, h_2,\cdots,h_n\} > 0$$

represent inner coupling matrices.

$$A = (a_{ij}) \in \mathbb{R}^{N\times N}$$

is the outer-coupling configuration matrix. If there exists a link from i to j $(i \neq j)$, then the corresponding entry a_{ij} should be greater than zero $(a_{ij} > 0)$; otherwise, $a_{ij} = 0$ $(i \neq j)$. Additionally, a_{ii} is defined as the negative sum of the out-weights (weights of outgoing edges) from node i, excluding the self-loop weight, i.e.,

$$a_{ii} = -\sum_{j=1, j\neq i}^{N} a_{ij}.$$

This paper considers the y-layer network as the following system involving impulses:

$$\begin{cases} \dot{y}_i(t) = f(y_i(t)) + c_1 \sum\limits_{j=1}^{N} a_{ij} H y_j(t) + c_2\Gamma(x_i(t) \\ \qquad -y_i(t)) + u_i(t), \quad t \in [t_{k-1}, t_k), \\ \Delta y_i(t_k) = D(y_i(t_k^-) - x_i(t_k^- - \tau)), \quad k \in \mathbb{Z}_+, \end{cases} \tag{2.2}$$

with the initial value $y_i(s) = \phi_i(s)$, $s \in [-\tau, 0]$ and the initial value for the x-layer network as $x_i(s) = \hat{\phi}_i(s)$, $s \in [-\tau, 0]$. Here, τ is a positive constant,

$$\Delta y_i(t_k) = y_i(t_k) - y_i(t_k^-),$$

and D denotes the impulse matrix. The sequence $\{t_k\}_{k\in\mathbb{Z}_+}$ denotes impulse time sequence, which strictly increases over the interval $\mathbb{R}_+$. We represent such a sequence by the set $\mathcal{F}$, and represent a subset of $\mathcal{F}$ by $\mathcal{F}_M$. The impulse time sequence of $\mathcal{F}_M$ satisfies

$$0 < t_i < t_{i+1} < \infty, \quad i = 1,\cdots,M-1,$$

abbreviated by $\{t_k\}^M$, where M represents the total number of impulse points. To achieve lag synchronization, external inputs $u_i(t)$ need to be applied to the nodes of the y-layer network.

Define the lag synchronization errors

$$\zeta_i(t) = y_i(t) - x_i(t-\tau).$$

The interlayer errors system is obtained

$$\begin{cases} \dot{\zeta}_i(t) = g(\zeta_i(t)) \doteq F(\zeta_i(t)) + c_1 \sum\limits_{j=1}^{N} a_{ij} H \zeta_j(t) - c_2 \Gamma \zeta_i(t) \\ \qquad + c_2 \Gamma(x_i(t) - y_i(t-\tau)) + u_i(t), \ \ t \neq t_k, \\ \zeta_i(t_k) = h(\zeta_i(t_k^-)) \doteq (I+D)\zeta_i(t_k^-), \ \ k \in \mathbb{Z}_+, \end{cases} \tag{2.3}$$

with the initial condition

$$\zeta_i(s) = \phi_i(s) - \hat{\phi}_i(s-\tau), \quad s \in [-\tau, 0]$$

and

$$F(\zeta_i(t)) = f(y_i(t)) - f(x_i(t-\tau)),$$

where $i = 1, \cdots, N$. Let

$$\zeta(t) = (\zeta_1^T(t), \zeta_2^T(t), \cdots, \zeta_N^T(t))^T,$$
$$\zeta(s) = \bar{\phi}(s) = (\zeta_1^T(s), \zeta_2^T(s), \cdots, \zeta_N^T(s))^T, \quad s \in [-\tau, 0].$$

Definition 2.1. *[38] For a given constant $\tau > 0$ and impulse time sequence $\{t_k\} \in \mathcal{F}$, network (2.1) is said to be a finite-time lag synchronization if there exists time $T > 0$ such that*

$$\lim_{t \to T} |\zeta_i(t)| = \lim_{t \to T} |y_i(t) - x_i(t-\tau)| = 0$$

and

$$|\zeta_i(t)| \equiv 0, \ \ \text{if } t \geq T, \ i = 1, 2, \cdots, N,$$

where the synchronizing time T is related to $\zeta_i(s)$ and $\mathcal{F}$.

Remark 2.1. Definition 2.1 explains the concept of finite-time lag synchronization, in which two-layer networks achieve finite-time lag synchronization from time 0 to T. Synchronization time T is related to the initial value condition and the impulse time sequence. Specifically, when $\tau = 0$, the network (2.1) can realize interlayer synchronization between layers within time T.

Definition 2.2. *For any vector*

$$w = (w_1, \cdots, w_n)^T \in \mathbb{R}^n$$

and constant α, the following definitions are given

$$S(w) = (sign(w_1), \cdots, sign(w_n))^T,$$
$$D(w) = diag\{|w_1|^\alpha, \cdots, |w_n|^\alpha\}.$$

Assumption 2.1. *Assuming the existence of certain constants $l_i > 0$, it can be stated that for a given nonlinear function $f_i(\cdot)$, the following inequality holds:*

$$|f_i(v_1) - f_i(v_2)| \leq l_i |v_1 - v_2|,$$

where $f_i(0) = 0$, $i = 1, \cdots, n$, and $v_1, v_2 \in \mathbb{R}$. Let

$$L = diag\{l_1, \cdots, l_n\}.$$

Lemma 2.1. *[42] (Synchronizing impulses) If there exist $\mathcal{K}$-class functions ψ_1 and ψ_2, locally Lipschitz continuous function $V(\zeta_i)$: $\mathbb{R}^n \to \mathbb{R}_+$, and some positive constants α, $0 < \mu < 1$, $\mu < \gamma < 1$, $0 < \lambda < 1$, such that*

$$\begin{aligned} \psi_1(|\zeta_i|) \leq V(\zeta_i) &\leq \psi_2(|\zeta_i|), \\ V(h(\zeta_i)) &\leq \mu^{\frac{1}{1-\lambda}} V(\zeta_i), \\ D^+ V[\zeta_i(t)] g(\zeta_i(t)) &\leq -\alpha V^\lambda(\zeta_i(t)), \quad t \neq t_k, \end{aligned} \tag{2.4}$$

holds, where

$$\zeta_i(t) = \zeta_i(t, \varphi)$$

is the solution of system (2.3) with $\varphi \in U$, $U \subseteq C_\tau$ as an open set containing origin, then system (2.3) is finite time stable over any class $\mathcal{F}$. In particular, let $\{t_k\}^M \in \mathcal{F}_M$ and

$$t_M \leq \gamma^{M-1} \frac{(\gamma - \mu)}{1 - \mu} \frac{V^{1-\lambda}(\varphi(0))}{\alpha(1-\lambda)}.$$

Thus, the settling time determined by initial value $\varphi \in U$ and impulse time sequence $\{t_k^M\} \in \mathcal{F}_M$ has the following boundary:

$$T(\{t_k\}^M, \varphi) \leq \gamma^M \frac{V^{1-\lambda}(\varphi(0))}{\alpha(1-\lambda)}.$$

In addition, system (2.3) can achieve globally finite time stable with impulse time sequence $\{t_k\} \in \mathcal{F}$, if $\psi_1 \in \mathcal{K}_\infty$ and $U = C_\tau$.

Lemma 2.2. *[42] (Desynchronizing impulses) Set*

$$\bar{U}_\sigma = \{\zeta_i \in \mathbb{R}^n : |\zeta_i| \leq \sigma\}$$

for positive contant σ. System (2.3) can achieve finite time stable over the class of impulse time sequence $\mathcal{F}$, if there exist $\mathcal{K}$-class functions ψ_1 and ψ_2, locally Lipschitz continuous function $V(\zeta_i)$: $\mathbb{R}^n \to \mathbb{R}_+$, and constants $\beta \in$

$[1,\infty)$, $\eta \in (0,1)$, $\alpha > 0$ *such that (2.4) holds, and the impulse time sequence* $\{t_k\} \in \mathcal{F}$ *satisfies*

$$\min_{k\in\mathbb{Z}_+}\left\{\frac{t_k}{\mu^{k-1}} \geq \frac{\psi_2^{1-\lambda}(\sigma)}{\alpha(1-\lambda)}\right\} := N_0 < +\infty.$$

Moreover, the bound of settling time is the following:

$$T(\{t_k\},\varphi) \leq \mu^{N_0-1}\frac{\psi_2^{1-\lambda}(\sigma)}{\alpha(1-\lambda)}, \quad \forall\varphi \in \bar{U}_\sigma, \forall\{t_k\} \in \mathcal{F},$$

where N_0 *depends on* $\{t_k\}$.

3. Main results

This section explores the design of various controllers to realize finite-time lag synchronization of the network (2.1), considering both synchronizing impulses and desynchronizing impulses.

3.1. Synchronizing impulses

Theorem 3.1. *Assume that there are some positive constants* $\delta,\beta \in (0,1)$, $\gamma \in (\beta,1)$, *constant* $-1 < \mu < 1$, *matrices* $P_{n\times n} > 0$, $Q_{n\times n} > 0$, *and real matrix* $W_{n\times n}$ *such that*

$$(i) \quad \begin{pmatrix} \Phi & I_N\otimes P \\ \star & -I_N\otimes Q \end{pmatrix} < 0,$$

$$(ii) \quad (I+D)^T P(I+D) \leq \beta^{\frac{2}{1-\mu}}P,$$

where

$$\Phi = I_N\otimes(LQL - 2c_2P\Gamma) + 2c_1A\otimes PH - I_N\otimes W - I_N\otimes W^T.$$

Then, the two-layer networks (2.1) realize finite-time lag synchronization with the class $\mathcal{F}$ *under the following controllers:*

$$u_i(t) = u_i^{(1)}(t) + u_i^{(2)}(t).$$

For later use, we define

$$U^{(2)}(t) = (u_1^{(2)T}(t), u_2^{(2)T}(t), \cdots, u_n^{(2)T}(t))^T.$$

Their expressions are the following:

$$\begin{cases} u_i^{(1)}(t) = -c_2\Gamma(x_i(t) - y_i(t-\tau)), \\ U^{(2)}(t) = -(I_N\otimes P^{-1})[(I_N\otimes W)\zeta(t) \\ \qquad +\frac{1}{2}\delta\lambda_{max}^{\frac{1+\mu}{2}}(P)D(\zeta(t))S(\zeta(t))]. \end{cases} \tag{3.1}$$

In particular, let $\{t_k\}^M \in \mathcal{F}_M$ *and*

$$t_M \leq \gamma^{M-1}\frac{(\gamma-\beta)}{1-\beta}\frac{2\lambda_{max}^{\frac{1-\mu}{2}}(P)|\bar{\phi}|^{1-\mu}}{\delta(1-\mu)},$$

then, the lag synchronization time satisfies

$$T(\{t_k\}^M,\bar{\phi}) \leq \gamma^M\frac{2\lambda_{max}^{\frac{1-\mu}{2}}(P)|\bar{\phi}|^{1-\mu}}{\delta(1-\mu)}, \tag{3.2}$$

which is determined by initial state $\bar{\phi} \in U$ *and impulse time sequence* $\{t_k^M\}$.

Proof. With the guidance of Lyapunov functions, we consider

$$V(t) = \zeta^T(t)(I_N\otimes P)\zeta(t) = \sum_{i=1}^N \zeta_i^T(t)P\zeta_i(t). \tag{3.3}$$

As $t \in [t_{k-1}, t_k)$ for $k \in \mathbb{Z}_+$, the Dini derivative of $V(t)$ along system (2.3) is

$$\begin{aligned} D^+V(t) =&2\sum_{i=1}^N \zeta_i^T(t)P\dot{\zeta}_i(t) \\ =&2\sum_{i=1}^N \zeta_i^T(t)P[F(\zeta_i(t)) + c_1\sum_{j=1}^N a_{ij}H\zeta_j(t) \\ &- c_2\Gamma\zeta_i(t) + c_2\Gamma(x_i(t) - y_i(t-\tau)) + u_i(t)] \qquad (3.4) \\ =&2\sum_{i=1}^N \zeta_i^T(t)PF(\zeta_i(t)) + 2c_1\sum_{i=1}^N\sum_{j=1}^N a_{ij}\zeta_i^T(t)PH\zeta_j(t) \\ &- 2c_2\sum_{i=1}^N \zeta_i^T(t)P\Gamma\zeta_i(t) + 2\sum_{i=1}^N \zeta_i^T(t)Pu_i^{(2)}(t). \end{aligned}$$

From Assumption 2.1, one may obtain

$$\begin{aligned} 2\sum_{i=1}^N \zeta_i^T(t)PF(\zeta_i(t)) \leq& \sum_{i=1}^N \zeta_i^T(t)PQ^{-1}P\zeta_i(t) \\ &+ \sum_{i=1}^N F^T(\zeta_i(t))QF(\zeta_i(t)) \\ \leq& \sum_{i=1}^N \zeta_i^T(t)PQ^{-1}P\zeta_i(t) \qquad (3.5) \\ &+ \sum_{i=1}^N \zeta_i^T(t)LQL\zeta_i(t) \\ =&\zeta^T(t)[I_N\otimes(PQ^{-1}P)]\zeta(t) \\ &+ \zeta^T(t)[I_N\otimes(LQL)]\zeta(t) \\ =&\zeta^T(t)[I_N\otimes(PQ^{-1}P + LQL)]\zeta(t). \end{aligned}$$

Therefore, combined with the above content (3.3)–(3.5), it holds that

$$\begin{aligned}
D^+V(t) \leq &\zeta^T(t)[I_N \otimes (PQ^{-1}P + LQL)]\zeta(t) \\
&+ 2\zeta^T(t)[c_1 A \otimes (PH)]\zeta(t) \\
&- 2c_2\zeta^T(t)[I_N \otimes (P\Gamma)]\zeta(t) + 2\zeta^T(t)(I_N \otimes P)U^{(2)}(t) \\
=&\zeta^T(t)[I_N \otimes (PQ^{-1}P + LQL - 2c_2 P\Gamma) \qquad (3.6)\\
&+ 2c_1 A \otimes (PH)]\zeta(t)) + 2\zeta^T(t)(I_N \otimes P)U^{(2)}(t) \\
\leq &- \delta(\zeta^T(t)(I_N \otimes P)\zeta(t))^{\frac{1+\mu}{2}} \\
= &- \delta V^{\frac{1+\mu}{2}}(t).
\end{aligned}$$

When $t = t_k$, $k \in \mathbb{Z}_+$, we can obtain

$$\begin{aligned}
V(t_k) &= \zeta^T(t_k)(I_N \otimes P)\zeta(t_k) \\
&= \zeta^T(t_k^-)\{I_N \otimes [(I + D)^T P(I + D)]\}\zeta(t_k^-) \\
&\leq \beta^{\frac{2}{1-\mu}}\zeta^T(t_k^-)(I_N \otimes P)\zeta(t_k^-) \\
&\leq \beta^{\frac{2}{1-\mu}} V(t_k^-).
\end{aligned} \tag{3.7}$$

It can be easily found that (3.6) and (3.7) satisfy Lemma 2.1. Therefore, the network (2.1) under the controller (3.1) achieves finite-time lag synchronization on any class of impulse time sequences. Moreover, the synchronizing time (3.2) is derived. □

In what follows, another Lyapunov function is constructed to obtain the finite-time lag synchronization of the network (2.1) under synchronizing impulses. The special case considered is

$$D = diag\{d_1, d_2, \cdots, d_n\} < 0, \quad d_j \in (-1, 0)$$

for all $j = 1, \cdots, n$.

Theorem 3.2. *Suppose that there are some positive constants $k_1, k_2, \delta, \gamma, \beta$, $0 < \beta < 1, \beta < \gamma < 1$, constant $-1 < \mu < 1$, and diagonal matrix $P_{n\times n} > 0$, such that*

$$\lambda_{max}(I + D) \leq \beta^{\frac{2}{1-\mu}},$$

then the network (2.1) can achieve finite-time lag synchronization on any class $\mathcal{F}$ with the following controller:

$$u_i(t) = u_i^{(1)}(t) + u_i^{(2)}(t) + u_i^{(3)}(t),$$

where

$$\begin{cases}
u_i^{(1)}(t) = -c_2\Gamma(x_i(t) - y_i(t - \tau)), \\
u_i^{(2)}(t) = \begin{pmatrix} \bar{u}_1^{(2)T}(t) \\ \bar{u}_2^{(2)T}(t) \\ \vdots \\ \bar{u}_n^{(2)T}(t) \end{pmatrix} \cdot W_i, \\
u_i^{(3)}(t) = -\frac{\delta}{2}P^{-1}S(\zeta_i(t))[2S^T(\zeta_i(t))P\zeta_i(t)]^{\frac{1+\mu}{2}},
\end{cases} \tag{3.8}$$

with

$$\begin{aligned}
\bar{u}_j^{(2)}(t) &= (u_{1j}, u_{2j}, \cdots, u_{Nj})^T \in \mathbb{R}^N \\
&= -\frac{k_1}{2}p_j l_j^2 S(\bar{\zeta}_j(t))\zeta_j^T(t)S(\bar{\zeta}_j(t)) - \frac{1}{2k_1 p_j}\bar{\zeta}_j(t) \\
&\quad - \frac{c_1 k_2}{2}\bar{h}AA^T S(\bar{\zeta}_j^T(t))\bar{\zeta}_j^T(t)S(\bar{\zeta}_j(t)) - \frac{c_1}{2k_2}\bar{h}\bar{\zeta}_j(t), \\
W_i &= (0 \ \cdots \ 1 \ \cdots \ 0)^T \in R^N,
\end{aligned}$$

where W_i is the vector with 0, except the ith element is 1,

$$\bar{\zeta}_j = (\zeta_{1j}, \cdots, \zeta_{Nj})^T \in \mathbb{R}^N$$

denotes the column vector composed of the jth row elements of all

$$\zeta_i(t)(i = 1, \cdots, N)$$

and

$$\bar{h} = \max_{1\leq j\leq n} h_j.$$

Particularly, let

$$\{t_k\}^M \in \mathcal{F}_M$$

and

$$t_M \leq \gamma^{M-1}\frac{(\gamma - \beta)}{1 - \beta}\frac{2^{\frac{3-\mu}{2}}\lambda_{max}^{\frac{1-\mu}{2}}|\bar{\phi}|^{1-\mu}}{\delta(1 - \mu)},$$

then, the lag synchronization time determined by initial value $\bar{\phi} \in U$ and impulse time sequence

$$\{t_k\}^M \in \mathcal{F}_M$$

has the bound of

$$T(\{t_k\}^M, \bar{\phi}) \leq \gamma^M \frac{2^{\frac{3-\mu}{2}}\lambda_{max}^{\frac{1-\mu}{2}}|\bar{\phi}|^{1-\mu}}{\delta(1 - \mu)}. \tag{3.9}$$

Proof. Consider the following Lyapunov function

$$V(t) = 2\sum_{i=1}^{N}\zeta_i^T(t)PS(\zeta_i(t)). \tag{3.10}$$

For $t \in [t_{k-}, t_k)$, the Dini-derivative of $V(t)$ along the solution of (2.3) is given by

$$\begin{aligned}
D^+V(t) =& 2\sum_{i=1}^{N}\zeta_i^T(t)P\dot{S}(\zeta_i(t)) + 2\sum_{i=1}^{N}\dot{e}_i^T(t)PS(\zeta_i(t))\\
=& 2\sum_{i=1}^{N}S^T(\zeta_i(t))P\dot{\zeta}_i(t)\\
=& 2\sum_{i=1}^{N}S^T(\zeta_i(t))P[F(\zeta_i(t)) + c_1\sum_{j=1}^{N}a_{ij}H\zeta_j(t)\\
&- c_2\Gamma\zeta_i(t) + c_2\Gamma(x_i(t) - y_i(t-\tau)) + u_i(t)]\\
\leq& 2\sum_{i=1}^{N}S^T(\zeta_i(t))PL\zeta_i(t)\\
&+ 2c_1\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}S^T(\zeta_i(t))PH\zeta_j(t) \qquad (3.11)\\
&- 2c_2\sum_{i=1}^{N}S^T(\zeta_i(t))P\Gamma\zeta_i(t)\\
&+ 2c_2\sum_{i=1}^{N}S^T(\zeta_i(t))P\Gamma(x_i(t) - y_i(t-\tau))\\
&+ 2\sum_{i=1}^{N}S^T(\zeta_i(t))Pu_i(t)\\
=& 2\sum_{i=1}^{N}S^T(\zeta_i(t))PL\zeta_i(t)\\
&+ 2c_1\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}S^T(\zeta_i(t))PH\zeta_j(t)\\
&- 2c_2\sum_{i=1}^{N}S^T(\zeta_i(t))P\Gamma\zeta_i(t) + 2\sum_{i=1}^{N}S^T(\zeta_i(t))Pu_i^{(2)}(t).
\end{aligned}$$

For all $\zeta_i(t) \in \mathbb{R}^n$, it can be seen that

$$\zeta_i^T(t)\zeta_i(t) \leq (\zeta_i^T(t)S(\zeta_i(t)))^2.$$

One obtains that when $|\bar{\zeta}_j(t)| \neq 0$,

$$\begin{aligned}
&2\sum_{i=1}^{N}S^T(\zeta_i(t))PL\zeta_i(t)\\
&\leq \sum_{i=1}^{N}[S^T(\zeta_i(t))P^2L^2S(\zeta_i(t))\cdot k_1\zeta_i^T(t)S(\zeta_i(t))\\
&\quad + \zeta_i^T(t)\zeta_i(t)\cdot k_1^{-1}\frac{1}{\zeta_i^T(t)S(\zeta_i(t))}] \qquad (3.12)\\
&\leq k_1\sum_{i=1}^{N}S^T(\zeta_i(t))P^2L^2S(\zeta_i(t))\cdot\zeta_i^T(t)S(\zeta_i(t))\\
&\quad + \frac{1}{k_1}\sum_{i=1}^{N}\zeta_i^T(t)S(\zeta_i(t)),
\end{aligned}$$

and

$$\begin{aligned}
&2c_1\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}S^T(\zeta_i(t))PH_1\zeta_j(t)\\
&= 2c_1\sum_{j=1}^{n}p_jh_jS^T(\bar{\zeta}_j(t))A\bar{\zeta}_j(t)\\
&\leq c_1\sum_{j=1}^{n}p_jh_j[S^T(\bar{\zeta}_j(t))AA^TS(\bar{\zeta}_j(t))\cdot k_2\bar{\zeta}_j^T(t)S(\bar{\zeta}_j(t))\\
&\quad + \bar{\zeta}_j^T(t)\bar{\zeta}_j\cdot k_2^{-1}\frac{1}{\bar{\zeta}_j(t)^TS(\bar{\zeta}_j(t))}] \qquad (3.13)\\
&\leq c_1\sum_{j=1}^{n}p_jh_j[S^T(\bar{\zeta}_j(t))AA^TS(\bar{\zeta}_j(t))\cdot k_2\bar{\zeta}_j^T(t)S(\bar{\zeta}_j(t))\\
&\quad + \frac{1}{k_2}\bar{\zeta}_j^T(t)S(\bar{\zeta}_j(t))].
\end{aligned}$$

Moreover, when $\bar{\zeta}_j = 0$, the following hold:

$$\begin{aligned}
&2\sum_{i=1}^{N}S^T(\zeta_i(t))PL\zeta_i(t)\\
&= k_1\sum_{i=1}^{N}S^T(\zeta_i(t))P^2L^2S(\zeta_i(t))\cdot\zeta_i^T(t)S(\zeta_i(t)) \qquad (3.14)\\
&\quad + \frac{1}{k_1}\sum_{i=1}^{N}\zeta_i^T(t)S(\zeta_i(t)) = 0
\end{aligned}$$

and

$$\begin{aligned}
&2c_1\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}S^T(\zeta_i(t))PH_1\zeta_j(t)\\
&= c_1\sum_{j=1}^{n}p_jh_j[S^T(\bar{\zeta}_j(t))AA^TS(\bar{\zeta}_j(t))\cdot k_2\bar{\zeta}_j^T(t)S(\bar{\zeta}_j(t)) \qquad (3.15)\\
&\quad + \frac{1}{k_2}\bar{\zeta}_j^T(t)S(\bar{\zeta}_j(t))] = 0.
\end{aligned}$$

From the definition of $S(x)$, one has

$$S^T(\zeta_i(t))S(\zeta_i(t)) = \begin{cases} \varrho \in \{1,\cdots,n\}, & |\zeta_i(t)| \neq 0,\\ 0, & |\zeta_i(t)| = 0. \end{cases}$$

Combining analysis and (3.10)–(3.15) within the interval $t \in [t_{k-1}, t_k)$, one has that

$$
\begin{aligned}
D^+V(t) \leq & k_1 \sum_{i=1}^{N} S^T(\zeta_i(t))P^2L^2S(\zeta_i(t)) \cdot \zeta_i^T(t)S(\zeta_i(t)) \\
& + \frac{1}{k_1} \sum_{i=1}^{N} \zeta_i^T(t)S(\zeta_i(t)) \\
& + c_1 \sum_{j=1}^{n} p_j h_j [S^T(\bar{\zeta}_j(t))AA^TS(\bar{\zeta}_j(t)) \cdot k_2 \bar{\zeta}_j^T(t)S(\bar{\zeta}_j(t)) \\
& + \frac{1}{k_2} \bar{\zeta}_j^T(t)S(\bar{\zeta}_j(t))] - 2c_2 \sum_{i=1}^{N} S^T(\zeta_i(t))P\Gamma\zeta_i(t) \\
& + 2\sum_{j=1}^{n} p_j S^T(\bar{\zeta}_j(t))\bar{u}_j^{(2)}(t) + 2\sum_{i=1}^{N} S^T(\zeta_i(t))Pu_i^{(3)}(t) \\
\leq & 2\sum_{i=1}^{N} S^T(\zeta_i(t))Pu_i^{(3)}(t) \\
= & \delta \sum_{i=1}^{N} S^T(\zeta_i(t))S(\zeta_i(t))[2S^T(\zeta_i(t))P\zeta_i(t)]^{\frac{1+\mu}{2}} \\
\leq & -\delta[2\sum_{i=1}^{N} S^T(\zeta_i(t))P\zeta_i(t)]^{\frac{1+\mu}{2}} \\
= & -\delta V(t)^{\frac{1+\mu}{2}}.
\end{aligned} \tag{3.16}
$$

When $t = t_k$, $k \in \mathbb{Z}_+$, we can obtain

$$
\begin{aligned}
V(t_k) &= 2\sum_{i=1}^{N} \zeta_i(t_k^-)^T(I+D)PS((I+D)\zeta_i(t_k^-)) \\
&\leq 2\lambda_{max}(I+D) \sum_{i=1}^{N} \zeta_i^T(t_k^-)PS(\zeta_i(t_k^-)) \\
&\leq 2\beta^{\frac{2}{1-\mu}} \sum_{i=1}^{N} \zeta_i^T(t_k^-)PS(\zeta_i(t_k^-)) \\
&= \beta^{\frac{2}{1-\mu}} V(t_k^-).
\end{aligned} \tag{3.17}
$$

Based on Lemma 2.1, combining (3.16) and (3.17) can deduce that the network (2.1) under the controller (3.8) and any class of impulse time sequence $\mathcal{F}$ can achieve finite-time lag synchronization. Meanwhile, the synchronizing time (3.9) is satisfied. □

Remark 3.1. In the demonstration of Theorem 3.1, inequalities (3.7) indicate that for a given μ, we need to find the smallest β to satisfy

$$(I+D)^TP(I+D) \leq \beta^{\frac{2}{1-\mu}}P$$

to be close enough. It is noteworthy that $\bar{\zeta}_j(t)^TS(\bar{\zeta}_j(t))$ is introduced in the proof of Theorem 3.2. In previous research, the sign function has been found to be crucial when addressing finite-time problems. In addition, by comparing the synchronizing time of Theorems 3.1 and 3.2, and without impulses, it has been observed that the presence of γ^M leads to a reduction in the synchronizing time, and the value of M is directly associated with the number of impulses. In other words, the synchronizing impulses play a role in promoting network synchronization.

3.2. Desynchronizing impulses

In the following, additional results on finite-time lag synchronization for two-layer complex networks are given with an emphasis on the impact of impulsive disturbance; the impulses produce opposite shocks, the desynchronizing impulses.

Theorem 3.3. *Suppose that there exist constants $\beta \in [1, \infty)$, $\delta > 0$, $\sigma > 0$, and $\mu \in (-1, 1)$, matrices $P_{n\times n} > 0$, $Q_{n\times n} > 0$, and real matrix $W_{n\times n}$ such that*

$$
\begin{aligned}
&(i) \quad \begin{pmatrix} \Phi & I_N \otimes P \\ \star & -I_N \otimes Q \end{pmatrix} < 0, \\
&(ii) \quad (I+D)^TP(I+D) \leq \beta^{\frac{2}{1-\mu}}P,
\end{aligned}
$$

where

$$\Phi = I_N \otimes (LQL - 2c_2P\Gamma) + 2c_1A \otimes PH - I_N \otimes W - I_N \otimes W^T.$$

Then, the two-layer network (2.1) for the class $\mathcal{F}$ with the controller (3.1) can achieve finite time lag synchronization, where impulse time sequence $\{t_k\} \in \mathcal{F}$ satisfies

$$\min_{j\in\mathbb{Z}_+} \left\{ \frac{t_j}{\beta^{j-1}} \geq \frac{2\lambda_{max}^{\frac{1-\mu}{2}}(P)\sigma^{1-\mu}}{\delta(1-\mu)} \right\} := N_0 < +\infty.$$

Furthermore, the synchronizing time has the following boundary:

$$T(\{t_k\}, \bar{\phi}) \leq \beta^{N_0-1} \frac{2\lambda_{max}^{\frac{1-\mu}{2}}(P)\sigma^{1-\mu}}{\delta(1-\mu)}, \quad \forall \bar{\phi} \in U_\sigma, \forall \{t_k\} \in \mathcal{F},$$

where N_0 depends on $\{t_k\}$.

Next, another finite-time lag synchronization under impulsive disturbance is presented based on the Lyapunov function form in Theorem 3.2 considering the special case

$$D = diag\{d_1, d_2, \cdots, d_n\} > 0, \quad j = 1, 2, \cdots, n.$$

Theorem 3.4. *Suppose that there exist positive constants $k_1, k_2, \delta, \sigma, \beta$, and $\beta > 1$, constant $\mu \in (-1, 1)$, and diagonal matrix $P_{n\times n} > 0$, such that*

$$\lambda_{max}(I + D) \le \beta^{\frac{2}{1-\mu}} I,$$

then the two-layer networks (2.1) achieve finite-time lag synchronization with any class $\mathcal{F}$ under the controller (3.8), where $\{t_k\} \in \mathcal{F}$ satisfies

$$\min_{j\in\mathbb{Z}_+} \left\{ \frac{t_j}{\beta^{j-1}} \ge \frac{2^{\frac{3-\mu}{2}} \lambda_{max}^{\frac{1-\mu}{2}}(P) \sigma^{\frac{1-\mu}{2}}}{\delta(1-\mu)} \right\} := N_0 < +\infty.$$

Furthermore, synchronizing time is bounded by

$$T(\{t_k\}, \bar{\phi}) \le \beta^{N_0 - 1} \frac{2^{\frac{3-\mu}{2}} \lambda_{max}^{\frac{1-\mu}{2}}(P) \sigma^{\frac{1-\mu}{2}}}{\delta(1-\mu)}, \quad \forall \bar{\phi} \in U_\sigma, \forall \{t_k\} \in \mathcal{F},$$

where N_0 depends on $\{t_k\}$.

Remark 3.2. It is worth noting that a special feedback control structure $\frac{\zeta_i}{|\zeta_i|^2}$ in u_i is introduced in the controller designing for existing results. The controller u_i has a clear drawback, which is defined even though $\zeta_i = 0$. It becomes challenging to judge whether u_i remains bounded when the errors of networks approach zero. As a result, these controllers face limitations in their application to finite-time lag synchronization problems. The controllers (3.1) and (3.8) designed in this article can effectively solve this problem and are also suitable for the case involving impulses.

The conclusions obtained from Theorems 3.3 and 3.4 show that the desynchronizing impulses have an inhibitory effect on synchronization and can prolong the synchronization time.

4. Numerical examples

This segment offers an example to demonstrate the efficacy of the proposed finite-time lag synchronization results. We consider two scenarios: one with synchronizing impulses and the other with desynchronizing impulses.

Example 4.1. The modeling of complex networks is often used in various fields, such as traffic analysis and disease spread. In this context, let's consider the two-layer complex networks (2.1) described below:

$$\begin{cases} \dot{x}_i(t) = f(x_i(t)) + c_1 \sum_{j=1}^{3} a_{ij} H x_j(t) + c_2 \Gamma(y_i(t) - x_i(t)), \\ \dot{y}_i(t) = f(y_i(t)) + c_1 \sum_{j=1}^{3} a_{ij} H y_j(t) + c_2 \Gamma(x_i(t) - y_i(t)), \end{cases} \tag{4.1}$$

with

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix},$$

$$H = \begin{pmatrix} 0.4 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.3 \end{pmatrix},$$

$$\Gamma = \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & 0.3 \end{pmatrix},$$

$c_1 = 0.3$, $c_2 = 0.5$, and

$$f(x_i(t)) = \begin{pmatrix} -a & a & 0 \\ b & -1 & 0 \\ 0 & 0 & -c \end{pmatrix} \begin{pmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \end{pmatrix} + \begin{pmatrix} 0 \\ -x_{i1}x_{i3} \\ x_{i1}x_{i2} \end{pmatrix}, \tag{4.2}$$

where $a = 10$, $b = 28$, $c = \frac{8}{3}$. In simulations, the initial conditions for the node states are taken as

$$x_1(0) = (1, 3, 0.5)^T, \; x_2(0) = (1, 3, -3)^T, \; x_3(0) = (-1, 2, 2)^T.$$

Next, consider the y-layer network involving impulses in the form of

$$\begin{cases} \dot{y}_i(t) = f(y_i(t)) + c_1 \sum_{j=1}^{3} a_{ij} H y_j(t) + c_2 \Gamma(x_i(t) - y_i(t)), \\ \qquad t \in [t_{k-1}, t_k), \\ \Delta y_i(t_k) = D(y_i(t_k^-) - x_i(t_k^- - \tau)), \quad k \in \mathbb{Z}_+, \end{cases} \tag{4.3}$$

with the initial value

$$y_1(0) = (-1, -2, -1.5)^T, \quad y_2(0) = (2, -3, 3)^T$$

and

$$y_3(0) = (-1, 0.5, 1)^T.$$

In the simulation, the lag synchronization of $\tau = 0.2$ is considered. Figure 1 shows the lag synchronization errors without control and impulses, then the x-layer state trajectories are shown in Figure 2.

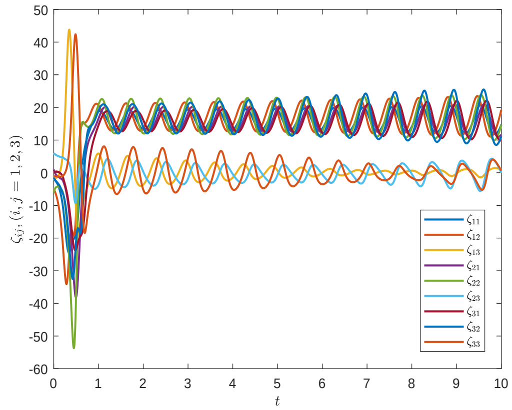

Figure 1. Trajectories of lag synchronization errors between (4.1) and (4.3) without control and impulses.

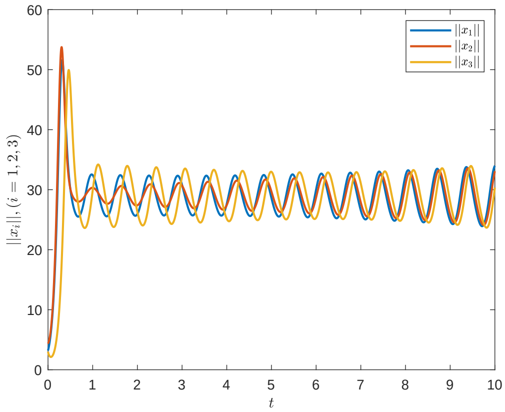

Figure 2. Trajectories of the x-layer network (4.1) without control and impulses.

Case 1. In what follows, consider when the impulses have a facilitating effect on lag synchronization. Select $\delta = 0.9$, $\mu = 0.2, \beta = 0.5$, and $\gamma = 0.7$ and note the impulse matrix

$$D = \begin{pmatrix} -0.6 & 0 & 0 \\ 0 & -0.6 & 0 \\ 0 & 0 & -0.6 \end{pmatrix}.$$

Based on Theorem 3.1, the feasible solution can be derived:

$$P = \begin{pmatrix} 0.3573 & 0 & 0 \\ 0 & 0.3538 & 0 \\ 0 & 0 & 0.3604 \end{pmatrix},$$

$$W = \begin{pmatrix} 0.9350 & 0 & 0 \\ 0 & 0.9394 & 0 \\ 0 & 0 & 0.9848 \end{pmatrix}.$$

Therefore, the errors system can achieve finite-time lag synchronization under the controller (3.1) over the impulse time sequence, where synchronization time is bounded by

$$T(\{t_k\}^M, \bar{\phi}) \leq 0.7^M \cdot 1.84 \cdot |\bar{\phi}|^{0.8}.$$

When $M = 2$, there is

$$T(\{t_k\}^M, \bar{\phi}) \leq 5.85.$$

Under the same conditions, when considering the network without impulses, the synchronizing time can be estimated as

$$T(\{t_k\}^M, \bar{\phi}) \leq 11.94.$$

The lag synchronization errors with controller (3.1) and without/with the synchronizing impulses are depicted in Figures 3 and 4. It can be seen that the synchronizing impulses can effectively shorten the network synchronization time. Figure 5 is given to show the corresponding state trajectories of the two-layer network with controller (3.1), from which it can be seen that the network is lag synchronization with $\tau = 0.2$.

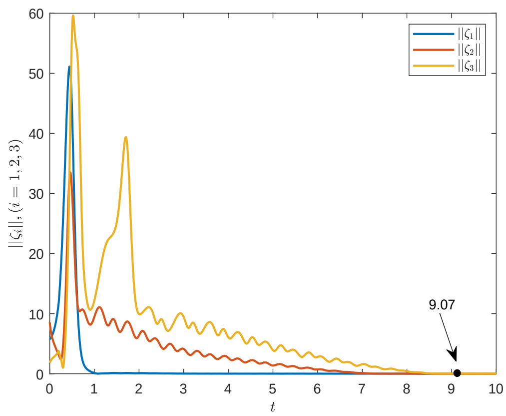

Figure 3. Trajectories of lag synchronization errors between (4.1) and (4.3) with control and without impulses.

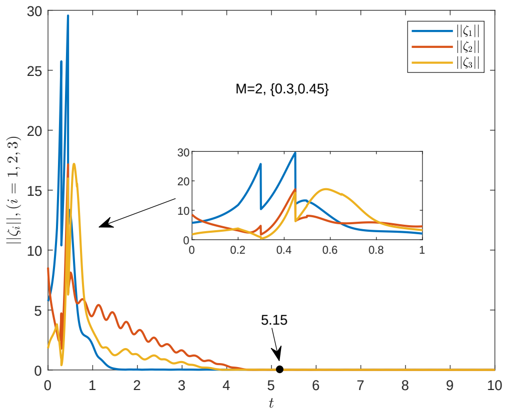

Figure 4. Trajectories of lag synchronization errors between (4.1) and (4.3) with control and impulses.

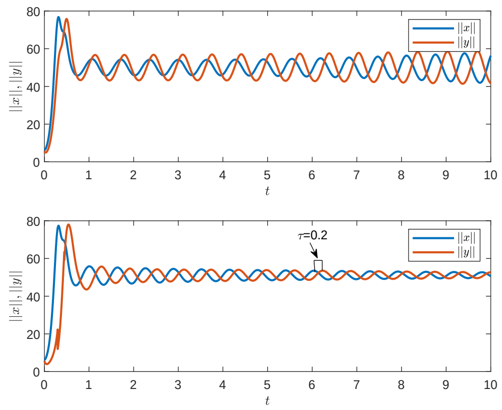

Figure 5. The upper graph shows the trajectories of the x-layer network and the y-layer network without impulses and control; the bottom graph shows the trajectories of the x-layer network and the y-layer network with synchronizing impulses and control.

Case 2. Next, the situation when the impulses have an inhibitory effect on the lag synchronization is explored. In this case, let's select $\mu = 0.4$, $\delta = 0.9$, and $\beta = 1.2$, and choose impulse matrix

$$D = \begin{pmatrix} 0.3 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.3 \end{pmatrix},$$

and impulse time sequence $t_k = 2.3k$. Based on Theorem 3.3, the following is the derivation of a feasible solution

$$P = \begin{pmatrix} 0.2535 & 0 & 0 \\ 0 & 0.2501 & 0 \\ 0 & 0 & 0.2563 \end{pmatrix},$$

$$W = \begin{pmatrix} 0.9337 & 0 & 0 \\ 0 & 0.9358 & 0 \\ 0 & 0 & 0.9698 \end{pmatrix}.$$

Hence, the error system can achieve finite-time lag synchronization with the controller (3.1), where synchronizing time is bounded by

$$T(\{t_k\}, \bar{\phi}) \leq 24.81.$$

Under the same conditions, when considering the network without impulses, the synchronizing time can be estimated as

$$T(\{t_k\}^M, \bar{\phi}) \leq 9.97.$$

The lag synchronization errors with controller (3.1) and without/with the desynchronizing impulses are shown in Figures 6 and 7. It can be found that the desynchronizing impulses can extend the network synchronizing time. As indicated in Figure 8, the corresponding state trajectories of the two-layer network with controller (3.1) are given, from which it can be seen that the network is lag synchronized with $\tau = 0.2$.

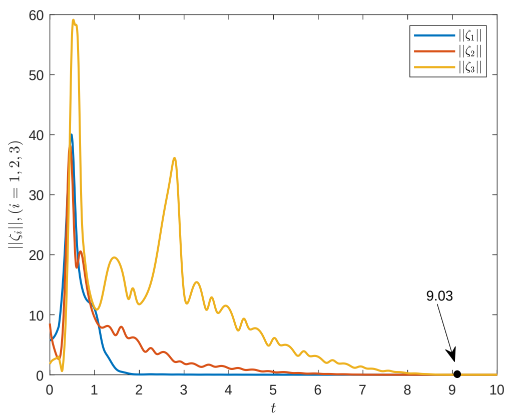

Figure 6. Trajectories of lag synchronization errors between (4.1) and (4.3) with control and without desynchronizing impulses.

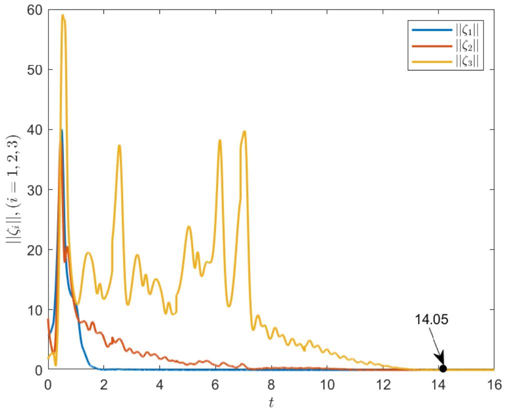

Figure 7. Trajectories of lag synchronization errors between (4.1) and (4.3) with control and desynchronizing impulses.

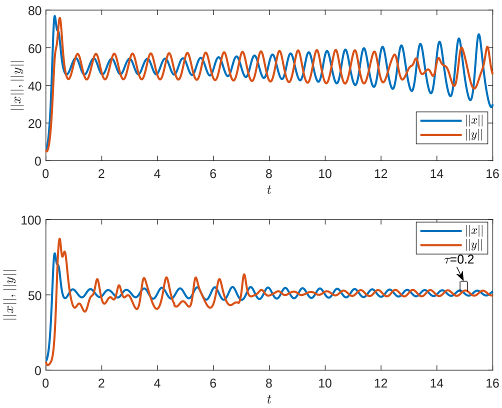

Figure 8. The upper graph shows the trajectories of the x-layer network and the y-layer network without impulses and control; the bottom graph shows the trajectories of the x-layer network and the y-layer network with desynchronizing impulses and control.

5. Conclusions

This article investigates finite-time lag synchronization problems for two-layer complex networks with impulses. Employing finite-time control and impulsive control theory, several conditions are derived to guarantee the finite-time lag synchronization of two-layer complex networks. The impact of synchronizing and desynchronizing impulses is analyzed, and multiple memory controllers are introduced to facilitate synchronization. Additionally, we calculate the synchronizing times of synchronizing and desynchronizing impulses affecting the model under consideration. A numerical example is presented to illustrate the practical significance of the proposed results, showcasing how impulses can alter the settling time of synchronization in two-layer complex networks. However, the structure of the two-layer network studied in this paper has a certain particularity, and the controller designed is related to the delay state of both layers. Therefore, realizing finite-time lag synchronization of multilayer networks with coupling delay is a future work.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare that there are no conflicts of interest in this paper.

References

1. A. Arenas, A. Díaz-Guilera, J. Kurths, Y. Moreno, C. Zhou, Synchronization in complex networks, *Phys. Reports*, **469** (2008), 93–153. https://doi.org/10.1016/j.physrep.2008.09.002
2. E. Estrada, *The structure of complex networks: theory and applications*, Oxford University Press, 2012. https://doi.org/10.1093/acprof:oso/9780199591756.001

3. W. Zhang, Y. Tang, Q. Miao, J. Fang, Synchronization of stochastic dynamical networks under impulsive control with time delays, *IEEE Trans. Neural Networks Learn. Syst.*, **25** (2013), 1758–1768. https://doi.org/10.1109/TNNLS.2013.2294727

4. X. Liu, T. Chen, Synchronization of nonlinear coupled networks via aperiodically intermittent pinning control, *IEEE Trans. Neural Networks Learn. Syst.*, **26** (2014), 113–126. https://doi.org/10.1109/TNNLS.2014.2311838

5. Z. Fan, X. Wu, B. Mao, J. Lü, Output discernibility of topological variations in linear dynamical networks, *IEEE Trans. Autom. Control*, 2024. https://doi.org/10.1109/TAC.2024.3366315

6. X. Liu, K. Zhang, Synchronization of linear dynamical networks on time scales: pinning control via delayed impulses, *Automatica*, **72** (2016), 147–152. https://doi.org/10.1016/j.automatica.2016.06.001

7. M. O. Jackson, *Social and economic networks*, Princeton: Princeton University Press, 2008. https://doi.org/10.2307/j.ctvcm4gh1

8. S. Boccaletti, G. Bianconi, R. Criado, C. I. del Genio, J. Gómez-Gardeñes, M. Romance, et al., The structure and dynamics of multilayer networks, *Phys. Reports*, **544** (2014), 1–122. https://doi.org/10.1016/J.PHYSREP.2014.07.001

9. G. D'Agostino, A. Scala, *Networks of networks: the last frontier of complexity*, Berlin: Springer, 2014. https://doi.org/10.1007/978-3-319-03518-5

10. P. J. Mucha, T. Richardson, K. Macon, M. A. Porter, P. J. Mucha, Community structure in time-dependent, multiscale, and multiplex networks, *Science*, **328** (2010), 876–878. https://doi.org/10.1126/science.1184819

11. A. Ouannas, Z. Odibat, Generalized synchronization of different dimensional chaotic dynamical systems in discrete time, *Nonlinear Dyn.*, **81** (2015), 765–771. https://doi.org/10.1007/s11071-015-2026-0

12. X. Liu, T. Chen, Synchronization of complex networks via aperiodically intermittent pinning control, *IEEE Trans. Autom. Control*, **60** (2015), 3316–3321. https://doi.org/10.1109/TAC.2015.2416912

13. X. Yang, J. Cao, Y. Long, W. Rui, Adaptive lag synchronization for competitive neural networks with mixed delays and uncertain hybrid perturbations, *IEEE Trans. Neural Networks*, **21** (2010), 1656–1667. https://doi.org/10.1109/TNN.2010.2068560

14. B. Mao, X. Wu, H. Liu, Y. Xu, J. Lü, Adaptive fuzzy tracking control with global prescribed-time prescribed performance for uncertain strict-feedback nonlinear systems, *IEEE Trans. Cybern.*, 2024. https://doi.org/10.1109/TCYB.2024.3366177

15. B. Blasius, A. Huppert, L. Stone, Complex dynamics and phase synchronization in spatially extended ecological systems, *Nature*, **399** (1999), 354–359. https://doi.org/10.1038/20676

16. X. He, Y. Wang, X. Li, Uncertain impulsive control for leader-following synchronization of complex networks, *Chaos Solitons Fract.*, **147** (2021), 110980. https://doi.org/10.1016/j.chaos.2021.110980

17. M. Li, X. Yang, X. Li, Delayed impulsive control for lag synchronization of delayed neural networks involving partial unmeasurable states, *IEEE Trans. Neural Networks Learn. Syst.*, **35** (2022), 783–791. https://doi.org/10.1109/TNNLS.2022.3177234

18. J. Zhou, T. Chen, L. Xiang, Chaotic lag synchronization of coupled delayed neural networks and its applications in secure communication, *Circuits Syst. Signal Process.*, **24** (2005), 599–613. https://doi.org/10.1007/s00034-005-2410-y

19. P. Wang, G. Wen, X. Yu, W. Yu, Synchronization of multi-layer networks: from node-to-node synchronization to complete synchronization, *IEEE Trans. Circuits Syst. I*, **66** (2018), 1141–1152. https://doi.org/10.1109/TCSI.2018.2877414

20. H. Liu, J. Li, Z. Li, Z. Zeng, J. Lü, Intralayer synchronization of multiplex dynamical networks via pinning impulsive control, *IEEE Trans. Cybern.*, **52** (2020), 2110–2122. https://doi.org/10.1109/TCYB.2020.3006032

21. D. Ning, J. Chen, M. Jiang, Pinning impulsive synchronization of two-layer heterogeneous delayed networks, *Phys. A*, **586** (2022), 126461. https://doi.org/10.1016/j.physa.2021.126461

22. X. Wu, X. Wu, C. Wang, B. Mao, J. Lu, J. Lü, et al., Synchronization in multiplex networks, *Phys. Reports*, **1060** (2024), 1–54. https://doi.org/10.1016/j.physrep.2024.01.005

23. S. Jalan, A. Singh, Cluster synchronization in multiplex networks, *Europhys. Lett.*, **113** (2016), 30002. https://doi.org/10.1209/0295-5075/113/30002

24. Y. Li, X. Wu, J. Lu, J. Lü, Synchronizability of duplex networks, *IEEE Trans. Circuits Syst. II*, **63** (2015), 206–210. https://doi.org/10.1109/TCSII.2015.2468924

25. X. Wu, Y. Li, J. Wei, J. Zhao, J. Feng, J. Lu, Inter-layer synchronization in two-layer networks via variable substitution control, *J. Franklin Inst.*, **357** (2020), 2371–2387. https://doi.org/10.1016/j.jfranklin.2019.12.019

26. R. Sevilla-Escoboza, I. Sendiña-Nadal, I. Leyva, R. Gutiérrez, J. M. Buldú, S. Boccaletti, Inter-layer synchronization in multiplex networks of identical layers, *Chaos*, **26** (2016), 065304. https://doi.org/10.1063/1.4952967

27. Y. Xu, X. Wu, B. Mao, J. Lü, C. Xie, Finite-time intra-layer and inter-layer quasi-synchronization of two-layer multi-weighted networks, *IEEE Trans. Circuits Syst. I*, **68** (2021), 1589–1598. https://doi.org/10.1109/TCSI.2021.3050988

28. C. Song, J. Zhou, J. Wang, Finite time inter-layer synchronization of duplex networks via event-dependent intermittent control, *IEEE Trans. Circuits Syst. II*, **69** (2022), 4889–4893. https://doi.org/10.1109/TCSII.2022.3187269

29. X. Li, L. Zhou, F. Tan, An image encryption scheme based on finite-time cluster synchronization of two-layer complex dynamic networks, *Soft Comput.*, **26** (2022), 511–525. https://doi.org/10.1007/s00500-021-06500-y

30. T. Jing, F. Chen, X. Zhang, Finite-time lag synchronization of time-varying delayed complex networks via periodically intermittent control and sliding mode control, *Neurocomputing*, **199** (2016), 178–184. https://doi.org/10.1016/j.neucom.2016.03.018

31. X. Yang, X. Li, P. Duan, Finite-time lag synchronization for uncertain complex networks involving impulsive disturbances, *Neural Comput. Appl.*, **34** (2022), 5097–5106. https://doi.org/10.1007/s00521-021-05987-8

32. B. Li, Finite-time synchronization for complex dynamical networks with hybrid coupling and time-varying delay, *Nonlinear Dyn.*, **76** (2014), 1603–1610. https://doi.org/10.1007/s11071-013-1232-x

33. D. Yang, X. Li, S. Song, Finite-time synchronization for delayed complex dynamical networks with synchronizing or desynchronizing impulses, *IEEE Trans. Neural Networks Learn. Syst.*, **33** (2020), 736–746. https://doi.org/10.1109/TNNLS.2020.3028835

34. X. Yang, J. Lu, Finite-time synchronization of coupled networks with Markovian topology and impulsive effects, *IEEE Trans. Autom. Control*, **61** (2015), 2256–2261. https://doi.org/10.1109/TAC.2015.2484328

35. X. Yang, J. Lam, D. W. Ho, Z. Feng, Fixed-time synchronization of complex networks with impulsive effects via nonchattering control, *IEEE Trans. Autom. Control*, **62** (2017), 5511–5521. https://doi.org/10.1109/TAC.2017.2691303

36. W. Zhang, Y. Tang, X. Wu, J. Fang, Synchronization of nonlinear dynamical networks with heterogeneous impulses, *IEEE Trans. Circuits Syst. I*, **61** (2013), 1220–1228. https://doi.org/10.1109/TCSI.2013.2286027

37. W. H. Chen, S. Luo, W. X. Zheng, Impulsive synchronization of reaction-diffusion neural networks with mixed delays and its application to image encryption, *IEEE Trans. Neural Networks Learn. Syst.*, **27** (2016), 2696–2710. https://doi.org/10.1109/TNNLS.2015.2512849

38. X. Wang, X. Liu, K. She, S. Zhong, Finite-time lag synchronization of master-slave complex dynamical networks with unknown signal propagation delays, *J. Franklin Inst.*, **354** (2017), 4913–4929. https://doi.org/10.1016/j.jfranklin.2017.05.004

39. U. E. Vincent, R. Guo, Finite-time synchronization for a class of chaotic and hyperchaotic systems via adaptive feedback controller, *Phys. Lett. A*, **375** (2011), 2322–2326. https://doi.org/10.1016/j.physleta.2011.04.041

40. G. Al-Mahbashi, M. M. Noorani, Finite-time lag synchronization of uncertain complex dynamical networks with disturbances via sliding mode control, *IEEE Access*, **7** (2019), 7082–7092. https://doi.org/10.1109/ACCESS.2018.2877707

41. X. Zhang, X. Li, J. Cao, F. Miaadi, Design of memory controllers for finite-time stabilization of delayed neural networks with uncertainty, *J. Franklin Inst.*, **355** (2018), 5394–5413. https://doi.org/10.1016/j.jfranklin.2018.05.037

42. X. Li, D. W. Ho, J. Cao, Finite-time stability and settling-time estimation of nonlinear impulsive systems, *Automatica*, **99** (2019), 361–368. https://doi.org/10.1016/j.automatica.2018.10.024