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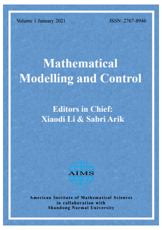

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Research article

Dynamical behavior of solutions of a free boundary problem

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Abstract: This paper is concerned with the spreading properties for a reaction-diffusion equation with free boundary condition. We obtained a complete description of the long-time dynamical behavior of this problem. By introducing a parameter σ in the initial data, we revealed a threshold value σ^* such that spreading happens when $\sigma > \sigma^*$ and vanishing happens when $\sigma \le \sigma^*$. There exists a unique $L^* > 0$ independent of the initial data such that $\sigma^* = 0$ if and only if the length of initial occupying interval is no smaller than $2L^*$. These theoretical results may have important implications for prediction and prevention of biological invasions.

Keywords: reaction-diffusion equation; free boundary problem; long time behavior; spreading phenomena

1. Introduction

In this paper, we consider the following free boundary problem:

$$\begin{cases} u_t = u_{xx} + F(x,u), & k(t) < x < h(t),\ t > 0, \\ u(t,x) = 0,\ h'(t) = -\mu_1 u_x(t,x), & t > 0,\ x = h(t), \\ u(t,x) = 0,\ k'(t) = -\mu_2 u_x(t,x), & t > 0,\ x = k(t), \\ -k(0) = h(0) = h_0,\ u(0,x) = u_0(x), & -h_0 \le x \le h_0, \end{cases} \tag{P}$$

where $u(t,x)$ denotes the population density of a species over a one dimensional space, the free boundaries $x = k(t)$ and $x = h(t)$ represent the spreading fronts, and μ_1, μ_2 are two given positive constants (see [1,2] on more background of such free boundary conditions). For some $h_0 > 0$, the initial function u_0 belongs to $\mathscr{X}(h_0)$, where

$$\mathscr{X}(h_0) := \Big\{\phi \in C^2([-h_0,h_0]) : \phi(-h_0) = \phi(h_0) = 0,\ \phi(x) > 0 \text{ in } (-h_0,h_0)\Big\}.$$

When $x > 0$, the nonlinear reaction term $F(x,u) \equiv f(u)$, where f is *globally Lipschitz*, satisfies

$$\begin{cases} f(0) = f(1) = 0 < f'(0), \quad f'(1) < 0, \\ (1-u)f(u) > 0, \quad \forall u > 0, u \ne 1, \end{cases} \tag{1.1}$$

and when $x < 0$, the nonlinear reaction term $F(x,u) \equiv g(u)$, where g is *globally Lipschitz*, satisfies

$$\begin{cases} g(0) = g(\theta) = g(1) = 0, & g'(0) < 0,\ g'(1) < 0, \\ g(u) < 0 \text{ in } (0,\theta), & g(u) > 0 \text{ in } (1,\infty), \end{cases} \tag{1.2}$$

for some $\theta \in (0,1)$, and $\int_0^1 g(s)ds > 0$. These two types of nonlinearities have been studied in [3,4].

We assume that the population density is continuous and population flux is conserved at $x = 0$. Then, the interface conditions at $x = 0$ are given by

$$\begin{cases} u(t,0-0) = u(t,0+0), & t > 0, \\ u_x(t,0-0) = u_x(t,0+0), & t > 0. \end{cases} \tag{1.3}$$

Throughout the paper, in addition to conditions (1.1) and (1.2) on f and g, we further assume that

(H) $\quad g(u) < f(u)$ for all $0 < u < 1$ and $\mu_2 \le \mu_1$.

Problem (P) with $F(x,u) \equiv f(u)$ or $F(x,u) \equiv g(u)$ for $x \in \mathbb{R}$ was studied in [1,5]. It is shown that there are a spreading-vanishing dichotomy for $F(x,u) \equiv f(u)$ and a spreading-transition-vanishing trichotomy for $F(x,u) \equiv g(u)$. Relevant works on the dynamics of free boundary problems in a spatial heterogeneity environment can be found in [6–11].

The study of the corresponding problems in bounded or unbounded intervals can be found, for example, in [12–14].

Our primary goal in this paper is to study the dynamics of the reaction-diffusion model (P) with (1.3). By a similar argument as in [1, 9], we have the following basic results:

(i) For any given $u_0 \in \mathscr{X}(h_0)$ for $h_0 > 0$, problem (P) admits a unique positive solution (u, k, h) with

$$u \in C^{1,2}((0,\infty)\times([k(t),h(t)]/\{0\}) \cap C^{\alpha/2,1+\alpha}((0,\infty)\times[k(t),h(t)])$$

and $k, h \in C^{1+\frac{\alpha}{2}}([0,\infty))$ for any $\alpha \in (0,1)$.

(ii) There exist two positive constants C_1 and C_2 such that

$$\begin{cases} 0 < u(t,x) \le C_1 & \text{for } t > 0,\ x \in [k(t), h(t)], \\ 0 < -k'(t),\ h'(t) \le C_2 & \text{for } t > 0. \end{cases}$$

Let us define

$$k_\infty := \lim_{t\to\infty} k(t) \quad \text{and} \quad h_\infty := \lim_{t\to\infty} h(t).$$

We are now in a position to give a description of the long-time dynamical behavior of problem (P) with (1.3), which is stated as follows.

Theorem 1.1. *Assume that* **(H)** *holds. Let* (u, k, h) *be a time-global solution of* (P) *with* (1.3) *and* $u_0 = \sigma\phi$ *for some* $\phi \in \mathscr{X}(h_0)$*,* $h_0 > 0$ *and* $\sigma \ge 0$*. Then, there is* $\sigma^* \in [0,\infty]$ *such that:*

(i) Vanishing happens when $0 \le \sigma \le \sigma^*$ *in the sense that* $[k_\infty, h_\infty]$ *is a bounded interval and*

$$\lim_{t\to\infty} \|u(t,\cdot)\|_{L^\infty([k(t),h(t)])} = 0.$$

(ii) Spreading happens when $\sigma > \sigma^*$ *in the sense that* $(k_\infty, h_\infty) = \mathbb{R}$ *and*

$$\lim_{t\to\infty} u(t,x) = 1 \quad \textit{locally uniformly in } \mathbb{R}.$$

(iii) $\sigma^* = 0$ *if and only if* $h_0 \ge L^*$*, where* L^* *is given in Lemma 2.2.*

Theorem 1.1 indicates that if $h_0 \ge L^*$, the species will survive regardless of the choice of the initial data; if $h_0 < L^*$, the species will survive only for large initial data. Based on the comparison principle, the proof of this theorem is given in Section 2.

2. Classification of dynamical behavior

This section covers the long-time dynamical behavior of (P) with (1.3) and the proof of Theorem 1.1. In the first subsection, we show some properties of the principal eigenvalues of two linear eigenvalue problems. In Subsection 2.2, we give a general convergence theorem. We give some conditions for vanishing and spreading in Subsection 2.3. Subsection 2.4 is devoted to the proof of Theorem 1.1.

2.1. Linear eigenvalue problems

First, for any given $L > 0$, let us consider the following eigenvalue problem:

$$\begin{cases} -\varphi'' - f'(0)\varphi = \lambda\varphi, & 0 < x < L, \\ -\varphi'' - g'(0)\varphi = \lambda\varphi, & -\infty < x < 0, \\ \varphi(-\infty) = \varphi(L) = 0, \\ \varphi(0-0) = \varphi(0+0), \\ \varphi'(0-0) = \varphi'(0+0), \end{cases} \tag{2.1}$$

and obtain the following result on the properties of its principal eigenvalue.

Lemma 2.1. *For any given* $L > 0$*, let* $\lambda_1(L)$ *be the principal eigenvalue of* (2.1)*. Then,* $\lambda_1(L) \in (-f'(0), -g'(0))$ *for any* $L > 0$*, and* $\lambda_1(L)$ *is decreasing with respect to* $L > 0$*. There exists*

$$L_* = \frac{1}{\sqrt{f'(0)}}\left(\arctan\sqrt{-\frac{g'(0)}{f'(0)}} + \frac{\pi}{2}\right), \tag{2.2}$$

such that $\lambda_1(L)$ *is negative (resp.* 0*, or positive) when* $L > L_*$ *(resp.* $L = L_*$*, or* $L < L_*$*).*

Proof. To simplify, we write $\lambda_1 = \lambda_1(L)$. Let $\varphi(x)$ be the corresponding positive eigenfunction. It follows from [15] that $\lambda_1 \in (-f'(0), -g'(0))$ for any $L > 0$. As $\varphi(-\infty) = 0 < \varphi(x)$ for $x < 0$, by the second equation of (2.1), we see that there is a constant $C_1 > 0$ such that

$$\varphi(x) = C_1 e^{\sqrt{-(\lambda_1 + g'(0))}x} \quad \text{for } x < 0.$$

It is direct to check that $\varphi'(0-0) > 0$ and

$$\frac{\varphi'(0-0)}{\varphi(0-0)} = \sqrt{-(\lambda_1 + g'(0))} > 0. \tag{2.3}$$

It follows from the first equation of (2.1) that $\varphi'' < 0$ in $[0, L]$. Combining this with

$$\varphi'(0+0) = \varphi'(0-0) > 0 > \varphi'(L),$$

we find a unique constant $a^* \in (0, L)$ such that $\varphi'(a^*) = 0$. Thanks to this, we can find a constant $C_2 > 0$ such that

$$\varphi(x) = C_2 \cos \sqrt{\lambda_1 + f'(0)}(x - a^*) \quad \text{in} \quad [0, L],$$

which implies that

$$\frac{\varphi'(0+0)}{\varphi(0+0)} = \sqrt{\lambda_1 + f'(0)} \tan \sqrt{\lambda_1 + f'(0)} a^*.$$

This, together with (2.3), produces that

$$a^* = \frac{1}{\sqrt{\lambda_1 + f'(0)}} \arctan \sqrt{-\frac{\lambda_1 + g'(0)}{\lambda_1 + f'(0)}}. \tag{2.4}$$

Moreover, it follows from $\varphi(L) = 0$ that

$$L - a^* = \frac{\pi}{2\sqrt{\lambda_1 + f'(0)}}.$$

Combining with (2.4), we can have

$$L = \frac{1}{\sqrt{\lambda_1 + f'(0)}} \left(\arctan \sqrt{-\frac{\lambda_1 + g'(0)}{\lambda_1 + f'(0)}} + \frac{\pi}{2} \right).$$

It is obvious that λ_1 is decreasing in $L > 0$. Moreover, we can check that when $L = L_*$, then $\lambda_1 = 0$. Thanks to the monotonicity of λ_1 in L, all the other assertions follows. □

For our purpose, we consider the following eigenvalue problem:

$$\begin{cases} -\varphi'' - f'(0)\varphi = \lambda\varphi, & 0 < x < L, \\ -\varphi'' - g'(0)\varphi = \lambda\varphi, & -l < x < 0, \\ \varphi(-l) = \varphi(L) = 0, \\ \varphi(0-0) = \varphi(0+0), \\ \varphi'(0-0) = \varphi'(0+0), \end{cases} \tag{2.5}$$

where l and L are two positive constants. We can obtain the following lemma.

Lemma 2.2. *Let L_* be given in Lemma 2.1. For any given $L > 0$, the principal eigenvalue $\lambda_1(L, l)$ of (2.5) is decreasing with respect to $l > 0$. When*

$$L \in (L_*, \frac{\pi}{\sqrt{f'(0)}}),$$

then there exists

$$l^*(L) = \frac{\ln\left[1 + \frac{2\sqrt{-g'(0)}}{\sqrt{f'(0)} \tan\left(\sqrt{f'(0)}L - \frac{\pi}{2}\right) - \sqrt{-g'(0)}}\right]}{2\sqrt{-g'(0)}}, \tag{2.6}$$

such that $\lambda_1(L, l)$ is negative (resp. 0, or positive) when $l > l^*(L)$ (resp. $l = l^*(L)$, or $l < l^*(L)$). Moreover, there exists

$$L^* \in \left(L_*, \frac{\pi}{\sqrt{f'(0)}}\right),$$

such that $l^*(L^*) = L^*$.

Proof. It is direct to see that for any given $L > 0$, $\lambda_1(L, l)$ is decreasing in $l > 0$. We check that if

$$L \geq \frac{\pi}{\sqrt{f'(0)}}, \quad \lambda_1(L, l) < 0$$

for all $l > 0$; if $L \leq L_*$, $\lambda_1(L, l) > 0$ for all $l > 0$; and if

$$L \in (L_*, \frac{\pi}{\sqrt{f'(0)}}), \quad \lambda_1(L, \infty) < 0 < \lambda_1(L, 0).$$

Combined with the monotonicity of $\lambda_1(L, l)$ in l, we obtain the existence and uniqueness of $l^*(L)$. Let us give the calculation of (2.6). When $l = l^*(L)$, it follows that

$$\begin{cases} -\varphi''(x) - f'(0)\varphi = 0, & 0 < x < L, \\ -\varphi''(x) - g'(0)\varphi = 0, & -l^*(L) < x < 0, \\ \varphi(-l^*(L)) = 0 = \varphi(L), \\ \varphi(0-0) = \varphi(0+0), \\ \varphi'(0-0) = \varphi'(0+0). \end{cases} \tag{2.7}$$

Inspired by [15], since

$$\varphi(-l^*(L)) = 0 < \varphi \quad \text{in} \quad (-l^*(L), 0),$$

we can find a constant $\tilde{C}_1 > 0$ such that

$$\varphi(x) = \tilde{C}_1 e^{-\sqrt{-g'(0)}x}(e^{2\sqrt{-g'(0)}(x + l^*(L))} - 1) \quad \text{in} \quad (-l^*(L), 0),$$

which implies that

$$\varphi'(0-0) = \tilde{C}_1 \sqrt{-g'(0)}(1 + e^{2\sqrt{-g'(0)}l^*(L)}) > 0$$

and

$$\frac{\varphi'(0-0)}{\varphi(0-0)} = \sqrt{-g'(0)} \cdot \frac{e^{2\sqrt{-g'(0)}l^*(L)} + 1}{e^{2\sqrt{-g'(0)}l^*(L)} - 1}. \tag{2.8}$$

By the second equation of (2.7), we have $\varphi''(x) < 0$ for $x \in (0, L)$. Combined with

$$\varphi'(0+0) = \varphi'(0-0) > 0 > \varphi'(L),$$

we find a unique $a_* \in (0, L)$ satisfying $\varphi'(a_*) = 0$. Thus, there is a constant $\tilde{C}_2 > 0$ such that

$$\varphi(x) = \tilde{C}_2 \cos[\sqrt{f'(0)}(x - a_*)] \quad \text{for} \quad x \in (0, L].$$

A direct calculation yields that

$$\frac{\varphi'(0+0)}{\varphi(0+0)} = \sqrt{f'(0)} \tan \sqrt{f'(0)} a_*$$

and

$$L - a_* = \frac{\pi}{2\sqrt{f'(0)}}.$$

Combined with (2.8), we obtain that

$$L - \frac{1}{\sqrt{f'(0)}} \arctan\left(\sqrt{-\frac{g'(0)}{f'(0)}} \cdot \frac{e^{2\sqrt{-g'(0)}l^*(L)} + 1}{e^{2\sqrt{-g'(0)}l^*(L)} - 1}\right) = \frac{\pi}{2\sqrt{f'(0)}}. \tag{2.9}$$

Thus, (2.6) follows. Moreover, it is direct to check that $l^*(L)$ is decreasing in

$$L \in (L_*, \frac{\pi}{\sqrt{f'(0)}})$$

and

$$\lim_{L \to L_*} l^*(L) = \infty$$

and

$$\lim_{L \to \frac{\pi}{\sqrt{f'(0)}}} l^*(L) = 0,$$

which implies the existence and uniqueness of L^*. The proof is complete now. □

2.2. A general convergence theorem

Let us consider the following problem

$$\begin{cases} U'' + f(U) = 0, & 0 < x < h_\infty, \\ U'' + g(U) = 0, & k_\infty < x < 0, \\ U(0-0) = U(0+0), \\ U'(0-0) = U'(0+0), \\ U(k_\infty) = 0 = U(h_\infty). \end{cases} \tag{2.10}$$

By a phase plane analysis, as in [15], we have the following result.

Lemma 2.3. *Assume that* **(H)** *holds, then all solutions* U *of* (2.10) *with* $(k_\infty, h_\infty) = \mathbb{R}$ *are* 0 *and* 1.

Now, by similar analysis to that in [5, 9], we can present the following general convergence result.

Theorem 2.4. *Assume that* **(H)** *holds and* (u, k, h) *is a solution of* (P) *with* $u_0 \in \mathscr{X}(h_0)$ *for* $h_0 > 0$. *Then,* u *converges to a solution* U *of* (2.10) *as* $t \to \infty$ *locally uniformly in* (k_∞, h_∞). *When* $(k_\infty, h_\infty) = \mathbb{R}$, U *is one of the following types:* 0, 1*; when* $h_\infty < \infty$ *or* $k_\infty > -\infty$, *then* $U \equiv 0$.

2.3. Vanishing and spreading phenomena

Let us start with the following condition for vanishing.

Lemma 2.5. *Assume that* **(H)** *holds. Let* (u, k, h) *be a solution of* (P) *with* (1.3) *and* $u_0 \in \mathscr{X}(h_0)$ *for* $h_0 > 0$. *If* $h_\infty < \infty$, *we have* $k_\infty > -\infty$ *and*

$$\lim_{t \to \infty} \|u\|_{L^\infty([k(t), h(t)])} = 0.$$

Proof. Thanks to **(H)**, it follows from [5, Lemma 2.8] and the comparison principle that

$$k_\infty > -\infty.$$

This, together with Theorem 2.4, yields that $u \to 0$ locally uniformly in $[k_\infty, h_\infty]$. Let us show that the convergence of u to 0 is uniform in $[k(t), h(t)]$. Set

$$C := 1 + \theta + \|u_0\|_{L^\infty([-h_0, h_0])},$$

then there is $C_1 > 0$ depending on C such that

$$f(u),\ g(u) \le C_1 \quad \text{for} \quad u \in [0, C].$$

Denote

$$w(t, x) := C[2M(x - k(t)) - M^2(x - k(t))^2]$$

for $(t, x) \in D_M$, where

$$D_M := \{(t, x) : t > 0, k(t) \le x \le k(t) + M^{-1}\}$$

with

$$M := \max\left\{h_0^{-1},\ \sqrt{\frac{C_1}{2C}},\ \frac{4\|u_0\|_{C^1([-h_0, h_0])}}{3C}\right\}.$$

It follows from the proof of [1, Lemma 2.2] that $u \le w$ in D_M. For any given $\epsilon > 0$, let

$$\delta := \min\{\frac{\epsilon}{2MC}, \frac{1}{M}\},$$

then there is $T_1 > 0$ such that

$$k_\infty < k(t) < k_\infty + \delta \le k_\infty + M^{-1} \quad \text{for} \ \ t > T_1.$$

Thus, we have that for $t > T_1$ and $x \in [k(t), k_\infty + \delta]$,

$$u(t, x) \le w(t, x) \le w(t, k_\infty + \delta) \le C(2M\delta - M^2\delta^2) < \epsilon.$$

Similarly, we can prove that there exists $T_2 > 0$ such that

$$u(t, x) < \epsilon \ \ \text{for} \ \ t > T_2, \ x \in [h_\infty - \delta, h(t)].$$

Moreover, u converges to 0 uniformly for $x \in [k_\infty + \delta, h_\infty - \delta]$ as $t \to \infty$, and there is $T \ge T_1 + T_2$ such that

$$u(t, x) < \epsilon \quad \text{for} \ \ t > T, \ x \in [k_\infty + \delta, h_\infty + \delta].$$

Let $\epsilon \to 0$, then, by standard theory for parabolic equations, we have that the convergence of u to 0 is uniform in $[k(t), h(t)]$, which ends the proof. □

Next we give the following condition for vanishing.

Lemma 2.6. *Let L^* be given in Lemma 2.2 and (u, k, h) be a solution of (P) with* (1.3) *and $u_0 \in \mathscr{X}(h_0)$ for $h_0 > 0$. If $h_0 < L^*$ and $\|u_0\|_{L^\infty}$ is sufficiently small, then vanishing happens, that is $h_\infty - k_\infty \le 2L^*$ and*

$$\lim_{t\to\infty} \|u(t, \cdot)\|_{L^\infty([k(t),h(t)])} = 0.$$

Proof. For any given $h_* \in (h_0, L^*)$, it follows from Lemma 2.2 that problem (2.5) with $L = l = h_*$, admits a positive principal eigenvalue λ_*, whose corresponding positive eigenfunction φ, can be normalized by $\|\varphi\|_{L^\infty} = 1$. Let x_0 and x_1 be the leftmost and rightmost local maximum point of $\varphi(\cdot)$. Set

$$\delta := \min\Big\{\frac{\lambda_*}{2}, \ \frac{h_*}{h_0} - 1, \ 1\Big\}, \ \ \eta := \max\Big\{-x_0, \ x_1, \ h_0, \ h_* - \frac{\delta}{4}h_0\Big\},$$

then

$$\varepsilon_0 := \min\{\varphi(\eta), \quad \varphi(-\eta)\} \le 1,$$

and there exists $\varepsilon_1 = \varepsilon_1(\delta) > 0$ small such that

$$-2(\mu_1 + \mu_2)\varepsilon_1[\varphi'(h_*) - \varphi'(-h_*)] < \delta^2 h_0$$

and

$$f(s) \le (f'(0) + \delta)s, \ \ g(s) \le (g'(0) + \delta)s, \ \ s \in [0, \varepsilon_1].$$

Define

$$w(t, x) := \varepsilon_0\varepsilon_1 e^{-\delta t}\varphi(x) \ \ \text{for} \ \ (t, x) \in [0, \infty) \times (-h_*, h_*).$$

A direct calculation shows that

$$w_x(t, 0 - 0) = w_x(t, 0 + 0), \ \ w(t, 0 - 0) = w(t, 0 + 0)$$

for $t > 0$, and

$$\begin{cases} w_t - w_{xx} - f(w) \ge (\lambda_* - 2\delta)w \ge 0, & t > 0, \ 0 < x < h_*, \\ w_t - w_{xx} - g(w) \ge (\lambda_* - 2\delta)w \ge 0, & t > 0, \ -h_* < x < 0. \end{cases}$$

If u_0 is chosen to be sufficiently small such that

$$u_0(x) \le \varepsilon_0\varepsilon_1\varphi(x) = w(0, x) \ \ \text{for} \ \ x \in [-h_0, h_0],$$

it follows from the comparison theorem that $u(t, x) \le w(t, x)$ for $(t, x) \in [0, \tau) \times [k(t), h(t)]$, where

$$\tau := \sup\{t > 0 : k(t) > -h_* \ \ \text{and} \ \ h(t) < h_*\}.$$

We claim that $\tau = \infty$. Once this claim is proved, we have

$$[k(t), h(t)] \subset [-h_*, h_*]$$

for all $t > 0$, and so vanishing happens by Lemma 2.5.

Let us prove $\tau = \infty$ by contradiction, and assume that $\tau < \infty$. Without loss of generality we may assume that $h(\tau) = h_*$. We define

$$\xi(t) := h_0\Big(1 + \delta - \frac{\delta}{2}e^{-\delta t}\Big), \quad \bar{u}(t, x) := \varepsilon_1 e^{-\delta t}\varphi(x - \xi(t) + h_*)$$

for $t \ge 0$,

$$x \in I(t) := [\eta + \xi(t) - h_*, \xi(t)].$$

It follows from the choice of η that

$$x - \xi(t) + h_* \ge x_1 \quad \text{and} \quad \eta + \xi(t) - h_* > h_0 \quad \text{for} \ \ t \ge 0, \ x \in I(t).$$

A direct calculation implies that for $t \ge 0, x \in I(t)$,

$$\bar{u}_t - \bar{u}_{xx} - f(\bar{u}) \ge (\lambda_* - 2\delta)\bar{u} - \varepsilon_1 e^{-\delta t}\xi'(t)\varphi'(x - \xi(t) + h_*) \ge 0,$$

where we have used

$$\xi'(t) > 0 \quad \text{and} \quad \varphi'(x - \xi(t) + h_*) \le 0$$

for $t \ge 0$ and $x \in I(t)$. Moreover, we can check that, for $t > 0$,

$$\xi'(t) = \frac{\delta^2 h_0}{2} e^{-\delta t} \ge -\mu_1 \varepsilon_1 e^{-\delta t} \varphi'(h_*) = -\mu \bar{u}_x(t, \xi(t)).$$

Now we prove that $h(t) \le \xi(t)$ for $t \in [0, \tau]$. The conclusion is true when

$$h(t) \le \eta + \xi(t) - h_*.$$

Consider the case where

$$\Psi := \{0 \le t \le \tau : h(t) > \eta + \xi(t) - h_*\} \ne \emptyset$$

consists of some intervals and $[\tau_1, \tau_2]$ is one of them. As

$$\eta + \xi(0) - h_* > h_0,$$

then,

$$\tau_1 > 0 \quad \text{and} \quad h(\tau_i) = \eta + \xi(\tau_i) - h_* \quad \text{for} \quad i = 1,\ 2.$$

It is direct to check that

$$\begin{aligned} u(t, \eta + \xi(t) - h_*) &\le w(t, \eta + \xi(t) - h_*) \\ &\le \varepsilon_0 \varepsilon_1 e^{-\delta t} \\ &\le \bar{u}(t, \eta + \xi(t) - h_*), \quad t \in [\tau_1, \tau_2]. \end{aligned}$$

Hence, $(\bar{u}, \xi)$ is an upper solution in $[\tau_1, \tau_2] \times [\eta + k(t) - h_*, h(t)]$ and by comparison we have $h(t) \le \xi(t)$ for $t \in [\tau_1, \tau_2]$. Thus, we have proved that $h(t) \le \xi(t)$ for $t \in [0, \tau]$, which yields that

$$h(\tau) \le \xi(\tau) < \xi(\infty) \le h_*,$$

contradicting our assumption $h(\tau) = h_*$. This proves $\tau = \infty$, which completes the proof of this lemma. □

Later we show the following condition for spreading.

Lemma 2.7. *Assume that* **(H)** *holds. Let L^* be given in Lemma 2.2 and (u, k, h) be a solution of (P) with (1.3) and $u_0 \in \mathscr{X}(h_0)$ for $h_0 > 0$. If $h_0 \ge L^*$, then spreading happens in the sense that*

$$(k_\infty, h_\infty) = \mathbb{R} \ \text{ and } \ \lim_{t \to \infty} u(t, x) = 1$$

locally uniformly in $\mathbb{R}$.

Proof. As $h_0 \ge L^*$ and $h'(t) > 0 > k'(t)$ for $t > 0$, then

$$[-L^*, L^*] \subset [k(1), h(1)].$$

It follows from Lemma 2.2 that problem (2.5) with $L = h(1)$ and $l = -k(1)$, admits a negative principal eigenvalue λ_1, whose corresponding eigenfunction φ_1, can be chosen positive and normalized by $\|\varphi_1\|_{L^\infty} = 1$. Set

$$\underline{u}(x) = \begin{cases} \rho \varphi_1(x), & x \in [k(1), h(1)], \\ 0, & x \notin [k(1), h(1)], \end{cases}$$

where the constant $\rho > 0$ can be chosen to be small such that

$$f(s) \ge (f'(0) + \lambda_1)s \quad \text{and} \quad g(s) \ge (g'(0) + \lambda_1)s \quad \text{for} \quad s \in [0, \rho].$$

A direct calculation shows that

$$\underline{u}(0-0) = \underline{u}(0+0), \quad \underline{u}_x(0-0) = \underline{u}_x(0+0)$$

and

$$\begin{cases} \underline{u}_t - \underline{u}_{xx} - f(\underline{u}) \le 0, & t > 0,\ 0 < x < h(1), \\ \underline{u}_t - \underline{u}_{xx} - g(\underline{u}) \le 0, & t > 0,\ k(1) < x < 0. \end{cases}$$

Since $u(2, \cdot) > 0$ in $[k(1), h(1)]$, we can choose ρ to be smaller if necessary satisfying

$$u(2, \cdot) > \underline{u}(\cdot) \quad \text{in} \quad [k(1), h(1)].$$

Thus $\underline{u}(x)$ is a subsolution of (P), and by comparison we have

$$u(t+2, x) > \underline{u}(x) \quad \text{for} \quad t > 0,\ x \in (k(1), h(1)).$$

This, together with Lemma 2.5, implies that $h_\infty = \infty$ and $u \not\to 0$. Combined with Theorem 2.4, we have $k_\infty = -\infty$ and $u \to 1$ locally uniformly in $\mathbb{R}$, which means that spreading happens. □

2.4. *The proof of Theorem 1.1*

It is easy to see that there are two possibilities: (i) $h_\infty < \infty$; (ii) $h_\infty = \infty$. In case (i), it follows from Lemma 2.5 that vanishing happens. For case (ii), it follows from Lemma 2.7 and its proof that spreading happens. Thus, we can obtain the spreading-vanishing dichotomy.

In the rest of this proof, let us show the sharp threshold behaviors. Define

$$\sigma^* := \sup\{\sigma_0 : \text{ vanishing happens for } \sigma \in (0, \sigma_0]\}.$$

When $h_0 \geq L^*$, it follows from Lemma 2.7 that $\sigma^* = 0$. When $h_0 < L^*$, by Lemma 2.6, we see that vanishing happens for small $\sigma > 0$. So, $\sigma^* \in (0, +\infty]$. If $\sigma^* = \infty$, vanishing happens for all $\sigma > 0$, which ends the proof. Let us consider the case that $\sigma^* < \infty$. We claim that vanishing happens for $\sigma = \sigma^*$. Otherwise it follows that spreading must happen for $\sigma = \sigma^*$, which yields that there is $t_0 > 0$ such that

$$(k(t_0), h(t_0)) \supset [-L^* - 1, L^* + 1].$$

Due to the continuous dependence of the solution on the initial values, there is $\epsilon > 0$ sufficiently small such that $(u_\epsilon, k_\epsilon, h_\epsilon)$, the solution of (1.1) with $u_0 = (\sigma^* - \epsilon)\phi$, satisfies

$$[k_\epsilon(t_0), h_\epsilon(t_0)] \supset [-L^*, L^*].$$

Combined with Lemma 2.7, we see that spreading happens to $(u_\epsilon, k_\epsilon, h_\epsilon)$, which is a contradiction. Thanks to this, we can use the comparison principle and the spreading-vanishing dichotomy to obtain that spreading happens for $\sigma > \sigma^*$ and vanishing happens for $\sigma \leq \sigma^*$ in this case, which completes the whole proof of Theorem 1.1. □

3. Conclusions

In this paper, we have studied the population dynamics of a single species in a one-dimensional environment which is modeled by the equation $u_t = u_{xx} + F(x, u)$ in the domain

$$\{(t, x) \in \mathbb{R}^2 : t \geq 0,\ x \in (k(t), h(t))\},$$

where $k(t)$ and $h(t)$ are the free boundaries. By choosing the initial data $\sigma\phi$ for some $\phi \in \mathscr{X}(h_0)$, $h_0 > 0$ and $\sigma \geq 0$, we find that there exists a critical value σ^* such that spreading happens when $\sigma > \sigma^*$ and vanishing happens when $\sigma \leq \sigma^*$.

In the current paper, we have assumed that the species live in the domain

$$\{(t, x) \in \mathbb{R}^2 : t \geq 0,\ x \in (k(t), h(t))\}.$$

Nevertheless, the habitat of a biological population, in general, can be rather complicated. For example, natural river systems are often in a spatial network structure such as dendritic trees. The network topology (i.e., the topological structure of a river network) can greatly influence the species persistence and extinction. It would be interesting to consider the population dynamics of a single species in a general river habitat. We plan to study this problem in future work.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest in this paper.

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