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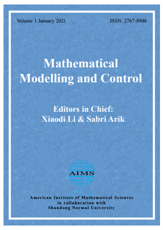

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Research article

Hopf algebra of labeled simple graphs arising from super-shuffle product

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Abstract: From the connections between permutations and labeled simple graphs, we generalized the super-shuffle product and the cut-box coproduct on permutations to labeled simple graphs. We then proved that the vector space spanned by labeled simple graphs is a Hopf algebra with these two operations.

Keywords: Hopf algebra; labeled simple graph; super-shuffle product; cut-box coproduct

1. Introduction

In 1964, Hopf first proposed Hopf algebra in order to study the properties of algebraic topology and algebraic groups [1]. In 1965, Milnor and Moore introduced the basic definitions and properties of Hopf algebras [2], then Chase and Sweedler did some relevant works and introduced common notations [3, 4]. After that, Hopf algebra has been used to study a lot of objects, such as posets [5], symmetric functions [6, 7], quantum groups [8], and Clifford algebras [9].

In 1979, Joni and Rota first studied Hopf algebras on combinatorial objects, such as polynomials and puzzles [10]. In 1994 and 1995, Schmitt studied incidence Hopf algebras and a Hopf algebra on graphs with an addition invariant and introduced a variety of examples of incidence Hopf algebras arising from families of graphs, matroids, and distributive lattices, many of which generalize well-known Hopf algebras [11, 12].

In 1997 and 1999, Connes and Kreimer studied Hopf algebra structures on rooted trees and rooted forests and their applications in renormalization in quantum field theories [13, 14]. This promotes the study of Hopf algebras on graphs. In 2020, Aval et al. mentioned a Hopf algebra on labeled graphs arising from the unshuffle coproduct [15]. For more Hopf algebras on graphs, please refer to [16–20].

Permutations are related to graphs closely. In 1995, Malvenuto and Reutenauer studied a Hopf algebra on permutations, where the product is the classic shuffle $\sqcup\!\sqcup$ [21]. In 2014, Vargas defined a commutative but non-cocommutative Hopf algebra on permutations by the super-shuffle product $\underline{\sqcup\!\sqcup}$ and the cut-box coproduct $\Delta_\diamond$ without a proof [22], which was done by Liu and Li in 2021 [23]. In 2020, Zhao and Li defined another commutative Hopf algebra structure on permutations and its duality and figured out closed-formulas of the antipodes [24]. It is well-known that permutations are elements of symmetric groups, which are widely used in various fields, such as the algebraic number theory [25].

A labeled simple graph is a simple graph with vertices labeled by distinct positive integers. In this paper, we generalize the super-shuffle product and the cut-box coproduct on permutations to labeled simple graphs. We prove that the vector space spanned by labeled simple graphs with these two operations is a Hopf algebra.

This paper is organized as follows. In Section 2, we review some basic concepts of Hopf algebra, give the definition of labeled simple graphs, and define the super-shuffle product and the cut-box coproduct on labeled simple graphs. In Section 3, we prove that the vector space spanned by labeled simple graphs is a graded algebra with the super-shuffle product and a graded coalgebra with the cut-box coproduct. Furthermore, we prove the compatibility of these operations, then the vector space is a Hopf algebra. Finally, we summarize our main conclusions in Section 4.

2. Basic definitions

2.1. Preliminaries

Here, we recall some basic definitions related to Hopf algebra and see [4] for more details. Let C be a $\mathbb{K}$-module over commutative ring $\mathbb{K}$.

Define $\mathbb{K}$-bilinear mappings m from $C \otimes C$ to C and μ from $\mathbb{K}$ to C, such that the diagrams in Figure 1 are commutative, then (C, m, μ) is a $\mathbb{K}$-*algebra*. Here, m and μ are called a *product* and a *unit*, respectively.

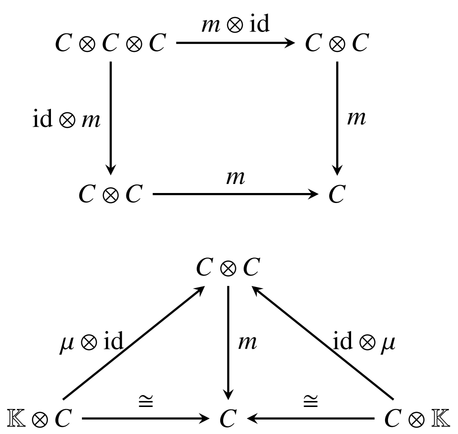


Figure 1. Associative law and unitary property.

Define $\mathbb{K}$-linear mappings Δ from C to $C \otimes C$ and ν from C to $\mathbb{K}$, such that the diagrams in Figure 2 are commutative, then (C, Δ, ν) is a $\mathbb{K}$-*coalgebra*. Here, Δ and ν are called a *coproduct* and a *co-unit*, respectively.

We say (C, m, μ, Δ, ν) is a *bialgebra* if (C, m, μ) is an algebra, (C, Δ, ν) is a coalgebra, and one of the following *compatibility* conditions holds:

(i) Δ and co-unit ν are algebra homomorphisms;

(ii) m and unit μ are coalgebra homomorphisms.

In fact, (i) and (ii) are equivalent; see [26] for details.

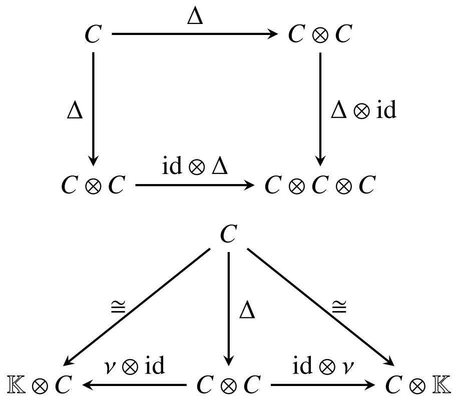


Figure 2. Coassociative law and co-unitary property.

A vector space C is *graded* if

$$C = \bigoplus_{n \geqslant 0} C_n$$

and we call it *connected* when $C_0 \cong \mathbb{K}$ [26]. The algebra (C, m, μ) is *graded* if the product m satisfies

$$m(C_i \otimes C_j) \subseteq C_{i+j}$$

and

$$\mu(\mathbb{K}) \subseteq C_0.$$

Similarly, the coalgebra (C, Δ, ν) is *graded* if the coproduct Δ satisfies

$$\Delta(C_n) \subseteq \bigoplus C_i \otimes C_{n-i}$$

and

$$\nu(C_n) = 0,$$

when $n \geqslant 1$. A bialgebra is *graded* when its algebra and coalgebra structures are both graded.

For bialgebra (C, m, μ, Δ, ν), we call $S\colon C \to C$ an *antipode* if it satisfies

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = \mu \circ \nu = m \circ (\mathrm{id} \otimes S) \circ \Delta,$$

i.e., the diagram in Figure 3 is commutative. A bialgebra is a *Hopf algebra* when it has an antipode.

Actually, a graded connected bialgebra must be a Hopf algebra [26].

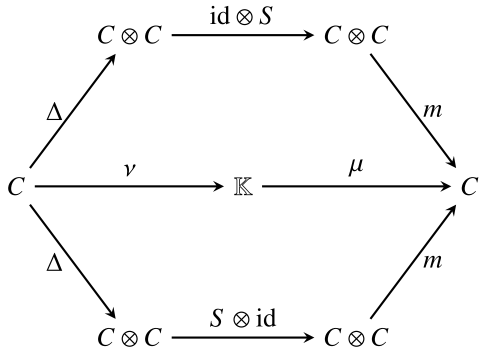


Figure 3. Antipode.

2.2. Main concepts

In this subsection, we recall some basic concepts of graph theory, which can be found in [27].

A *labeled simple graph* $\Gamma = (V, E)$ is a finite graph with no cycles and no multiple edges whose vertices are distinct positive integers, where V is the set of all vertices of Γ, also denoted by $V(\Gamma)$, and E is the set of all edges of Γ, also denoted by $E(\Gamma)$. Obviously, $E \subseteq V \times V$. If $(i_1, i_2) \in E$, then $i_1 \neq i_2$ and $(i_2, i_1) \notin E$, since the graph Γ has no cycles and no multiple edges. In particular, Γ is the empty graph when $V = \emptyset$, denoted by ϵ.

Let $\Gamma = (V, E)$ and $I \subseteq V$. Define the *restriction of* Γ *on* I by $\Gamma_I = (I, E_I)$, where

$$E_I = \{(i, j) | i, j \in I, (i, j) \in E\},$$

and we call Γ_I a *subgraph of* Γ. If I is a nontrivial subset of V, we call Γ_I a *true subgraph of* Γ. If the vertex sets of two subgraphs of Γ are disjoint, then we say that the subgraphs are *disjoint subgraphs*. If

$$\Gamma_1 = (V_1, E_1), \quad \Gamma_2 = (V_2, E_2)$$

and

$$V_1 \cap V_2 = \emptyset,$$

then denote

$$\Gamma_1 \cup \Gamma_2 = (V_1 \cup V_2, E_1 \cup E_2).$$

Obviously, there are no edges between V_1 and V_2.

We introduce the following notations for convenience:

$$[n] = \begin{cases} \{1, 2, \dots, n\}, & n > 0, \\ \emptyset, & n = 0, \end{cases}$$

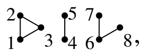
and

$$[i, j] = \begin{cases} \{i, i+1, \dots, j\}, & i \leqslant j, \\ \emptyset, & i > j. \end{cases}$$

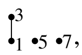
Example 1. The labeled simple graph

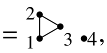
$$\Gamma = ([8], \{(1, 2), (1, 3), (2, 3), (4, 5), (6, 7), (6, 8)\})$$

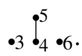
can be represented as the graph

$\Gamma =$  ,

then

$\Gamma_{\{1,3,5,7\}} = (\{1, 3, 5, 7\}, \{(1, 3)\}) =$  ,

$\Gamma_{[4]} = ([4], \{(1, 2), (1, 3), (2, 3)\}) =$  ,

$\Gamma_{[3,6]} = ([3, 6], \{(4, 5)\}) =$  .

Let

$$H_n = \{\Gamma \mid \Gamma = ([n], E) \text{ is a labeled simple graph}\},$$

and $\mathcal{H}_n$ be the vector space spanned by H_n over field $\mathbb{K}$, for a nonnegative integer n. In particular, $H_0 = \{\epsilon\}$ and $\mathcal{H}_0 = \mathbb{K}H_0$. Denote

$$H = \bigcup_{n=0}^{\infty} H_n \quad \text{and} \quad \mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n.$$

Let $\Gamma = (V, E)$ be a nonempty labeled simple graph, where

$$V = \{v_1, v_2, \cdots, v_n\}.$$

Define the *restructure of* $\Gamma = (V, E)$ *by* $\hat{V}$ to be $\hat{\Gamma} = (\hat{V}, \hat{E})$, where

$$\hat{V} = \{\hat{v}_1, \hat{v}_2, \cdots, \hat{v}_n\}$$

is a set of distinct positve integers satisfying

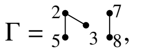
$$\hat{v}_i < \hat{v}_j \Leftrightarrow v_i < v_j,$$

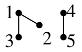
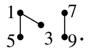
and $\hat{E}$ satisfies

$$(\hat{v}_i, \hat{v}_j) \in \hat{E} \Leftrightarrow (v_i, v_j) \in E,$$

for any $1 \leqslant i, j \leqslant n$.

Example 2. For



the restructure of Γ by $[5]$ is  and the restructure of Γ by $\{1, 3, 5, 7, 9\}$ is .

Let I be the set $\{i_1, i_2, \cdots, i_n\}$ of distinct positive intergers with $i_1 < i_2 < \cdots < i_n$. We define a mapping st_I from I to $[|I|]$ to be the *standardization of I* satisfying $\mathrm{st}_I(i_a) = a$ for $1 \leqslant a \leqslant n$. For $x, y \in I$, we have $\mathrm{st}_I(x) < \mathrm{st}_I(y)$ if, and only if, $x < y$. For a subset S of I, denote

$$\mathrm{st}_I(S) = \{\mathrm{st}_I(i) | i \in S\}.$$

In general, the standardizations of a number in different sets are different. For example, let $I_1 = \{6, 7, 9\}$ and $I_2 = \{1, 3, 7, 9, 11\}$, then $\mathrm{st}_{I_1}(7) = 2$ and $\mathrm{st}_{I_2}(7) = 3$. For convenience, we omit the subscript of the set.

Define the *standard form* of $\Gamma = (V, E)$ by $\mathrm{st}(\Gamma) = (\mathrm{st}(V), \mathrm{st}(E))$, where $\mathrm{st}(V) = [|V|]$ and $\mathrm{st}(E)$ satisfies

$$(\mathrm{st}(v_1), \mathrm{st}(v_2)) \in \mathrm{st}(E) \Leftrightarrow (v_1, v_2) \in E.$$

Obviously, the above standardizations are of the vertex set V, so we omit the subscript. In particular, we have $\mathrm{st}(\epsilon) = \epsilon$. Thus, $\mathrm{st}(\cdot)$ is a mapping from the set of all labeled simple graphs to H. In fact, the standard form of $\Gamma = (V, E)$ is the restructure of Γ by $[|V|]$.

In addition, for a positive integer n, let $\Gamma^{\uparrow n}$ be the restructure of Γ by the set

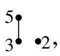
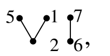
$$V^{\uparrow n} := \{v + n | v \in V\}.$$

Similarly, let $\Gamma^{\downarrow n}$ be the restructure of Γ by the set

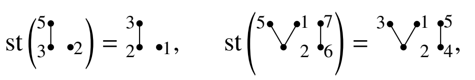
$$V^{\downarrow n} := \{v - n | v \in V\}$$

provided n is less than the minimum of V.

Example 3. For labeled simple graphs

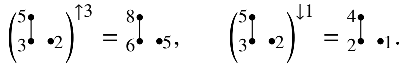
  and 

their standard forms are



$$\mathrm{st}\,(\bullet_5) = \bullet_1,$$

and

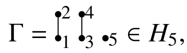


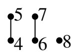
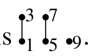
For nonempty Γ in H, the standard form of any restructure of Γ must be Γ, i.e.,

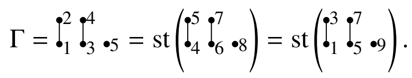
$$\mathrm{st}(\hat{\Gamma}) = (\mathrm{st}(\hat{V}), \mathrm{st}(\hat{E})) = (V, E) = \Gamma,$$

where the $\hat{\Gamma}$ is a restructure of Γ. Conversely, if the standard form of a labeled simple graph is Γ, then it must be a restructure of Γ.

Example 4. For



the restructure of Γ by $[4, 8]$ is  and the restructure of Γ by $\{1, 3, 5, 7, 9\}$ is . We have



For $\Gamma = ([n], E)$ in H_n, we call i a *split* of Γ if

$$\Gamma_{[i]} \cup \Gamma_{[i+1,n]} = \Gamma,$$

where $0 \leqslant i \leqslant n$. Obviously, i is a split of Γ if, and only if, there are no edges between $[i]$ and $[i + 1, n]$ in Γ. By the definition, 0 and n, called *trivial splits*, are always splits of labeled simple graphs in H_n when $n \geqslant 1$. We call Γ *indecomposible* if it is nonempty and only has trivial splits.

For $\Gamma = ([n], E)$ in H_n, $n \geqslant 1$, assume that $\{i_0, i_1, \cdots, i_s\}$ is the set of all splits of Γ, where

$$0 = i_0 < i_1 < \cdots < i_s = n,$$

then we call $\Gamma_{[i_{k-1}+1, i_k]}$ an *atom* of Γ, $1 \leqslant k \leqslant s$. Obviously, the standard form of an atom is indecomposible since there is no split of Γ in $[i_{k-1} + 1, i_k]$ for $1 \leqslant k \leqslant s$. Let

$$\Gamma_k = \mathrm{st}(\Gamma_{[i_{k-1}+1, i_k]})$$

for $1 \leqslant k \leqslant s$. We define the *decomposition* of Γ by

$$\Gamma = \Gamma_1 \diamond \Gamma_2 \diamond \cdots \diamond \Gamma_s.$$

Actually, if $j_k := i_k - i_{k-1}$, then $\Gamma_k \in H_{j_k}$ for $1 \leqslant k \leqslant s$, and

$$\Gamma = \Gamma_1 \diamond \cdots \diamond \Gamma_s = \Gamma_1 \cup \Gamma_2^{\uparrow i_1} \cup \cdots \cup \Gamma_s^{\uparrow i_{s-1}}.$$

In particular, when $\Gamma = \epsilon$, its decomposition is itself.

Example 5. (1) The set of splits of $\overset{2}{\underset{1}{\mid}}\,\overset{5}{\underset{3}{\mid}}\,\bullet_4$ is $\{0, 2, 5\}$ and its decomposition is

$$\overset{2}{\underset{1}{\mid}}\,\overset{5}{\underset{3}{\mid}}\,\bullet_4 = \mathrm{st}\left(\overset{2}{\underset{1}{\mid}}\right) \diamond \mathrm{st}\left(\overset{5}{\underset{3}{\mid}}\,\bullet_4\right) = \overset{2}{\underset{1}{\mid}} \diamond \overset{3}{\underset{1}{\mid}}\,\bullet_2.$$

The atoms of $\overset{2}{\underset{1}{\mid}}\,\overset{5}{\underset{3}{\mid}}\,\bullet_4$ are $\overset{2}{\underset{1}{\mid}}$ and $\overset{5}{\underset{3}{\mid}}\,\bullet_4$.

(2) The set of splits of $\overset{1}{\underset{2}{\mid}}\,\overset{4}{\underset{3}{\mid}}\,\bullet_5$ is $\{0, 2, 4, 5\}$, so its decomposition is

$$\overset{1}{\underset{2}{\mid}}\,\overset{4}{\underset{3}{\mid}}\,\bullet_5 = \mathrm{st}\left(\overset{1}{\underset{2}{\mid}}\right) \diamond \mathrm{st}\left(\overset{4}{\underset{3}{\mid}}\right) \diamond \mathrm{st}\,(\bullet_5) = \overset{1}{\underset{2}{\mid}} \diamond \overset{2}{\underset{1}{\mid}} \diamond \bullet_1.$$

The atoms of $\overset{1}{\underset{2}{\mid}}\,\overset{4}{\underset{3}{\mid}}\,\bullet_5$ are $\overset{1}{\underset{2}{\mid}}$, $\overset{4}{\underset{3}{\mid}}$, and $\bullet_5$.

(3) The set of splits of ${}_1\overset{2}{\triangleright}3$ is $\{0, 3\}$, so it is indecomposible. Its decomposition is itself, and so is its atom.

Define the *cut-box coproduct* Δ on $\mathcal{H}$ by

$$\begin{aligned}\Delta(\Gamma) &= \sum_{j=0}^{s} \Gamma_1 \diamond \cdots \diamond \Gamma_j \otimes \Gamma_{j+1} \diamond \cdots \diamond \Gamma_s \\ &= \sum_{j=0}^{s} \mathrm{st}(\Gamma_{[1,i_j]}) \otimes \mathrm{st}(\Gamma_{[i_j+1,i_s]}),\end{aligned}$$

for nonempty

$$\Gamma = \Gamma_1 \diamond \Gamma_2 \diamond \cdots \diamond \Gamma_s$$

in H_n with splits

$$0 = i_0 < i_1 < \cdots < i_s = n \text{ and } \Delta(\epsilon) = \epsilon \otimes \epsilon.$$

Define the *co-unit* ν from $\mathcal{H}$ to $\mathbb{K}$ by

$$\nu(\Gamma) = \begin{cases} 1, & \Gamma = \epsilon, \\ 0, & \text{otherwise}, \end{cases}$$

for Γ in H.

Example 6.

$$\begin{aligned}\Delta({}_1\overset{2}{\triangleright}3) &= \epsilon \otimes {}_1\overset{2}{\triangleright}3 + {}_1\overset{2}{\triangleright}3 \otimes \epsilon, \\ \Delta(\overset{2}{\underset{1}{\mid}}\,\overset{5}{\underset{3}{\mid}}\,\bullet_4) &= \Delta(\overset{2}{\underset{1}{\mid}} \diamond \overset{3}{\underset{1}{\mid}}\,\bullet_2) \\ &= \epsilon \otimes \overset{2}{\underset{1}{\mid}} \diamond \overset{3}{\underset{1}{\mid}}\,\bullet_2 + \overset{2}{\underset{1}{\mid}} \otimes \overset{3}{\underset{1}{\mid}}\,\bullet_2 + \overset{2}{\underset{1}{\mid}} \diamond \overset{3}{\underset{1}{\mid}}\,\bullet_2 \otimes \epsilon \\ &= \epsilon \otimes \overset{2}{\underset{1}{\mid}}\,\overset{5}{\underset{3}{\mid}}\,\bullet_4 + \overset{2}{\underset{1}{\mid}} \otimes \overset{3}{\underset{1}{\mid}}\,\bullet_2 + \overset{2}{\underset{1}{\mid}}\,\overset{5}{\underset{3}{\mid}}\,\bullet_4 \otimes \epsilon,\end{aligned}$$

$$\begin{aligned}\Delta(\overset{1}{\underset{2}{\mid}}\,\overset{4}{\underset{3}{\mid}}\,\bullet_5) &= \Delta(\overset{1}{\underset{2}{\mid}} \diamond \overset{2}{\underset{1}{\mid}} \diamond \bullet_1) \\ &= \epsilon \otimes \overset{1}{\underset{2}{\mid}} \diamond \overset{2}{\underset{1}{\mid}} \diamond \bullet_1 + \overset{1}{\underset{2}{\mid}} \otimes \overset{2}{\underset{1}{\mid}} \diamond \bullet_1 \\ &\quad + \overset{1}{\underset{2}{\mid}} \diamond \overset{2}{\underset{1}{\mid}} \otimes \bullet_1 + \overset{1}{\underset{2}{\mid}} \diamond \overset{2}{\underset{1}{\mid}} \diamond \bullet_1 \otimes \epsilon \\ &= \epsilon \otimes \overset{1}{\underset{2}{\mid}}\,\overset{4}{\underset{3}{\mid}}\,\bullet_5 + \overset{1}{\underset{2}{\mid}} \otimes \overset{2}{\underset{1}{\mid}}\,\bullet_3 + \overset{1}{\underset{2}{\mid}}\,\overset{4}{\underset{3}{\mid}} \otimes \bullet_1 + \overset{1}{\underset{2}{\mid}}\,\overset{4}{\underset{3}{\mid}}\,\bullet_5 \otimes \epsilon.\end{aligned}$$

Theorem 2.1. *$(\mathcal{H}, \Delta, \nu)$ is a graded coalgebra.*

Proof. It is easy to verify that ν is a co-unit. Obviously,

$$(\mathrm{id} \otimes \Delta) \circ \Delta(\epsilon) = \epsilon \otimes \epsilon \otimes \epsilon = (\Delta \otimes \mathrm{id}) \circ \Delta(\epsilon).$$

Suppose $\Gamma = ([n], E)$ with $n \geqslant 1$, and its decomposition is

$$\Gamma = \Gamma_1 \diamond \Gamma_2 \diamond \cdots \diamond \Gamma_s,$$

then,

$$\begin{aligned}&(\mathrm{id} \otimes \Delta) \circ \Delta(\Gamma) \\ &= (\mathrm{id} \otimes \Delta) \circ \Delta(\Gamma_1 \diamond \Gamma_2 \diamond \cdots \diamond \Gamma_s) \\ &= (\mathrm{id} \otimes \Delta) \sum_{j=0}^{s} \Gamma_1 \diamond \cdots \diamond \Gamma_j \otimes \Gamma_{j+1} \diamond \cdots \diamond \Gamma_s \\ &= \sum_{j=0}^{s} \Gamma_1 \diamond \cdots \diamond \Gamma_j \otimes (\sum_{k=j}^{s} \Gamma_{j+1} \diamond \cdots \diamond \Gamma_k \otimes \Gamma_{k+1} \diamond \cdots \diamond \Gamma_s) \\ &= \sum_{0 \leqslant j \leqslant k \leqslant s} \Gamma_1 \diamond \cdots \diamond \Gamma_j \otimes \Gamma_{j+1} \diamond \cdots \diamond \Gamma_k \otimes \Gamma_{k+1} \diamond \cdots \diamond \Gamma_s \\ &= \sum_{k=0}^{s} (\sum_{j=0}^{k} \Gamma_1 \diamond \cdots \diamond \Gamma_j \otimes \Gamma_{j+1} \diamond \cdots \diamond \Gamma_k) \otimes \Gamma_{k+1} \diamond \cdots \diamond \Gamma_s \\ &= (\Delta \otimes \mathrm{id}) \sum_{k=0}^{s} \Gamma_1 \diamond \cdots \diamond \Gamma_k \otimes \Gamma_{k+1} \diamond \cdots \diamond \Gamma_s \\ &= (\Delta \otimes \mathrm{id}) \circ \Delta(\Gamma),\end{aligned}$$

where

$$\Gamma_{k+1} \diamond \cdots \diamond \Gamma_k = \epsilon$$

for $0 \leqslant k \leqslant s$. So, Δ satisfies the coassociative law. Hence, $(\mathcal{H}, \Delta, \nu)$ is a coalgebra.

By the definition of the coproduct Δ, we have

$$\Delta(\mathcal{H}_n) \subseteq \bigoplus \mathcal{H}_i \otimes \mathcal{H}_{n-i}$$

and

$$\nu(\mathcal{H}_n) = 0,$$

when $n \geqslant 1$. So, $(\mathcal{H}, \Delta, \nu)$ is a graded coalgebra. □

Define the *super-shuffle product* $*$ on $\mathcal{H}$ by

$$\Gamma_1 * \Gamma_2 = \sum_{\substack{I,\, J:\, |I|=m,\, |J|=n \\ I\cup J=[m+n]=V(\Gamma) \\ \mathrm{st}(\Gamma_I)=\Gamma_1,\, \mathrm{st}(\Gamma_J)=\Gamma_2}} \Gamma \tag{1}$$

for Γ_1 in H_m and Γ_2 in H_n. Sometimes, we denote it by $*(\Gamma_1, \Gamma_2)$. Obviously, the product $*$ is commutative on $\mathcal{H}$. Define the *unit* μ from $\mathbb{K}$ to $\mathcal{H}$ by $\mu(1) = \epsilon$.

Actually, Γ_I is the restructure of Γ_1 by I, and Γ_J is the restructure of Γ_2 by J in (1). Given I and J satisfying $|I| = m$, $|J| = n$, and $I \cup J = [m+n]$, Γ traverses all graphs in H_{m+n}, which is a union of the restructure of $\Gamma_1 = (V_1, E_1)$ by I, the restructure of $\Gamma_2 = (V_2, E_2)$ by J, and some edges between $\hat{V}_1$ and $\hat{V}_2$. That is, Γ traverses the set

$$P_{I,J} = \{(\hat{V}_1 \cup \hat{V}_2, \hat{E}_1 \cup \hat{E}_2 \cup C) | \hat{V}_1 = I,$$

$$\hat{V}_2 = J, C \subseteq \hat{V}_1 \times \hat{V}_2\}.$$

So, we rewrite (1) as

$$\Gamma_1 * \Gamma_2 = \sum_{\substack{I,\, J:\, |I|=m,\, |J|=n \\ I\cup J=[m+n]}} \sum_{\Gamma\in P_{I,J}} \Gamma. \tag{2}$$

That is, each term Γ in $\Gamma_1 * \Gamma_2$ is a graph by adding some edges between $\hat{V}_1$ and $\hat{V}_2$ to $\hat{\Gamma}_1 \cup \hat{\Gamma}_2$, where

$$\hat{V}_1 \cup \hat{V}_2 = [m+n],$$

i.e.,

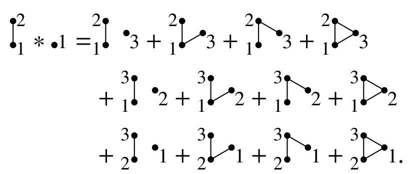
$$\Gamma = (\hat{V}_1 \cup \hat{V}_2, \hat{E}_1 \cup \hat{E}_2 \cup C), \tag{3}$$

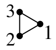
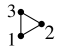
where C is a set of edges between $\hat{V}_1$ and $\hat{V}_2$. Conversely, $(\hat{V}_1, \hat{V}_2, C)$ can uniquely determine a term in $\Gamma_1 * \Gamma_2$, where

$$\hat{V}_1 \cup \hat{V}_2 = [m+n]$$

and C is a set of edges between $\hat{V}_1$ and $\hat{V}_2$. We consider two terms in $\Gamma_1 * \Gamma_2$ the same if, and only if, their corresponding $\hat{V}_1$, $\hat{V}_2$ and C are the same. Thus, each term in $\Gamma_1 * \Gamma_2$ is unique.

Example 7.



Here, we color the vertices of the term Γ in $\Gamma_1 * \Gamma_2$ restricted to Γ_1 red and to Γ_2 blue, respectively. In this example, although  and  are the same as graphs, we consider that they are different in $\Gamma_1 * \Gamma_2$ because their $\hat{V}_1$ and $\hat{V}_2$ are not the same. So, each term in $\Gamma_1 * \Gamma_2$ is unique.

In order to represent vertices in each term of $\Gamma_1 * \Gamma_2$ before restructure, we name the vertices in Γ_1 and Γ_2, respectively, as

$$V(\Gamma_1) = \{v_{11}, v_{12}, \cdots, v_{1m}\}$$

and

$$V(\Gamma_2) = \{v_{21}, v_{22}, \cdots, v_{2n}\},$$

where $v_{11} < v_{12} < \cdots < v_{1m}$ and $v_{21} < v_{22} < \cdots < v_{2n}$. Although v_{11} and v_{21} are both equal to 1, we consider that they are different because they belong to different graphs, then the vertex set of a term in $\Gamma_1 * \Gamma_2$ is

$$\hat{V}_1 \cup \hat{V}_2 = \{\hat{v}_{11}, \cdots, \hat{v}_{1m}, \hat{v}_{21}, \cdots, \hat{v}_{2n}\} = [m+n].$$

Theorem 2.2. *$(\mathcal{H}, *, \mu)$ is a graded algebra.*

Proof. It is easy to verify that μ is a unit. Suppose

$$\Gamma_1 = ([n_1], E_1), \quad \Gamma_2 = ([n_2], E_2)$$

and

$$\Gamma_3 = ([n_3], E_3)$$

in H. For any term Γ in $(\Gamma_1 * \Gamma_2) * \Gamma_3$, it corresponds to two disjoint subsets J and K of $[n_1+n_2+n_3]$ with $|J| = n_1+n_2$ and $|K| = n_3$, such that $\mathrm{st}(\Gamma_J)$ is a term in $\Gamma_1 * \Gamma_2$ and $\mathrm{st}(\Gamma_K) = \Gamma_3$. It means

$$(\Gamma_1 * \Gamma_2) * \Gamma_3 = \sum_{\substack{J,\, K:\, |J|=n_1+n_2,\, |K|=n_3 \\ J\cup K=[n_1+n_2+n_3]=V(\Gamma)}} \sum_{\substack{\mathrm{st}(\Gamma_K)=\Gamma_3 \\ \mathrm{st}(\Gamma_J) \text{ is a term in } \Gamma_1 * \Gamma_2}} \Gamma. \tag{4}$$

For a fixed J in (4), $\mathrm{st}(\Gamma_J)$ corresponds to two disjoint subsets P and Q of $[n_1+n_2]$ with $|P| = n_1$ and $|Q| = n_2$, such that

$$\mathrm{st}(\mathrm{st}(\Gamma_J)_P) = \Gamma_1$$

and

$$\mathrm{st}(\mathrm{st}(\Gamma_J)_Q) = \Gamma_2.$$

Therefore, there is a subset M of J with $|M| = n_1$ corresponding to P, i.e., $\mathrm{st}_J(M) = P$, such that

$$\mathrm{st}(\Gamma_M) = \mathrm{st}(\mathrm{st}(\Gamma_J)_P) = \Gamma_1.$$

Similarly, there is a subset N of J with $|N| = n_2$ corresponding to Q, i.e., $\mathrm{st}_J(N) = Q$, such that

$$\mathrm{st}(\Gamma_N) = \mathrm{st}(\mathrm{st}(\Gamma_J)_Q) = \Gamma_2.$$

That means (4) can be rewritten as

$$(\Gamma_1 * \Gamma_2) * \Gamma_3 = \sum_{\substack{J,\,K:\ |J|=n_1+n_2,\ |K|=n_3 \\ J\cup K=[n_1+n_2+n_3]=V(\Gamma)}} \sum_{\substack{M,\,N:\ |M|=n_1,\ |N|=n_2 \\ M\cup N=J \\ \mathrm{st}(\Gamma_M)=\Gamma_1,\ \mathrm{st}(\Gamma_N)=\Gamma_2,\ \mathrm{st}(\Gamma_K)=\Gamma_3}} \Gamma. \tag{5}$$

For a fixed subset J in $[n_1 + n_2 + n_3]$ with cardinality $n_1 + n_2$, P traverses all subsets with cardinality n_1 in $[n_1 + n_2]$ since $\mathrm{st}(\Gamma_J)$ traverses all terms in $\Gamma_1 * \Gamma_2$. Meanwhile, M traverses all subsets with cardinality n_1 in J. Therefore, M traverses all subsets with cardinality n_1 in $[n_1+n_2+n_3]$ since J traverses all subsets with cardinality n_1+n_2 in $[n_1+n_2+n_3]$. At the same time, N traverses all subsets with cardinality n_2 in $[n_1 + n_2 + n_3]$ from $J = M \cup N$. Thus, (5) can be rewritten as

$$(\Gamma_1 * \Gamma_2) * \Gamma_3 = \sum_{\substack{M,\,N,\,K:\ |M|=n_1,\ |N|=n_2,\ |K|=n_3 \\ M\cup N\cup K=[n_1+n_2+n_3]=V(\Gamma) \\ \mathrm{st}(\Gamma_M)=\Gamma_1,\ \mathrm{st}(\Gamma_N)=\Gamma_2,\ \mathrm{st}(\Gamma_K)=\Gamma_3}} \Gamma. \tag{6}$$

Similarly, $\Gamma_1 * (\Gamma_2 * \Gamma_3)$ is equal to (6). Hence, $*$ satisfies the associative law and $(\mathcal{H}, *, \mu)$ is an algebra.

By the definition of the product $*$, we have

$$\mathcal{H}_i * \mathcal{H}_j \subseteq \mathcal{H}_{i+j}$$

and

$$\mu(\mathbb{K}) \subseteq \mathcal{H}_0.$$

So, $(\mathcal{H}, *, \mu)$ is a graded algebra. □

3. Main theorems

In this section, we will prove that $(\mathcal{H}, *, \mu, \Delta, \nu)$ is a Hopf algebra. Now, we give two lemmas.

Let $\Gamma_1 = (V_1, E_1)$ in H_m and $\Gamma_2 = (V_2, E_2)$ in H_n be nonempty graphs and Γ be a term in $\Gamma_1 * \Gamma_2$. Thus, Γ can be represented by $(\hat{V}_1 \cup \hat{V}_2, \hat{E}_1 \cup \hat{E}_2 \cup C)$ from (3), where

$$\hat{\Gamma}_1 := (\hat{V}_1, \hat{E}_1)$$

is the restructure of Γ_1 by $\hat{V}_1$ and

$$\hat{\Gamma}_2 := (\hat{V}_2, \hat{E}_2)$$

is the restructure of Γ_2 by $\hat{V}_2$. Obviously,

$$\hat{V}_1 \cup \hat{V}_2 = [m + n].$$

Lemma 3.1. *Each atom of Γ in $\Gamma_1 * \Gamma_2$ can only contain subgraphs of $\hat{\Gamma}_1$ or $\hat{\Gamma}_2$ corresponding to some complete atoms in Γ_1 or Γ_2.*

Proof. Let

$$\Gamma_1 = (\{v_{11}, \cdots, v_{1m}\}, E_1)$$

and

$$\Gamma_2 = (\{v_{21}, \cdots, v_{2n}\}, E_2)$$

be nonempty in H, where $v_{11} < \cdots < v_{1m}$ and $v_{21} < \cdots < v_{2n}$. Consider a term Γ in $\Gamma_1 * \Gamma_2$. Suppose $\Gamma_{[i,j]}$ is an atom of Γ containing a nonempty subgraph of $\hat{\Gamma}_{1\{\hat{v}_{1k}\}_{k=p}^{q}}$, where $\hat{\Gamma}_{1\{\hat{v}_{1k}\}_{k=p}^{q}}$ corresponds to the atom $\Gamma_{1\{v_{1k}\}_{k=p}^{q}}$ of Γ_1.

When $p = q$, there is only one element in $\{\hat{v}_{1k}\}_{k=p}^{q}$, then $\Gamma_{[i,j]}$ contains the complete atom $\hat{\Gamma}_{1\{\hat{v}_{1k}\}_{k=p}^{q}}$. Hence, the conclusion holds.

When $1 \leqslant p < q \leqslant m$, $\{v_{1k}\}_{k=p}^{q}$ contains at least two vertices. Suppose that $\Gamma_{[i,j]}$ contains a true subgraph of $\hat{\Gamma}_{1\{\hat{v}_{1k}\}_{k=p}^{q}}$. In fact, since $\{\hat{v}_{1k}\}_{k=p}^{q}$ maintains the order relationship in $\{v_{1k}\}_{k=p}^{q}$, the vertices of this true subgraph correspond to a true subinterval in $\{v_{1k}\}_{k=p}^{q}$. Let

$$\omega = \min\{k \mid \hat{v}_{1k} \in [i, j], p \leqslant k \leqslant q\}$$

and

$$\Omega = \max\{k \mid \hat{v}_{1k} \in [i, j], p \leqslant k \leqslant q\}.$$

We have $i \leqslant \hat{v}_{1\omega} \leqslant \hat{v}_{1\Omega} \leqslant j$, then

$$\{\hat{v}_{1k}\}_{k=\omega}^{\Omega} \subseteq [i, j]$$

and $\omega \neq p$ or $\Omega \neq q$ because $\Gamma_{[i,j]}$ contains a true subgraph of $\hat{\Gamma}_{1\{v_{1k}\}_{k=p}^{q}}$.

If $\omega \neq p$, then $\omega > p$. From

$$1 \leqslant \max_{1\leqslant k\leqslant \omega-1}\{\hat{v}_{1k}\} < i \leqslant \min_{\omega\leqslant k\leqslant m}\{\hat{v}_{1k}\},$$

$$\{\hat{v}_{1k}\}_{k=1}^{\omega-1} \subseteq [i - 1]$$

and

$$\{\hat{v}_{1k}\}_{k=\omega}^{m} \subseteq [i, m + n].$$

From $\Gamma_{[i,j]}$ is an atom of Γ and $i - 1$ is a split of Γ, there are no edges between $[i - 1]$ and $[i, m + n]$ in Γ. Therefore,

$$\Gamma_{\{\hat{v}_{1k}\}_{k=1}^{m}} = \Gamma_{\{\hat{v}_{1k}\}_{k=1}^{\omega-1}} \cup \Gamma_{\{\hat{v}_{1k}\}_{k=\omega}^{m}}.$$

By the definition of $*$, we have

$$\mathrm{st}(\Gamma_{\{\hat{v}_{1k}\}_{k=1}^{m}}) = \Gamma_1$$

and

$$\mathrm{st}_{\hat{V}_1}(\hat{v}_{1k}) = v_{1k} = k.$$

Hence,

$$\begin{aligned}\Gamma_1 &= \mathrm{st}(\Gamma_{\{\hat{v}_{1k}\}_{k=1}^{m}})\\ &= \mathrm{st}(\Gamma_{\{\hat{v}_{1k}\}_{k=1}^{\omega-1}} \cup \Gamma_{\{\hat{v}_{1k}\}_{k=\omega}^{m}})\\ &= \Gamma_{1[\omega-1]} \cup \Gamma_{1[\omega,m]},\end{aligned}$$

where $p-1 < \omega-1 < q$, i.e., $\omega-1$ is a split of Γ_1. However, there is no split of Γ_1 between $p-1$ and q, since $\Gamma_{1\{v_{1k}\}_{k=p}^{q}}$ is an atom of Γ_1, which is a contradiction. Similarly, when $\Omega \neq q$, we have $p-1 < \Omega < q$ and Ω is a split of Γ_1, a contradiction.

Thus, if an atom of Γ contains a true subgraph in $\hat{\Gamma}_1$, then this subgraph must correspond to some complete atoms of Γ_1. Similarly, we can prove the conclusion holds for atoms in Γ_2. □

For simplicity, we restate Lemma 3.1 as: for any term Γ in $\Gamma_1 * \Gamma_2$, any atom of Γ can only contain some complete *original* atoms of Γ_1 or Γ_2.

Remark 3.1. For Γ in H_n, suppose its decomposition is

$$\Gamma = \Gamma_1 \diamond \Gamma_2 \diamond \cdots \diamond \Gamma_s$$

with splits

$$0 = i_0 < i_1 < \cdots < i_s = n,$$

then

$$\begin{aligned}\Delta(\Gamma) &= \sum_{k=0}^{s} \Gamma_1 \diamond \cdots \diamond \Gamma_k \otimes \Gamma_{k+1} \diamond \cdots \diamond \Gamma_s\\ &= \sum_{k=0}^{s} \mathrm{st}(\Gamma_{[1,i_k]}) \otimes \mathrm{st}(\Gamma_{[i_k+1,i_s]}).\end{aligned}$$

If $\Theta_1 \otimes \Theta_2$ is the term in $\Delta(\Gamma)$, then Θ_1 is a standard form of the first k atoms of Γ for some $0 \leqslant k \leqslant s$. Let Γ be a term in $\Gamma_1 * \Gamma_2$ and $\Theta_1 \otimes \Theta_2$ be a term in $\Delta(\Gamma)$. From Lemma 3.1, Θ_1 only contains the standard forms of some complete original atoms of Γ_1 or Γ_2. If the first few atoms of a labeled simple graph contain l vertices, then these vertices must be $[l]$. So, if Θ_1 contains i original atoms of Γ_1, then they must be the first i atoms of Γ_1. Similarly, if Θ_1 contains j original atoms of Γ_2, then they must be the first j atoms of Γ_2.

Let Γ_1 and Γ_2 be nonempty in H. Suppose their decompositions are

$$\Gamma_1 = \Gamma_{11} \diamond \cdots \diamond \Gamma_{1s}$$

and

$$\Gamma_2 = \Gamma_{21} \diamond \cdots \diamond \Gamma_{2t}.$$

Define $\Delta_{ij}(\Gamma_1 * \Gamma_2)$ to be the sum of all terms $\Theta_1 \otimes \Theta_2$ in $\Delta(\Gamma_1 * \Gamma_2)$, where Θ_1 contains the first i complete original atoms in Γ_1 and the first j complete original atoms in Γ_2, $0 \leqslant i \leqslant s$, and $0 \leqslant j \leqslant t$.

Lemma 3.2. *Let Γ_1 and Γ_2 be nonempty in H. Assume their decompositions are*

$$\Gamma_1 = \Gamma_{11} \diamond \cdots \diamond \Gamma_{1s} \ \text{ and } \ \Gamma_2 = \Gamma_{21} \diamond \cdots \diamond \Gamma_{2t},$$

then,

$$\begin{aligned}\Delta_{ij}(\Gamma_1 * \Gamma_2) =& (\Gamma_{11} \diamond \cdots \diamond \Gamma_{1i} * \Gamma_{21} \diamond \cdots \diamond \Gamma_{2j})\\ &\otimes (\Gamma_{1,i+1} \diamond \cdots \diamond \Gamma_{1s} * \Gamma_{2,j+1} \diamond \cdots \diamond \Gamma_{2t}),\end{aligned}$$

for $0 \leqslant i \leqslant s$ and $0 \leqslant j \leqslant t$.

Proof. Denote

$$V(\Gamma_{11} \diamond \cdots \diamond \Gamma_{1i}) = V_{11},$$

$$|V_{11}| = h_1, \quad V_1 \backslash V_{11} = V_{12},$$

and

$$(E_1)_{V_{11}} = E_{11}, \quad (E_1)_{V_{12}} = E_{12}.$$

Similarly,

$$V(\Gamma_{21} \diamond \cdots \diamond \Gamma_{2j}) = V_{21}, \quad |V_{21}| = h_2, \quad V_2 \backslash V_{21} = V_{22},$$

and

$$(E_2)_{V_{21}} = E_{21}, \quad (E_2)_{V_{22}} = E_{22}.$$

Obviously,

$$\Gamma_1 = (V_{11} \cup V_{12}, E_{11} \cup E_{12})$$

and

$$\Gamma_2 = (V_{21} \cup V_{22}, E_{21} \cup E_{22}).$$

Next, we denote $\hat{V}_{11}$ as the subset corresponding to V_{11} in $\hat{V}_1$ and $(\hat{V}_{11}, \hat{E}_{11})$ as the restructure (V_{11}, E_{11}) by $\hat{V}_{11}$. Similarly, we have $\hat{V}_{12}$, $\hat{E}_{12}$, $\hat{V}_{21}$, $\hat{E}_{21}$, $\hat{V}_{22}$, and $\hat{E}_{22}$.

By (3), each term Γ in $\Gamma_1 * \Gamma_2$,

$$\Gamma = (\hat{V}_1 \cup \hat{V}_2, \hat{E}_1 \cup \hat{E}_2 \cup C),$$

where C is a set of edges between $\hat{V}_1$ and $\hat{V}_2$. Let $\Theta_1 \otimes \Theta_2$ be a term in $\Delta(\Gamma)$ and in $\Delta_{ij}(\Gamma_1 * \Gamma_2)$. By the definition of Δ_{ij}, $h_1 + h_2$ is a split of Γ,

$$\hat{V}_{11} \cup \hat{V}_{21} = [h_1 + h_2],$$

$$\hat{V}_{12} \cup \hat{V}_{22} = [h_1 + h_2 + 1, m + n]$$

and

$$\Theta_1 = \mathrm{st}(\Gamma_{[h_1+h_2]}) = \mathrm{st}(\Gamma_{\hat{V}_{11} \cup \hat{V}_{21}}).$$

Since $h_1 + h_2$ is a split of Γ, there are no edges between $[h_1 + h_2]$ and $[h_1 + h_2 + 1, m + n]$ in Γ. Thus, there are no edges between $\hat{V}_{11}$ and $\hat{V}_{22}$ and no edges between $\hat{V}_{12}$ and $\hat{V}_{21}$. Therefore, C is $C_1 \cup C_2$, where

$$C_1 \subseteq \hat{V}_{11} \times \hat{V}_{21}$$

and

$$C_2 \subseteq \hat{V}_{12} \times \hat{V}_{22}.$$

Hence,

$$\Gamma = (\hat{V}_{11} \cup \hat{V}_{12} \cup \hat{V}_{21} \cup \hat{V}_{22}, \hat{E}_{11} \cup \hat{E}_{12} \cup \hat{E}_{21} \cup \hat{E}_{22} \cup C_1 \cup C_2). \quad (7)$$

Therefore,

$$\Theta_1 = \mathrm{st}[(\hat{V}_{11} \cup \hat{V}_{21}, \hat{E}_{11} \cup \hat{E}_{21} \cup C_1)]$$

and

$$\Theta_2 = \mathrm{st}[(\hat{V}_{12} \cup \hat{V}_{22}, \hat{E}_{12} \cup \hat{E}_{22} \cup C_2)],$$

then Θ_1 is a term in

$$\Gamma_{11} \diamond \cdots \diamond \Gamma_{1i} * \Gamma_{21} \diamond \cdots \diamond \Gamma_{2j}$$

for

$$\mathrm{st}((\Theta_1)_{\hat{V}_{11}}) = \mathrm{st}(\hat{V}_{11}, \hat{E}_{11}) = \Gamma_{11} \diamond \cdots \diamond \Gamma_{1i}$$

and

$$\mathrm{st}((\Theta_1)_{\hat{V}_{21}}) = \mathrm{st}(\hat{V}_{21}, \hat{E}_{21}) = \Gamma_{21} \diamond \cdots \diamond \Gamma_{2j}.$$

Similarly, Θ_2 is a term in

$$\Gamma_{1,i+1} \diamond \cdots \diamond \Gamma_{1s} * \Gamma_{2,j+1} \diamond \cdots \diamond \Gamma_{2t}.$$

For given Θ_1 in $\Gamma_{11} \diamond \cdots \diamond \Gamma_{1i} * \Gamma_{21} \diamond \cdots \diamond \Gamma_{2j}$, let

$$S = \left\{ \Gamma \middle| \begin{array}{l} \Gamma \text{ is a term in } \Gamma_1 * \Gamma_2 \text{ and } \exists\, \Theta_2 \text{ s.t. } \Theta_1 \otimes \Theta_2 \\ \text{is a term in } \Delta(\Gamma) \text{ and in } \Delta_{ij}(\Gamma_1 * \Gamma_2) \end{array} \right\}.$$

For each Γ in S,

$$\Gamma = (\hat{V}_{11} \cup \hat{V}_{12} \cup \hat{V}_{21} \cup \hat{V}_{22}, \hat{E}_{11} \cup \hat{E}_{12} \cup \hat{E}_{21} \cup \hat{E}_{22} \cup C_1 \cup C_2)$$

by (7), where $\hat{V}_{11}$, $\hat{V}_{21}$, and C_1 are fixed, since Θ_1 is fixed.

By (3), when Γ traverses all terms in $\Gamma_1 * \Gamma_2$, its $\hat{V}_1$ and $\hat{V}_2$ traverse all disjoint subsets of $[m+n]$ with cardinalities m and n, respectively, and C traverses all subsets in $\hat{V}_1 \times \hat{V}_2$ for fixed $\hat{V}_1$ and $\hat{V}_2$. Thus, $\hat{V}_{12}$ and $\hat{V}_{22}$ of Γ in S traverse all disjoint subsets of $[h_1 + h_2 + 1, m + n]$ with cardinalities $m - h_1$ and $n - h_2$, respectively. Also, C_2 of Γ in S traverses all subsets of $\hat{V}_{12} \times \hat{V}_{22}$ for fixed $\hat{V}_{12}$ and $\hat{V}_{22}$. Correspondingly, for $\hat{V}_{12}$ and $\hat{V}_{22}$ of Γ in S, $\mathrm{st}_{\hat{V}_{12} \cup \hat{V}_{22}}(\hat{V}_{12})$ and $\mathrm{st}_{\hat{V}_{12} \cup \hat{V}_{21}}(\hat{V}_{22})$ traverse all disjoint subsets of $[m + n - h_1 - h_2]$ with cardinalities $m - h_1$ and $n - h_2$, respectively. Also, $\mathrm{st}_{\hat{V}_{12} \cup \hat{V}_{22}}(C_2)$ traverses all subsets in

$$\mathrm{st}_{\hat{V}_{12} \cup \hat{V}_{22}}(\hat{V}_{12}) \times \mathrm{st}_{\hat{V}_{12} \cup \hat{V}_{21}}(\hat{V}_{22})$$

for fixed $\hat{V}_{12}$ and $\hat{V}_{22}$. From the arguments above and (2), for fixed Θ_1 in $\Gamma_{11} \diamond \cdots \diamond \Gamma_{1i} * \Gamma_{21} \diamond \cdots \diamond \Gamma_{2j}$, Θ_2 traverses all terms in $\Gamma_{1,i+1} \diamond \cdots \diamond \Gamma_{1s} * \Gamma_{2,j+1} \diamond \cdots \diamond \Gamma_{2t}$. Similarly, for fixed Θ_2 in $\Gamma_{1,i+1} \diamond \cdots \diamond \Gamma_{1s} * \Gamma_{2,j+1} \diamond \cdots \diamond \Gamma_{2t}$, Θ_1 traverses all terms in $\Gamma_{11} \diamond \cdots \diamond \Gamma_{1i} * \Gamma_{21} \diamond \cdots \diamond \Gamma_{2j}$. Therefore,

$$\begin{aligned} \Delta_{ij}(\Gamma_1 * \Gamma_2) =& (\Gamma_{11} \diamond \cdots \diamond \Gamma_{1i} * \Gamma_{21} \diamond \cdots \diamond \Gamma_{2j}) \\ & \otimes (\Gamma_{1,i+1} \diamond \cdots \diamond \Gamma_{1s} * \Gamma_{2,j+1} \diamond \cdots \diamond \Gamma_{2t}). \end{aligned}$$

□

Next, we show that $(\mathcal{H}, *, \mu, \Delta, \nu)$ is a Hopf algebra.

Theorem 3.1. *$(\mathcal{H}, *, \mu, \Delta, \nu)$ is a bialgebra.*

Proof. It is easy to verify ν is an algebra homomorphism. We only need to prove Δ is an algebra homomorphism, i.e.,

$$\Delta(\Gamma_1 * \Gamma_2) = \Delta(\Gamma_1) * \Delta(\Gamma_2) \quad (8)$$

for Γ_1 and Γ_2 in H.

If Γ_1 or Γ_2 is an empty graph, then (8) holds. Suppose

$$\Gamma_1 = (V_1, E_1) = \Gamma_{11} \diamond \cdots \diamond \Gamma_{1s}$$

in H_m and

$$\Gamma_2 = (V_2, E_2) = \Gamma_{21} \diamond \cdots \diamond \Gamma_{2t}$$

in H_n are nonempty graphs. Let

$$\Delta_i(\Gamma_1 * \Gamma_2) = \sum_{j=0}^{t} \Delta_{ij}(\Gamma_1 * \Gamma_2).$$

By Lemma 3.2, we have

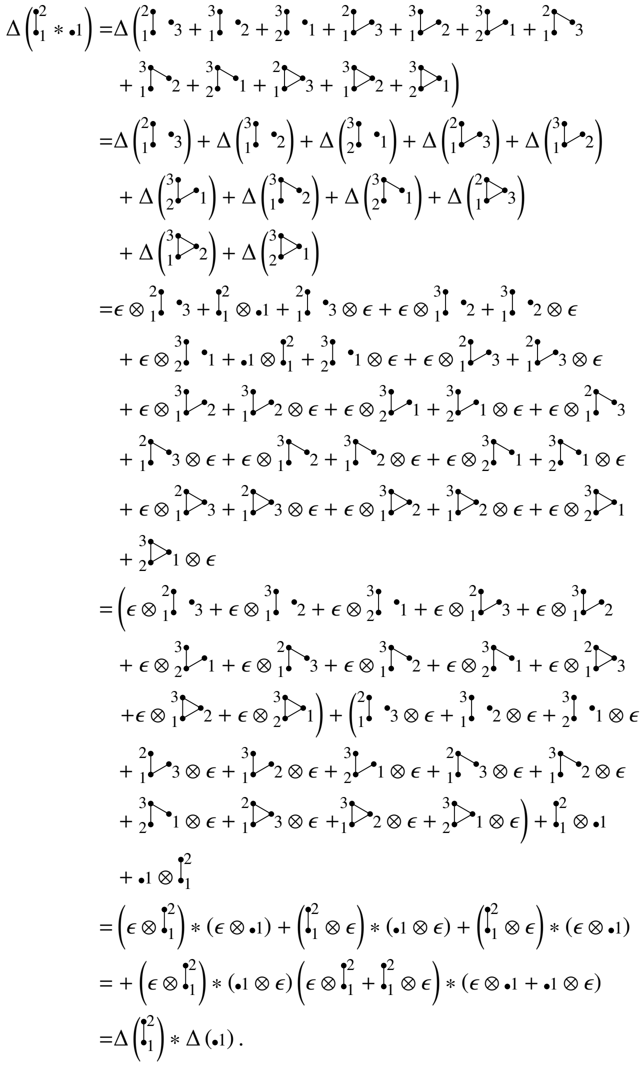
$$\begin{aligned}
\Delta_i(\Gamma_1 * \Gamma_2) =& \Gamma_{11} \diamond \cdots \diamond \Gamma_{1i} \otimes (\Gamma_{1,i+1} \diamond \cdots \diamond \Gamma_{1s} * \Gamma_{21} \diamond \cdots \diamond \Gamma_{2t}) \\
& + (\Gamma_{11} \diamond \cdots \diamond \Gamma_{1i} * \Gamma_{21}) \\
& \otimes (\Gamma_{1,i+1} \diamond \cdots \diamond \Gamma_{1s} * \Gamma_{22} \diamond \cdots \diamond \Gamma_{2t}) \\
& + (\Gamma_{11} \diamond \cdots \diamond \Gamma_{1i} * \Gamma_{21} \diamond \Gamma_{22}) \\
& \otimes (\Gamma_{1,i+1} \diamond \cdots \diamond \Gamma_{1s} * \Gamma_{23} \diamond \cdots \diamond \Gamma_{2t}) \\
& + \cdots \\
& + (\Gamma_{11} \diamond \cdots \diamond \Gamma_{1i} * \Gamma_{21} \diamond \cdots \diamond \Gamma_{2t}) \otimes \Gamma_{1,i+1} \diamond \cdots \diamond \Gamma_{1s} \\
=& \sum_{j=0}^{t} (\Gamma_{11} \diamond \cdots \diamond \Gamma_{1i} * \Gamma_{21} \diamond \cdots \diamond \Gamma_{2j}) \\
& \otimes (\Gamma_{1,i+1} \diamond \cdots \diamond \Gamma_{1s} * \Gamma_{2,j+1} \diamond \cdots \diamond \Gamma_{2t}) \\
=& (\Gamma_{11} \diamond \cdots \diamond \Gamma_{1i} \otimes \Gamma_{1,i+1} \diamond \cdots \diamond \Gamma_{1s}) \\
& * (\sum_{j=0}^{t} \Gamma_{21} \diamond \cdots \diamond \Gamma_{2j} \otimes \Gamma_{2,j+1} \diamond \cdots \diamond \Gamma_{2t}) \\
=& (\Gamma_{11} \diamond \cdots \diamond \Gamma_{1i} \otimes \Gamma_{1,i+1} \diamond \cdots \diamond \Gamma_{1s}) * \Delta(\Gamma_2).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\Delta(\Gamma_1 * \Gamma_2) =& \Delta_0(\Gamma_1 * \Gamma_2) + \Delta_1(\Gamma_1 * \Gamma_2) + \cdots + \Delta_s(\Gamma_1 * \Gamma_2) \\
=& (\sum_{i=0}^{s} \Gamma_{11} \diamond \cdots \diamond \Gamma_{1i} \otimes \Gamma_{1,i+1} \diamond \cdots \diamond \Gamma_{1s}) * \Delta(\Gamma_2) \\
=& \Delta(\Gamma_1) * \Delta(\Gamma_2).
\end{aligned}$$

Hence, $(\mathcal{H}, *, \mu, \Delta, \nu)$ is a bialgebra. □

Example 8.



Corollary 3.1. *$(\mathcal{H}, *, \mu, \Delta, \nu)$ is a Hopf algebra.*

Proof. Since $(\mathcal{H}, *, \mu, \Delta, \nu)$ is a graded connected bialgebra, it is a Hopf algebra. □

4. Conclusions

Many combinatorial objects have Hopf algebra structures. The labeled simple graphs are important combinatorial objects. In this paper, we generalize the super-shuffle product and the cut-box coproduct on permutations to labeled simple graphs. We prove that the vector space spanned by labeled simple graphs with the super-shuffle product and the cut-box coproduct is a Hopf algebra. In the future, we will study the duality of the Hopf algebra $(H, *, \mu, \Delta, \nu)$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest in this paper.

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