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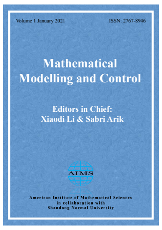

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Research article

The new Topp-Leone exponentied exponential model for modeling financial data

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Abstract: We proposed in this article a new three-parameter distribution, which is referred as the Topp-Leone exponentiated exponential model is proposed. It is used in modeling claim and risk data applied in actuarial and insurance studies. The probability density function of the suggested distribution can be unimodel and positively skewed. Different distributional and mathematical properties of the TL-EE model were provided. Furthermore, we established a maximum likelihood estimation method for estimating the unknown parameters involved in the model, and some actuarial measures were calculated. Also, the potential of these actuarial statistics were provided via numerical simulation experiments. Finally, two real datasets of insurance losses were analyzed to prove the performance and superiority of the suggested model among all its competitors distributions.

Keywords: actuarial measures;insurance losses; coefficient of skewness; simulation experiments; Topp-Leone exponentied exponential

1. Introduction

Statistical distributions are a fundamental tool for understanding and predicting phenomena in the real world. Many researchers have been interested in developing a novel family of, which have more efficient in modeling numerous kinds of skewed data in different fields, such as environmental, biomedical, finance, and insurance (for example, see Meraou and raqab [1], Hashemi et al. [2], and Abdelghani et al. [3]). For further information on analyzing experimental data, one may refer to Naderi et al. [4], Nadarajah [5], Chakraborty et al. [6], and the references cited therein.

Various statistical distributions have been considered in this context and in the last decades. These new models are generated using various methods, like adding one or more parameters to the already existing model, and it provides great pliancy in fitting asymmetric data in practice. For more information, on may refer to some recently generated distributions, namely, the new version of the beta power transformed family for [7], Marshal-Olkin generated family for [8], alpha power transformed for [9], extended alpha power transformed family for [10], Kumaraswamy Marshal-Olkin family for [11], bivariate Kavya-Manoharan transformation family for [12], new hyperbolic sine-generator for [13], and Kavya-Manoharan Weibull-G family for [14].

Choosing the best statistical distribution for modeling data is a critical and challenging task. There are situations where the modeled probability distributions fail to choose the best fit to the different types of datasets, precisely in risk measurement, economic, financial, actuarial sciences, and insurance losses, there is no claim since of the heavier tails. Indeed, this is a desirable case for the benefit of insurance companies. Some record values from numerous areas such as insurance, actuarial sciences, and economics, exist with observations far from the data's mean. Outliers or heavier tails can cause this. Classical distributions fail to model this type of dataset. Thus, heavy-tailed distributions are needed to describe the datasets appropriately. Heavy-tailed distributions available in the literature are: heavy-tailed log-logistic distribution by [15], heavy-tailed beta-power transformed Weibull distribution by [16], alpha power inverse Weibull distribution by [17], and a heavy-tailed and over dispersed collective risk model by [18]. For more details, see [19–26]. An approach to modeling the number of claims, in this case, is to use the Topp-Leone generated (TL-G) family of distributions. Recently, [27] derived this new version of the generation family. Its cumulative distribution function (CDF) can be expressed as

$$G(y;\eta,\lambda) = (1-(1-H(y;\eta))^2)^\lambda,\ y \in R, \lambda > 0. \tag{1.1}$$

The associated probability density function (PDF) of TL-G is given by

$$g(y;\eta,\lambda) = 2\lambda h(y;\eta)(1-H(y;\eta))(1-(1-H(y;\eta))^2)^{\lambda-1}, \tag{1.2}$$

where $h(y;\eta)$ and $H(y;\eta)$ represent PDF and CDF of the parent model. In the literature, The TL-G family has been studied by different authors, for example the TL Weibull distribution has been considered by [28], which he takes the Weibull distribution baseline distribution and studying different statistical properties. [29] has proposed alpha power inverted TL distribution and studied different distributional properties. TL Frèchet distribution is studied by [30]. In the same way, TL modified Weibull distribution is derived by [31], and presents the type II TL Bur XII distribution [32]. They showed that the new distribution is more flexible than other generalizations of the Bur distribution.

It's worth noting that the extended exponential (EE) distribution is the commonly used distribution for analyzing skewed data and complementary risk scenarios. It is as a new generation of an exponential (Exp) distribution, which can be used in various practical cases, including fitting the claim severity in actuarial science, following the work of [33–45].

The research [46] introduced the EE model. The corresponding PDF and CDF of EE distribution are

$$\psi(t;\alpha,\theta) = \alpha\theta e^{-\theta t}(1-e^{-\theta t})^{\alpha-1}, \quad t > 0 \tag{1.3}$$

and

$$\Psi(t;\alpha,\theta) = (1-e^{-\theta t})^\alpha, \tag{1.4}$$

where, $\alpha > 0$ is the shape parameter and $\theta > 0$ is the scale parameter.

The objective of this work is: Firstly, we define a novel version of the EE distribution which is more efficient to modeling the complex, skewed and asymmetric datasets. The new extension model is the TL extended exponential (TL-EE) model and it has three parameters. It can be positively skewed and unimodal. The unknown parameters of the TL-EE model have been estimated using the maximum likelihood estimation (MLE) technique. Second main objective is devoted to exploring three well-known actuarial measures for the TL-EE model including value at risk (VaR), tail value at risk (TVaR), and tail variance premium (TVP). These indicator risks have been a great potential in portfolio optimization under uncertainty.

The rest of this article is structured as follows. Section 2 introduces the new TL-EE model and derives various distributional properties. Also, different basic statistical properties for the suggested model were presented in Section 3. We discuss the model's parameters for TL-EE distribution in Section 4. Brief simulation experiments have been conducted in Section 5. Further, various actuarial measures from the TL-EE model have been derived in Section 6. Finally, two real datasets representing the insurance losses is performed in Section 7 for examine the potential of the proposed TL-EE model. In the last section, closing remarks are devoted.

2. TL exponentied exponential model

Here, the proposed TL-EE model with some distributional properties are derived. Let Z be a random variable that follows a TL-EE with parameters θ, α and λ ($Z \sim$ TL-EE(θ,α,λ)). According to Eqs (1.1)–(1.4), the CDF and PDF of the TL-EE model are obtained, respectively, to be

$$F_{\text{TL-EE}}(z;\theta,\alpha,\lambda) = (1-(1-(1-e^{-\theta z})^\alpha)^2)^\lambda, \tag{2.1}$$

whether $z > 0,\ \theta,\alpha,\lambda > 0$, and

$$f_{\text{TL-EE}}(z;\theta,\alpha,\lambda) = 2\lambda\alpha\theta e^{-\theta z}(1-e^{-\theta z})^{\alpha-1}(1-(1-e^{-\theta z})^\alpha)(1-(1-(1-e^{-\theta z})^\alpha)^2)^{\lambda-1}. \tag{2.2}$$

It is clear that from Eq (2.1), if $\alpha = 1$, the TL-EE model becomes the TL-Exponential distribution. For various values of the model parameters, Figures 1 and 2 sketched graphs of the PDF and CDF of the TL-EE model, respectively. The PDF of the TL-EE model is decreasing if $\alpha < 1$ and is unimodal if $\alpha > 1$.

Now, the survival with hazard rate functions (SF, HR) of the TL-EE distribution is expressed by

$$S_{\text{TL-EE}}(z;\theta,\alpha,\lambda) = 1-(1-(1-(1-e^{-\theta z})^{\alpha})^{2})^{\lambda}$$

and

$$h_{\text{TL-EE}}(z;\theta,\alpha,\lambda) = \frac{2\lambda\alpha\theta e^{-\theta z}(1-e^{-\theta z})^{\alpha-1}(1-(1-e^{-\theta z})^{\alpha})}{(1-(1-(1-e^{-\theta z})^{\alpha})^{2})}.$$

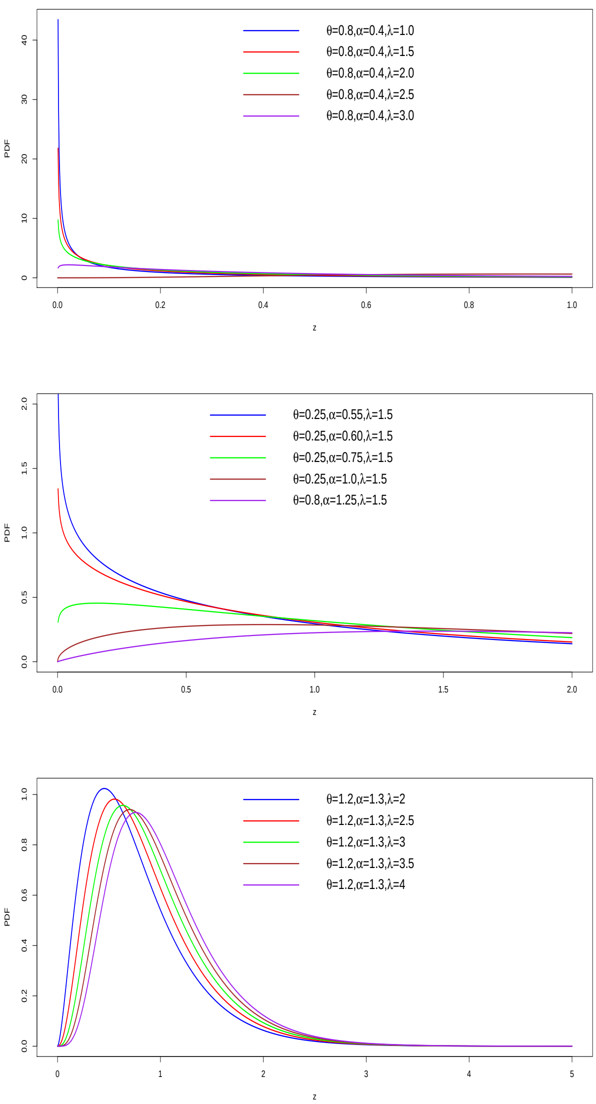

Figure 1. Density plots for the TL-EE model under different selected parameter values.

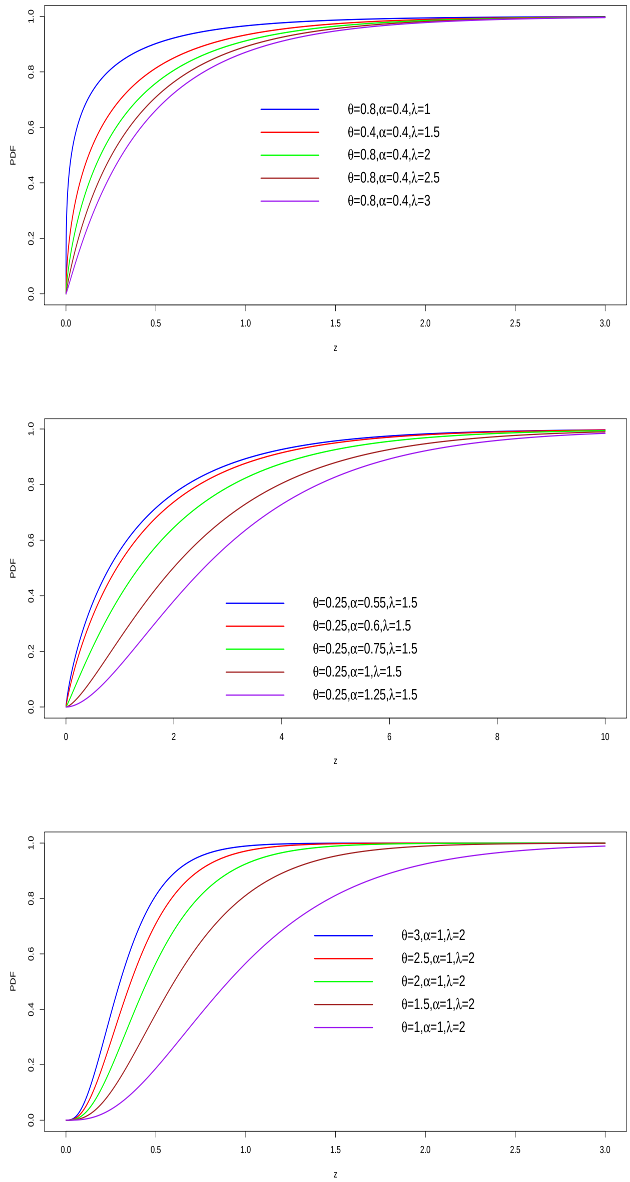

Figure 2. Graphs for CDF of the TL-EE model using several parametric values θ, α, and λ.

Figures 3 depicts the graphs of HR of the TL-EE model using numerous selected records of parameters. From Figure 3, it can be seen that the HR of the TL-EE model is increasing with shape parameter $\alpha > 1$ and decreasing with shape parameter $\alpha < 1$.

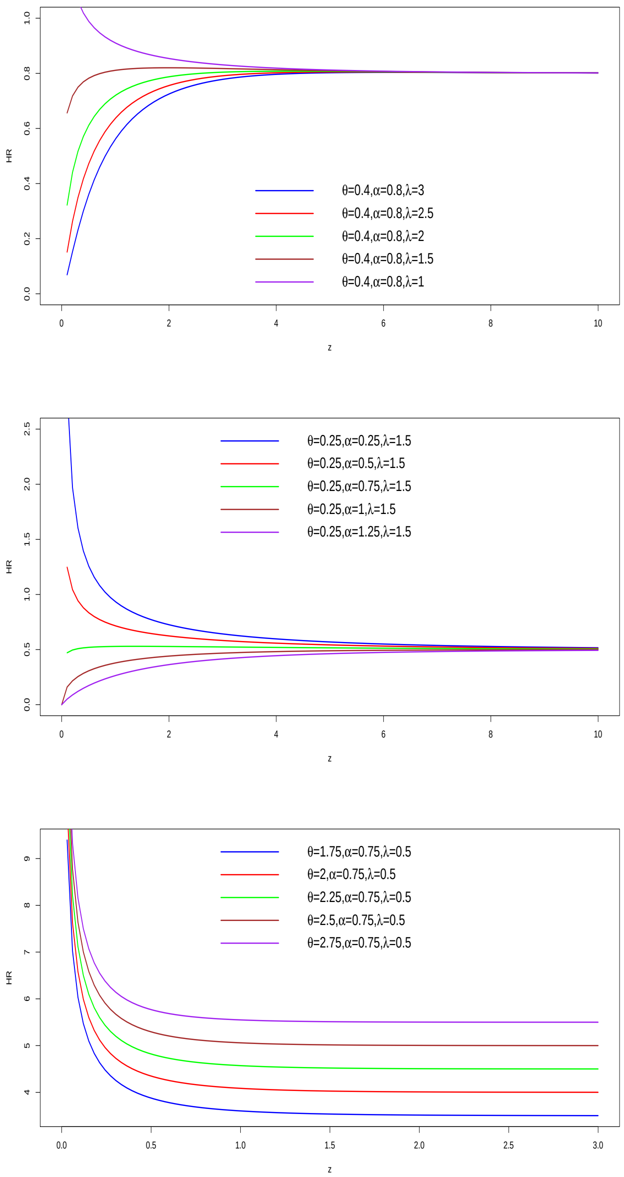

Figure 3. Graphs for HR of the TL-EE model using several parametric values θ, α, and λ.

3. Statistical properties of TL-EE model

In this section, the quantile function, skewness, kurtosis and moment-generating function of the proposed model are established.

Let $Z \sim$ TL-EE$(\theta, \alpha, \lambda)$. The quantile function of Z

$$z_u = Q(u) = F^{-1}_{\text{TL-EE}}(u), \quad 0 < u < 1$$

is obtained as follows:

$$z_u = -\frac{1}{\theta}\log\left\{1 - \left[1 - \left(1 - u^{1/\lambda}\right)^{1/2}\right]^{1/\alpha}\right\}. \tag{3.1}$$

By using (3.1), a random sample from the TL-EE model is obtained with u follows a uniform random number (0,1).

By taking

$$u = \frac{1}{4}, \frac{1}{2}, \quad \text{and} \quad \frac{3}{4}$$

in (3.1), we can be illustrated the 1^{st} quantile, median, and 3^{rd} quantile, respectively.

The skewness (Skew) and the kurtosis (Kurt) measures of Z are obtained to be

$$\mathcal{S} = \frac{z_{1/4} + z_{3/4} - 2z_{1/2}}{z_{3/4} - z_{1/4}}$$

and

$$\mathcal{K} = \frac{z_{7/8} - z_{5/8} + z_{3/8} - z_{1/8}}{z_{6/8} - z_{2/8}},$$

The rth moment of Z can be expressed by

$$\begin{aligned}\mu_r' &= \int_0^\infty z^r f_{\text{TL-EE}}(z;\theta,\alpha,\lambda)dz \\ &= 2\alpha\lambda\theta^{-r}\sum_{i=0}^{\infty}\pi_i(\alpha,\lambda)\,\Phi_i(t;r,\alpha),\end{aligned} \tag{3.2}$$

where

$$\pi_i(\alpha,\lambda) = (-1)^i\binom{\alpha-1}{i}\binom{\lambda-1}{i}$$

and

$$\Phi_i(t;r,\alpha) = \int_0^1 (\log(t))^r\, t^{i-1}\,(1-(1-t)^\alpha)^{2i+1}dt.$$

Proof.

$$\mu_r' = \int_0^\infty z^r 2\lambda\alpha\theta e^{-\theta z}(1-e^{-\theta z})^{\alpha-1}(1-(1-e^{-\theta z})^\alpha)$$
$$(1-(1-(1-e^{-\theta z})^\alpha)^2)^{\lambda-1}\,dz.$$

By taking $t = e^{-\theta z}$, then

$$\mu_r' = 2\lambda\alpha\theta^{-r}\int_0^1 \frac{(\log(t))^r}{t}(1-t)^{\alpha-1}(1-(1-t)^\alpha)$$
$$(1-(1-(1-t)^\alpha)^2)^{\lambda-1}\,dt.$$

Using the series representation

$$(1-y)^{a-1} = \sum_{i=0}^{\infty}(-1)^i\binom{a-1}{i}y^i.$$

The formula of the rth moment of Z can be written as

$$\mu'_r = 2\lambda\alpha\theta^{-r}\int_0^1 \sum_{i=0}^{\infty}(-1)^i\binom{\alpha-1}{i}\binom{\lambda-1}{i}\log(t))^r t^{i-1}$$
$$(1-(1-t)^{\alpha})\,(1-(1-t)^{\alpha})^{2i}\,dt$$
$$= 2\lambda\alpha\theta^{-r}\int_0^1 \sum_{i=0}^{\infty}(-1)^i\binom{\alpha-1}{i}\binom{\lambda-1}{i}\log(t))^r t^{i-1}$$
$$(1-(1-t)^{\alpha})^{2i+1}\,dt$$
$$= 2\alpha\lambda\theta^{-r}\sum_{i=0}^{\infty}\pi_i(\alpha,\lambda)\,\Phi_i(t;r,\alpha).$$

From (3.2), the mean, 2^{nd} moment, and variance of Z can be written as

$$\mu'_1 = \frac{2\alpha\lambda}{\theta}\sum_{i=0}^{\infty}\pi_i(\alpha,\lambda)\,\Phi_i(t;1,\alpha),$$

$$\mu'_2 = \frac{2\alpha\lambda}{\theta^2}\sum_{i=0}^{\infty}\pi_i(\alpha,\lambda)\,\Phi_i(t;2,\alpha),$$

and

$$Var(Z) = \mu_1'^2 - \mu'_2.$$

Consequently, the coefficient of variation (CV) of Z is obtained to be

$$CV(Z) = \frac{\sqrt{Var(Z)}}{\mu'_1}.$$

At the end, The moment-generating function of Z can be expressed by

$$M_Z(l) = \mathrm{E}[e^{lz}]$$
$$= 2\alpha\lambda\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\frac{\pi_i(\alpha,\lambda)\,l^k}{k!\,\theta^k}\,\Phi_i(t;k,\alpha). \quad (3.3)$$

The proof is completed. □

Proof.

$$M_Z(l) = \int_0^{\infty} e^{lz}\,2\lambda\alpha\theta e^{-\theta z}(1-e^{-\theta z})^{\alpha-1}(1-(1-e^{-\theta z})^{\alpha})$$
$$(1-(1-(1-e^{-\theta z})^{\alpha})^2)^{\lambda-1}dz.$$

By using the series representation

$$e^{lz} = \sum_{i=0}^{\infty}\frac{l^k z^k}{k!}$$

and taking

$$t = e^{-\theta z},$$

we have

$$M_Z(l) = 2\lambda\alpha\int_0^1 \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}(-1)^i\binom{\alpha-1}{i}\binom{\lambda-1}{i}\frac{l^k\ \log(t))^k}{k!}t^{i-1}$$
$$(1-(1-t)^{\alpha})^{2i+1}dt$$
$$= 2\alpha\lambda\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\frac{\pi_i(\alpha,\lambda)\,l^k}{k!\,\theta^k}\,\Phi_i(t;k,\alpha).$$

The cumulant generating function (CGF) $K_Z(l)$ of the TL-EE distribution can be resulted as

$$K_Z(l) = \log\left\{2\alpha\lambda\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\frac{\pi_i(\alpha,\lambda)\,l^k}{k!\,\theta^k}\,\Phi_i(t;k,\alpha)\right\}. \quad (3.4)$$

The characteristic function $\phi_Z(l)$ of the TL-EE model can be concluded from $M_Z(l)$ and it is given as

$$M_Z(l) = \mathrm{E}[e^{ilz}]$$
$$= 2\alpha\lambda\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\frac{\pi_i(\alpha,\lambda)il^k}{k!\,\theta^k}\,\Phi_i(t;k,\alpha), \quad (3.5)$$

where

$$i = \sqrt{-1}$$

is the imaginary unit. □

The different statistic measures of TL-EE distribution are recorded in Tables 1 and 2, whereas Figures 4 and 5 plotted the 3D curves by applying various selected values of the parameters. From the numerical results of Tables 1 and 2, we have:

(1) The measures of Mean and Var of Z decrease when θ tends to be increases with fixed α and λ, and CV, $\mathcal{S}$, and $\mathcal{K}$ measures have constants values, which indicate that these measures are free of parameter of θ.

(2) If α tends to be increases with θ and λ are fixed, the Mean and Var measures augment, whereas the CV, $\mathcal{S}$ and $\mathcal{K}$ decrease.

(3) The proposed TL-EE model is more efficient in analyzing more datasets.

Table 1. Various statistical measures for theTL-EE model at λ=1.5.

	θ	μ_1'	Var	CV	$\mathcal{S}$	$\mathcal{K}$
α=0.25	0.5	0.2460	0.2550	2.0523	4.2055	25.407
	1	0.1230	0.0637	2.0523	4.2055	25.407
	1.5	0.0820	0.0283	2.0523	4.2055	25.407
	2	0.0615	0.0159	2.0523	4.2055	25.407
α=0.5	0.5	0.6196	0.6258	1.2768	2.5999	9.7552
	1	0.3098	0.1564	1.2768	2.5999	9.7552
	1.5	0.2065	0.0695	1.2768	2.5999	9.7552
	2	0.1549	0.0391	1.2768	2.5999	9.7552
α=0.75	0.5	0.9702	0.9287	0.9932	2.0249	5.9692
	1	0.4851	0.2321	0.9932	2.0249	5.9692
	1.5	0.3234	0.1031	0.9932	2.0249	5.9692
	2	0.2425	0.0580	0.9932	2.0249	5.9692
α=1	0.5	1.2813	1.1606	0.8407	1.7251	4.3798
	1	0.6406	0.2901	0.8407	1.7251	4.3798
	1.5	0.4271	0.1289	0.8407	1.7251	4.3798
	2	0.3203	0.0725	0.8407	1.7251	4.3798

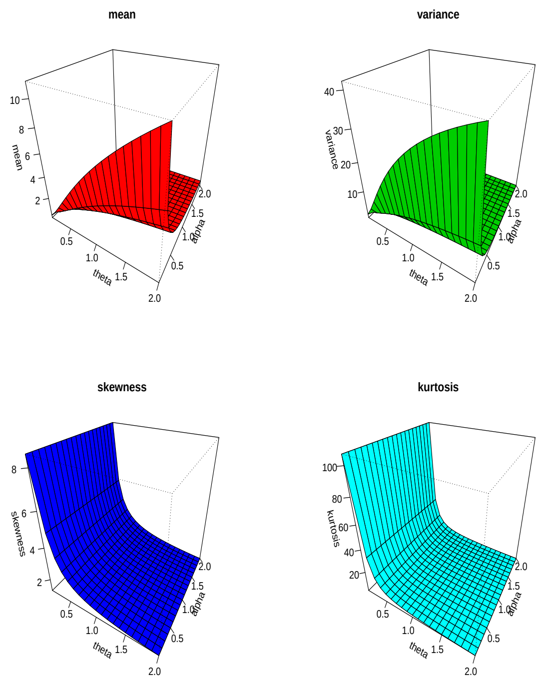

Figure 4. 3D curves for various statistical properties of the TL-EE model in Table 1.

Table 2. Various statistical measures for theTL-EE model at λ=3.

	θ	μ_1'	Var	CV	$\mathcal{S}$	$\mathcal{K}$
α=0.25	0.5	0.4325	0.4196	1.4976	3.1710	14.660
	1	0.2162	0.1049	1.4976	3.1710	14.660
	1.5	0.1441	0.0466	1.4976	3.1710	14.66
	2	0.1081	0.0262	1.4976	3.1710	14.660
α=0.5	0.5	0.9848	0.8709	0.9476	2.0392	6.1814
	1	0.4924	0.2177	0.9476	2.0392	6.1814
	1.5	0.3282	0.0967	0.9476	2.0392	6.1814
	2	0.2462	0.0544	0.9476	2.0392	6.1814
α=0.75	0.5	1.4485	1.1686	0.7462	1.6458	4.1135
	1	0.7242	0.2921	0.7462	1.6458	4.1135
	1.5	0.4828	0.1298	0.7462	1.6458	4.1135
	2	0.3621	0.0730	0.7462	1.6458	4.1135
α=1	0.5	1.8340	1.3685	0.6378	1.4452	3.2372
	1	0.9170	0.3421	0.6378	1.4452	3.2372
	1.5	0.6113	0.1520	0.6378	1.4452	3.2372
	2	0.4585	0.0855	0.6378	1.4452	3.2372

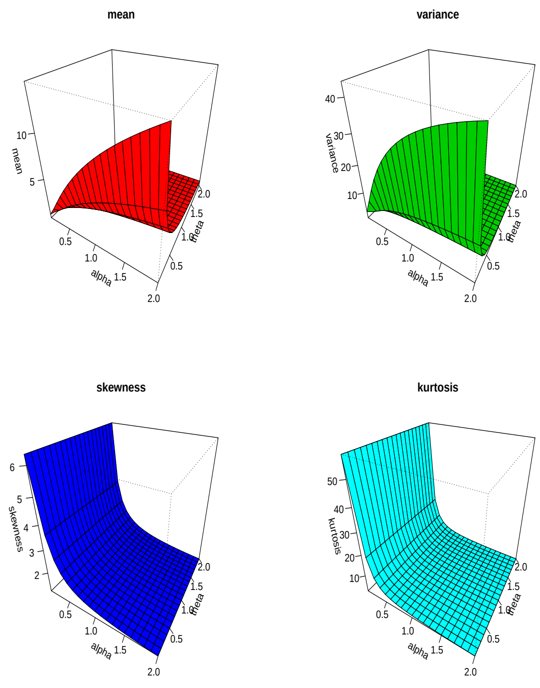

Figure 5. 3D curves for various statistical properties of the TL-EE model in Table 2.

4. Order statistics

Let $Z_{1:n} \leq Z_{2:n} \leq \ldots \leq Z_{n:n}$ denote the order statistic of a random sample of size n $Z_1, Z_2, \ldots, Z_n$ from the TL-EE model. The PDF of k^{th} order statistic of $Z_{k:n}$ can be formulated as

$$g_{Z_{k:n}}(x) = \frac{n!}{(k-1)!(n-k)!}[F(z)]^{k-1}[1-F(z)]^{n-k}f(z). \quad (4.1)$$

With replacing (4.1), we get

$$\begin{aligned} g_{Z_{k:n}}(x) = &\frac{2\lambda\alpha\theta e^{-\theta z} n!}{(k-1)!(n-k)!}\Big[1-(1-(1-e^{-\theta z})^{\alpha})^{2}\Big]^{\lambda k-1} \\ &\times \Big[1-(1-(1-(1-e^{-\theta z})^{\alpha})^{2})^{\lambda}\Big]^{n-k} \\ &\times (1-e^{-\theta z})^{\alpha-1}(1-(1-e^{-\theta z})^{\alpha}). \end{aligned} \quad (4.2)$$

Specifically, we can obtain the PDF of the first and latest order statistics

$$Z_{1:n} = \min\{Z_1, Z_2, ..., Z_n\} \text{ and } Z_{n:n} = \max\{Z_1, Z_2, ..., Z_n\}$$

and they are given, respectively, by

$$\begin{aligned} g_{Z_{1:n}}(x) = &2n\lambda\alpha\theta e^{-\theta z}\Big[1-(1-(1-e^{-\theta z})^{\alpha})^{2}\Big]^{\lambda-1} \\ &\times \Big[1-(1-(1-(1-e^{-\theta z})^{\alpha})^{2})^{\lambda}\Big]^{n-1} \\ &\times (1-e^{-\theta z})^{\alpha-1}(1-(1-e^{-\theta z})^{\alpha}) \end{aligned} \quad (4.3)$$

and

$$\begin{aligned} g_{Z_{n:n}}(x) = &2n\lambda\alpha\theta e^{-\theta z}\Big[1-(1-(1-e^{-\theta z})^{\alpha})^{2}\Big]^{\lambda n-1} \\ &\times (1-e^{-\theta z})^{\alpha-1}(1-(1-e^{-\theta z})^{\alpha}). \end{aligned} \quad (4.4)$$

At the end, the rth order moment of $Z_{r:n}$ for the TL-EE model is written as follows

$$E(Z_{k:n}^{r}) = \int_0^{\infty} z^r \cdot g_{Z_{k:n}}(z)\mathrm{d}z,$$

where $g_{Z_{k:n}}(z)$ is given in (4.2).

5. Rényi entropy

Rényi entropy is a very important tool in information measure. It is introduced as

$$R(p) = \frac{1}{1-p}\log\left(\int_0^{\infty}[f(z;\theta,\alpha,\lambda)]^p dz\right), \quad p \neq 1, p > 0.$$

For the TL-EE model, we get

$$\begin{aligned} I = &\int_0^{\infty}[f(z;\theta,\alpha,\lambda)]^p = (2\alpha\lambda\theta)^p\int_0^{\infty} e^{-p\theta z}(1-e^{-\theta z})^{p(\alpha-1)} \\ &\times (1-(1-e^{-\theta z})^{p})^{\alpha}dz \times (1-(1-(1-e^{-\theta z})^{\alpha})^{2})^{p(\lambda-1)}. \end{aligned}$$

By taking $t = e^{-\theta z}$, we have

$$\begin{aligned} I = &2^p\alpha^p\lambda^p\theta^{p-1}\int_0^1 t^p(1-t)^{p(\alpha-1)}(1-(1-t)^p)^{\alpha} \\ &\times (1-(1-(1-t)^{\alpha})^{2})^{p(\lambda-1)}dt. \end{aligned}$$

Using the series representation,

$$(1-y)^{a-1} = \sum_{i=0}^{\infty}(-1)^i\binom{a-1}{i}y^i.$$

So, we have

$$\begin{aligned} I = &2^p\alpha^p\lambda^p\theta^{p-1}\int_0^1\sum_{i=0}^{\infty}(-1)^i\binom{p(\alpha-1)}{i}t^{p+i}(1-(1-t)^p)^{\alpha} \\ &\times (1-(1-(1-t)^{\alpha})^{2})^{p(\lambda-1)}dt \\ = &\frac{2^p\alpha^p\lambda^p\theta^{p-1}}{1-p}\sum_{i=0}^{\infty}\eta_i(\alpha,p)\Psi_i(t,\alpha,\lambda,p) \end{aligned}$$

with

$$\eta_i(\alpha,p) = (-1)^i\binom{p(\alpha-1)}{i}$$

and

$$\Psi_i(t,\alpha,\lambda,p) = \int_0^1 t^{p+i}(1-(1-t)^p)^{\alpha}(1-(1-(1-t)^{\alpha})^{2})^{p(\lambda-1)}dt.$$

Consequently, Rényi entropy of the TL-EE distribution is given by

$$R(p) = \frac{1}{1-p}\log\left\{\frac{2^p\alpha^p\lambda^p\theta^{p-1}}{1-p}\sum_{i=0}^{\infty}\eta_i(\alpha,p)\Psi_i(t,\alpha,\lambda,p)\right\}.$$

6. Maximum likelihood estimation

Suppose $z_1, z_2, \ldots, z_n$ are n observations from the TL-EE model. Furthermore, the log-likelihood function corresponding to Eq (2.2) can be written as

$$\begin{aligned} LL(\Omega) = &n\log(2\lambda\alpha\theta) - \theta\sum_{i=1}^{n} z_i + (\alpha-1)\sum_{i=1}^{n}\log(1-e^{-\theta z_i}) \\ &+ (\lambda-1)\sum_{i=1}^{n}\log(1-(1-(1-e^{-\theta z_i})^{\alpha})^{2}) \\ &+ \sum_{i=1}^{n}\log(1-(1-e^{-\theta z_i})^{\alpha}). \end{aligned} \quad (6.1)$$

Here, $\Omega = (\theta, \alpha, \lambda)$. To get the MLEs of θ, α, and λ, we maximize Eq (6.1) with respect to the unknown parameters and then equate the result to zero. That is,

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \frac{\alpha z_i e^{-\theta z_i}(1-e^{-\theta z_i})^{(\alpha-1)}}{(1-(1-e^{-\theta z_i})^{\alpha})} - 2\sum_{i=1}^{n} \frac{(1-(1-e^{-\theta z_i})^{\alpha})\alpha z_i e^{-\theta z_i}(1-e^{-\theta z_i})^{(}\alpha-1)}{1-(1-(1-e^{-\theta z_i})^{\alpha})^2} - \sum_{i=1}^{n} z_i + \sum_{i=1}^{n} \frac{z_i e^{-\theta z_i}}{1-e^{-\theta z_i}} = 0, \tag{6.2}$$

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} \frac{(1-e^{-\theta z_i})^{\alpha}\log(1-e^{-\theta z_i})}{(1-(1-e^{-\theta z_i})^{\alpha})} - 2\sum_{i=1}^{n} \frac{((1-e^{-\theta z_i})^{\alpha}-1)(1-e^{-\theta z_i})^{\alpha}\log(1-e^{-\theta z_i})}{(1-(1-(1-e^{-\theta z_i})^{\alpha})^2)} + \sum_{i=1}^{n} \log(1-e^{-\theta z_i}) = 0, \tag{6.3}$$

and

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} \log(1-(1-(1-e^{-\theta z_i})^{\alpha})^2) = 0. \tag{6.4}$$

Clearly, from Eqs (6.2)–(6.4) the final estimates cannot be resulted in explicit form. To overcome this problem, numerous approximate techniques such as bisection method, fixed point, and Newton-Raphson are produced to obtain the final estimates of Ω.

7. Simulation analysis

Here, we perform an Monte Carlo Markov Chain simulation analysis to examine the potential of the MLEs of the proposed model for different sample sizes $n = \{100, 250, 500, 705, 1000\}$ using the quantile function of the proposed model and for different parameter sets

$$\text{Set1} = (\theta = 0.6,\ \alpha = 0.8,\ \lambda = 0.5),$$

$$\text{Set2} = (\theta = 0.25,\ \alpha = 0.5,\ \lambda = 0.7),$$

$$\text{Set3} = (\theta = 0.75,\ \alpha = 1.0,\ \lambda = 0.9)$$

and

$$\text{Set4} = (\theta = 0.8,\ \alpha = 0.25,\ \lambda = 1.2).$$

The above steps are provided to obtain a random sample from the TL-EE model:

Step 1. Drawn u from uniform with interval $[0, 1]$.

Step 2. Obtain z as

$$z = -\frac{1}{\theta}\log\left\{1-\left[1-\left(1-u^{1/\lambda}\right)^{1/2}\right]^{1/\alpha}\right\}.$$

We computed the average estimate (AE), the average mean square errors (MSEs), the average biases (AB), and associated mean relative errors (MREs) based on M =1000 times. These values are given as follows:

$$\text{AE} = \frac{1}{M}\sum_{i=1}^{M}\widehat{\boldsymbol{\varepsilon}}, \qquad \text{AB} = \frac{1}{M}\sum_{i=1}^{M}|\widehat{\boldsymbol{\varepsilon}}-\boldsymbol{\varepsilon}|,$$

$$\text{MSE} = \frac{1}{M}\sum_{i=1}^{M}(\widehat{\boldsymbol{\varepsilon}}-\boldsymbol{\varepsilon})^2, \quad \text{MRE} = \frac{1}{M}\sum_{i=1}^{M}|\widehat{\boldsymbol{\varepsilon}}-\boldsymbol{\varepsilon}|/\boldsymbol{\varepsilon},$$

where

$$\boldsymbol{\varepsilon} = (\theta, \alpha, \lambda).$$

Tables 3–6 summarize the results of the simulation study of the TL-EE model. It is noted from the results that a the AEs of all parameters converge to the actual values of parameters. Also, the ABs, MSEs, and MREs decrease as n tends to be increase based on MLE technique which ensure that the estimates of unknown parameters are consistent and asymptotically unbiased.

Table 3. AEs, ABs, MSEs, and MREs for the proposed TL-EE model using Set1.

Simple size	Est.	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$
100	AE	0.6932	1.3583	1.0299
	AB	0.0932	0.5583	0.5299
	MSE	0.0664	1.4037	1.3624
	MRE	0.1553	0.6979	1.0598
250	AE	0.6341	1.0291	0.6319
	AB	0.0341	0.2291	0.1319
	MSE	0.0225	0.4660	0.3173
	MRE	0.0569	0.2864	0.2638
500	AE	0.6255	0.9456	0.5500
	AB	0.0255	0.1456	0.0500
	MSE	0.0130	0.2324	0.0889
	MRE	0.0426	0.1820	0.1001
750	AE	0.6075	0.8852	0.5434
	AB	0.0075	0.0852	0.0434
	MSE	0.0054	0.1477	0.0581
	MRE	0.0125	0.1065	0.0869
1000	AE	0.6047	0.8728	0.5401
	AB	0.0047	0.0728	0.0401
	MSE	0.0043	0.1318	0.0519
	MRE	0.0078	0.0911	0.0803

Table 4. AEs, ABs, MSEs, and MREs for the proposed TL-EE model using Set2.

Simple size	Est.	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$
100	AE	0.3099	1.0043	0.8253
	AB	0.0599	0.5043	0.1253
	MSE	0.0143	0.8638	1.5698
	MRE	0.2396	1.0086	0.1790
250	AE	0.2668	0.6964	0.8021
	AB	0.0168	0.1964	0.1021
	MSE	0.0052	0.2768	0.4464
	MRE	0.0674	0.3928	0.1636
500	AE	0.2485	0.5278	0.8099
	AB	0.0014	0.0278	0.1099
	MSE	0.0019	0.0493	0.1588
	MRE	0.0056	0.0557	0.1570
750	AE	0.2552	0.5480	0.7556
	AB	0.0052	0.04806	0.0556
	MSE	0.0016	0.0596	0.1034
	MRE	0.0208	0.0961	0.0795
1000	AE	0.2553	0.5312	0.7357
	AB	0.0053	0.0312	0.03574
	MSE	0.0011	0.0268	0.0693
	MRE	0.0210	0.0624	0.0510

Table 5. AEs, ABs, MSEs, and MREs for the proposed TL-EE model using Set3.

Simple size	Est.	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$
100	AE	0.8480	2.1978	1.9580
	AB	0.0980	1.1978	1.0580
	MSE	0.0665	1.2910	1.7854
	MRE	0.1307	1.1978	1.1756
250	AE	0.8246	1.5339	1.3104
	AB	0.0346	0.5339	0.4104
	MSE	0.0302	1.0916	1.6425
	MRE	0.0861	0.5339	0.4560
500	AE	0.7731	1.2048	1.1657
	AB	0.0231	0.2048	0.2657
	MSE	0.0130	0.6911	1.1267
	MRE	0.0275	0.2048	0.2953
750	AE	0.7656	1.1924	0.9820
	AB	0.0156	0.1924	0.0820
	MSE	0.0087	0.4560	0.3348
	MRE	0.0208	0.1924	0.0911
1000	AE	0.7653	1.1837	0.9436
	AB	0.0153	0.1837	0.0436
	MSE	0.0082	0.4025	0.1537
	MRE	0.0204	0.1837	0.0485

Table 6. TAEs, ABs, MSEs, and MREs for the proposed TL-EE model using Set4.

Simple size	Est.	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$
100	AE	0.9913	0.4293	2.4947
	AB	0.1913	0.1793	1.2947
	MSE	0.1833	0.2143	1.9780
	MRE	0.2391	0.7172	1.0789
250	AE	0.8291	0.3066	1.5424
	AB	0.0291	0.0566	0.3424
	MSE	0.0882	0.06599	1.4114
500	AE	0.8240	0.2643	1.3341
	AB	0.0240	0.0143	0.1341
	MSE	0.0217	0.0087	0.3792
	MRE	0.0300	0.0572	0.1118
750	AE	0.8172	0.2658	1.2881
	AB	0.0172	0.0140	0.0881
	MSE	0.0163	0.0070	0.2778
	MRE	0.0208	0.0653	0.0734
1000	AE	0.8163	0.2650	1.2237
	AB	0.0163	0.0150	0.02376
	MSE	0.0137	0.0048	0.1518
	MRE	0.0204	0.0602	0.0198

8. Actuarial measures

The exposure market risk in a portfolio of instruments is one of the most important tasks of actuarial sciences. In the last decades, various characteristics of risk measures have been considered for example, one may refer to [47–49]). This part of the study introduces the final expression of VaR, TVaR, and TVP measures for the TL-EE model.

8.1. VaR

The VaR at level significance q defined the quantile function of proposed model, which introducing the percentage loss in the portfolio value. Let Z follows the TL-EE distribution. The expression of VaR, denoted by T_1, for the TL-EE is given by

$$\mathrm{VaR}_q(Z) = \inf\{z : F(z) > q\} = F_Z^{-1}(q),$$

WHERE $0 < q < 1$.

Hence, the VaR of Z is

$$T_1 = -\frac{1}{\theta} \log\left\{1 - \left[1 - \left(1 - q^{1/\lambda}\right)^{1/2}\right]^{1/\alpha}\right\}. \tag{8.1}$$

8.2. TVaR

The TVaR or conditional tail expectation quantifies the average of losses above the VaR for some given confidence level. The TVaR, denoted by T_2 for the proposed TL-EE model, is determined by using the following relation

$$\begin{aligned} T_2 &= E[Z \mid Z > \mathrm{VaR}_q(Z)] \\ &= \frac{1}{1-q}\int_{\mathrm{VaR}_q}^{\infty} z f_Z(z;\theta,\alpha,\lambda)\, dz \\ &= \frac{2\alpha\lambda}{\theta(1-q)}\sum_{i=0}^{\infty}\pi_i(\alpha,\lambda)\,\Phi_i(t;1,\alpha) \end{aligned}$$

with

$$\Phi_i(t;1,\alpha) = \int_{\mathrm{VaR}_q}^{\infty} \log(t)\, t^{i-1}\,(1-(1-t)^{\alpha})^{2i+1} dt.$$

8.3. TVP

The TVP has great importance in the portfolio sector. The expression of TVP, denoted T_4, of the TL-EE model is defined by the following equation:

$$T_4 = T_2 + \delta T_3,$$

where $0 < \delta < 1$ and

$$T_3 = \frac{1}{(1-q)}\int_{\mathrm{VaR}_q}^{\infty} z^2 f_Z(z;\theta,\alpha,\lambda)\, dz - (T_2)^2.$$

8.4. Numerical computations for actuarial measures

The numerical experiments for T_1, T_2, and T_3 of the TL-EE model and other well known distributions such as EE and Exp for various parametric values are shown in this part. The parameters were calculated using the ML approach. The three risk metrics were computed using the 1000 replications of process, and the final results were summarized in Tables 7 and 8.

Table 7. The values of T_1, T_2, and T_4 for the TL-EE and different fitting models.

Model	Par	q	T_1	T_2	T_4
TL-EE	$\theta = 2.5$	0.60	1.6234	1.8534	1.8810
	$\alpha = 4.25$	0.65	1.6583	1.8839	1.9132
	$\lambda = 100$	0.70	1.6969	1.9183	1.9493
		0.75	1.7408	1.9583	1.9910
		0.80	1.7926	2.0064	2.0407
		0.85	1.8570	2.0673	2.1032
		0.90	1.9449	2.1519	2.1892
		0.95	2.0903	2.2941	2.3329
EE	$\theta = 2.5$	1.3107	1.4156	1.2374	1.4161
	$\alpha = 4.25$	0.65	0.9357	1.3689	1.4812
		0.70	1.0078	1.4353	1.5546
		0.75	1.0905	1.5127	1.6391
		0.80	1.1891	1.6063	1.7397
		0.85	1.3131	1.7256	1.8658
		0.90	1.4838	1.8919	2.0388
		0.95	1.7692	2.1731	2.3266
Exp	$\theta = 2.5$	0.60	0.3665	0.7665	0.8625
		0.65	0.4199	0.8199	0.9239
		0.70	0.4815	0.8815	0.9935
		0.75	0.5545	0.9545	1.0745
		0.80	0.6437	1.0437	1.1717
		0.85	0.7588	1.1588	1.2948
		0.90	0.9210	1.3210	1.4650
		0.95	1.1982	1.5982	1.7502

Table 8. The values of T_1, T_2, and T_4 for the TL-EE and different fitting models.

Model	Par	q	T_1	T_2	T_4
TL-EE	$\theta = 4.0$	0.60	1.0991	1.2434	1.2543
	$\alpha = 6.5$	0.65	1.1211	1.2625	1.2740
	$\lambda = 85$	0.70	1.1453	1.2841	1.2963
		0.75	1.1729	1.3091	1.3220
		0.80	1.2054	1.3393	1.3527
		0.85	1.2458	1.3775	1.3915
		0.90	1.3008	1.4304	1.4450
		0.95	1.3919	1.5194	1.5346
EE	$\theta = 4.0$	0.60	0.6456	0.9230	0.9644
	$\alpha = 6.5$	0.65	0.6867	0.9598	1.0040
		0.70	0.7325	1.00159	1.0485
		0.75	0.7849	1.0502	1.0999
		0.80	0.8472	1.1091	1.1614
		0.85	0.9253	1.1839	1.2388
		0.90	1.0325	1.2881	1.3456
		0.95	1.2114	1.4642	1.5242
Exp	$\theta = 4.0$	0.60	0.2290	0.4790	0.5165
		0.65	0.2624	0.5124	0.5530
		0.70	0.3009	0.5509	0.5947
		0.75	0.3465	0.5965	0.6434
		0.80	0.4023	0.6523	0.7023
		0.85	0.4742	0.7242	0.7774
		0.90	0.5756	0.8256	0.8818
		0.95	0.7489	0.9989	1.0583

For visual comparisons, we presented the findings visually as shown in Figures 6 and 7. From these results and Figures, we conclude that the TL-EE model is heavier than EE and Exp distributions which make it very suitable to fit heavy tailed datasets.

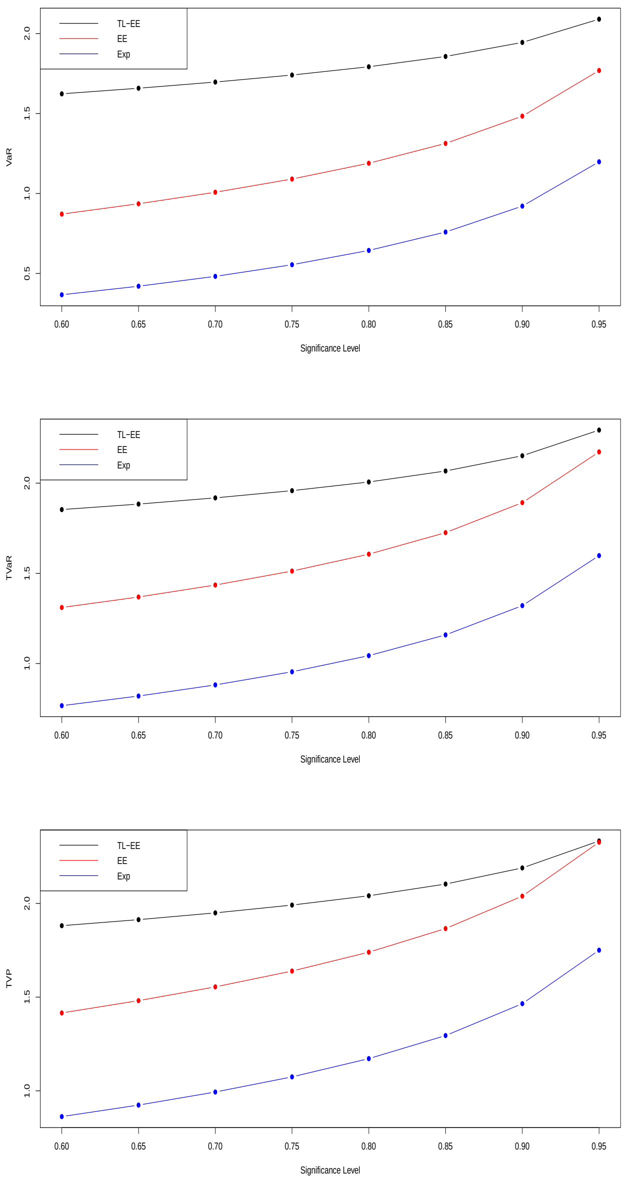

Figure 6. Curves for value of T_1, T_2, and T_4 in Table 7.

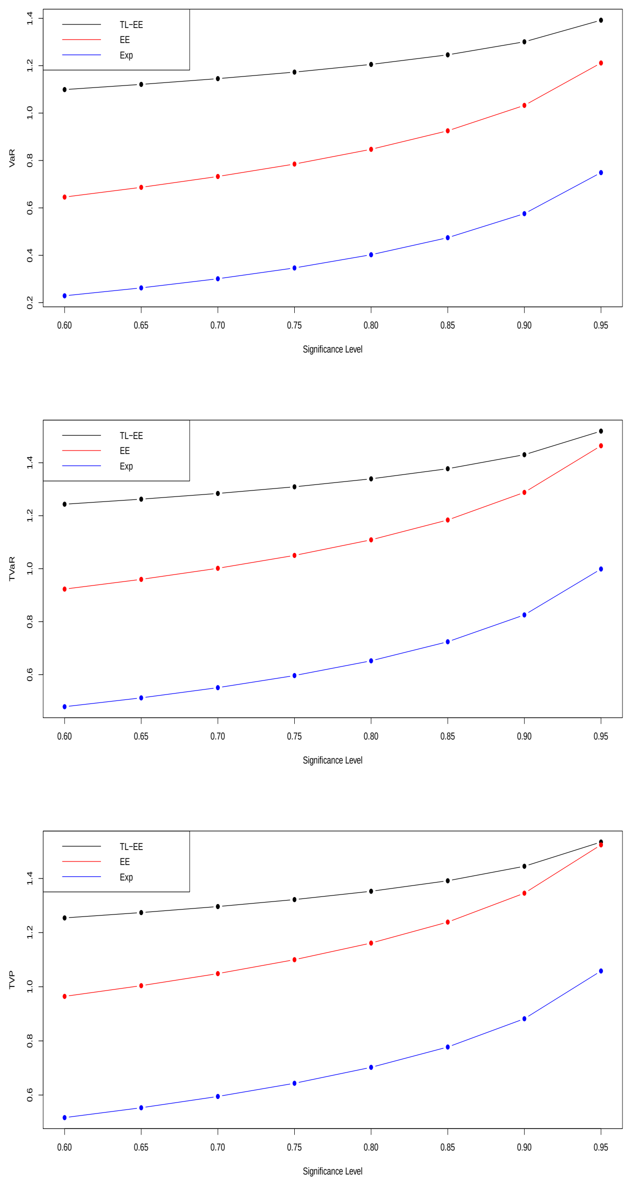

Figure 7. Curves for value of T_1, T_2, and T_4 in Table 8.

9. Real datasets

9.1. Norwegian fire insurance dataset

In this section, we consider the data from fire losses, and it contains records of all the total loss amounts in thousands of Norwegian Krone (NKR) from 1972 to 1992. The recorded datasets are taken from *norfire* function () containing in the **CASdatasets** package [50], and it was also studied by [51].

Table 9 reported different characteristic measures of the proposed dataset. The skewness measure refers that the recorded data is positively skewed.

Table 9. Basic statistics of Norwegian fire insurance dataset.

Mean	Q_2	Sd	Q_1	Q_3	Skew	Kurt
520.0	573.5	134.7	676.0	697.1	772.5	982.0

The adaptability of the TL-EE model is made by discussing its efficacy to that of other analogous models like EE, Exp, Poisson Lomax (PL) (see [52]), Poisson exponential (PE) (see [53]), Weibull (Wei), Lindley (Lin), zero truncated Poisson gamma (ZTPGA) (see [3]), Power Lindley (PLin) (see [54]), Exponential geometric (EG) (see [55]), and Two parameters Mira (TPM) (see [56]) distributions. The PDFs of the suggested models are:

(1) PL distribution:

$$f(z;\alpha,\lambda,\theta)=\frac{\alpha\theta}{\lambda(e^{\theta}-1)}\,(1+z/\lambda)^{\alpha-1}\,e^{\theta(1-(1+z/\lambda)^{-\alpha})};$$

where $z,\ \alpha,\lambda,\theta>0$.

(2) PE distribution:

$$f(z;\lambda,\theta)=\frac{\lambda\theta}{e^{\theta}-1}\,e^{-\lambda z+\theta(1-e^{-\lambda z})};\ \text{ where } z>0,\ \ \lambda,\theta>0.$$

(3) Exp distribution:

$$f(z;\lambda)=\lambda e^{-\lambda z};\ \text{ where } z>0,\ \lambda>0.$$

(4) Lin distribution:

$$f(z;\lambda)=\frac{\lambda^2}{\lambda+1}(1+z)e^{-\lambda z};\ \text{ where } z>0,\ \lambda>0.$$

(5) Wei distribution:

$$f(z;\mu,\sigma)=\frac{\mu}{\sigma}\left(\frac{z}{\sigma}\right)^{\mu-1}e^{-\left(\frac{z}{\sigma}\right)^{\mu}};\ \text{ where } z>0,\ \mu,\sigma>0.$$

(6) ZTPGA:

$$f(z,\alpha,\lambda,\theta)=\frac{\theta\lambda^{\alpha}z^{\alpha-1}e^{-\lambda z}}{(e^{\theta}-1)\Gamma(\alpha)}e^{\theta H_{\alpha,\lambda}(z)};$$

where $zx>0,\ \alpha,\lambda,\theta>0$, and $H_{\alpha,\lambda}(z)$ is the CDF of gamma distribution.

(7) PLin:

$$f(z,\alpha,\beta)=\frac{\alpha\beta^2}{\beta+1}(1+z^{\alpha})\,z^{\alpha-1}\,e^{-\beta z^{\alpha}};\ \text{ where } z>0,\ \alpha,\beta>0.$$

(8) EG:

$$f(z,\lambda,p)=\frac{p\lambda e^{-\lambda z}}{(p+(1-p)e^{-\lambda x})^2};$$

where $z>0,\ \lambda>0,\ 0<p<1$.

(9) TPM:

$$f(z,\alpha,\delta)=\frac{\delta^3\left(\alpha z^2+2\right)e^{-\delta z}}{2\left(\alpha+\delta^2\right)};\ \text{ where } z>0,\ \ \alpha,\delta>0.$$

The results of the MLEs and the corresponding log-likelihood (ll) function of TL-EE model with proposed compared distributions are reported in Table 10. Now, To determination about the best model for modeling the dataset, numerous measures including Akaike information criterion ($\mathcal{A}_1$), correction Akaike information criterion ($\mathcal{A}_2$), Hannan Quinn information criterion ($\mathcal{A}_3$), Bayesian Information criterion ($\mathcal{B}_1$) and Kolmogorov-Smirnov ($\mathcal{KS}$) statistics with associated $\mathcal{P}$-values are computed. The results of all these measures are recorded in Table 11. From these results, we can deduce that the TL-EE model is a more suitable candidate distribution for analyzing the Norwegian fire insurance dataset. The estimated PDF, CDF, and survival function of the TL-EE model with the empirical dataset, the scaled total time on the test (TTT), the probability-probability (PP), and box plots for the Norwegian fire insurance dataset are sketched in Figure 8 and 9. These figures confirms this conclusion.

Table 10. The estimated parameters with corresponding ll values of fitted models.

Distribution	Par			ll
TL-EE	$\hat{\theta}$=0.0048	$\hat{\alpha}$=2.8201	$\hat{\lambda}$=67.987	-344.054
EE	$\hat{\theta}$=0.0064	$\hat{\alpha}$=55.743		-349.589
PL	$\hat{\alpha}$ = 470.909	$\hat{\lambda}$=590.227	$\hat{\theta}$=8.4518	-348.5498
PE	$\hat{\lambda}$=0.0065	$\hat{\theta}$=59.178		-349.538
Wei	$\hat{\mu}$=5.5134	$\hat{\sigma}$= 753.625		-349.229
Exp	$\hat{\lambda}$=0.0014			-415.080
Lin	$\hat{\lambda}$=0.0028			-394.886
ZTPGA	$\hat{\alpha}$=5.0374	$\hat{\lambda}$=0.0146	$\hat{\theta}$=21.882	-344.680
PLin	$\hat{\alpha}$=0.0786	$\hat{\beta}$= 2.1419		-380.426
EG	$\hat{\alpha}$=0.0043	$\hat{p}$= 0.0858		-380.322
TPM	$\hat{\alpha}$=0.0043	$\hat{\delta}$= 0.8077		-383.885

Table 11. Comparison criterion and goodness-of-fit statistics for Norwegian fire dataset.

Model	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{B}_1$	$\mathcal{KS}$	$\mathcal{P}$-value
TL-EE	694.109	694.580	696.438	700.131	0.1018	0.618
EE	703.178	704.409	705.731	709.193	0.1320	0.292
PL	703.099	703.570	705.428	709.121	0.1190	0.417
PE	703.077	703.308	704.630	707.092	0.1357	0.262
Wei	702.458	702.689	704.011	706.473	0.1212	0.393
Exp	832.160	832.236	832.937	834.168	0.5257	1.2×10^{-13}
Lin	791.772	791.847	792.548	793.779	0.4396	1.1×10^{-9}
ZTPGA	695.360	695.831	697.689	701.382	0.1046	0.5839
PLin	764.852	765.082	766.404	768.866	0.3852	2.5×10^{-06}
EG	764.645	764.876	766.198	768.660	0.38.26	7.5×10^{-06}
TPM	771.770	772.001	773.323	775.785	0.3873	1.36×10^{-07}

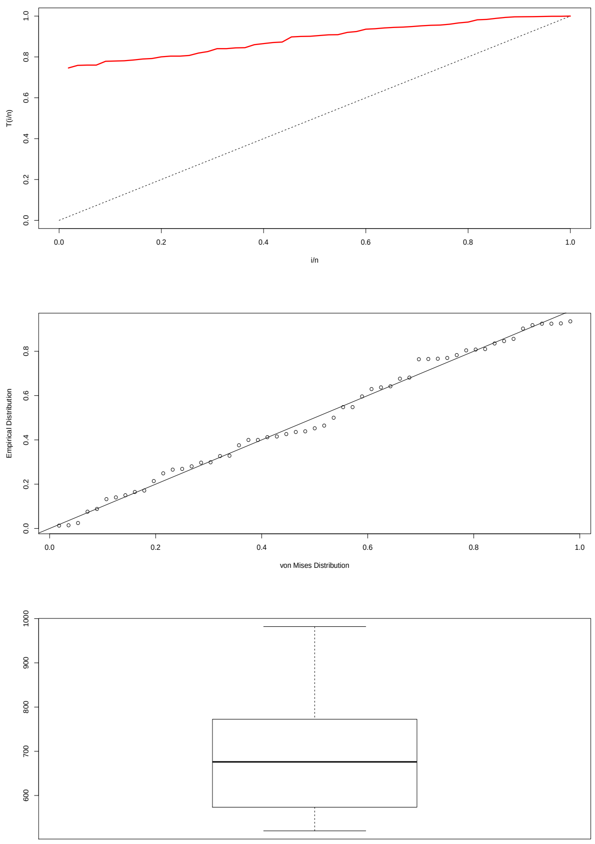

Figure 8. TTT, PP and box plots of the TL-EE for the dataset.

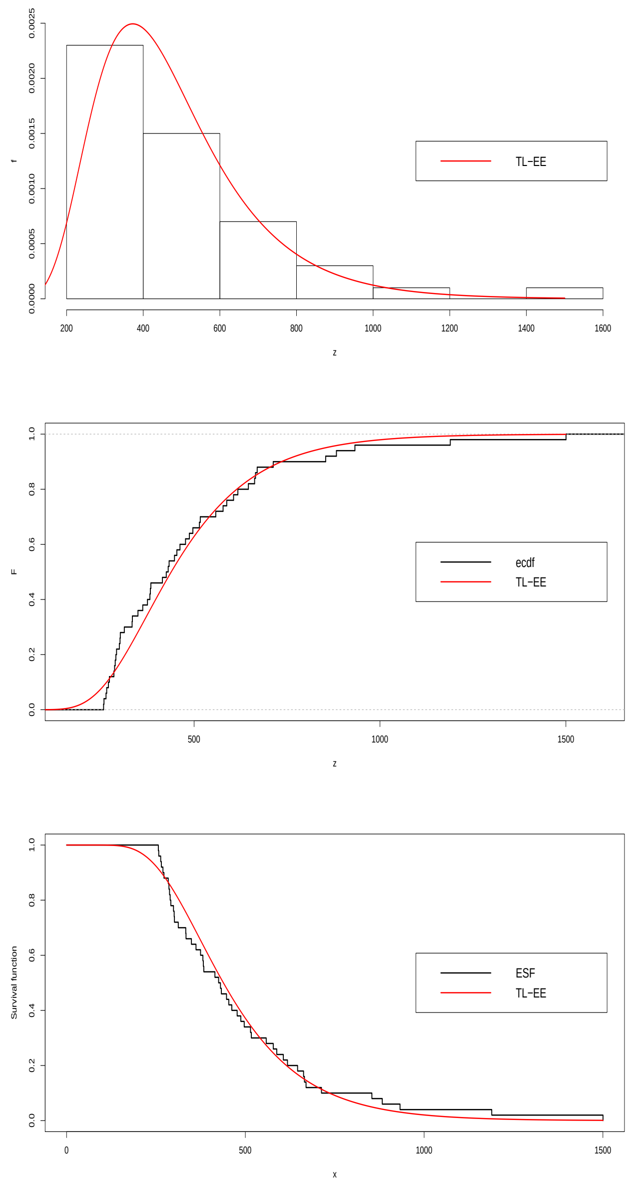

Figure 9. Graphs of the estimated PDF, CDF, and SF of the TL-EE for Norwegian fire insurance data.

Further, we compute the risk measures of the TL-EE, EE, and Exp models using the Norwegian fire insurance dataset. The results are displayed in Table 12, and it can be concluded that the values of actuarial measures of the TL-EE distribution are very close to their corresponding empirical values for the Norwegian fire dataset. The suggested TL-EE model is very suitable to fit and analyze the Norwegian fire dataset. Figure 10 confirms this conclusion.

Table 12. Results of T_1, T_2, and T_4 using the Norwegian fire insurance.

Model	Par	q	T_1	T_2	T_4
Empirical		0.85	875.8	925.7	982.0
		0.90	917.8	939.6	1285.2
		0.95	933.6	955.6	1335.8
		0.99	1041.1	1194.8	1571.9
TL-EE	$\hat{\theta}$ =0.0048	0.85	825.0	932.4	10280.6
	$\hat{\alpha}$ =2.8201	0.90	869.9	975.6	10714.9
	$\hat{\lambda}$ =67.987	0.95	944.2	1048.2	11156.3
		0.99	1111.6	1214.3	11601.9
EE	$\hat{\alpha}$ =55.743	0.85	910.8	1073.0	22573.1
	$\hat{\theta}$ =0.0064	0.90	978.3	138.4	23600.3
		0.95	1090.5	1248.5	24654.0
		0.99	1344.7	1501.1	25649.0
Exp	$\hat{\lambda}$ = 0.0014	0.85	1322.4	2019.5	415065.0
		0.90	1605.1	2302.2	439644.4
		0.95	2088.2	2785.3	464424.4
		0.99	33210.2	3907.3	484983.8

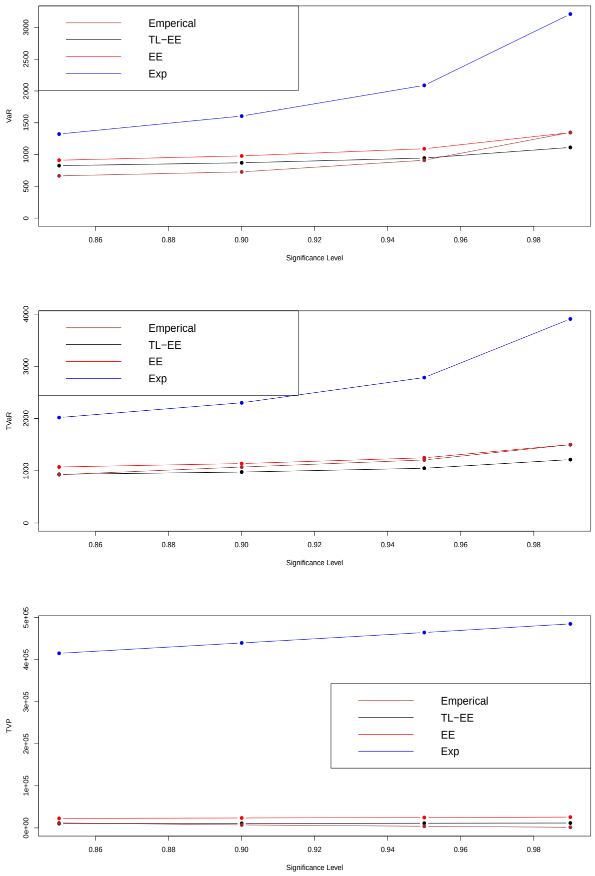

Figure 10. Graphical plots of the computational value of VaR, TVaR, and TVP using the dataset.

9.2. Medical insurance claim data

In this subsection, we presented the dataset from a group medical insurance and it contains values of all the claim amounts exceeding 25,000 USD over the period 1991. The values of dataset were taken from http://www.soa.org. Before progressing further, let us provide some basic statistics of the observed value of dataset, which is provided in Table 13. The skewness measure give first indicator that the proposed data is skewed to the right, and the plot of TTT, PP, and box are displayed in Figure 11.

Table 13. Basic statistics of the medical insurance claim dataset.

Mean	Q_2	Sd	Q_1	Q_3	Skew	Kurt
487.2	427.8	248.0	301.1	585.4	1.932	4.471

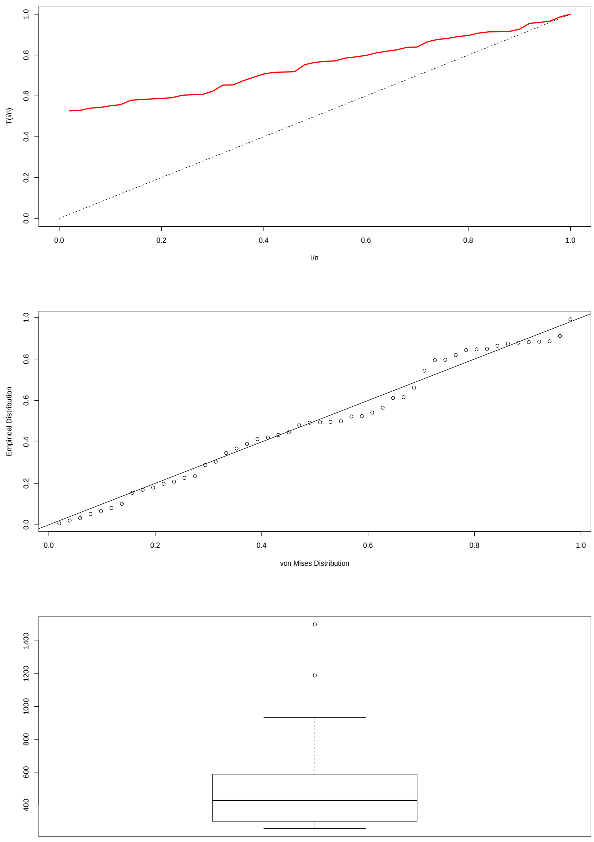

Figure 11. TTT, PP, and box plots of the medical insurance claim data.

The TL-EE distribution and the other proposed models are estimated by using the MLE method. Table 14 summarized the results of the MLEs and the corresponding ll function. Furthermore, to select the model which fits the dataset well, we compute various statistical measures such as $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{B}_1$, $\mathcal{KS}$, and $\mathcal{P}$-value, and the are reported in Table 15. From these results, we can deduce that the TL-EE model is a more suitable candidate distribution for analyzing the dataset. The estimated PDF, CDF, and survival function of the TL-EE model with the empirical dataset are drawn in Figure 12. This figure ensures this conclusion.

Table 14. The estimated parameters with corresponding ll values of fitted models.

Distribution	Par			ll
TL-EE	$\hat{\theta}$=0.0029	$\hat{\alpha}$=0.4591	$\hat{\lambda}$=38.326	-331.867
EE	$\hat{\theta}$=0.0049	$\hat{\alpha}$=5.7608		-334.355
PL	$\hat{\alpha}$ = 133.818	$\hat{\lambda}$=190.851	$\hat{\theta}$=3.9591	-332.767
PE	$\hat{\lambda}$=0.0068	$\hat{\theta}$=14.346		-333.323
Wei	$\hat{\mu}$=2.1057	$\hat{\sigma}$= 552.523		-340.802
Exp	$\hat{\lambda}$=0.0020			-359.435
Lin	$\hat{\lambda}$=0.0040			-345.044
ZTPGA	$\hat{\alpha}$=1.8306	$\hat{\lambda}$=0.0077	$\hat{\theta}$=6.1417	-335.129
PLin	$\hat{\alpha}$=1.5245	$\hat{\beta}$= 0.0001		-337.759
EG	$\hat{\alpha}$=0.0076	$\hat{p}$= 0.0338		-339.999
TPM	$\hat{\alpha}$=0.0062	$\hat{\delta}$= 0.3874		-338.991

Table 15. Comparison criterion and goodness-of-fit statistics for the medical insurance claim dataset.

Model	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{B}_1$	$\mathcal{KS}$	$\mathcal{P}$-value
TL-EE	669.734	670.256	671.919	675.470	0.1089	0.5567
EE	672.711	672.967	674.167	676.535	0.1470	0.2082
PL	671.534	672.055	673.718	677.270	0.1124	0.5162
PE	670.646	670.901	672.102	674.470	0.1181	0.4535
Wei	685.604	685.859	687.060	689.428	0.1803	0.0678
Exp	720.871	720.954	721.599	722.783	0.4094	4.2×10^{-08}
Lin	692.089	692.172	692.817	694.001	0.2846	0.0004
ZTPGA	676.258	676.780	678.442	681.994	0.1104	0.5383
PLin	679.518	679.773	680.974	683.342	0.1520	0.1784
EG	683.998	684.253	685.454	687.822	0.1707	0.0961
TPM	681.982	682.237	683.438	685.806	0.2147	0.0167

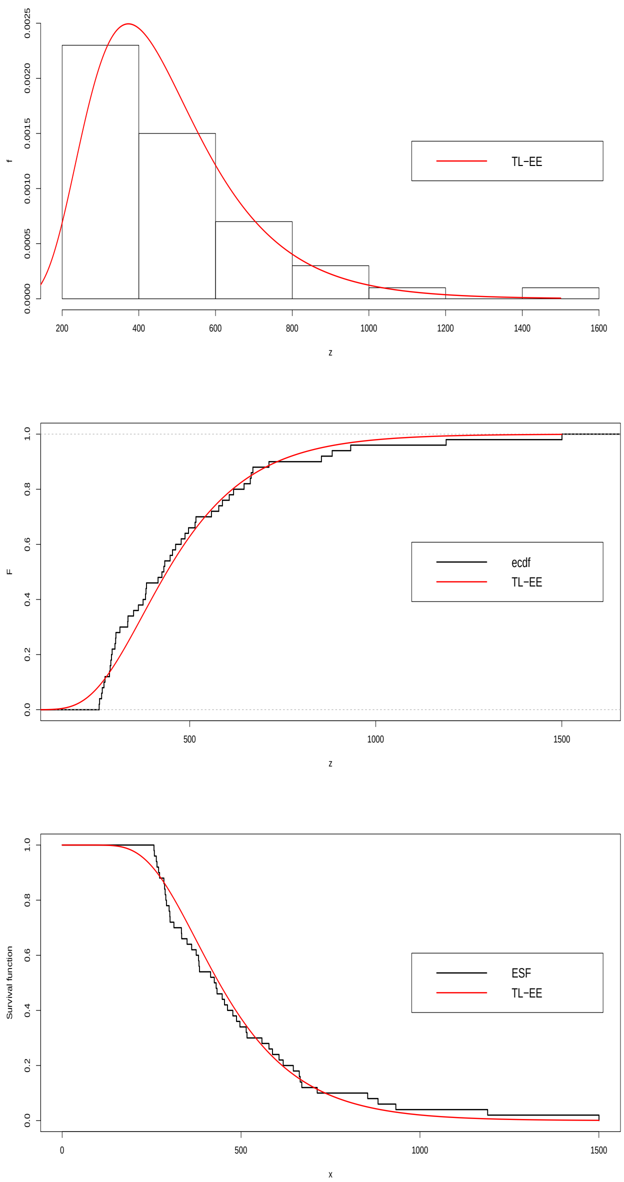

Figure 12. Estimated PDF, CDF and SF using second dataset for the TL-EE model.

Finally, we provide the results of risk measures of the TL-EE, EE, and Exp models using the medical insurance claim dataset. The results are given in Table 16, and it can be concluded that the values of actuarial measures of the TL-EE model are very close to their corresponding empirical values for the medical insurance claim dataset. Hence, the proposed TL-EE model is very suitable to fit and analyze the medical insurance claim dataset. Figure 13 ensures this result.

Table 16. Results of T_1, T_2, and T_4 using the medical insurance claim data set.

Model	Par	q	T_1	T_2	T_4
Empirical		0.85	664.3	925.8	11899.1
		0.90	726.9	1071.7	7070.6
		0.95	910.2	1207.3	3901.6
		0.99	1347.8	1500.3	1500.3
TL-EE	$\hat{\theta}$ =0.0029	0.85	687.2	942.0	26230.5
	$\hat{\alpha}$ =0.4591	0.90	759.2	1027.0	27617.5
	$\hat{\lambda}$ =38.326	0.95	879.9	1170.6	29096.2
		0.99	1156.6	1500.8	30579.1
EE	$\hat{\alpha}$ =0.0049	0.85	731.0	861.8	37517.5
	$\hat{\theta}$ =5.7608	0.90	818.4	932.3	39320.7
		0.95	964.4	1051.9	41155.2
		0.99	1296.3	1328.3	42819.2
Exp	$\hat{\lambda}$ = 0.0020	0.85	948.5	1448.5	213948.6
		0.90	1151.2	1651.2	226651.3
		0.95	1497.8	1997.8	239499.0
		0.99	2302.5	2802.5	250302.6

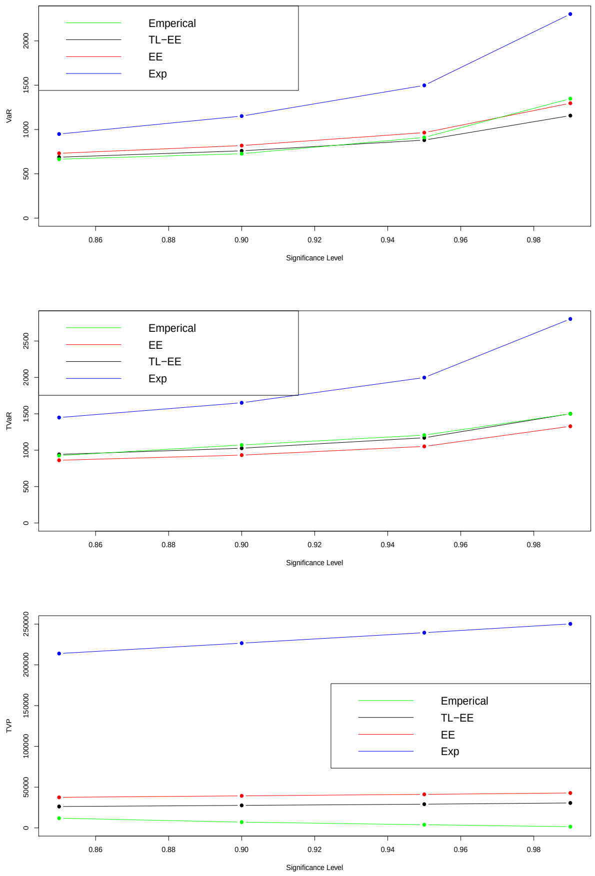

Figure 13. Graphical plots of the computational value of VaR, TVaR, and TVP using the dataset.

10. Conclusions

This article uses a new class of distributions based on a TL family of distributions. It is named a TL-EE model. Different distributional and mathematical properties of this model are provided including, rth moment, moment generating and characteristic function as well as distribution of order statistic and Reny entropy. The MLE was used to estimate the parameters of the proposed distribution. Various actuarial measures are computed, and brief simulation analyses are illustrated to see the efficiency of the TL-EE in risk insurance. In the end, we have illustrated two financial datasets, and it is checked that the TL-EE model regularly outperforms other competitor distributions. In future work, we will use various censored methods, including progressive censoring shames under different types along with illustration of accelerated life tests with numerous kinds of stress load.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that there are is conflict of interest in this paper.

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Appendix

```
## plot of PDF
pdf.TL_EE=function(par,x){
theta=par[1]
```

```
alpha=par[2]
lambda=par[3]
2*lambda*(alpha*theta*exp(-theta*x)*
(1-exp(-theta*x))^(alpha-1))*
(1-(1-exp(-theta*x))^alpha)*
(1-(1-(1-exp(-theta*x))^alpha)^2)^(lambda-1)
}
t=seq(0,5,len=1000)
plot(t,pdf.TL_EE(c(1.2,1.3,2.0),t),col="blue",
xlab="z",ylab="PDF",type="l",lwd=2,lty=1)

## plot of CDF
cdf.TL_EE <- function(par,x){
theta = par[1]
alpha=par[2]
lambda=par[3]
(1-(1-(1-exp(-theta*x))^alpha)^2)^lambda
}
t=seq(0,3,len=1000)
plot(t,cdf.TL_EE(c(0.8,0.4,0.75),t),col="blue",
xlab="z",ylab="PDF",type="l",lwd=2,lty=1)

## plot of HR
hazard.TL_EE=function(par,x){
theta=par[1]
alpha=par[2]
lambda=par[3]
2*lambda*(alpha*theta*exp(-theta*x)*
(1-exp(-theta*x))^(alpha-1))*
(1-(1-exp(-theta*x))^alpha)*
(1-(1-(1-exp(-theta*x))^alpha)^2)^(lambda-1)/
(1-((1-(1-(1-exp(-theta*x))^alpha)^2)^lambda))
}
t=seq(0,10,len=1000)
plot(t,hazard.TL_EE(c(0.4,0.8,3),t),col="blue",
xlab="z",ylab="HR",type="l",lwd=2,lty=1)

## Application
q=data
result=goodness.fit(pdf = pdf.TL_EE,
cdf =cdf.TL_EE,method = "BFGS",
starts = c(1/mean(q),0.2,0.5),
data = q,domain = c(0,Inf),mle = NULL)

## Estimated PDF, CDF and SF plots
par(mfrow = c(1, 3))
x = seq(0, max(q), len = 1000)
hist(q,prob=T,xlab="z",main="",ylab="f")
lines(x, pdf.TL_EE(par = result$mle,x),
col="red",lty=1,lwd=2)
plot(ecdf(q),verticals=TRUE, do.points=FALSE,
lwd=2,main="",ylab="F",xlab="z")
lines(x, cdf.TL_EE(par = result$mle,x),
col="red",lty=1,lwd=2)
plot(Surv(q),lwd=2,lty=1,xlab="x",
ylab="Survival function",conf.int=F)
lines(x,esf.TL_EE(par = re$mle,x),
col="red",type="l",lwd=2,lty=1)

## TTT, PP and box plots
par(mfrow = c(1, 3))
TTT(q,lwd=2.5,grid=F,lty=2,col="red")
pp.plot(q,ref.line=T)
boxplot(q)
```