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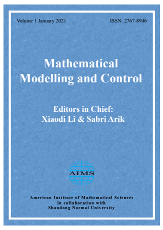

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Research article

Filter design for continuous-discrete Takagi-Sugeno fuzzy system with finite frequency specifications

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Abstract: This paper was concerned with the problem of filter design for the continuous-discrete system in the Takagi-Sugeno (T-S) fuzzy model. In a known finite frequency (FF) domain, an FF H_∞ performance was defined for the nonlinear continuous-discrete system. With the designed filter, sufficient conditions were then established for the filtering error system to be asymptotically stable and having a prescribed FF H_∞ performance. After that, a systematic method for the filter design was proposed. Finally, an example was provided to check effectiveness of the derived results.

Keywords: continuous-discrete systems; Takagi-Sugeno (T-S) fuzzy scheme; filter design; finite frequency (FF)

1. Introduction

In the last few decades, many researchers have investigated two-dimensional (2-D) systems including continuous-continuous, discrete-discrete and continuous-discrete settings, as these systems have great applications in engineering fields such as process control, multi-dimensional digital filtering and image processing; see, for instance, [1] and reference therein. Recently, due to the continuous-discrete systems having an advantage of describing vehicle platoon model [2], linear repetitive processes [3], iterative learning control systems [4] and others, the study of continuous-discrete systems has been a major topic. Now, the main research is about stability problem [5] and feedback control [6].

In practice, many systems have complex characteristics and nonlinearities, which can not be fully described by the linear system models [7–9]. Utilizing the Takagi-Sugeno (T-S) fuzzy model [10] to approximate the nonlinear systems via fuzzy sets and fuzzy reasoning is an efficient approach. It is well known that the T-S fuzzy model has the favourable ability to approximate any nonlinear system [11], which provides sufficient tools and techniques for researchers to design complex nonlinear systems [12]. To this day, some designers have tried to investigate the continuous-discrete nonlinear systems via using fuzzy-model-based control methods [13]. Now, the T-S fuzzy model has been an efficient tool to solve nonlinear problems by adopting the mature linear system theories.

On the other hand, the practical systems are often disturbed by the noise signals; therefore, the filtering issues for nonlinear systems have been studied extensively by the T-S fuzzy model approach [14]. In order to describe systems more accurately, filtering problem is considered to estimate the system state by using the known information under the noisy disturbance. One commonly studied scheme for 2-D systems is H_∞ filtering [15, 16].

Among most of the existing research results on the filtering problem, frequency of the disturbances is usually assumed to occupy the entire frequency domain, which actually brings over-design compared to the finite frequency (FF) case since the useful information of the FF range of disturbances are not fully utilized. In some practical engineering applications, the disturbance signals are intrinsically dominated within FF ranges, which could be known in advance [17]. It is more reasonable and useful to design controllers or filters according to the frequency characteristics of disturbances. The generalized Kalman-Yakubovich-Popov lemma proposed in [18] presented an efficient way in inspecting the FF specifications of the controlled system, which can be directly utilized to derive certain linear matrix inequality (LMI) conditions so as to design appropriate filters. The 2-D generalized Kalman-Yakubovich-Popov lemmas proposed in [19, 20] are useful tools when dealing with the problems of H_∞ control, H_∞ filtering and fault detection observer/filter design for linear 2-D systems in FF domains, which have been studied in [21, 22].

It should be noted that the system properties formalized as frequency domain inequalities are inapplicable to nonlinear systems. Due to this reason, it cannot describe the system representations accurately through the method of frequency-domain analysis. Stimulated by the concept of time domain interpretations of the frequency domain inequalities [23], the authors of [24] dealt with the problem of filtering design for nonlinear systems in the T-S fuzzy model by introducing the FF H_∞ index. Based on the above theories, the filter design method and FF static output feedback H_∞ controller design method have been proposed for Roesser-type 2-D discrete systems in the T-S fuzzy model in [25, 26], respectively. The most striking characteristic of continuous-discrete systems lies in that there are a differential equation and a difference equation with respect to the continuous and discrete variables, respectively [27]. Therefore, the results proposed for discrete and continuous 2-D systems are unable to be used for the addressed systems directly in this paper. To the authors' best knowledge, there has been no results on FF H_∞ filtering for nonlinear continuous-discrete systems. Thus, it is necessary to develop the FF filtering theory specially for continuous-discrete T-S fuzzy systems. Inspired by these works, the authors aim to design the H_∞ filter for continuous-discrete T-S fuzzy systems in this paper.

This paper aims to design a filter for the continuous-discrete system in the T-S fuzzy model with FF disturbances. The main contributions of this work are summarized as follows:

1) Motivated by the time-domain interpretations of the frequency domain inequalities [23], definition of the FF H_∞ performance for continuous-discrete systems is proposed, which contains the frequency information of the disturbance input and generalizes the standard H_∞ performance.

2) Based on the Parseval's theorem for continuous-discrete systems, the FF H_∞ performance analysis results have been obtained, which generalizes the FF bounded real lemma for linear systems [20].

3) Sufficient conditions for the existence of the desired filter and a systematic method for the filter design are developed to ensure the asymptotic stability and FF H_∞ performance of the filtering error system. By setting the matrix and decision variables dependent of the membership function, the proposed results are applicable to linear systems, which means that the prosed results are very general.

The paper proceeds as follows. The preliminaries and problem formulation are presented in Section 2. Main results, including FF H_∞ performance analysis and filter design are given in Section 3. An illustrative example demonstrates the theoretical results potency in Section 4. Finally, some conclusions are summarized in Section 5.

2. Preliminaries and problem formulation

Notations: $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}_n$ stand for the sets of all real numbers, complex numbers and Hermitian $n\times n$ dimensional matrices, respectively. A^{-1}, A^T and A^* denote the inverse, the transpose and the complex conjugate transpose of matrix A, respectively.

$$\mathrm{He}(A) := (A + A^*)/2$$

and $\mathrm{tr}(A)$ stand for the Hermitian and the trace of square matrix A, respectively. The notation $A > 0$ ($A \geq 0$) means that A is positive definite (positive semi-definite). N_A denotes a matrix whose columns form a basis of the null-space of matrix A. The symbol "$\star$" represents the term originated by conjugate symmetry in a matrix. The L_2 norm of a 2-D signal $w(t,k)$ is given by

$$\|w\|_2^2 = \sum_{k=0}^{\infty}\int_{t=0}^{\infty} w^T(t,k)w(t,k)\mathrm{d}t.$$

$w(t,k)$ is said to belong to $L_2\{[0,\infty),\lfloor 0,\infty)\}$ if $\|w\|_2 < \infty$, where $\lfloor 0,\infty) := \{0,1,2,\cdots\}$.

2.1. System description

Consider the continuous-discrete T-S fuzzy system with its i-th rule as follows:

Rule i: **IF** $\theta_1(t,k)$ is $M_{i1},\cdots$, and $\theta_L(t,k)$ is M_{iL}, **THEN**

$$\begin{aligned}\tilde{x}(t,k) &= A_i x(t,k) + B_i w(t,k),\\ y(t,k) &= C_i x(t,k) + D_i w(t,k),\\ z(t,k) &= L_i x(t,k) + E_i w(t,k)\end{aligned} \tag{2.1}$$

with

$$\tilde{x}(t,k) = \begin{bmatrix} \frac{\partial x^h(t,k)}{\partial t} \\ x^v(t,k+1)\end{bmatrix},\quad x(t,k) = \begin{bmatrix} x^h(t,k) \\ x^v(t,k)\end{bmatrix},$$

where $i \in \underline{N} = \{1,2,\cdots,N\}$, $t \in [0,\infty)$ and $k \in \lfloor 0,\infty)$; $x^h(\cdot,\cdot) \in \mathbb{R}^{n_h}$ and $x^v(\cdot,\cdot) \in \mathbb{R}^{n_v}$ are state vectors that vary in the horizontal and the vertical directions, respectively; $x(\cdot,\cdot)$ denotes the whole state in $\mathbb{R}^n$ with $n = n_h + n_v$; $w(\cdot,\cdot) \in \mathbb{R}^{n_w}$, $y(\cdot,\cdot) \in \mathbb{R}^{n_y}$ and $z(\cdot,\cdot) \in \mathbb{R}^{n_z}$ are the disturbance input, the measured output and the signal to be estimated, respectively. $\theta_p(t,k)$, $p = 1,2,..,L$, are the premise variables; M_{ip} is the fuzzy set; L is the number of premise variables; N is the number of **IF-THEN** rules; A_i, B_i, C_i, D_i, L_i and E_i are known matrices of appropriate dimensions. The energy of the disturbance $w(t,k)$ is assumed to be dominated in a known rectangular FF region introduced later.

Via using the inference product, the singleton fuzzifer and the center-average defuzzifer, (2.1) can be described by

$$\begin{aligned}\tilde{x}(t,k) &= A(\mu)x(t,k) + B(\mu)w(t,k),\\ y(t,k) &= C(\mu)x(t,k) + D(\mu)w(t,k),\\ z(t,k) &= L(\mu)x(t,k) + E(\mu)w(t,k)\end{aligned} \tag{2.2}$$

with

$$\begin{bmatrix} A(\mu) & B(\mu) \\ C(\mu) & D(\mu) \\ L(\mu) & E(\mu)\end{bmatrix} = \sum_{i=1}^{N}\mu_i(t,k)\begin{bmatrix} A_i & B_i \\ C_i & D_i \\ L_i & E_i\end{bmatrix},$$

$$\mu_i(t,k) = \frac{\beta_i(t,k)}{\sum_{i=1}^{N}\beta_i(t,k)},\quad \beta_i(t,k) = \prod_{p=1}^{L} M_{ip}(\theta_p(t,k)), \tag{2.3}$$

where $M_{ip}(\theta_p(t,k))$ is the degree of the membership function of $\theta_p(t,k)$ in M_{ip}. Assume that $M_{ip}(\theta_p(t,k)) \geq 0$ for all $i \in \underline{N}$ and $p \in \underline{L} = \{1,2,\cdots,L\}$, then for all $t \geqslant 0$ and $k \in \lfloor 0,\infty)$, the normalized membership function $\mu_i(t,k)$ satisfies

$$\mu_i(t,k) \geq 0 \ \text{ and } \ \sum_{i=1}^{N}\mu_i(t,k) = 1.$$

The boundary conditions (BCs) associated with (2.1) are

$$x^h(t,k) = \begin{cases} h_0, & t=0,\ 0 \leq k \leq z_2, \\ 0, & t=0,\ \forall k > z_2,\end{cases}$$

$$x^v(t,k) = \begin{cases} v_0, & k=0,\ 0 \leq t \leq z_1, \\ 0, & k=0,\ \forall t > z_1,\end{cases}$$

where $h_0 \in \mathbb{R}^{n_h}$ and $v_0 \in \mathbb{R}^{n_v}$ are given vectors, z_1 is a positive scalar and z_2 is a positive integer.

2.2. Fuzzy filter

In order to estimate the signal $z(t,k)$, a fuzzy Roesser-type filter is designed as follows:

Filter rule i: **I**F $\theta_1(t,k)$ is $M_{i1},\cdots$, and $\theta_L(t,k)$ is M_{iL}, **T**HEN

$$\begin{aligned}\tilde{x}_F(t,k) &= A_{F,i}x_F(t,k) + B_{F,i}y(t,k),\\ z_F(t,k) &= C_{F,i}x_F(t,k) + D_{F,i}y(t,k)\end{aligned} \tag{2.4}$$

with $x_F^h(0,k) = 0$, $x_F^v(t,0) = 0$ for $t \geq 0$ and $k \in \lfloor 0,\infty)$,

$$\tilde{x}_F(t,k) = \begin{bmatrix} \frac{\partial x_F^h(t,k)}{\partial t} \\ x_F^v(t,k+1)\end{bmatrix},\quad x_F(t,k) = \begin{bmatrix} x_F^h(t,k) \\ x_F^v(t,k)\end{bmatrix},$$

where $x_F^h(\cdot,\cdot) \in \mathbb{R}^{n_h}$ and $x_F^v(\cdot,\cdot) \in \mathbb{R}^{n_v}$ are the horizontal and the vertical filter states, respectively; $z_F(\cdot,\cdot) \in \mathbb{R}^{n_z}$ is an estimation of $z(\cdot,\cdot)$; $A_{F,i}$, $B_{F,i}$, $C_{F,i}$ and $D_{F,i}$ ($i \in \underline{N}$) are appropriately dimensioned filter matrices to be determined.

The defuzzified output of system (2.4) is obtained as

$$\begin{aligned}\tilde{x}_F(t,k) &= A_F(\mu)x_F(t,k) + B_F(\mu)y(t,k),\\ z_F(t,k) &= C_F(\mu)x_F(t,k) + D_F(\mu)y(t,k),\end{aligned} \tag{2.5}$$

where

$$\begin{bmatrix} A_F(\mu) & B_F(\mu) \\ C_F(\mu) & D_F(\mu)\end{bmatrix} = \sum_{i=1}^{N}\mu_i(t,k)\begin{bmatrix} A_{F,i} & B_{F,i} \\ C_{F,i} & D_{F,i}\end{bmatrix}. \tag{2.6}$$

Defining

$$\hat{z}(t,k) = z(t,k) - z_F(t,k),$$

$$\hat{x}^h(t,k) = \begin{bmatrix} x^h(t,k) \\ x_F^h(t,k) \end{bmatrix}, \quad \hat{x}^v(t,k) = \begin{bmatrix} x^v(t,k) \\ x_F^v(t,k) \end{bmatrix}$$

and augmenting the filter states with system (2.2), the following filtering error system (FES) can then be obtained

$$\begin{aligned} \tilde{\hat{x}}(t,k) &= \hat{A}(\mu)\hat{x}(t,k) + \hat{B}(\mu)w(t,k), \\ \hat{z}(t,k) &= \hat{C}(\mu)\hat{x}(t,k) + \hat{D}(\mu)w(t,k), \end{aligned} \tag{2.7}$$

where

$$\hat{D}(\mu) = E(\mu) - D_F(\mu)D(\mu),$$

$$\tilde{\hat{x}} = \begin{bmatrix} \frac{\partial \hat{x}^h(t,k)}{\partial t} \\ \hat{x}^v(t,k+1) \end{bmatrix}, \quad \hat{x}(t,k) = \begin{bmatrix} \hat{x}^h(t,k) \\ \hat{x}^v(t,k) \end{bmatrix},$$

$$\hat{A}(\mu) = \Pi \begin{bmatrix} A(\mu) & 0 \\ B_F(\mu)C(\mu) & A_F(\mu) \end{bmatrix} \Pi^T,$$

$$\hat{C}(\mu) = \begin{bmatrix} L(\mu) - D_F(\mu)C(\mu) & -C_F(\mu) \end{bmatrix} \Pi^T,$$

$$\hat{B}(\mu) = \Pi \begin{bmatrix} B(\mu) \\ B_F(\mu)D(\mu) \end{bmatrix},$$

$$\Pi = \begin{bmatrix} I_{n_h} & 0 & 0 & 0 \\ 0 & 0 & I_{n_h} & 0 \\ 0 & I_{n_v} & 0 & 0 \\ 0 & 0 & 0 & I_{n_v} \end{bmatrix}.$$

2.3. Problem statement

In practice, effects of the noises are usually dominated in FF ranges. Based on the practical situation, denoting the Fourier transform of $w(t,k)$ as $W(\mathrm{j}w_h, e^{\mathrm{j}w_v})$ with j being the imaginary unit, a rectangular FF domain R for the noise signal $w(t,k)$ is given as follows:

$$R = \{(w_h, w_v) : w_{h1} \le w_h \le w_{h2}, w_{v1} \le w_v \le w_{v2}\}, \tag{2.8}$$

where $w_{h1},\ w_{h2} \in \mathbb{R}$ and $w_{v1},\ w_{v2} \in (-\pi, \pi]$ are the lower and upper bounds of frequency variables w_h and w_v respectively.

Definition 2.1. *For the FF domain R and a given positive scalar γ, the FES (2.7) is said to have an FF H_∞ index γ if the following two conditions are satisfied:*

(i) When $w(t,k) \equiv 0$, the FES (2.7) is AS;
(ii) Under the zero BCs, the inequality

$$\|\hat{z}\|_2^2 < \gamma^2 \|w\|_2^2 \tag{2.9}$$

holds for all solutions of (2.7) with

$$w(\cdot,\cdot) \in L_2\{[0,\infty), \lfloor 0,\infty)\}\backslash\{0\}$$

such that

$$\sum_{k=0}^{\infty} \int_0^{\infty} (w_{h1}\hat{x}^h(t,k) + \mathrm{j}\frac{\partial \hat{x}^h(t,k)}{\partial t}) \times (w_{h2}\hat{x}^h(t,k) + \mathrm{j}\frac{\partial \hat{x}^h(t,k)}{\partial t})^* \mathrm{d}t < 0, \tag{2.10}$$

$$e^{\mathrm{j}(w_v^a)} \int_0^{\infty} \Big[\sum_{k=0}^{\infty} (\hat{x}^v(t,k+1) - e^{\mathrm{j}w_{v1}}\hat{x}^v(t,k)) \times (\hat{x}^v(t,k+1) - e^{\mathrm{j}w_{v2}}\hat{x}^v(t,k))^*\Big] \mathrm{d}t < 0, \tag{2.11}$$

where

$$w_v^a = (w_{v2} - w_{v1})/2.$$

The FF filtering problem to be addressed in this paper is to design a fuzzy filter in the form of (2.4), such that the FES (2.7) is AS and has an FF H_∞ performance level γ. To get the main results, we still need the following lemma.

Lemma 2.1. ([28]) If the 2-D complex vector functions $x(t,k)$ and $y(t,k)$ belong to $L_2\{[0,\infty), \lfloor 0,\infty)\}$, then the corresponding Laplace-Z transforms $X(s,z)$ and $Y(s,z)$ exist and the following relation holds, provided that $x(t,k)$ and $y(t,k)$ have the same dimensions

$$\begin{aligned} &\sum_{k=0}^{\infty} \int_0^{\infty} \mathrm{tr}(x(t,k)y^*(t,k))\mathrm{d}t \\ &= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} \mathrm{tr}(X(\mathrm{j}w_h, e^{\mathrm{j}w_v})Y^*(\mathrm{j}w_h, e^{\mathrm{j}w_v}))\mathrm{d}w_h \mathrm{d}w_v. \end{aligned}$$

3. Main results

3.1. FF performance analysis

In the following, one sufficient condition is presented for the FES (2.7) having an FF H_∞ performance.

Theorem 3.1. *Assume that the FES (2.7) is asymptotically stable (AS). For the FF domain R and the scalar $\gamma > 0$, the FES (2.7) has an FF H_∞ performance γ if there exist symmetric matrices*

$$\hat{P} = \mathrm{diag}\{\hat{P}^h, \hat{P}^v\} \in \mathbb{R}^{2n\times 2n}$$

and

$$\hat{Q} = \mathrm{diag}\{\hat{Q}^h, \hat{Q}^v\} \in \mathbb{R}^{2n\times 2n},$$

such that $\hat{P}^h > 0$, $\hat{Q}^v > 0$ *and*

$$\begin{bmatrix} \hat{A}(\mu) & \hat{B}(\mu) \\ I & 0 \end{bmatrix}^T \begin{bmatrix} -\hat{P} & -\hat{\Lambda}^*\hat{P}\hat{W} + \hat{\Lambda}^*\hat{Q} \\ -\hat{W}^*\hat{P}\hat{\Lambda} + \hat{Q}\hat{\Lambda} & \hat{L}_h\hat{P} - \hat{W}^*\hat{P}\hat{W} - \hat{L}_v\hat{Q} \end{bmatrix} \times \begin{bmatrix} \hat{A}(\mu) & \hat{B}(\mu) \\ I & 0 \end{bmatrix} + \begin{bmatrix} \hat{C}^T(\mu)\hat{C}(\mu) & \hat{C}^T(\mu)\hat{D}(\mu) \\ \hat{D}^T(\mu)\hat{C}(\mu) & -\gamma^2 I + \hat{D}^T(\mu)\hat{D}(\mu) \end{bmatrix} < 0 \tag{3.1}$$

with

$$\hat{\Lambda} = \mathrm{diag}\{I_{2n_h}, e^{-\mathrm{j}(w_v^c)}I_{2n_v}\},\ \ \hat{W} = \mathrm{diag}\{-\mathrm{j}(w_h^c)I_{2n_h}, 0_{2n_v}\},$$

$$\hat{L}_v = \mathrm{diag}\{0, 2\cos(w_v^a)I_{2n_v}\},\ \ \hat{L}_h = \mathrm{diag}\{(w_h^a)^2 I_{2n_h}, I_{2n_v}\},$$

$w_h^c = (w_{h1} + w_{h2})/2$, $w_h^a = (w_{h2} - w_{h1})/2$, $w_v^c = (w_{v1} + w_{v2})/2$.

Proof. Multiplying the inequality (3.1) by $[\hat{x}^T(t,k)\ \ w^T(t,k)]$ from the left and by its transpose from the right with

$$w(\cdot,\cdot) \in L_2\{[0,\infty), [0,\infty)\}\backslash\{0\},$$

we have

$$\begin{aligned}
&\begin{bmatrix} \tilde{\hat{x}}(t,k) \\ \hat{x}(t,k) \end{bmatrix}^T \begin{bmatrix} -\hat{P} & -\hat{\Lambda}^*\hat{P}\hat{W} + \hat{\Lambda}^*\hat{Q} \\ -\hat{W}^*\hat{P}\hat{\Lambda} + \hat{Q}\hat{\Lambda} & \hat{L}_h\hat{P} - \hat{W}^*\hat{P}\hat{W} - \hat{L}_v\hat{Q} \end{bmatrix} \times \begin{bmatrix} \tilde{\hat{x}}(t,k) \\ \hat{x}(t,k) \end{bmatrix} + \Phi(t,k) \\
&= \Big(\frac{\partial \hat{x}^h(t,k)}{\partial t}\Big)^T \hat{Q}^h \hat{x}^h(t,k) + (\hat{x}^h(t,k))^T \hat{Q}^h \frac{\partial \hat{x}^h(t,k)}{\partial t} \\
&\quad - (\hat{x}^v(t,k+1))^T \hat{P}^v \hat{x}^v(t,k+1) + (\hat{x}^v(t,k))^T \hat{P}^v \hat{x}^v(t,k) \\
&\quad + \Phi(t,k) - \mathrm{tr}(\hat{Q}^v[-e^{-\mathrm{j}(w_v^c)}\hat{x}^v(t,k+1)(\hat{x}^v(t,k))^T \\
&\quad - e^{\mathrm{j}(w_v^c)}\hat{x}^v(t,k)(\hat{x}^v(t,k+1))^T + 2\hat{x}^v(t,k)\cos(w_v^a) \\
&\quad \times (\hat{x}^v(t,k))^T]) - \mathrm{tr}(\hat{P}^h \mathrm{He}((w_{h1}\hat{x}^h(t,k) + \mathrm{j}\frac{\partial \hat{x}^h(t,k)}{\partial t}) \\
&\quad \times (w_{h2}\hat{x}^h(t,k) + \mathrm{j}\frac{\partial \hat{x}^h(t,k)}{\partial t})^*)) \\
&< 0
\end{aligned} \tag{3.2}$$

with

$$\Phi(t,k) = \begin{bmatrix} \hat{x}(t,k) \\ w(t,k) \end{bmatrix}^T \begin{bmatrix} \hat{C}^T(\mu)\hat{C}(\mu) & \hat{C}^T(\mu)\hat{D}(\mu) \\ \hat{D}^T(\mu)\hat{C}(\mu) & -\gamma^2 I + \hat{D}^T(\mu)\hat{D}(\mu) \end{bmatrix} \times \begin{bmatrix} \hat{x}(t,k) \\ w(t,k) \end{bmatrix}.$$

Taking integration for t from zero to ∞ and summation for integer k from zero to ∞, in view of the zero BCs, we have

$$\sum_{k=0}^{\infty}\int_0^{\infty} \Phi(t,k)\mathrm{d}t - \mathrm{tr}(\mathrm{He}(\hat{P}^h S^h)) - \mathrm{tr}(\hat{Q}^v S^v) < 0, \tag{3.3}$$

where

$$\begin{aligned}
S^h :=& \sum_{k=0}^{\infty}\int_0^{\infty} (w_{h1}\hat{x}^h(t,k) + \mathrm{j}\frac{\partial \hat{x}^h(t,k)}{\partial t}) \\
&\times (w_{h2}\hat{x}^h(t,k) + \mathrm{j}\frac{\partial \hat{x}^h(t,k)}{\partial t})^*\mathrm{d}t, \\
S^v :=& \int_0^{\infty}\sum_{k=0}^{\infty}\Big\{ - e^{-\mathrm{j}(w_v^c)}\hat{x}^v(t,k+1)(\hat{x}^v(t,k))^T - e^{\mathrm{j}(w_v^c)}\hat{x}^v(t,k) \\
&\times (\hat{x}^v(t,k+1))^T + \hat{x}^v(t,k)2\cos(w_v^a)(\hat{x}^v(t,k))^T\Big\}\mathrm{d}t.
\end{aligned}$$

Clearly, S^v is Hermitian, and it follows from condition (2.10) that $S^h < 0$. Due to the zero BCs, it is true that

$$\begin{aligned}
S^v =& e^{\mathrm{j}(w_v^a)}\int_0^{\infty}\sum_{k=0}^{\infty}\Big[(\hat{x}^v(t,k+1) - e^{\mathrm{j}w_{v1}}\hat{x}^v(t,k)) \\
&\times (\hat{x}^v(t,k+1) - e^{\mathrm{j}w_{v2}}\hat{x}^v(t,k))^*\Big]\mathrm{d}t,
\end{aligned}$$

and combining with constraint (2.11) infers that $S^v < 0$. Applying Lemma 2.1 to S^h, we have

$$\begin{aligned}
S^h =& \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\infty}^{\infty}(w_{h1} - w_h)(w_{h2} - w_h)\hat{X}^h(\mathrm{j}w_h, e^{\mathrm{j}w_v}) \\
&\times (\hat{X}^h(\mathrm{j}w_h, e^{\mathrm{j}w_v}))^*\mathrm{d}w_h\mathrm{d}w_v,
\end{aligned}$$

which directly guarantees that S^h is also Hermitian, where $\hat{X}^h(\mathrm{j}w_h, e^{\mathrm{j}w_v})$ denotes the 2-D Laplace-Z transform of $\hat{x}^h(t,k)$. Additionally, since $\hat{P}^h > 0$ and $\hat{Q}^v > 0$, $-\mathrm{tr}(\mathrm{He}(\hat{P}^h S^h)) - \mathrm{tr}(\hat{Q}^v S^v)$ is also positive, then it follows from (3.3) that

$$\sum_{k=0}^{\infty}\int_0^{\infty}\Phi(t,k)\mathrm{d}t < 0,$$

which means that FES (2.7) has an FF H_∞ performance γ according to Definition 2.1. This completes the proof. □

To further achieve the main result, the following lemma is needed to guarantee the asymptotic stability of FES (2.7).

Lemma 3.1. *[28] The FES (2.7) with zero input is AS if positive definite matrices* $\bar{P}^h \in \mathbb{R}^{2n_h \times 2n_h}$ *and* $\bar{P}^v \in \mathbb{R}^{2n_v \times 2n_v}$ *exist such that*

$$\begin{bmatrix} \hat{A}(\mu) \\ I \end{bmatrix}^T \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} \\ \bar{Q}_{12} & \bar{Q}_{22} \end{bmatrix} \begin{bmatrix} \hat{A}(\mu) \\ I \end{bmatrix} < 0 \tag{3.4}$$

holds, where

$$\bar{Q}_{11} = \text{diag}\{0, \bar{P}^v\},\quad \bar{Q}_{12} = \text{diag}\{\bar{P}^h, 0\}$$

and

$$\bar{Q}_{22} = \text{diag}\{0, -\bar{P}^v\}.$$

Lemma 3.1 gives a sufficient condition ensuring the considered system to be AS. To facilitate the filter design, it is necessary to decouple the product terms between the matrix variables $\hat{P}$, $\hat{Q}$ and the system matrices $\hat{A}(\mu)$, $\hat{B}(\mu)$ in (3.1). Thus, we need to present an alternative of Theorem 3.1.

Theorem 3.2. *For the FF domain R and the scalar $\gamma > 0$, the FES (2.7) has an FF H_∞ performance γ if there exist matrices $\hat{G}(\mu)$, $\hat{F}(\mu)$, $\bar{G}(\mu)$, $\bar{F}(\mu) \in \mathbb{R}^{2n\times 2n}$ and $\hat{H}(\mu) \in \mathbb{R}^{n_w\times 2n}$, symmetric matrices*

$$\hat{P} = \text{diag}\{\hat{P}^h, \hat{P}^v\} \in \mathbb{R}^{2n\times 2n},\quad \bar{P}^h \in \mathbb{R}^{2n_h\times 2n_h},\quad \bar{P}^v \in \mathbb{R}^{2n_v\times 2n_v}$$

and

$$\hat{Q} = \text{diag}\{\hat{Q}^h, \hat{Q}^v\} \in \mathbb{R}^{2n\times 2n},$$

such that $\hat{P}^h > 0$, $\hat{Q}^v > 0$, $\bar{P}^h > 0$, $\bar{P}^v > 0$ and

$$\begin{bmatrix} \Phi_{11}(\mu) & \Phi_{12}(\mu) & \Phi_{13}(\mu) & 0 \\ \star & \Phi_{22}(\mu) & \Phi_{23}(\mu) & \hat{C}^T(\mu) \\ \star & \star & \Phi_{33}(\mu) & \hat{D}^T(\mu) \\ \star & \star & \star & -I_{n_z} \end{bmatrix} < 0, \tag{3.5}$$

$$\begin{bmatrix} \bar{Q}_{11} - 2He(\bar{G}^T(\mu)) & \bar{Q}_{12} + \bar{G}(\mu)\hat{A}(\mu) - \bar{F}^T(\mu) \\ \star & \bar{Q}_{22} + 2\text{He}(\hat{A}^T(\mu)\bar{F}^T(\mu)) \end{bmatrix} < 0, \tag{3.6}$$

where

$$\Phi_{11}(\mu) = -\hat{P} - \hat{G}(\mu) - \hat{G}^T(\mu),$$
$$\Phi_{12}(\mu) = -\hat{\Lambda}^*\hat{P}\hat{W} + \hat{\Lambda}^*\hat{Q} + \hat{G}(\mu)\hat{A}(\mu) - \hat{F}^T(\mu),$$
$$\Phi_{13}(\mu) = \hat{G}(\mu)\hat{B}(\mu) - \hat{H}^T(\mu),$$
$$\Phi_{22}(\mu) = \hat{L}_h\hat{P} - \hat{L}_v\hat{Q} - \hat{W}^*\hat{P}\hat{W} + 2\text{He}(\hat{F}(\mu)\hat{A}(\mu)),$$
$$\Phi_{23}(\mu) = \hat{F}(\mu)\hat{B}(\mu) + \hat{A}^T(\mu)\hat{H}^T(\mu)$$
$$\Phi_{33}(\mu) = -\gamma^2 I + 2\text{He}(\hat{H}(\mu)\hat{B}(\mu)),$$

and the other notations are the same as defined in Theorem 3.1 and Lemma 3.1.

Proof. Define

$$\Gamma(\mu) = \begin{bmatrix} -\hat{P} & -\hat{\Lambda}^*\hat{P}\hat{W} + \hat{\Lambda}^*\hat{Q} & 0 \\ -\hat{W}^*\hat{P}\hat{\Lambda} + \hat{Q}\hat{\Lambda} & \Gamma_{22}(\mu) & \hat{C}^T(\mu)\hat{D}(\mu) \\ 0 & \hat{D}^T(\mu)\hat{C}(\mu) & \Gamma_{33}(\mu) \end{bmatrix},$$
$$U(\mu) = \begin{bmatrix} -I_{2n} & \hat{A}(\mu) & \hat{B}(\mu) \end{bmatrix},$$
$$Y(\mu) = \begin{bmatrix} \hat{G}^T(\mu) & \hat{F}^T(\mu) & \hat{H}^T(\mu) \end{bmatrix}^T,$$

where

$$\Gamma_{22}(\mu) = \hat{L}_h\hat{P} - \hat{L}_v\hat{Q} - \hat{W}^*\hat{P}\hat{W} + \hat{C}^T(\mu)\hat{C}(\mu)$$

and

$$\Gamma_{33}(\mu) = -\gamma^2 I + \hat{D}^T(\mu)\hat{D}(\mu).$$

Then, we have

$$\begin{aligned}\Psi(\mu) &:= Y(\mu)U(\mu) + U^T(\mu)Y^T(\mu) + \Gamma(\mu) \\ &= \begin{bmatrix} \Psi_{11}(\mu) & \Psi_{12}(\mu) & \Psi_{13}(\mu) \\ \star & \Psi_{22}(\mu) & \Psi_{23}(\mu) \\ \star & \star & \Psi_{33}(\mu) \end{bmatrix}\end{aligned} \tag{3.7}$$

with

$$\Psi_{11}(\mu) = -\hat{P} - \hat{G}(\mu) - \hat{G}^T(\mu),$$
$$\Psi_{12}(\mu) = -\hat{\Lambda}^*\hat{P}\hat{W} + \hat{\Lambda}^*\hat{Q} + \hat{G}(\mu)\hat{A}(\mu) - \hat{F}^T(\mu),$$
$$\Psi_{22}(\mu) = \hat{L}_h\hat{P} - \hat{L}_v\hat{Q} - \hat{W}^*\hat{P}\hat{W} + \hat{C}^T(\mu)\hat{C}(\mu) + 2\text{He}(\hat{F}(\mu)\hat{A}(\mu)),$$
$$\Psi_{13}(\mu) = \hat{G}(\mu)\hat{B}(\mu) - \hat{H}^T(\mu),$$
$$\Psi_{23}(\mu) = \hat{C}^T(\mu)\hat{D}(\mu) + \hat{F}(\mu)\hat{B}(\mu) + \hat{A}^T(\mu)\hat{H}^T(\mu)$$
$$\Psi_{33}(\mu) = -\gamma^2 I + \hat{D}^T(\mu)\hat{D}(\mu) + 2\text{He}(\hat{H}(\mu)\hat{B}(\mu)).$$

By Schur complement lemma, $\Psi(\mu) < 0$ is equivalent to inequality (3.5), then by taking

$$N_{U(\mu)} = \begin{bmatrix} \hat{A}(\mu) & \hat{B}(\mu) \\ I_{2n} & 0 \\ 0 & I_{n_w} \end{bmatrix},$$

pre- and post-multiplying both sides of $\Psi(\mu) < 0$ by $N^T_{U(\mu)}$ and $N_{U(\mu)}$, respectively, and noting that $N_{U(\mu)}$ is column full rank, it is known that inequality (3.1) does hold.

In addition, we set

$$\bar{\Gamma}(\mu) = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} \\ \bar{Q}_{12} & \bar{Q}_{22} \end{bmatrix},\quad \bar{Y}(\mu) = \begin{bmatrix} \bar{G}(\mu) \\ \bar{F}(\mu) \end{bmatrix},$$
$$\bar{U}(\mu) = \begin{bmatrix} -I_{2n} & \hat{A}(\mu) \end{bmatrix},$$

where $\bar{Q}_{11}$, $\bar{Q}_{12}$ and $\bar{Q}_{22}$ are defined in Lemma 3.1. The constraint condition (3.6) is exactly

$$\bar{\Psi}(\mu) := \bar{Y}(\mu)\bar{U}(\mu) + \bar{U}^T(\mu)\bar{Y}^T(\mu) + \bar{\Gamma}(\mu) < 0.$$

Similarly, taking

$$\bar{N}_{U(\mu)} = [\hat{A}^T(\mu)I_{2n}]^T,$$

pre- and post-multiplying both sides of $\bar{\Psi}(\mu) < 0$ by $N^T_{\bar{U}(\mu)}$ and $N_{\bar{U}(\mu)}$, respectively, and noting that $N_{\bar{U}(\mu)}$ is column full rank, it follows that

$$\begin{bmatrix} \hat{A}(\mu) \\ I_{2n} \end{bmatrix}^T \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} \\ \bar{Q}_{12} & \bar{Q}_{22} \end{bmatrix} \begin{bmatrix} \hat{A}(\mu) \\ I_{2n} \end{bmatrix} < 0.$$

From Lemma 3.1, we know that the FES (2.7) with $w(\cdot,\cdot) \equiv 0$ is AS. The proof is completed by resorting to Theorem 3.1 and Definition 2.1. □

Remark 3.1. *The slack matrices $\hat{G}(\mu)$, $\hat{F}(\mu)$, $\bar{G}(\mu)$, $\bar{F}(\mu)$ and $\hat{H}(\mu)$ are introduced to decouple the products between the Lyapunov matrices $\hat{P}$, $\bar{P}^h$, $\bar{P}^v$, $\hat{Q}$ and the system matrices, which aids in the filter design while also bringing certain conservatism.*

3.2. *Fuzzy filter design*

The following result can be derived via specifying the structure of the slack matrices in Theorem 3.2 and coverting the filter design problem to a group of LMI constraints, which are numerically tractable.

Theorem 3.3. *For the FF domain R and the scalar $\gamma > 0$, the FES (2.7) has an FF H_∞ performance γ if there exist matrices*

$$V_{AF,i} \in \mathbb{R}^{n\times n},\ \ V_{BF,i} \in \mathbb{R}^{n\times n_y},\ C_{F,i} \in \mathbb{R}^{n_z\times n},\ D_{F,i} \in \mathbb{R}^{n_z\times n_y},$$
$$H_{1,i} \in \mathbb{R}^{n_w\times n},\ \ P_2 = \mathrm{diag}\{P^h_2, P^v_2\} \in \mathbb{R}^{n\times n},$$
$$Q_2 = \mathrm{diag}\{Q^h_2, Q^v_2\} \in \mathbb{R}^{n\times n},$$
$$G_{l,i}, F_{l,i}, \bar{G}_{l,i}, \bar{F}_{l,i} \in \mathbb{R}^{n\times n},\ \ (l = 1, 2), V \in \mathbb{R}^{n\times n}$$

and symmetric matrices

$$P_f = \mathrm{diag}\{P^h_f, P^v_f\} \in \mathbb{R}^{n\times n},$$
$$Q_f = \mathrm{diag}\{Q^h_f, Q^v_f\} \in \mathbb{R}^{n\times n},\ \ (f = 1, 3),$$
$$P^h_s \in \mathbb{R}^{n_h\times n_h},\ \ \bar{P}^v_s \in \mathbb{R}^{n_v\times n_v},\ \ (s = 1, 2, 3),$$

such that

$$\begin{bmatrix} \mathrm{diag}\{\bar{P}^h_1, \bar{P}^v_1\} & \mathrm{diag}\{\bar{P}^h_2, \bar{P}^v_2\} \\ \star & \mathrm{diag}\{\bar{P}^h_3, \bar{P}^v_3\} \end{bmatrix} > 0, \tag{3.8}$$

$$\begin{bmatrix} P^h_1 & P^h_2 \\ \star & P^h_3 \end{bmatrix} > 0,\quad \begin{bmatrix} Q^v_1 & Q^v_2 \\ \star & Q^v_3 \end{bmatrix} > 0, \tag{3.9}$$

$$\begin{bmatrix} T_{11,ii} & T_{12,ij} & T_{13,ij} & 0 \\ \star & T_{22,ij} & T_{23,ij} & \tilde{C}^T_{i,j} \\ \star & \star & T_{33,ij} & \hat{D}^T_{i,j} \\ \star & \star & \star & -I_{n_z} \end{bmatrix} < 0, \tag{3.10}$$

$$\begin{bmatrix} L_{11,ii} & L_{12,ij} \\ \star & L_{22,ij} \end{bmatrix} < 0 \tag{3.11}$$

hold for all $i, j \in \underline{N}$, where

$$T_{11,ii} = \begin{bmatrix} -P_1 - G_{1,i} - G^T_{1,i} & -P_2 - V - G^T_{2,i} \\ \star & -P_3 - V - V^T \end{bmatrix},$$

$$T_{13,ij} = \begin{bmatrix} G_{1,i}B_j + V_{BF,i}D_j - H^T_{1,i} \\ G_{2,i}B_j + V_{BF,i}D_j \end{bmatrix},$$

$$T_{12,ij} = \begin{bmatrix} -\Lambda^* P_1 W + \Lambda^* Q_1 + G_{1,i}A_j + V_{BF,i}C_j - F^T_{1,i} & -\Lambda^* P_2 W + \Lambda^* Q_2 + V_{AF,i} - F^T_{2,i} \\ -\Lambda^* P^T_2 W + \Lambda^* Q^T_2 + G_{2,i}A_j + V_{BF,i}C_j & -\Lambda^* P_3 W + \Lambda^* Q_3 + V_{AF,i} \end{bmatrix},$$

$$T_{22,ij} = \begin{bmatrix} L_h P_1 - L_v Q_1 - W^* P_1 W + 2\mathrm{He}(F_{1,i}A_j) & L_h P_2 - L_v Q_2 - W^* P_2 W + A^T_j F^T_{2,i} \\ \star & L_h P_3 - W^* P_3 W - L_v Q_3 \end{bmatrix},$$

$$T_{23,ij} = \begin{bmatrix} F_{1,i}B_j + A^T_j H^T_{1,i} \\ F_{2,i}B_j \end{bmatrix},$$

$$T_{33,ij} = -\gamma^2 I_{n_w} + 2\mathrm{He}(H_{1,i}B_j),$$

$$\tilde{C}_{ij} = \begin{bmatrix} L_j - D_{F,i}C_j & -C_{F,i} \end{bmatrix},$$

$$\hat{D}_{ij} = E_j - D_{F,i}D_j,$$

$$L_{11,ii} = \begin{bmatrix} \mathrm{diag}\{0,\ \bar{P}^v_1\} - \bar{G}_{1,i} - \bar{G}^T_{1,i} & \mathrm{diag}\{0,\ \bar{P}^v_2\} - V - \bar{G}^T_{2,i} \\ \star & \mathrm{diag}\{0,\ \bar{P}^v_3\} - V - V^T \end{bmatrix},$$

$$L_{12,ij} = \begin{bmatrix} \mathrm{diag}\{\bar{P}^h_1, 0\} + \bar{G}_{1,i}A_j + V_{BF,i}C_j - \bar{F}^T_{1,i} & \mathrm{diag}\{\bar{P}^h_2, 0\} + V_{AF,i} - \bar{F}^T_{2,i} \\ \mathrm{diag}\{\bar{P}^h_2, 0\} + \bar{G}_{2,i}A_j + V_{BF,i}C_j - V^T & \mathrm{diag}\{\bar{P}^h_3, 0\} + V_{AF,i} - V^T \end{bmatrix},$$

$$L_{22,ij} = \begin{bmatrix} -\mathrm{diag}\{0,\ \bar{P}_1^v\} + 2\mathrm{He}(\bar{F}_{1,i}A_j + V_{BF,i}C_j) & -\mathrm{diag}\{0,\ \bar{P}_2^v\} + V_{AF,i} + A_j^T\bar{F}_{2,i}^T + C_j^T V_{BF,i}^T \\ \star & -\mathrm{diag}\{0,\ \bar{P}_3^v\} + 2\mathrm{He}(V_{AF,i}) \end{bmatrix}.$$

Moreover, the coefficients of filter (2.4) are designed as

$$A_{F,i} = V^{-1}V_{AF,i}, \quad B_{F,i} = V^{-1}V_{BF,i}. \tag{3.12}$$

Proof. Parameterize the slack matrices $\hat{G}(\mu)$, $\hat{F}(\mu)$, $\bar{G}(\mu)$, $\bar{F}(\mu)$ and $\hat{H}(\mu)$ as

$$\hat{G}(\mu) = \Pi\begin{bmatrix} G_1(\mu) & V \\ G_2(\mu) & V \end{bmatrix}\Pi^T,\ \hat{F}(\mu) = \Pi\begin{bmatrix} F_1(\mu) & 0 \\ F_2(\mu) & 0 \end{bmatrix}\Pi^T,$$
$$\bar{G}(\mu) = \Pi\begin{bmatrix} \bar{G}_1(\mu) & V \\ \bar{G}_2(\mu) & V \end{bmatrix}\Pi^T,\ \bar{F}(\mu) = \Pi\begin{bmatrix} \bar{F}_1(\mu) & V \\ \bar{F}_2(\mu) & V \end{bmatrix}\Pi^T,$$
$$\hat{H}(\mu) = \begin{bmatrix} H_1(\mu) & 0 \end{bmatrix}\Pi^T \tag{3.13}$$

with Π given in (2.7). Define matrix variables $\hat{P}$, $\bar{P}^h$, $\bar{P}^v$ and $\hat{Q}$ as follows:

$$\hat{P} = \Pi\begin{bmatrix} P_1 & P_2 \\ \star & P_3 \end{bmatrix}\Pi^T, \quad \hat{Q} = \Pi\begin{bmatrix} Q_1 & Q_2 \\ \star & Q_3 \end{bmatrix}\Pi^T,$$
$$\bar{Q}_{12} = \Pi\begin{bmatrix} \mathrm{diag}\{\bar{P}_1^h, 0\} & \mathrm{diag}\{\bar{P}_2^h, 0\} \\ \star & \mathrm{diag}\{\bar{P}_3^h, 0\} \end{bmatrix}\Pi^T,$$
$$\bar{Q}_{11} = -\bar{Q}_{22} = \Pi\begin{bmatrix} \mathrm{diag}\{0, \bar{P}_1^v\} & \mathrm{diag}\{0, \bar{P}_2^v\} \\ \star & \mathrm{diag}\{0, \bar{P}_3^v\} \end{bmatrix}\Pi^T, \tag{3.14}$$

i.e.,

$$\bar{P}^v = \begin{bmatrix} \bar{P}_1^v & \bar{P}_2^v \\ \star & \bar{P}_3^v \end{bmatrix}, \quad \bar{P}^h = \begin{bmatrix} \bar{P}_1^h & \bar{P}_2^h \\ \star & \bar{P}_3^h \end{bmatrix}.$$

In addition, we can obtain the following equalities

$$\hat{\Lambda} = \Pi\begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda \end{bmatrix}\Pi^T, \quad \hat{L}_v = \Pi\begin{bmatrix} L_v & 0 \\ 0 & L_v \end{bmatrix}\Pi^T,$$
$$\hat{L}_h = \Pi\begin{bmatrix} L_h & 0 \\ 0 & L_h \end{bmatrix}\Pi^T, \quad \hat{W} = \Pi\begin{bmatrix} W & 0 \\ 0 & W \end{bmatrix}\Pi^T. \tag{3.15}$$

Due to $\Pi^T\Pi = I$, taking

$$V_{AF}(\mu) = VA_F(\mu), \quad V_{BF}(\mu) = VB_F(\mu)$$

and putting (3.8), (3.9), (3.13)–(3.15) into (3.5) and (3.6), respectively, we have

$$\begin{bmatrix} \Phi_{11}(\mu) & \Phi_{12}(\mu) & \Phi_{13}(\mu) & 0 \\ \star & \Phi_{22}(\mu) & \Phi_{23}(\mu) & \hat{C}^T(\mu) \\ \star & \star & \Phi_{33}(\mu) & \hat{D}^T(\mu) \\ \star & \star & \star & -I_{n_z} \end{bmatrix} < 0, \tag{3.16}$$

$$\begin{bmatrix} \Delta_{11}(\mu) & \Delta_{12}(\mu) \\ \star & \Delta_{22}(\mu) \end{bmatrix} < 0 \tag{3.17}$$

with

$$\Phi_{11}(\mu) = \Pi\Xi_{11}(\mu)\Pi^T,\ \Phi_{12}(\mu) = \Pi\Xi_{12}(\mu)\Pi^T,$$
$$\Phi_{13}(\mu) = \Pi\Xi_{13}(\mu),\quad \Phi_{22}(\mu) = \Pi\Xi_{22}(\mu)\Pi^T,$$
$$\Phi_{23}(\mu) = \Pi\Xi_{23}(\mu),\quad \Phi_{33}(\mu) = \Xi_{33}(\mu),$$
$$\hat{C}(\mu) = \tilde{C}(\mu)\Pi^T,\quad \Delta_{11}(\mu) = \Pi L_{11}(\mu)\Pi^T$$
$$\Delta_{12}(\mu) = \Pi L_{12}(\mu)\Pi^T,\ \Delta_{22}(\mu) = \Pi L_{22}(\mu)\Pi^T.$$

Define

$$V_{AF}(\mu) = \sum_{i=1}^{N}\mu_i(t,k)V_{AF,i},\quad V_{BF}(\mu) = \sum_{i=1}^{N}\mu_i(t,k)V_{BF,i},$$
$$H_1(\mu) = \sum_{i=1}^{N}\mu_i(t,k)H_{1,i},\quad G_l(\mu) = \sum_{i=1}^{N}\mu_i(t,k)G_{l,i},$$
$$F_l(\mu) = \sum_{i=1}^{N}\mu_i(t,k)F_{l,i},\quad \bar{G}_l(\mu) = \sum_{i=1}^{N}\mu_i(t,k)\bar{G}_{l,i},$$
$$\bar{F}_l(\mu) = \sum_{i=1}^{N}\mu_i(t,k)\bar{F}_{l,i}$$

with $l = 1,\ 2$. Pre- and post-multiplying both sides of (3.16) by $\mathrm{diag}\{\Pi^T, \Pi^T, I, I\}$ and $\mathrm{diag}\{\Pi, \Pi, I, I\}$, respectively, (3.16) is equivalent to condition (3.10). Similarly, pre- and post-multiplying both sides of (3.17) by $\mathrm{diag}\{\Pi^T, \Pi^T\}$ and $\mathrm{diag}\{\Pi, \Pi\}$, respectively, (3.17) is equivalent to condition (3.11). It is noted that (3.11) guarantees $-V - V^T < 0$, which means that V is nonsingular and the filter coefficients could be calculated by (3.12). This concludes the proof. □

From the above result, it is known that by solving the convex optimization problem,

$$\min\ \gamma^2\ \ s.t.\ \text{LMIs (3.8)} - (3.11), \tag{3.18}$$

a sub-optimal filter with filter gains designed in (3.12) could be achieved with FF H_∞ performance γ^{**}, which is the minimum value of γ.

Remark 3.2. *The conditions $\bar{P}^h > 0$, $\bar{P}^v > 0$, $\hat{P}^h > 0$ and $\hat{Q}^v > 0$ in Theorem 3.2 are implied by (3.8) and (3.9) in Theorem 3.3, utilizing the equivalent variable transformation.*

4. Simulation example

The filter design method will be applied to a practical process of gas absorption, which is represented by the following nonlinear differential equation

$$\begin{aligned}\frac{\partial^2\varphi(p,q)}{\partial p\partial q} =& a_1\frac{\partial\varphi(p,q)}{\partial q} + a_2\frac{\partial\varphi(p,q)}{\partial p} + bw(p,q)\\ &+ a_0(1-0.25\sin^2(\varphi(p,q)))\varphi(p,q),\end{aligned} \tag{4.1}$$

where $\varphi(p,q)$ is the variable function; a_0, a_1, a_2 and b are real coefficients; $w(p,q)$ is the disturbance input; the BCs are

$$\varphi(p,0) = \varphi_1(p) \text{ and } \varphi(0,q) = \varphi_2(q).$$

Define

$$\varsigma(p,q) = \frac{\partial\varphi(p,q)}{\partial q} - a_2\varphi(p,q),\ x^h(t,k) = \varsigma(t,k) := \varsigma(t,kT),$$
$$x^v(t,k) = \varphi(t,k) := \varphi(t,kT),\ \frac{\partial x^v(t,k)}{\partial k} \cong \frac{x^v(t,k+1) - x^v(t,k)}{T},$$

where T is the difference step. Next, the following continuous-discrete model can be obtained

$$\begin{aligned}\begin{bmatrix} \frac{\partial x^h(t,k)}{\partial t} \\ x^v(t,k+1) \end{bmatrix} =& \begin{bmatrix} a_1 & a_1a_2 + a_0 - 0.25a_0\sin^2(x^v(t,k))) \\ T & 1+Ta_2 \end{bmatrix}\\ &\times \begin{bmatrix} x^h(t,k) \\ x^v(t,k) \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} w(t,k)\end{aligned} \tag{4.2}$$

with BCs

$$x^h(0,k) = \varphi(0,k+1)/T - (1/T + a_2)\varphi(0,k)$$

and

$$x^v(t,0) = \varphi(t,0).$$

Take the membership functions

$$\mu_1(t,k) = 1 - \sin^2(x^v(t,k)) \text{ and } \mu_2(t,k) = \sin^2(x^v(t,k))$$

in consideration of two **IF-THEN** rules. System (4.2) could be further approximated by the following continuous-discrete T-S fuzzy system

IF $\sin^2(x^v(t,k))$ is about zero, **THEN**

$$\begin{bmatrix} \frac{\partial x^h(t,k)}{\partial t} \\ x^v(t,k+1) \end{bmatrix} = A_1 \begin{bmatrix} x^h(t,k) \\ x^v(t,k) \end{bmatrix} + B_1 w(t,k). \tag{4.3}$$

IF $\sin^2(x^v(t,k))$ is about one, **THEN**

$$\begin{bmatrix} \frac{\partial x^h(t,k)}{\partial t} \\ x^v(t,k+1) \end{bmatrix} = A_2 \begin{bmatrix} x^h(t,k) \\ x^v(t,k) \end{bmatrix} + B_2 w(t,k), \tag{4.4}$$

where

$$A_1 = \begin{bmatrix} a_1 & a_1a_2 + a_0 \\ T & 1+a_2T \end{bmatrix},\ B_1 = B_2 = \begin{bmatrix} b \\ 0 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} a_1 & a_1a_2 + 0.75a_0 \\ T & 1+a_2T \end{bmatrix}.$$

Letting $a_0 = -2.4$, $a_1 = -0.4$, $a_2 = -5.5$, $b = -0.1$ and $T = 0.2$, the other matrix parameters are also taken as follows for the purpose of discussion: $E_1 = 1.5$, $E_2 = 1$,

$$A_1 = \begin{bmatrix} -0.4 & -0.2 \\ 0.2 & -0.1 \end{bmatrix},\ B_1 = B_2 = \begin{bmatrix} -0.1 \\ 0 \end{bmatrix},$$
$$D_1 = 0.1,\ C_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}^T,$$
$$A_2 = \begin{bmatrix} -0.4 & 0.4 \\ 0.2 & -0.1 \end{bmatrix},\ C_2 = \begin{bmatrix} 0.3 \\ 0.3 \end{bmatrix}^T,$$
$$L_1 = \begin{bmatrix} 1 & -1 \end{bmatrix},\ D_2 = 0.7,\ L_2 = \begin{bmatrix} -1.2 & 1 \end{bmatrix}.$$

The aim is to design a fuzzy continuous-discrete filter in the form of (2.5) to guarantee the asymptotic stability and an FF H_∞ performance of the FES (2.7). The FF domain of the disturbance input signal is taken from interval $[0.5, 13.06] \times [0, 1.57]$. Via solving the sub-optimal problem (3.18), the sub-optimal FF H_∞ performance level $\gamma^{**} = 1.9112$ is obtained and the corresponding solution to the LMIs is available, which is partially presented for space consideration

$$V_{AF,1} = \begin{bmatrix} -1.5152 & 0.0243 \\ 0.0213 & -0.2846 \end{bmatrix},\ V_{BF,1} = \begin{bmatrix} 0.0694 \\ 0.1927 \end{bmatrix},$$
$$V_{AF,2} = \begin{bmatrix} -1.5209 & 0.0243 \\ 0.0235 & -0.2846 \end{bmatrix},\ V_{BF,2} = \begin{bmatrix} 0.0458 \\ 0.2574 \end{bmatrix}.$$

By (3.12), the obtained matrix parameters of the FF H_∞ filter are designed as follows: $D_{F,1} = -0.1157$, $D_{F,2} = 1.0978$,

$$A_{F,1} = \begin{bmatrix} -0.5950 & 0.0078 \\ 0.0203 & -0.1779 \end{bmatrix}, \quad B_{F,1} = \begin{bmatrix} 0.0249 \\ 0.1201 \end{bmatrix},$$

$$A_{F,2} = \begin{bmatrix} -0.0214 & 0.0003 \\ 0.0003 & -0.0040 \end{bmatrix}, \quad B_{F,2} = \begin{bmatrix} 0.0006 \\ 0.0036 \end{bmatrix},$$

$$C_{F,1} = \begin{bmatrix} 0.1303 & 0.0021 \end{bmatrix}, \quad C_{F,2} = \begin{bmatrix} 0.0620 & 0.0258 \end{bmatrix}.$$

In the following, simulation results are provided. First, Figures 1–5 show the trajectories of $\hat{x}_1^h(t,k)$, $\hat{x}_2^h(t,k)$, $\hat{x}_1^v(t,k)$, $\hat{x}_2^v(t,k)$ and $\hat{z}(t,k)$ of the FES without disturbance input, respectively, where the BCs are

$$\hat{x}^h(t,k) = \begin{cases} [\ 0.2 \quad 0\ ]^T, & t = 0,\ 0 \le k \le 20, \\ [\ 0 \quad 0\ ]^T, & t = 0,\ k > 20, \end{cases}$$

$$\hat{x}^v(t,k) = \begin{cases} [\ 0.2 \quad 0\ ]^T, & 0 \le t \le 4,\ k = 0, \\ [\ 0 \quad 0\ ]^T, & t > 4,\ k = 0. \end{cases}$$

Figures 1–4 further show that the FES is AS. Next, take the disturbance input as follows:

$$w(t,k) = 0.8\cos(6.78(t-0.5)) \times [u(t) - u(t-1)]\frac{\sin(1.57k)}{\pi k}, \tag{4.5}$$

where $u(t)$ is the unit step function. Applying the Fourier transformation, the FF domain of the disturbance input signal (4.5) is $[0.5, 13.06] \times [0, 1.57]$. Under the zero BCs, it can be calculated that $\|\hat{z}\|_2/\|w\|_2 \doteq 1.5338$ with the disturbance input (4.5). Thus, the constraint (2.9) is satisfied, which means that the FES satisfies a prescribed FF H_∞ performance $\gamma^{**} = 1.9112$.

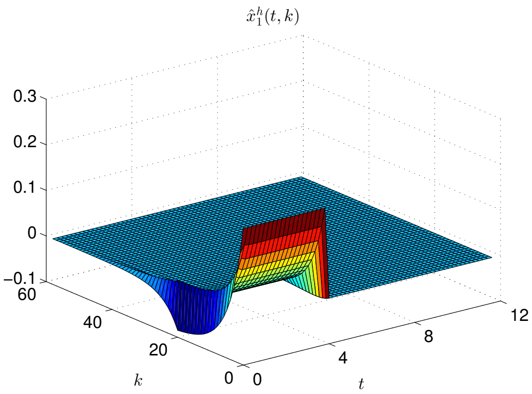

Figure 1. The trajectory of $\hat{x}_1^h(t,k)$.

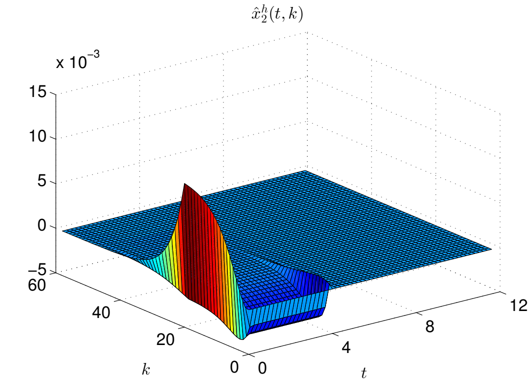

Figure 2. The trajectory of $\hat{x}_2^h(t,k)$.

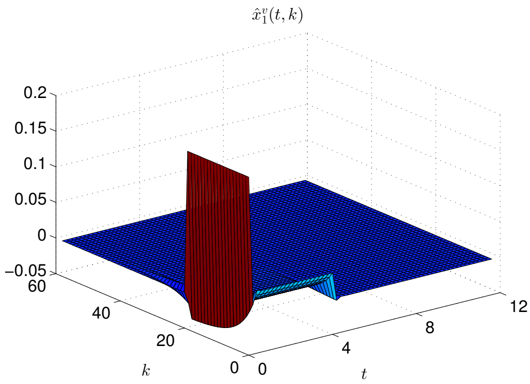

Figure 3. The trajectory of $\hat{x}_1^v(t,k)$.

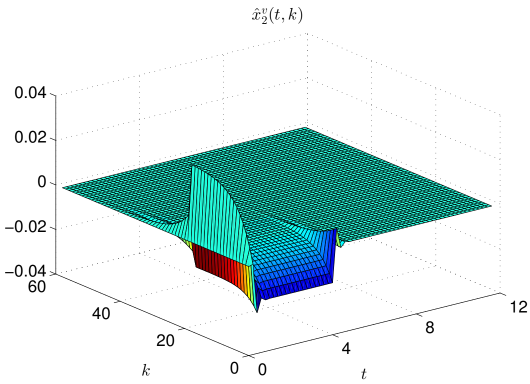

Figure 4. The trajectory of $\hat{x}_2^v(t,k)$.

Finally, to investigate the relationship between the size of the FF domain and the conservativeness of the proposed result, the obtained sub-optimal values of γ based on Theorem 3.3 for different FF sets of disturbances are displayed in Table 1. From Table 1, we find that, for a fixed continuous component interval, γ^{**} becomes larger as

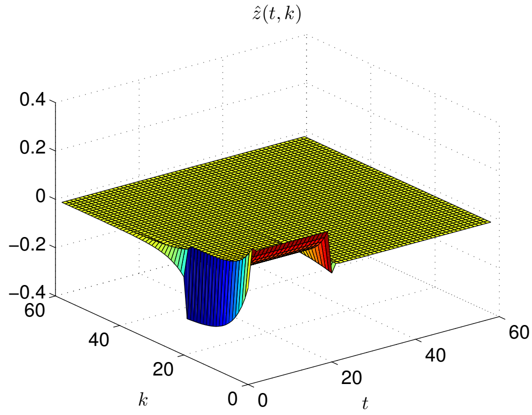

Figure 5. The trajectory of $\hat{z}(t,k)$ with $w(\cdot,\cdot) \equiv 0$.

the discrete one becomes wider; while for a fixed discrete component interval, if the continuous part is wider, then the obtained γ^{**} is larger. That is, a larger FF domain leads to a more conservative filtering performance.

Table 1. The value of γ^{**} and feasibility of the LMIs with different FF domains.

Frequency domain	γ^{**}	Feasibility
$[5.2, 8.7] \times [0, 1.57]$	1.4013	✓
$[0.5, 13.06] \times [0, 1.57]$	1.9112	✓
$[0.5, 13.06] \times [0.5, 1.2]$	1.3135	✓
$[0, 130.6] \times [-2, 3]$	Null	×
$[0, 1.3 \times 10^8] \times [-\pi, \pi]$	Null	×

5. Conclusions

The filter design problem has been concerned for the continuous-discrete T-S fuzzy systems in the Roesser model with FF disturbances. The systematic method has been proposed for the filter design, with which the FES was AS and had an FF H_∞ performance. A simulation case further showed validity of the results discussed above. Applying the similar techniques to FF fault detection problems and extending the proposed technique to the 2-D T-S fuzzy systems with delays are important and challenging research topics that deserve more future efforts.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare that there are no conflicts of interest in this paper.

References

1. M. S. Branicky, V. S. Borkar, S. K. Mitter, A unified framework for hybrid control: model and optimal control theory, *IEEE Trans. Autom. Control*, **43** (1998), 31–45. https://doi.org/10.1109/9.654885
2. S. Knorn, *A two-dimensional systems stability analysis of vehicle platoons*, Ph. D. thesis, National University of Ireland Maynooth, 2013.
3. L. Wu, H. Gao, C. Wang, Quasi sliding mode control of differential linear repetitive processes with unknown input disturbance, *IEEE Trans. Ind. Electron.*, **58** (2011), 3059–3068. https://doi.org/10.1109/TIE.2010.2072891
4. D. A. Bristow, M. Tharayil, A. G. Alleyne, A survey of iterative learning control, *IEEE Control Syst. Mag.*, **26** (2006), 96–114. https://doi.org/10.1109/MCS.2006.1636313
5. L. Wang, W. Wang, J. Gao, W. Chen, Stability and robust stabilization of 2-D continuous-discrete systems in Roesser model based on KYP lemma, *Multidimens. Syst. Signal Process.*, **28** (2017), 251–264. https://doi.org/10.1007/s11045-015-0355-2
6. G. Chesi, R. H. Middleton, LMI-based fixed order output feedback synthesis for two-dimensional mixed continuous-discrete-time systems, *IEEE Trans. Autom. Control*, **63** (2018), 960–972. https://doi.org/10.1109/TAC.2017.2727685

7. X. Wang, Y. Sun, D. Ding, Adaptive dynamic programming for networked control systems under communication constraints: a survey of trends and techniques, *Int. J. Network Dyn. Intell.*, **1** (2022), 85–98. https://doi.org/10.53941/ijndi0101008

8. X. Liu, S. Zhong, Stability analysis of delayed switched cascade nonlinear systems with uniform switching signals, *Math. Modell. Control*, **1** (2021), 90–101. https://doi.org/10.3934/mmc.2021007

9. Y. Su, H. Cai, J. Huang, The cooperative output regulation by the distributed observer approach, *Int. J. Network Dyn. Intell.*, **1** (2022), 20–35. https://doi.org/10.53941/ijndi0101003

10. T. Takagi, M. Sugeno, Fuzzy identification of systems and its applications to modelling and control, *IEEE Trans. Syst. Man Cybern.*, **SMC-15** (1985), 116–132. https://doi.org/10.1109/TSMC.1985.6313399

11. G. Feng, *Analysis and synthesis of fuzzy control systems: a model-based approach*, Boca Raton: CRC Press, Inc., 2010.

12. J. Wang, J. Liang, C. T. Zhang, Dissipativity analysis and synthesis for positive Roesser systems under the switched mechanism and Takagi-Sugeno fuzzy rules, *Inf. Sci.*, **546** (2021), 234–252. https://doi.org/10.1016/j.ins.2020.08.034

13. J. Liang, J. Wang, T. Huang, l_1 filtering for continuous-discrete T-S fuzzy positive Roesser model, *J. Franklin Inst.*, **355** (2018), 7281–7305. https://doi.org/10.1016/j.jfranklin.2018.07.017

14. Y. Luo, Z. Wang, J. Liang, G. Wei, F. E. Alsaadi, H_∞ control for 2-D fuzzy systems with interval time-varying delays and missing measurements, *IEEE Trans. Cybern.*, **47** (2017), 365–377. https://doi.org/10.1109/TCYB.2016.2514846

15. L. Li, W. Wang, X. Li, New approach to H_∞ filtering of two-dimensional T-S fuzzy systems, *Int. J. Robust Nonlinear Control*, **23** (2013), 1900–2012. https://doi.org/10.1002/rnc.2866

16. X. Ren, T. Hou, Pareto optimal filter design with hybrid H_2/H_∞ optimization, *Math. Modell. Control*, **3** (2023), 80–87. https://doi.org/10.3934/mmc.2023008

17. Y. Chen, W. Zhang, H. Gao, Finite frequency H_∞ control for building under earthquake excitation, *Mechatronics*, **20** (2010), 128–142. https://doi.org/10.1016/j.mechatronics.2009.11.001

18. T. Iwasaki, S. Hara, Generalized KYP lemma: unified frequency domain inequalities with design applications, *IEEE Trans. Autom. Control*, **50** (2005), 41–59. https://doi.org/10.1109/TAC.2004.840475(410)

19. X. Li, H. Gao, C. Wang, Generalized Kalman-Yakubovich-Popov lemma for 2-D FM LSS model, *IEEE Trans. Autom. Control*, **57** (2012), 3090–3103. https://doi.org/10.1109/TAC.2012.2200370

20. L. Wang, W. Wang, G. Zhang, W. Chen, Generalised Kalman-Yakubovich-Popov lemma with its application in finite frequency positive realness control for two-dimensional continuous-discrete systems in the Roesser model form, *IET Control Theory Appl.*, **9** (2015), 1676–1682. https://doi.org/10.1049/iet-cta.2014.0875

21. H. Gao, X. Li, H_∞ filtering for discrete-time state-delayed systems with finite frequency specifications, *IEEE Trans. Autom. Control*, **56** (2011), 2935–2941. https://doi.org/10.1109/TAC.2011.2159909

22. G. Wang, H. Xu, L. Wang, J. Yao, Robust H_∞ filtering for uncertain two-dimensional continuous-discrete state-delay systems in finite frequency domains, *IET Control Theory Appl.*, **12** (2018), 2316–2327. https://doi.org/10.1049/iet-cta.2018.5671

23. T. Iwasaki, S. Hara, A. L. Fradkov, Time domain interpretations of frequency domain inequalities on (semi) finite ranges, *Syst. Control Lett.*, **54** (2005), 681–691. https://doi.org/10.1016/j.sysconle.2004.11.007

24. D. W. Ding, G. H. Yang, Fuzzy filter design for nonlinear systems in finite-frequency domain, *IEEE Trans. Fuzzy Syst.*, **18** (2010), 935–945. https://doi.org/10.1109/TFUZZ.2010.2058807

25. Z. Duan, J. Shen, I. Ghous, J. Fu, H_∞ filtering for discrete-time 2D T-S fuzzy systems with finite frequency disturbances, *IET Control Theory Appl.*, **13** (2019), 1983–1994. https://doi.org/10.1049/iet-cta.2018.5918

26. M. Wang, G. Feng, J. Qiu, Finite-frequency fuzzy output feedback controller design for Roesser-type two-dimensional nonlinear systems, *IEEE Trans. Fuzzy Syst.*, **29** (2021), 861–873. https://doi.org/10.1109/TFUZZ.2020.2966155

27. D. Franke, 2D-analysis of Hybrid systems, In: P. M. Frank, *Advances in control*, Springer, 1999, 293–299.

28. Z. Duan, J. Liang, Z. Xiang, H_∞ control for continuous-discrete systems in T-S fuzzy model with finite frequency specifications two-dimensional nonlinear systems, *Discrete Contin. Dyn. Syst.*, **15** (2022), 3155–3172. https://doi.org/10.3934/dcdss.2022064

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