Research article

Solving interval type-2 fuzzy relation equations via semi-tensor product of interval matrices

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Abstract: This paper mainly studied the problem of solving interval type-2 fuzzy relation equations $\widetilde{A} \circ \widetilde{X} = \widetilde{B}$. First, to solve the interval type-2 fuzzy relation equations, we extend the semi-tensor product of matrices to interval matrices and give its specific definition. Second, the interval type-2 fuzzy relation equation was divided into two parts: primary fuzzy matrix equation $\widetilde{A}_\mu \circ \widetilde{X}_\mu = \widetilde{B}_\mu$ and secondary fuzzy matrix equation $\widetilde{A}_f \circ \widetilde{X}_f = \widetilde{B}_f$. Since all elements of $\widetilde{X}_f$ equal to one, only the principal fuzzy matrix equation needs to be considered. Furthermore, it was proved that all solutions can be obtained from the parameter set solutions if the primary fuzzy matrix equation is solvable. Finally, with semi-tensor product of interval matrices, the primary fuzzy matrix equation was transformed into an algebraic equation and the specific algorithm for solving an interval type-2 fuzzy relation equation was proposed.

Keywords: interval type-2 fuzzy relation equations; semi-tensor product; semi-tensor product of interval matrices; primary fuzzy matrix equation

1. Introduction

The concept of type-2 fuzzy sets (T2 FSs) was first proposed by Zadeh [1] and a detailed introduction was given in [2]. T2 FSs are an extension of the type-1 fuzzy set (T1 FS) and further considers the fuzziness of the fuzzy set. Since the definition of T2 FSs was proposed, most scholars have mainly studied the operations and properties of T2 FSs [3, 4]. Until the 1990s, Prof. Mendel redefined T2 FSs and proposed the type-2 fuzzy logic system (T2 FLS) [5, 6].

As an extension of T1 FSs, T2 FSs overcome the limitations of T1 FSs in dealing with the uncertainties of actual objects. However the definition of T2 FSs is complicated, and the corresponding graph must be a spatial graph. Due to the complexity of the expression of T2 FSs, Mendel introduces the definition of interval T2 FSs (IT2 FSs) as a special case of T2 FSs. The secondary membership grade of IT2 FSs is constant one, which is more simpler than T2 FSs. In general, IT2 fuzzy logic system (IT2 FLS) are used in most theories and applications [7, 8].

In the field of fuzzy control, solving fuzzy relation equations (FREs) plays an important role in the design of fuzzy controller and fuzzy logic reasoning. Most algorithms for solving FREs can obtain some specific solutions, such as the minimum or maximum solution [9], or can describe the solution theoretically [10]. In the existing methods, most of them are used for solving type-1 fuzzy relation equations (T1 FREs), while few methods are used for solving interval type-2 fuzzy relation equations (IT2 FREs). The main work of this paper is to propose a new method to obtain the entire solution set of IT2 FREs.

On the other hand, Prof. Cheng proposed a new matrix product-semi-tensor product (STP) of matrices, which is the generalization of the conventional matrix product and

retains almost all the main properties of the conventional matrix product. As a novel mathematical technique for handling logical operations, STP has been successfully applied to logical systems [11–14] and, based on this, a new algorithm for solving FREs has been devised. For example, in T1 FREs, the STP is used to solve fuzzy relation equalities and fuzzy relation inequalities [15–18]. In type-2 fuzzy relational equations (T2 FREs), only some simple algorithms have been proposed to study the solution of type-2 single-valued fuzzy relation equations and type-2 symmetry-valued fuzzy relation equations [19, 20]. However, the ordinary STP cannot be used directly to solve IT2 FREs. Therefore, we extend the STP to interval matrices and propose the STP of interval matrices, then discuss the solutions of IT2 FREs.

In the rest of this paper, section two introduces the basic concepts of the STP of interval matrices. Section three mainly gives the relevant definitions of interval-valued logic and gives its matrix representation. Section four discusses the solvability of IT2 FREs and designs an algorithm to solve IT2 FREs. Section five explains the viability of the proposed algorithm with a numerical example. Section six gives a brief summary of the paper.

2. Preliminaries

2.1. Basic concept

First, in order to express conveniently, we introduce some notations used throughout the paper.

- $$I[0, 1] := \left\{ \left[\underline{\alpha}, \overline{\alpha} \right] | 0 \le \underline{\alpha} \le \overline{\alpha} \le 1 \right\},$$
  where $\underline{\alpha}, \overline{\alpha} \in R$. If $\alpha = [\underline{\alpha}, \underline{\alpha}]$ (or $\alpha = [\overline{\alpha}, \overline{\alpha}]$), this is a point interval and α degenerates into a real number.

- $$I([0, 1]^m) := \left\{ \left[\underline{A}, \overline{A} \right] | \underline{A} \le \overline{A} \right\}.$$
  $\underline{A} = (\underline{\alpha}_i)_m$ and $\overline{A} = (\overline{\alpha}_i)_m$ are two m-dimensional vectors and $\left[\underline{a}_i, \overline{a}_i \right] \in I[0, 1]$, $i = 1 \cdots m$.

- $$I([0, 1]^{m \times n}) := \left\{ \left[\underline{A}, \overline{A} \right] | \underline{A} \le \overline{A} \right\}.$$
  $\underline{A} = (\underline{\alpha}_{ij})_{m \times n}$ and $\overline{A} = (\overline{\alpha}_{ij})_{m \times n}$ are two $m \times n$ dimensional matrices and $\left[\underline{\alpha}_{ij}, \overline{\alpha}_{ij} \right] \in I[0, 1]$, $i = 1 \cdots m$, $j = 1 \cdots n$.

- δ_r^i : the ith column of unit matrix I_n.
- $\left[\delta_r^i, \delta_r^j \right]$: a bounded closed interval, where δ_n^i represents its lower bound and δ_n^j represents its upper bound, abbreviated as $\delta_r[i, j]$.
- $Col_i(M)$: the ith column of interval matrix M.
- $Row_j(M)$: the jth row of interval matrix M.

Next, we define $\wedge, \vee$ and $\neg$ in $I[0, 1]$.

Definition 2.1. *[21] (1) Let*

$$\alpha = [\underline{\alpha}, \overline{\alpha}], \ \beta = [\underline{\beta}, \overline{\beta}] \in I[0, 1],$$

then,

$$\alpha \vee \beta = \left[\max\left(\underline{\alpha}, \underline{\beta} \right), \max\left(\overline{\alpha}, \overline{\beta} \right) \right], \tag{2.1}$$

$$\alpha \wedge \beta = \left[\min\left(\underline{\alpha}, \underline{\beta} \right), \min\left(\overline{\alpha}, \overline{\beta} \right) \right]. \tag{2.2}$$

(2) Let

$$A = \left[\underline{a}_{ij}, \overline{a}_{ij} \right] \in I([0, 1]^{m \times n}),$$
$$B = \left[\underline{b}_{jk}, \overline{b}_{jk} \right] \in I([0, 1]^{n \times p}),$$

then their max-min composition operation is defined as

$$A \circ B = C = \left[\underline{c}_{ik}, \overline{c}_{ik} \right] \in I([0, 1]^{m \times p}), \tag{2.3}$$

where

$$\left[\underline{c}_{ik}, \overline{c}_{ik} \right] = \left(\left[\underline{a}_{i1}, \overline{a}_{i1} \right] \wedge \left[\underline{b}_{1k}, \overline{b}_{1k} \right] \right) \vee \left(\left[\underline{a}_{i2}, \overline{a}_{i2} \right] \wedge \left[\underline{b}_{2k}, \overline{b}_{2k} \right] \right)$$
$$\vee \cdots \vee \left(\left[\underline{a}_{in}, \overline{a}_{in} \right] \wedge \left[\underline{b}_{nk}, \overline{b}_{nk} \right] \right),$$

where $i = 1, \cdots, m$, $j = 1, \cdots, k$.

Definition 2.2. *[22] Let*

$$A = \left[\underline{a}_{ij}, \overline{a}_{ij} \right],$$
$$B = \left[\underline{b}_{ij}, \overline{b}_{ij} \right] \in I([0, 1]^{m \times n}),$$

then the partial order $\ge, \le$ and $=$ are defined as
(1) If $\underline{a}_{ij} \ge \underline{b}_{ij}, \overline{a}_{ij} \ge \overline{b}_{ij}$, we say $A \ge B$.
(2) If $\underline{a}_{ij} \le \underline{b}_{ij}, \overline{a}_{ij} \le \overline{b}_{ij}$, we say $A \le B$.
(3) If $\underline{a}_{ij} = \underline{b}_{ij}, \overline{a}_{ij} = \overline{b}_{ij}$, we say $A = B$.

Property 2.1. *[15] Let*

$$A = \left[\underline{a}_{ij}, \overline{a}_{ij} \right], \ B = \left[\underline{b}_{ij}, \overline{b}_{ij} \right] \in I([0, 1]^{m \times n}),$$
$$C = \left[\underline{c}_{jk}, \overline{c}_{jk} \right], \ D = \left[\underline{d}_{jk}, \overline{d}_{jk} \right] \in I([0, 1]^{n \times p}).$$

Assume $A \le B$ and $C \le D$, then

$$A \circ C \le B \circ D.$$

Definition 2.3. *[22] (1) Let*

$$\alpha = [\underline{\alpha}, \overline{\alpha}], \ \ \beta = [\underline{\beta}, \overline{\beta}] \in I\,[0, 1]\,,$$

then the four operations of intervals α and β are as follows.

1) Addition operation

$$\alpha + \beta = [\underline{\alpha}, \overline{\alpha}] + [\underline{\beta}, \overline{\beta}] = \left[\underline{\alpha}+\underline{\beta}, \overline{\alpha} + \overline{\beta}\right]. \quad (2.4)$$

2) Subtraction operation

$$\alpha - \beta = [\underline{\alpha}, \overline{\alpha}] - [\underline{\beta}, \overline{\beta}] = \left[\underline{\alpha} - \overline{\beta}, \overline{\alpha} - \underline{\beta}\right]. \quad (2.5)$$

3) Multiplication operation

$$\alpha \times \beta = [\underline{\alpha}, \overline{\alpha}] \times [\underline{\beta}, \overline{\beta}] = [\underline{\alpha}\underline{\beta}, \overline{\alpha}\overline{\beta}]. \quad (2.6)$$

4) Division operation

$$\alpha/\beta = [\underline{\alpha}, \overline{\alpha}]/[\underline{\beta}, \overline{\beta}] = \Big[\min\left(\underline{\alpha}/\underline{\beta}, \underline{\alpha}/\overline{\beta}, \ \overline{\alpha}/\underline{\beta}, \right.$$
$$\left. \overline{\alpha}/\overline{\beta}\right), \max\left(\underline{\alpha}/\underline{\beta}, \underline{\alpha}/\overline{\beta}, \overline{\alpha}/\underline{\beta}, \overline{\alpha}/\overline{\beta}\right)\Big]. \quad (2.7)$$

Note that $0 \notin \beta = [\underline{\beta}, \overline{\beta}]$.

(2) If

$$\alpha = [\underline{\alpha}, \overline{\alpha}] \in I\,[0, 1]\,,$$

$$A = [\underline{a}_{ij}, \overline{a}_{ij}]_{m \times n} \in I\left([0, 1]^{m \times n}\right),$$

then the product of interval α and interval matrix A is

$$\alpha \times A := \left([\underline{\alpha}, \overline{\alpha}] \times \left[\underline{a}_{ij}, \overline{a}_{ij}\right]\right)_{m \times n}. \quad (2.8)$$

(3) If

$$A = \left[\underline{a}_{ij}, \overline{a}_{ij}\right]_{m \times n} \in I\left([0, 1]^{m \times n}\right),$$
$$B = \left[\underline{b}_{jk}, \overline{b}_{jk}\right]_{n \times p} \in I\left([0, 1]^{n \times p}\right),$$

then the product of interval matrices A and B is

$$A \times B = C = \left[\underline{c}_{ik}, \overline{c}_{ik}\right]_{n \times p}$$
$$= \left[\begin{array}{ccc} \left[\underline{c}_{11}, \overline{c}_{11}\right] & \cdots & \left[\underline{c}_{1p}, \overline{c}_{1p}\right] \\ \vdots & \ddots & \vdots \\ \left[\underline{c}_{n1}, \overline{c}_{n1}\right] & \cdots & \left[\underline{c}_{np}, \overline{c}_{np}\right] \end{array}\right], \quad (2.9)$$

where

$$\left[\underline{c}_{ik}, \overline{c}_{ik}\right] = \sum_{j=1}^{m} \left[\underline{a}_{ij}, \overline{a}_{ij}\right] \times \left[\underline{b}_{jk}, \overline{b}_{jk}\right]$$
$$= \left[\underline{a}_{i1}, \overline{a}_{i1}\right] \times \left[\underline{b}_{1k}, \overline{b}_{1k}\right] + \left[\underline{a}_{i2}, \overline{a}_{i2}\right]$$
$$\times \left[\underline{b}_{2k}, \overline{b}_{2k}\right] + \cdots + \left[\underline{a}_{im}, \overline{a}_{im}\right] \times \left[\underline{b}_{mk}, \overline{b}_{mk}\right].$$

Based on Definition 2.3, we give the relevant definition of the STP of interval matrices.

Definition 2.4. *(1) If*

$$A = \left[\underline{a}_{ij}, \overline{a}_{ij}\right]_{m \times n} \in I\left([0, 1]^{m \times n}\right),$$

$$B = \left[\underline{b}_{kl}, \overline{b}_{kl}\right]_{p \times q} \in I\left([0, 1]^{p \times q}\right),$$

then the kronecker product of interval matrices A and B is

$$A \otimes B = \left[\begin{array}{ccc} a_{11} \times B & \cdots & a_{1n} \times B \\ \vdots & \ddots & \vdots \\ a_{m1} \times B & \cdots & a_{mn} \times B \end{array}\right]. \quad (2.10)$$

(2) If

$$A = \left[\underline{a}_{ij}, \overline{a}_{ij}\right]_{m \times n} \in I\left([0, 1]^{m \times n}\right),$$
$$B = \left[\underline{b}_{kl}, \overline{b}_{kl}\right]_{p \times q} \in I\left([0, 1]^{p \times q}\right),$$

then the STP of interval matrices A and B is

$$A \ltimes B = \left(A \otimes I_{\frac{t}{n}}\right) \times \left(B \otimes I_{\frac{t}{p}}\right), \quad (2.11)$$

where $t = lcm\,(n, p)$ is the least common multiple of n and p.

(3) If

$$A = \left[\underline{a}_{ij}, \overline{a}_{ij}\right]_{m \times n} \in I\left([0, 1]^{m \times n}\right),$$
$$B = \left[\underline{b}_{kl}, \overline{b}_{kl}\right]_{p \times n} \in I\left([0, 1]^{p \times n}\right),$$

then the khatri-rao product of interval matrices A and B is

$$A * B = [Col_1\,(A) \ltimes Col_1\,(B) \ \ Col_2\,(A) \ltimes Col_2\,(B)$$
$$\cdots Col_n\,(A) \ltimes Col_n\,(B)]. \quad (2.12)$$

Remark 2.1. *In Definition 2.4, if $n = p$, then the STP of interval matrices degenerates to the ordinary interval matrix multiplication. Therefore, the STP of interval matrices is a generalization of interval matrices multiplication. In the context, the STP of interval matrices is $\ltimes$, which is omitted by default.*

Example 2.1. *Given the interval matrices A and B,*

$$A = \left[\begin{array}{cc} [0.2, 0.4] & [0.4, 0.5] \\ [0.6, 1.0] & [0.8, 0.9] \end{array}\right], \ B = \left[\begin{array}{cc} [0, 1] & [0.8, 0.9] \\ [0.2, 0.3] & [1, 1] \\ [0.4, 0.6] & [0.7, 0.9] \\ [0.6, 0.7] & [0.3, 0.4] \end{array}\right].$$

The kronecker product of the interval matrix A and the unit interval matrix I_2 is

$$A \otimes I_2 = \begin{bmatrix} [0.2, 0.4] & [0,0] & [0.4, 0.5] & [0,0] \\ [0,0] & [0.2, 0.4] & [0,0] & [0.4, 0.5] \\ [0.6, 1.0] & [0,0] & [0.8, 0.9] & [0,0] \\ [0,0] & [0.6, 1.0] & [0,0] & [0.8, 0.9] \end{bmatrix}.$$

The STP of interval matrices A and B is

$$A \ltimes B \overset{\Delta}{=} (A \otimes I_2) \times B$$

$$= \begin{bmatrix} [0.16, 0.70] & [0.44, 0.81] \\ [0.28, 0.47] & [0.32, 0.60] \\ [0.32, 1.54] & [1.04, 1.71] \\ [0.60, 0.93] & [0.84, 1.36] \end{bmatrix}.$$

The khatri-rao product of interval matrices A and B is

$$A * B = \begin{bmatrix} Col_1(A) \times Col_1(B) & Col_2(A) \times Col_2(B) \end{bmatrix}$$

$$= \begin{bmatrix} [0.00, 0.40] & [0.04, 0.12] & [0.08, 0.24] & [0.12, 0.28] \\ [0.32, 0.45] & [0.40, 0.50] & [0.28, 0.45] & [0.12, 0.20] \end{bmatrix}$$

$$\begin{bmatrix} [0.00, 1.00] & [0.12, 0.30] & [0.24, 0.60] & [0.36, 0.70] \\ [0.64, 0.81] & [0.80, 0.90] & [0.56, 0.81] & [0.24, 0.32] \end{bmatrix}^T.$$

According to the definition of STP of interval matrices, we can get the following properties.

Property 2.2. *(1) Let $A, B \in I([0,1]^{m \times n}), C \in I([0,1]^{p \times q})$, then*

$$(A + B) \ltimes C = A \ltimes C + B \ltimes C,$$
$$C \ltimes (A + B) = C \ltimes A + C \ltimes B. \tag{2.13}$$

(2) Let $A \in I([0,1]^{m \times n}), B \in I([0,1]^{p \times q})$ and $C \in I([0,1]^{r \times s})$, then

$$(A \ltimes B) \ltimes C = A \ltimes (B \ltimes C). \tag{2.14}$$

(3) Let $A \in I([0,1]^{m \times n})$, $C \in I([0,1]^s)$ and $R \in I([0,1]^s)$ are column and row interval vectors, respectively, then

$$C \ltimes A = (I_s \otimes A) \ltimes C,$$
$$R \ltimes A = (A \otimes I_s) \ltimes R. \tag{2.15}$$

2.2. Problem formulation

Let the interval type-2 fuzzy relation $\widetilde{R} \in F(V \times W)$, where the domain $V = \{v_1, v_2, \cdots, v_n\}$ and $W = \{w_1, w_2, \cdots, w_p\}$,

then the matrix form of interval type-2 fuzzy relation $\widetilde{R}$ can be defined as

$$M_{\widetilde{R}} = \begin{bmatrix} \frac{f_{\widetilde{R}}(v_1, w_1)}{\mu_{\widetilde{R}}(v_1, w_1)} & \cdots & \frac{f_{\widetilde{R}}(v_1, w_p)}{\mu_{\widetilde{R}}(v_1, w_p)} \\ \vdots & \ddots & \vdots \\ \frac{f_{\widetilde{R}}(v_n, w_1)}{\mu_{\widetilde{R}}(v_n, w_1)} & \cdots & \frac{f_{\widetilde{R}}(v_n, w_p)}{\mu_{\widetilde{R}}(v_n, w_p)} \end{bmatrix}. \tag{2.16}$$

$\mu_{\widetilde{R}}(v_i, w_k)$ and $f_{\widetilde{R}}(v_i, w_k)$ represent the primary membership grade and secondary membership grade of IT2 FSs, respectively. For primary membership grade, it is composed of upper membership grade and lower membership grade; that is,

$$\mu_{\widetilde{R}}(v_i, w_k) = \left[\underline{\mu}_{\widetilde{R}}(v_i, w_k), \overline{\mu}_{\widetilde{R}}(v_i, w_k) \right].$$

The secondary membership grade of IT2 FSs equals one; that is, $f_{\widetilde{R}}(v_i, w_k) = 1$, then the matrix form of interval type-2 fuzzy relation $\widetilde{R}$ can be further described as

$$M_{\widetilde{R}} = \begin{bmatrix} \frac{1}{[\underline{\mu}_{\widetilde{R}}(v_1, w_1), \overline{\mu}_{\widetilde{R}}(v_1, w_1)]} & \cdots & \frac{1}{[\underline{\mu}_{\widetilde{R}}(v_1, w_s), \overline{\mu}_{\widetilde{R}}(v_1, w_p)]} \\ \vdots & \ddots & \vdots \\ \frac{1}{[\underline{\mu}_{\widetilde{R}}(v_n, w_1), \overline{\mu}_{\widetilde{R}}(v_n, w_1)]} & \cdots & \frac{1}{[\underline{\mu}_{\widetilde{R}}(v_n, w_s), \overline{\mu}_{\widetilde{R}}(v_n, w_p)]} \end{bmatrix}. \tag{2.17}$$

Two common types of FREs exist in practical application [20]. One type is that the fuzzy relation is unknown, which is commonly used for designing fuzzy controllers. The other type is that the fuzzy input is unknown, which is commonly used for diagnosing diseases based on the symptom similarity. In terms of the aforementioned situations, it can be assumed that there are similar two types of IT2 FREs, as shown in Figures 1 and 2.
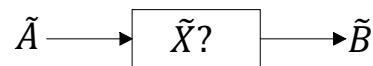
$$\tilde{A} \longrightarrow \boxed{\tilde{X}?} \longrightarrow \tilde{B}$$

Figure 1. Interval type-2 fuzzy relation unknown.

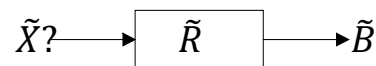$$\tilde{X}? \longrightarrow \boxed{\tilde{R}} \longrightarrow \tilde{B}$$

Figure 2. Interval type-2 fuzzy input unknown.

Type 1: assume $\widetilde{A} \in F(U \times V), \widetilde{B} \in F(U \times W)$. We seek an interval type-2 fuzzy relation $\widetilde{X} \in F(V \times W)$ such that it satisfies

$$\widetilde{A} \circ \widetilde{X} = \widetilde{B}. \tag{2.18}$$

Type 2: assume $\widetilde{R} \in F(V \times W), \widetilde{B} \in F(U \times W)$. We seek an interval type-2 fuzzy input $\widetilde{X} \in F(U \times V)$ such that it satisfies

$$\widetilde{X} \circ \widetilde{R} = \widetilde{B}. \tag{2.19}$$

Remark 2.2. *Take a transpose of both sides of (2.19) to get $\widetilde{R}^T \circ \widetilde{X}^T = \widetilde{B}^T$. (2.19) is equivalent to (2.18), so we only need to consider the solvability of (2.18).*

3. Interval-valued logic and its matrix representation

Definition 3.1. *(1)*

$$I_f = \{ \left[\underline{\alpha}, \overline{\alpha} \right] | 0 \le \underline{\alpha} \le \overline{\alpha} \le 1 \}$$

is called the domain of interval-valued fuzzy logic, and the interval-valued fuzzy logic variable is $P \in I_f$. When $\alpha = [0,0]$ (or $\alpha = [1,1]$), α degenerates into a classical logic variable.

(2)

$$I_k = \left\{ \left[\underline{\alpha}_1, \overline{\alpha}_1 \right], \left[\underline{\alpha}_2, \overline{\alpha}_2 \right], \cdots, \left[\underline{\alpha}_k, \overline{\alpha}_k \right] \right\}, \quad \left[\underline{\alpha}_i, \overline{\alpha}_i \right] \in I_f,$$

$i = 1, \cdots, k$, then I_k is called the domain of k-valued interval-valued fuzzy logic.

(3) Mapping

$$f : \underbrace{I_k \times I_k \times \cdots \times I_k}_{r} \to I_k$$

is called r-ary k-valued interval-valued logic function.

If
$$I_k = \left\{ \left[\underline{\alpha}_1, \overline{\alpha}_1 \right], \left[\underline{\alpha}_2, \overline{\alpha}_2 \right], \cdots, \left[\underline{\alpha}_k, \overline{\alpha}_k \right] \right\},$$

put the different upper and lower bounds of all interval-valued fuzzy logic variables in I_k into the ordered set Θ. If Θ does not contain zero and one, it needs to add zero or one:

$$\Theta = \{a_p | p = 1, \cdots, s; 0 \le a_1 < a_2 < \cdots < a_s \le 1\}.$$

In order to facilitate matrix calculation, each variable in I^k is represented as an interval vector. If $\underline{\alpha}_i = a_m$

$(1 \le m \le s, m \in Z^+)$ and $\overline{\alpha}_i = a_n$ $(1 \le n \le s, n \in Z^+)$, then the lower bound $\underline{\alpha}_i$ can be represented by vector δ_s^m and the upper bound $\overline{\alpha}_i$ can be represented by vector δ_s^n. Therefore,

$$\left[\underline{\alpha}_i, \overline{\alpha}_i \right] \sim [\delta_s^m, \delta_s^n] = \delta_s[m, n].$$

Similar to the proof of theorem in paper [23], we can obtain Theorem 3.1.

Theorem 3.1. *f is a r-ary k-valued interval-valued logic function, then there exists a unique structural matrix M_f, whose algebraic form is*

$$f(x_1, x_2, \cdots, x_r) = M_f \ltimes_{i=1}^{r} \left[\underline{x}_i, \overline{x}_i \right]. \tag{3.1}$$

Remark 3.1. *Structure matrix is also a special interval matrix that can be used to replace $\wedge, \vee$ and $\neg$ for algebraic operations.*

In the following, we give the structure matrix of $\wedge, \vee$ and $\neg$.

Let

$$I_k = \left\{ \left[\underline{\alpha}_1, \overline{\alpha}_1 \right], \left[\underline{\alpha}_2, \overline{\alpha}_2 \right], \cdots, \left[\underline{\alpha}_k, \overline{\alpha}_k \right] \right\}.$$

The ordered set Θ generated by I_k contains s different elements. To simply represent the structure matrix of $\wedge, \vee$ and $\neg$, we introduce a set of s-dimensional vectors

$$U_v = \left(1 \ 2 \cdots v - 1 \underbrace{v \cdots v}_{s-v+1} \right),$$

$$V_v = \left(\underbrace{v \cdots v}_{v} \ v + 1 \ v + 2 \cdots s \right), \quad v = 1, \cdots, s.$$

(1) The structure matrix of $\vee$:

$$M_d^s = \left[\underline{M}_d^s, \overline{M}_d^s \right], \quad \underline{M}_d^s = \overline{M}_d^s = \delta_s \left[U_1 U_2 \cdots U_s \right].$$

When s = 3, we have

$$M_d^3 = \delta_3 \left[\ [1,1] \quad [1,1] \quad [1,1] \quad [1,1] \quad [2,2] \right.$$
$$\left. [2,2] \quad [1,1] \quad [2,2] \quad [3,3] \ \right].$$

(2) The structure matrix of $\wedge$:

$$M_c^s = \left[\underline{M}_c^s, \overline{M}_c^s \right], \quad \underline{M}_c^s = \overline{M}_c^s = \delta_s \left[U_1 U_2 \cdots U_s \right].$$

When s = 3, we have

$$M_c^3 = \delta_3 \left[\ [1,1] \quad [2,2] \quad [3,3] \quad [2,2] \quad [2,2] \right.$$
$$\left. [3,3] \quad [3,3] \quad [3,3] \quad [3,3] \ \right].$$

4. Main results

4.1. Decomposition of IT2 FRE

Definition 4.1. *[20] In (2.17), the matrix constructed by the primary membership grade $\mu_{\widetilde{R}}(v_j, w_k)$ is called primary fuzzy matrix of interval type-2 fuzzy relation, denoted as $\widetilde{R}_\mu\left(\mu_{\widetilde{R}}(v_j, w_k)\right)$ and abbreviated as $\widetilde{R}_\mu$:*

$$\widetilde{R}_\mu = \begin{bmatrix} \left[\underline{\mu}_{\widetilde{R}}(v_1, w_1), \overline{\mu}_{\widetilde{R}}(v_1, w_1)\right] & \cdots & \left[\underline{\mu}_{\widetilde{R}}(v_1, w_p), \overline{\mu}_{\widetilde{R}}(v_1, w_p)\right] \\ \vdots & \ddots & \vdots \\ \left[\underline{\mu}_{\widetilde{R}}(v_n, w_1), \overline{\mu}_{\widetilde{R}}(v_n, w_1)\right] & \cdots & \left[\underline{\mu}_{\widetilde{R}}(v_n, w_p), \overline{\mu}_{\widetilde{R}}(v_n, w_p)\right] \end{bmatrix}.$$

Similarly, in (2.17), the matrix constructed by the secondary membership grade $f_{\widetilde{R}}(v_j, w_k)$ is called secondary fuzzy matrix of interval type-2 fuzzy relation, denoted as $\widetilde{R}_f\left(f_{\widetilde{R}}(v_j, w_k)\right)$ and abbreviated as $\widetilde{R}_f$:

$$\widetilde{R}_f = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}.$$

Clearly, (2.18) is composed of the primary fuzzy matrix equation and secondary fuzzy matrix equation.

Definition 4.2. *The IT2 FRE (2.18) can be divided into two parts: primary fuzzy matrix equation and secondary fuzzy matrix equation.*

(1) The primary fuzzy matrix equation is

$$\widetilde{A}_\mu \circ \widetilde{X}_\mu = \widetilde{B}_\mu, \tag{4.1}$$

where $\widetilde{A}_\mu \in I([0,1]^{m \times n})$, $\widetilde{B}_\mu \in I([0,1]^{m \times p})$, $\widetilde{X}_\mu \in I([0,1]^{n \times p})$ and $\widetilde{X}_\mu$ is unknown.
If

$$\widetilde{X}_\mu = \left[\underline{X}_\mu, \overline{X}_\mu\right] \in I([0,1]^{n \times p})$$

satisfies (4.1), then we call that $\widetilde{X}_\mu$ is the solution of (4.1). $\underline{X}_\mu$, $\overline{X}_\mu$ are lower and upper bound matrices of $\widetilde{X}_\mu$, respectively.
If

$$\widetilde{H}_\mu = \left[\underline{H}_\mu, \overline{H}_\mu\right]$$

is a solution of (4.1), and for any solution $\widetilde{X}_\mu$ of (4.1), there is $\widetilde{X}_\mu \leq \widetilde{H}_\mu$, then $\widetilde{H}_\mu$ is called the maximum solution of (4.1).
If

$$\widetilde{J}_\mu = \left[\underline{J}_\mu, \overline{J}_\mu\right]$$

is a solution of (4.1), and for any solution $\widetilde{X}_\mu$ of (4.1), there is $\widetilde{X}_\mu \geq \widetilde{J}_\mu$, then $\widetilde{J}_\mu$ is called the minimal solution of (4.1).
If

$$\widetilde{Q}_\mu = \left[\underline{Q}_\mu, \overline{Q}_\mu\right]$$

is a solution of (4.1), and for any solution $\widetilde{X}_\mu$ of (4.1), as long as $\widetilde{X}_\mu \leq \widetilde{Q}_\mu$ is satisfied, there is $\widetilde{X}_\mu = \widetilde{Q}_\mu$, then $\widetilde{Q}_\mu$ is called the minimum solution of (4.1).

(2) The secondary fuzzy matrix equation is

$$\widetilde{A}_f \circ \widetilde{X}_f = \widetilde{B}_f, \tag{4.2}$$

where $\widetilde{A}_f \in \mathcal{M}_{m \times n}$, $\widetilde{B}_f \in \mathcal{M}_{m \times p}$, $\widetilde{X}_f \in \mathcal{M}_{n \times p}$ and $\widetilde{X}_f$ is unknown.

The matrix $\widetilde{X}_f$ satisfying (4.2) is called the solution of this equation. In (4.2), the elements of $\widetilde{A}_f$ and $\widetilde{B}_f$ are all one, then the elements of $\widetilde{X}_f$ are all one.

4.2. Solvability of primary fuzzy matrix equation

The primary fuzzy matrix Eq (4.1) is equivalent to

$$\begin{cases} \underline{A}_\mu \circ \underline{X}_\mu = \underline{B}_\mu, \\ \overline{A}_\mu \circ \overline{X}_\mu = \overline{B}_\mu, \\ \underline{X}_\mu \leq \overline{X}_\mu. \end{cases} \tag{4.3}$$

The conditions for the establishment of (4.3) are relatively difficult, so we first need to determine whether (4.1) has solutions.

Lemma 4.1. *[24] Let*

$$A = \left(a_{ij}\right)_{m \times n}, \quad B = (b_{ik})_{m \times p}.$$

The T1 FRE $A \circ X = B$ has solutions if, and only if, $A^T \alpha B$ is a solution of this equation and $A^T \alpha B$ is the maximum solution of this equation. The α composition operation between fuzzy matrices is

$$A^T \alpha B = \wedge_{i=1}^{n} \left(a_{ji}\right) \alpha (b_{ik}),$$

where $(a_{ki})\alpha(b_{ij}) = \begin{cases} b_{ij}, a_{ki} > b_{ij}, \\ 1, a_{ki} \leq b_{ij}. \end{cases}$

Theorem 4.1. *If the primary fuzzy matrix Eq (4.1) has solutions then*

$$\widetilde{H}_\mu = \left[\underline{h}_{ik}, \overline{h}_{ik}\right]_{n \times p} = \begin{cases} \left[\underline{H}_\mu, \overline{H}_\mu\right], & \forall \underline{h}_{ik} \leq \overline{h}_{ik}, \\ \left[\underline{H}_\mu', \overline{H}_\mu\right], & \exists \underline{h}_{ik} > \overline{h}_{ik}. \end{cases} \tag{4.4}$$

is a solution of this equation and $\widetilde{H}_\mu$ is the maximum solution of this equation.

In (4.4),

$$\underline{H} = \underline{A}^T \underline{\alpha} \underline{B} = \left(\underline{h}_{ik}\right)_{n \times p}, \quad \overline{H} = \overline{A}^T \overline{\alpha} \overline{B} = \left(\overline{h}_{ik}\right)_{n \times p},$$

when

$$\forall \underline{h}_{ik} \leq \overline{h}_{ik}, \quad \underline{H} = \left(\underline{h}_{ik}\right)_{n \times p}, \quad \overline{H} = \left(\overline{h}_{ik}\right)_{n \times p}.$$

When $\exists \underline{h}_{ik} > \overline{h}_{ik}$, we replace all elements of $\underline{H}$ that do not satisfy $\underline{h}_{ik} \leq \overline{h}_{ik}$ with $\overline{h}_{ik}$; thus, generating a new lower bound matrix $\underline{H}_\mu'$.

Proof. The primary fuzzy matrix Eq (4.1) has solutions, then T1 FREs

$$\underline{A}_\mu \circ \underline{X}_\mu = \underline{B}_\mu \text{ and } \overline{A}_\mu \circ \overline{X}_\mu = \overline{B}_\mu$$

must have solutions. Lemma 4.1 implies that $\underline{H}_\mu$ and $\overline{H}_\mu$ are solutions of T1 FREs

$$\underline{A}_\mu \circ \underline{X}_\mu = \underline{B}_\mu \text{ and } \overline{A}_\mu \circ \overline{X}_\mu = \overline{B}_\mu,$$

respectively. $\underline{H}_\mu$ and $\overline{H}_\mu$ must exist in either of the following two cases.

(1) For $\forall \underline{h}_{ij} \leq \overline{h}_{ij}$, we known that $\underline{H}_\mu \leq \overline{H}_\mu$. $\underline{H}_\mu$ and $\overline{H}_\mu$ are solutions of T1 FREs

$$\underline{A}_\mu \circ \underline{X}_\mu = \underline{B}_\mu \text{ and } \overline{A}_\mu \circ \overline{X}_\mu = \overline{B}_\mu,$$

respectively. Hence,

$$\widetilde{H}_\mu = \left[\underline{H}_\mu, \overline{H}_\mu\right]$$

satisfies (4.3) and $\widetilde{H}_\mu$ is a solution of the primary fuzzy matrix equation.

From Lemma 4.1, it follows that $\underline{H}_\mu$ and $\overline{H}_\mu$ are maximum solutions of T1 FREs

$$\underline{A}_\mu \circ \underline{X}_\mu = \underline{B}_\mu \text{ and } \overline{A}_\mu \circ \overline{X}_\mu = \overline{B}_\mu,$$

respectively. Clearly, $\underline{X}_\mu \leq \underline{H}_\mu$ and $\overline{X}_\mu \leq \overline{H}_\mu$, so

$$\widetilde{H}_\mu = \left[\underline{H}_\mu, \overline{H}_\mu\right]$$

is the maximum solution of the primary fuzzy matrix equation.

(2) For $\exists \underline{h}_{ij} > \overline{h}_{ij}$, we know that the newly generated matrix is $\underline{H}_\mu'$ and the matrix satisfies

$$\widetilde{A}_\mu \circ \underline{H}_\mu' = \widetilde{B}_\mu \text{ and } \underline{H}_\mu' \leq \overline{H}_\mu.$$

According to

$$\widetilde{A}_\mu \circ \underline{H}_\mu' = \widetilde{B}_\mu,$$

$\underline{H}_\mu'$ is a solution of T1 FRE

$$\underline{A}_\mu \circ \underline{X}_\mu = \underline{B}_\mu.$$

From Lemma 4.1, it follows that $\overline{H}_\mu$ is a solution of T1 FRE $\overline{A}_\mu \circ \overline{X}_\mu = \overline{B}_\mu$, respectively. Hence,

$$\widetilde{H}_\mu = \left[\underline{H}_\mu', \overline{H}_\mu\right]$$

satisfies (4.3) and $\widetilde{H}_\mu$ is a solution of the primary fuzzy matrix equation.

From Lemma 4.1, it is known that $\underline{H}_\mu$ and $\overline{H}_\mu$ are, respectively, maximum solutions of T1 FREs

$$\underline{A}_\mu \circ \underline{X}_\mu = \underline{B}_\mu \text{ and } \overline{A}_\mu \circ \overline{X}_\mu = \overline{B}_\mu.$$

According to the requirement that $\underline{X}_\mu \leq \overline{X}_\mu$, we construct a new matrix $\underline{H}_\mu'$ based on $\underline{H}_\mu$. Clearly, $\underline{X}_\mu \leq \underline{H}_\mu'$ and $\overline{X}_\mu \leq \overline{H}_\mu$, so

$$\widetilde{H}_\mu = \left[\underline{H}_\mu', \overline{H}_\mu\right]$$

is the maximum solution of the primary fuzzy matrix equation.

In summary, $\widetilde{H}_\mu$ is a solution of the primary fuzzy matrix Eq (4.1) and is the maximum solution of this equation. $\qquad \square$

If the primary fuzzy matrix Eq (4.1) has solutions, the next step is to explore how to construct parameter set solutions $I^*\left(\widetilde{X}_\mu\right)$ and $I_*\left(\widetilde{X}_\mu\right)$ of this equation.

First, take all the elements in $\widetilde{A}_\mu$ and $\widetilde{B}_\mu$ and place the different upper and lower bounds of these elements in the the ordered set Θ:

$$\Theta = \{\xi_i | i = 1, \cdots, r; 0 = \xi_1 < \xi_2 < \cdots < \xi_r = 1\}.$$

Construct an ordered interval-valued set Ψ by the ordered set Θ, defined as

$$\Psi = \{[\xi_1, \xi_1], [\xi_1, \xi_2], \cdots, [\xi_1, \xi_r]; [\xi_2, \xi_2],$$
$$[\xi_2, \xi_3], \cdots, [\xi_2, \xi_r]; \cdots; [\xi_r, \xi_r]\}.$$

Next, according to the order interval-valued set Ψ, we define two mappings necessary to construct the parameter set solution $I^*\left(\widetilde{X}_\mu\right)$ and $I_*\left(\widetilde{X}_\mu\right)$ of primary fuzzy matrix Eq (4.1).

Definition 4.3. *Assuming* $x \in I_f, \left[\xi_i, \xi_j\right] \in \Psi$.

(1) $I_*: \left[\underline{x}, \overline{x}\right] \to \Psi$ *is*

$$
\begin{aligned}
I_*(x) &= I_*\left(\left[\underline{x}, \overline{x}\right]\right) \\
&= \max\left\{\left[\xi_i, \xi_j\right] \in \Psi \middle| \xi_i \le \underline{x}, \xi_j \le \overline{x}\right\}.
\end{aligned}
\tag{4.5}
$$

(2) $I^*: \left[\underline{x}, \overline{x}\right] \to \Psi$ *is*

$$
\begin{aligned}
I^*(x) &= I^*\left(\left[\underline{x}, \overline{x}\right]\right) \\
&= \min\left\{\left[\xi_i, \xi_j\right] \in \Psi \middle| \xi_i \ge \underline{x}, \xi_j \ge \overline{x}\right\}.
\end{aligned}
\tag{4.6}
$$

Note that 1) When $\underline{x} = \xi_i \in \Xi$, $\overline{x} = \xi_j \in \Xi$,

$$
I_*(x) = I^*(x) = \left[\xi_i, \xi_j\right].
$$

2) When $\underline{x} \notin \Xi$, $\overline{x} = \xi_j \in \Xi$, there exists a unique i such that $\xi_i < x < \xi_{i+1}$, then

$$
I_*(x) = \left[\xi_i, \xi_j\right], I^*(x) = \left[\xi_{i+1}, \xi_j\right].
$$

3) When $\underline{x} = \xi_i \in \Xi$, $\overline{x} \notin \Xi$, there exists a unique j such that $\xi_j < x < \xi_{j+1}$, then

$$
I_*(x) = \left[\xi_i, \xi_j\right], I^*(x) = \left[\xi_i, \xi_{j+1}\right].
$$

4) When $\underline{x} \notin \Xi$, $\overline{x} \notin \Xi$, there exists a unique i and j such that $\xi_i < x < \xi_{i+1}, \xi_j < x < \xi_{j+1}$, then

$$
I_*(x) = \left[\xi_i, \xi_j\right], I^*(x) = \left[\xi_{i+1}, \xi_{j+1}\right].
$$

By Definition 4.3, it is not difficult to derive the following properties.

Property 4.1. *Let*

$$
\widetilde{A}_\mu = \left[\underline{a}_{ij}, \overline{a}_{ij}\right] \in I\left([0,1]^{m \times n}\right),
$$

$$
\widetilde{B}_\mu = \left[\underline{b}_{ik}, \overline{b}_{ik}\right] \in I\left([0,1]^{m \times p}\right),
$$

then,

(1) $I_*(a_{ij}) = I^*(a_{ij}) = a_{ij}; I_*(b_{ik}) = I^*(b_{ik}) = b_{ik}$.
(2) $I_*(\widetilde{A}_\mu) = I^*(\widetilde{A}_\mu = \widetilde{A}_\mu; I_*(\widetilde{B}_\mu) = I^*(\widetilde{B}_\mu) = \widetilde{B}_\mu$.
(3) $I_*(\widetilde{A}_\mu \circ \widetilde{X}_\mu) = I_*(\widetilde{B}_\mu) = \widetilde{B}_\mu; I^*(\widetilde{A}_\mu \circ \widetilde{X}_\mu) = I^*(\widetilde{B}_\mu) = \widetilde{B}_\mu$.
(4) $\widetilde{X}_\mu \le I^*(\widetilde{X}_\mu), I_*(\widetilde{X}_\mu) \le \widetilde{X}_\mu$.

Property 4.2. *Let* $x, y \in I_f$, $x_i, y_i \in I_f$, $i = 1, \cdots, n$, *then*

(1) $I_*(x) \vee I_*(y) = I_*(x \vee y)$; $I^*(x) \vee I^*(y) = I^*(x \vee y)$.
(2) $I_*(x) \wedge I_*(y) = I_*(x \wedge y)$; $I^*(x) \wedge I^*(y) = I^*(x \wedge y)$.
(3) $\bigvee\limits_{i=1}^{n} \left[I_*(x_i) \wedge I_*(y_i)\right] = I_*\left[\bigvee\limits_{i=1}^{n}(x_i \wedge y_i)\right]$.
(4) $\bigvee\limits_{i=1}^{n} \left[I^*(x_i) \wedge I^*(y_i)\right] = I^*\left[\bigvee\limits_{i=1}^{n}(x_i \wedge y_i)\right]$.

Property 4.3. *Let*

$$
\widetilde{A}_\mu = \left[\underline{a}_{ij}, \overline{a}_{ij}\right] \in I\left([0,1]^{m \times n}\right),
$$

$$
\widetilde{X}_\mu = \left[\underline{x}_{jk}, \overline{x}_{jk}\right] \in I\left([0,1]^{n \times p}\right),
$$

then,

(1) $I_*(\widetilde{A}_\mu \circ \widetilde{X}_\mu) = I_*(\widetilde{A}_\mu) \circ I_*(\widetilde{X}_\mu)$.
(2) $I^*(\widetilde{A}_\mu \circ \widetilde{X}_\mu) = I^*(\widetilde{A}_\mu) \circ I^*(\widetilde{X}_\mu)$.

Theorem 4.2. $\widetilde{X}_\mu$ *is a solution of the primary fuzzy matrix Eq (4.1) if, and only if,* $I^*\left(\widetilde{X}_\mu\right)$ *is a solution of the primary fuzzy matrix equation.*

Proof. (Necessity) Assuming that $\widetilde{X}_\mu$ is a solution of the primary fuzzy matrix equation, it is clear that $\widetilde{A}_\mu \circ \widetilde{X}_\mu = \widetilde{B}_\mu$. By Property 4.1, it follows that

$$
I^*\left(\widetilde{A}_\mu \circ \widetilde{X}_\mu\right) = I^*\left(\widetilde{B}_\mu\right) = \widetilde{B}_\mu.
\tag{4.7}
$$

According to the Property 4.3, we know that

$$
I^*\left(\widetilde{A}_\mu \circ \widetilde{X}_\mu\right) = I^*\left(\widetilde{A}_\mu\right) \circ I^*\left(\widetilde{X}_\mu\right).
$$

From (4.7) we have

$$
I^*\left(\widetilde{A}_\mu\right) \circ I^*\left(\widetilde{X}_\mu\right) = \widetilde{B}_\mu.
\tag{4.8}
$$

By the Property 4.1, it is not difficult to obtain $I^*\left(\widetilde{A}_\mu\right) = \widetilde{A}_\mu$. From (4.8) we have

$$
\widetilde{A}_\mu \circ I^*\left(\widetilde{X}_\mu\right) = \widetilde{B}_\mu.
\tag{4.9}
$$

Formula (4.9) shows that $I^*\left(\widetilde{X}_\mu\right)$ is a solution of the primary fuzzy matrix equation.

(Sufficiency) Assuming that $I^*\left(\widetilde{X}_\mu\right)$ is a solution of the primary fuzzy matrix equation, it is clear that $\widetilde{A}_\mu \circ I^*\left(\widetilde{X}_\mu\right) = \widetilde{B}_\mu$. By Property 4.1, it follows that

$$
\widetilde{X}_\mu \le I^*(\widetilde{X}_\mu).
\tag{4.10}
$$

Using Property 2.1, we can get

$$\widetilde{B}_\mu \leq \widetilde{A}_\mu \circ \widetilde{X}_\mu \leq \widetilde{A}_\mu \circ I^*(\widetilde{X}_\mu). \qquad (4.11)$$

Formula (4.11) shows that $\widetilde{X}_\mu$ is a solution of the primary fuzzy matrix equation.

Therefore, the conclusion is correct. □

Similarly, $\widetilde{X}_\mu$ is a solution of the primary fuzzy matrix Eq (4.1) if, and only if, $I_*(\widetilde{X}_\mu)$ is a solution of the primary fuzzy matrix equation.

By Theorem 4.2, we can obtain the following corollary.

Corollary 4.1. *(1) The interval matrix $\widetilde{H}_\mu$ is the maximum solution of primary fuzzy matrix Eq (4.1) if, and only if, $I^*(\widetilde{H}_\mu)$ is the maximum solution of this equation.*

(2) The interval matrix $\widetilde{J}_\mu$ is the minimum solution of primary fuzzy matrix Eq (4.1) if, and only if, $I_*(\widetilde{J}_\mu)$ is the minimum solution of this equation.

(3) The interval matrix $\widetilde{Q}_\mu$ is the minimal solution of primary fuzzy matrix Eq (4.1) if, and only if, $I_*(\widetilde{Q}_\mu)$ is the minimal solution of this equation.

If the primary fuzzy matrix Eq (4.1) has solutions, we next explore how to obtain parameter set solutions of this equation. By Theorem 4.2, the ordered interval-valued set Ψ is sufficient to inscribe the entire parameter set solutions of the primary fuzzy matrix equation.

First, the primary fuzzy matrix equation can be rewritten to

$$\widetilde{A}_\mu \circ Col_k(\widetilde{X}_\mu) = Col_k(\widetilde{B}_\mu), \qquad (4.12)$$

where $k = 1, \cdots, p$.

In (4.12), the kth equality is equivalent to

$$\left(\left[\underline{a}_{i1}, \overline{a}_{i1} \right] \wedge \left[\underline{x}_{1k}, \overline{x}_{1k} \right] \right) \vee \left(\left[\underline{a}_{i2}, \overline{a}_{i2} \right] \wedge \left[\underline{x}_{2k}, \overline{x}_{2k} \right] \right) \\ \vee \cdots \vee \left(\left[\underline{a}_{in}, \overline{a}_{in} \right] \wedge \left[\underline{x}_{nk}, \overline{x}_{nk} \right] \right) = \left[\underline{b}_{ik}, \overline{b}_{ik} \right], \qquad (4.13)$$

where $i = 1, \cdots, m$.

Second, the logical form of the primary fuzzy matrix equation is converted to algebraic form.

For simplicity of presentation, let

$$a_{ij} = \left[\underline{a}_{ij}, \overline{a}_{ij} \right], x_{jk} = \left[\underline{x}_{jk}, \overline{x}_{jk} \right], \quad j = 1, \cdots, n.$$

With the help of Theorem 3.1, the left hand side (LHS) of (4.13) can be expressed in algebraic form:

$$LHS = \left(M_d^s \right)^{n-1} \left[(M_c^s a_{i1} x_{1k}) (M_c^s a_{i2} x_{2k}) \\ \cdots (M_c^s a_{in} x_{nk}) \right], \qquad (4.14)$$

where $i = 1, \cdots, m$.

By Property 2.2, we know that

$$x_{1k}(M_c^s a_{i2} x_{2k}) = (I_s \otimes M_c^s a_{i2}) x_{1k} x_{2k}. \qquad (4.15)$$

According to (4.15), (4.14) is simplified to

$$LHS = \left(M_d^s \right)^{n-1} \left[(M_c^s a_{i1}) (I_s \otimes M_c^s a_{i2}) x_{1k} x_{2k} \\ (M_c^s a_{i3} x_{3k}) \cdots (M_c^s a_{in} x_{nk}) \right]. \qquad (4.16)$$

From Property 2.2, it follows that

$$x_{1k} x_{2k}(M_c^s a_{i3} x_{3k}) = (I_{s^2} \otimes M_c^s a_{i3}) x_{1k} x_{2k} x_{3k}. \qquad (4.17)$$

According to (4.17), (4.16) is further simplified to

$$LHS = \left(M_d^s \right)^{n-1} \left[(M_c^s a_{i1}) (I_s \otimes M_c^s a_{i2}) \\ (I_{s^2} \otimes M_c^s a_{i2}) x_{1k} x_{2k} x_{3k} \cdots (M_c^s a_{in} x_{nk}) \right]. \qquad (4.18)$$

Repeating the process of (4.15)–(4.18), (4.14) is finally expressed as

$$\begin{aligned} LHS &= \left(M_d^s \right)^{n-1} \left[(M_c^s a_{i1}) (I_s \otimes M_c^s a_{i2}) (I_{s^2} \otimes M_c^s a_{i2}) \\ &\quad \cdots (I_{s^{n-1}} \otimes M_c^s a_{in}) \ltimes_{j=1}^n x_{jk} \right] \\ &= \left(M_d^s \right)^{n-1} \left[\left(M_c^s \left[\underline{a}_{i1}, \overline{a}_{i1} \right] \right) \left(I_s \otimes M_c^s \left[\underline{a}_{i2}, \overline{a}_{i2} \right] \right) \\ &\quad \cdots \left(I_{s^{n-1}} \otimes M_c^s \left[\underline{a}_{in}, \overline{a}_{in} \right] \right) \ltimes_{j=1}^n \left[\underline{x}_{jk}, \overline{x}_{jk} \right] \\ &:= L_i[\underline{x}_k, \overline{x}_k], \end{aligned} \qquad (4.19)$$

where $i = 1, \cdots, m$, and

$$\begin{aligned} L_i &= \left(M_d^s \right)^{n-1} M_c^s \left[\underline{a}_{i1}, \overline{a}_{i1} \right] \left(I_s \otimes M_c^s \left[\underline{a}_{i2}, \overline{a}_{i2} \right] \right) \\ &\quad \cdots (I_{s^{n-1}} \otimes M_c^s [a_{in}, a_{in}]) \ltimes_{j=1}^n \left[\underline{x}_{jk}, \overline{x}_{jk} \right], \\ [\underline{x}_k, \overline{x}_k] &= \ltimes_{j=1}^n \left[\underline{x}_{jk}, \overline{x}_{jk} \right], \end{aligned}$$

then (4.19) can be simplified to

$$L_i[\underline{x}_k, \overline{x}_k] = [\underline{b}_{ik}, \overline{b}_{ik}], \qquad (4.20)$$

where $i = 1, \cdots, m$.

Equation (4.20) is equivalent to

$$L[\underline{x}_k, \overline{x}_k] = [\underline{b}_k, \overline{b}_k], \qquad (4.21)$$

where

$$L = L_1 * L_2 * \cdots * L_m,$$

$$[\underline{b}_k, \overline{b}_k] = \ltimes_{i=1}^{m}[\underline{b}_{ik}, \overline{b}_{ik}],$$

where "*" denotes the khatri-rao product of interval matrices.

According to the above procedure, the value of the kth row of $\widetilde{X}_\mu$ can be determined. Let $k = 1, 2, \cdots, p$, and we can obtain the parameter set solutions of the primary fuzzy matrix equation.

A specific algorithm for solving all solutions of IT2 FRE (2.18) is given in the following.

Algorithm 4.1. *The following steps are used to solve the solution set of IT2 FRE (2.18).*

Step 1. *Decompose IT2 FRE (2.18) to construct the primary fuzzy matrix Eq (4.1).*

Step 2. *Use Theorem 4.1 to determine if there are solutions to the primary fuzzy matrix equation. If the primary fuzzy matrix equation has solutions, then proceed as follows; otherwise, IT2 FRE (equ:IT2 FRE(a)) has no solution.*

Step 3. *Construct an ordered set Θ from $\widetilde{A}_\mu$ and $\widetilde{B}_\mu$*

$$\Theta = \{\xi_i | i = 1, \cdots, r; 0 = \xi_1 < \xi_2 < \cdots < \xi_r = 1\}.$$

We specify

$$\xi_i \sim \delta_r^i, \xi_j \sim \delta_r^j, \quad [\xi_i, \xi_j] = \delta_r[i, j].$$

The elements in $\widetilde{A}_\mu$ and $\widetilde{B}_\mu$ can be represented as vectors to facilitate algebraic operations.

Step 4. *Construct (4.12) and convert it into the form of (4.21) to solve for the parameter set solutions of $Col_k(\widetilde{X}_\mu)$.*

Step 5. *Let $k = 1, 2, \cdots, p$, and we can get all parameter set solutions of $(\widetilde{X}_\mu)$. Determine the maximum and minimum (or minimal) solutions of the primary fuzzy matrix equation.*

Step 6. *Finally, based on the solution set of the primary fuzzy matrix equation and secondary fuzzy matrix equation, the solution set $\widetilde{X}$ of IT2 FRE is constructed.*

5. Application

Consider the following IT2 FRE,

$$\widetilde{X} \circ \widetilde{R} = \widetilde{B}, \qquad (5.1)$$

where

$$\widetilde{X} = \begin{bmatrix} \frac{1}{[\underline{x}_{11}, \overline{x}_{11}]} & \frac{1}{[\underline{x}_{12}, \overline{x}_{12}]} \\ \frac{1}{[\underline{x}_{21}, \overline{x}_{21}]} & \frac{1}{[\underline{x}_{22}, \overline{x}_{22}]} \end{bmatrix}, \quad \widetilde{R} = \begin{bmatrix} \frac{1}{[0.3, 0.7]} & \frac{1}{[0.2, 0.3]} \\ \frac{1}{[0.1, 0.5]} & \frac{1}{[0.5, 0.7]} \end{bmatrix},$$

$$\widetilde{B} = \begin{bmatrix} \frac{1}{[0.1, 0.5]} & \frac{1}{[0.1, 0.3]} \\ \frac{1}{[0.2, 0.5]} & \frac{1}{[0.5, 0.7]} \end{bmatrix}.$$

First, taking a transpose on both sides of (5.1), we get

$$\widetilde{R}^T \circ \widetilde{X}^T = \widetilde{B}^T. \qquad (5.2)$$

By decomposing IT2 FRE (5.2), we can obtain the primary fuzzy matrix equation and the secondary fuzzy matrix equation. From Definition 4.1, we only need to solve the primary fuzzy matrix equation to obtain the solution set of IT2 FRE. The primary fuzzy matrix equation of (5.2) can be expressed as

$$\widetilde{R}_\mu^T \circ \widetilde{X}_\mu^T = \widetilde{B}_\mu^T, \qquad (5.3)$$

where

$$\widetilde{R}_\mu^T = \begin{bmatrix} [0.3, 0.7] & [0.1, 0.5] \\ [0.2, 0.3] & [0.5, 0.7] \end{bmatrix}, \quad \widetilde{X}_\mu^T = \begin{bmatrix} [\underline{x}_{11}, \overline{x}_{11}] & [\underline{x}_{21}, \overline{x}_{21}] \\ [\underline{x}_{12}, \overline{x}_{12}] & [\underline{x}_{22}, \overline{x}_{22}] \end{bmatrix},$$

$$\widetilde{B}_\mu^T = \begin{bmatrix} [0.1, 0.5] & [0.2, 0.5] \\ [0.1, 0.3] & [0.5, 0.7] \end{bmatrix}.$$

Next, use Theorem 4.1 to determine if (5.3) has solutions.

$$\underline{H}_\mu = \left(\underline{A}_\mu^T\right)^T \alpha \left(\underline{B}_\mu^T\right)$$

$$= \begin{bmatrix} 0.3 & 0.1 \\ 0.2 & 0.5 \end{bmatrix}^T \alpha \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & 1 \end{bmatrix},$$

$$\overline{H}_\mu = \left(\overline{A}_\mu^T\right)^T \alpha \left(\overline{B}_\mu^T\right)$$

$$= \begin{bmatrix} 0.7 & 0.5 \\ 0.3 & 0.7 \end{bmatrix}^T \alpha \begin{bmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 0.3 & 1 \end{bmatrix},$$

$$\widetilde{H}_\mu = \begin{bmatrix} [0.1, 0.5] & [0.2, 0.5] \\ [0.1, 0.3] & [1, 1] \end{bmatrix},$$

$$\widetilde{R}_\mu^T \circ \widetilde{H}_\mu = \begin{bmatrix} [0.1, 0.5] & [0.2, 0.5] \\ [0.1, 0.3] & [0.5, 0.7] \end{bmatrix} = \widetilde{B}_\mu^T.$$

According to the above calculation, $\underline{H}_\mu \leq \overline{H}_\mu$, $\widetilde{H}_\mu$ is a solution of (5.3) and $\widetilde{H}_\mu$ is the maximum solution of this equation.

Let

$$Col_1\left(\widetilde{X}_\mu^T\right) = \left[[\underline{x}_{11}, \overline{x}_{11}] \, [\underline{x}_{21}, \overline{x}_{21}]\right]^T,$$

which needs to satisfy the following logical equation.

$$\begin{cases} \left([0.3, 0.7] \wedge [\underline{x}_{11}, \overline{x}_{11}]\right) \vee \left([0.1, 0.5] \wedge [\underline{x}_{21}, \overline{x}_{21}]\right) = [0.1, 0.5], \\ \left([0.2, 0.3] \wedge [\underline{x}_{11}, \overline{x}_{11}]\right) \vee \left([0.5, 0.7] \wedge [\underline{x}_{21}, \overline{x}_{21}]\right) = [0.1, 0.3]. \end{cases} \quad (5.4)$$

However, solving (5.4) directly is relatively difficult, so it needs to be converted into algebraic form.

Construct the ordered set Θ based on $\widetilde{A}_\mu$ and $\widetilde{B}_\mu$:

$$\Theta = \{0, 0.1, 0.2, 0.3, 0.5, 1\}.$$

Represent the elements in Θ as vectors

$$1 \sim \delta_7^1; \quad 0.7 \sim \delta_7^2; \quad 0.5 \sim \delta_7^3; \quad 0.3 \sim \delta_7^4;$$
$$0.2 \sim \delta_7^5; \quad 0.1 \sim \delta_7^6; \quad 0 \sim \delta_7^7.$$

Convert (5.4) into an algebraic equation

$$\begin{cases} M_d^7\left(M_c^7\delta_7\,[4,2]\,[\underline{x}_{11}, \overline{x}_{11}]\right)\left(M_c^7\delta_7\,[6,3]\,[\underline{x}_{21}, \overline{x}_{21}]\right) = \delta_7\,[6,3], \\ M_d^7\left(M_c^7\delta_7\,[5,4]\,[\underline{x}_{11}, \overline{x}_{11}]\right)\left(M_c^7\delta_7\,[3,2]\,[\underline{x}_{21}, \overline{x}_{21}]\right) = \delta_7\,[6,4]. \end{cases}$$

Let

$$\left[\underline{x}_1, \overline{x}_1\right] = \left[\underline{x}_{11}, \overline{x}_{11}\right] \ltimes \left[\underline{x}_{21}, \overline{x}_{21}\right],$$

which is equivalent to

$$\begin{cases} M_d^7 M_c^7 \delta_7\,[4,2]\left(I_7 \otimes M_c^7\delta_7\,[6,3]\right)\left[\underline{x}_1, \overline{x}_1\right] = \delta_7\,[6,3], \\ M_d^7 M_c^7 \delta_7\,[5,4]\left(I_7 \otimes M_c^7\delta_7\,[3,2]\right)\left[\underline{x}_1, \overline{x}_1\right] = \delta_7\,[6,4]. \end{cases}$$

Let

$$L_1 = M_d^7 M_c^7 \left[\delta_7^4, \delta_7^2\right]\left(I_7 \otimes M_c^7 \left[\delta_7^6, \delta_7^3\right]\right),$$

$$L_2 = M_d^7 M_c^7 \left[\delta_7^5, \delta_7^4\right]\left(I_7 \otimes M_c^7 \left[\delta_7^3, \delta_7^2\right]\right).$$

This leads to

$$L \ltimes [\underline{x}_1, \overline{x}_1] = [\underline{b}_1, \overline{b}_1]. \quad (5.5)$$

The MATLAB program provided in the literature [15] is improved so that it can calculate the STP of the interval matrix. (5.5) is calculated as

$$\begin{aligned} L =\,& L_1 * L_2 \\ =\,& \delta_{49}\,[[24,9][24,9][24,10][25,11][26,11][26,11][26,11] \\ & [24,9][24,9][24,10][25,11][26,11][26,11][26,11] \\ & [24,16][24,16][24,17][25,18][26,18][26,18][26,18] \\ & [24,16][24,16][24,17][25,25][26,25][26,25][26,25] \\ & [31,16][31,16][31,17][32,25][33,33][33,33][33,33] \\ & [38,16][38,16][38,17][39,25][40,33][41,41][41,41] \\ & [38,16][38,16][38,17][39,25][40,33][41,41][49,49]], \end{aligned}$$

$$[\underline{b}_1, \overline{b}_1] = \left[\delta_7^6, \delta_7^3\right] \ltimes \left[\delta_7^6, \delta_7^4\right] = \left[\delta_{49}^{41}, \delta_{49}^{18}\right].$$

Solving for (5.5), we get

$$[\underline{x}_1, \overline{x}_1] = [\delta_{49}^i, \delta_{49}^j],$$

where $i = 41, 42, 48$, $j = 18, 19, 20, 21$.

From the values of $[\underline{x}_1, \overline{x}_1]$, there are $3 \times 4 = 12$ parameter set solutions for $Col_1\left(\widetilde{X}_\mu^T\right)$, two of which do not satisfy $\underline{x}_{i1} \leq \overline{x}_{i1}$ $(i = 1, 2)$; then

$$\left(\widetilde{X}_\mu^T\right)_1^1 = \delta_7[[6,3]\,[6,4]]^T \sim [[0.1,0.5][0.1,0.3]]^T;$$
$$\left(\widetilde{X}_\mu^T\right)_1^2 = \delta_7[[6,3]\,[6,5]]^T \sim [[0.1,0.5][0.1,0.2]]^T;$$
$$\left(\widetilde{X}_\mu^T\right)_1^3 = \delta_7[[6,3]\,[6,6]]^T \sim [[0.1,0.5][0.1,0.1]]^T;$$
$$\left(\widetilde{X}_\mu^T\right)_1^4 = \delta_7[[6,3]\,[7,4]]^T \sim [[0.1,0.5][0,0.3]]^T;$$
$$\left(\widetilde{X}_\mu^T\right)_1^5 = \delta_7[[6,3]\,[7,5]]^T \sim [[0.1,0.5][0,0.2]]^T;$$
$$\left(\widetilde{X}_\mu^T\right)_1^6 = \delta_7[[6,3]\,[7,6]]^T \sim [[0.1,0.5][0,0.1]]^T;$$
$$\left(\widetilde{X}_\mu^T\right)_1^7 = \delta_7[[6,3]\,[7,7]]^T \sim [[0.1,0.5][0,0.0]]^T;$$
$$\left(\widetilde{X}_\mu^T\right)_1^8 = \delta_7[[7,3]\,[6,4]]^T \sim [[0,0.5][0.1,0.3]]^T;$$
$$\left(\widetilde{X}_\mu^T\right)_1^9 = \delta_7[[7,3]\,[6,5]]^T \sim [[0,0.5][0.1,0.2]]^T;$$
$$\left(\widetilde{X}_\mu^T\right)_1^{10} = \delta_7[[7,3]\,[6,6]]^T \sim [[0,0.5][0.1,0.1]]^T.$$

Assuming

$$Col_2\left(\widetilde{X}_\mu^T\right) = \left[[\underline{x}_{12}, \overline{x}_{12}]\,[\underline{x}_{22}, \overline{x}_{22}]\right]^T,$$

we have

$$L \ltimes [\underline{x}_2, \overline{x}_2] = [\underline{b}_2, \overline{b}_2], \quad (5.6)$$

where the value of L has been obtained in (5.5),

$$[\underline{b}_2, \overline{b}_2] = \delta_7\,[5,3] \ltimes \delta_7\,[3,2] = \delta_{49}\,[31,16].$$

Solving for (5.6), we get

$$[\underline{x}_2, \overline{x}_2] = [\delta_{49}^i, \delta_{49}^j],$$

where $i = 29, 30, 31$, $j = 15, 16, 22, 23, 29, 30, 36$.

Depending on the value of $[\underline{x}_2, \overline{x}_2]$, it follows that $Col_2\left(\widetilde{X}_\mu^T\right)$ has $3 \times 7 = 21$ parameter set solutions, six of which do not satisfy $\underline{x}_{i2} \le \overline{x}_{i2}$ ($i = 1, 2$); then

$$\left(\widetilde{X}_\mu^T\right)_2^1 = \delta_7[[5, 3][1, 1]]^T \sim [[0.2, 0.5][1, 1]]^T;$$
$$\left(\widetilde{X}_\mu^T\right)_2^2 = \delta_7[[5, 4][1, 1]]^T \sim [[0.2, 0.3][1, 1]]^T;$$
$$\left(\widetilde{X}_\mu^T\right)_2^3 = \delta_7[[5, 5][1, 1]]^T \sim [[0.2, 0.2][1, 1]]^T;$$
$$\left(\widetilde{X}_\mu^T\right)_2^4 = \delta_7[[5, 3][2, 1]]^T \sim [[0.2, 0.5][0.7, 1]]^T;$$
$$\left(\widetilde{X}_\mu^T\right)_2^5 = \delta_7[[5, 3][2, 2]]^T \sim [[0.2, 0.5][0.7, 0.7]]^T;$$
$$\left(\widetilde{X}_\mu^T\right)_2^6 = \delta_7[[5, 4][2, 1]]^T \sim [[0.2, 0.3][0.7, 1]]^T;$$
$$\left(\widetilde{X}_\mu^T\right)_2^7 = \delta_7[[5, 4][2, 2]]^T \sim [[0.2, 0.3][0.7, 0.7]]^T;$$
$$\left(\widetilde{X}_\mu^T\right)_2^8 = \delta_7[[5, 5][2, 1]]^T \sim [[0.2, 0.2][0.7, 1]]^T;$$
$$\left(\widetilde{X}_\mu^T\right)_2^9 = \delta_7[[5, 5][2, 2]]^T \sim [[0.2, 0.2][0.7, 0.7]]^T;$$
$$\left(\widetilde{X}_\mu^T\right)_2^{10} = \delta_7[[5, 3][3, 1]]^T \sim [[0.2, 0.5][0.5, 1]]^T;$$
$$\left(\widetilde{X}_\mu^T\right)_2^{11} = \delta_7[[5, 3][3, 2]]^T \sim [[0.2, 0.5][0.5, 0.7]]^T;$$
$$\left(\widetilde{X}_\mu^T\right)_2^{12} = \delta_7[[5, 4][3, 1]]^T \sim [[0.2, 0.3][0.5, 1]]^T;$$
$$\left(\widetilde{X}_\mu^T\right)_2^{13} = \delta_7[[5, 4][3, 2]]^T \sim [[0.2, 0.3][0.5, 0.7]]^T;$$
$$\left(\widetilde{X}_\mu^T\right)_2^{14} = \delta_7[[5, 5][3, 1]]^T \sim [[0.2, 0.2][0.5, 1]]^T;$$
$$\left(\widetilde{X}_\mu^T\right)_2^{15} = \delta_7[[5, 5][3, 2]]^T \sim [[0.2, 0.2][0.5, 0.7]]^T.$$

In summary, we can conclude that:

(1) The primary fuzzy matrix Eq (5.3) has a total of $10 \times 15 = 150$ parameter set solutions.

(2) The maximum solution of this equation is

$$\widetilde{H}_\mu = \left[\left(\widetilde{X}_\mu^T\right)_1^1, \left(\widetilde{X}_\mu^T\right)_2^1\right] = \left[\begin{array}{cc} [0.1, 0.5] & [0.2, 0.5] \\ [0.1, 0.3] & [1, 1] \end{array}\right].$$

(3) The equation has no minimum solution and only two minimal solutions,

$$\left(\widetilde{Q}_\mu\right)_1 = \left[\left(X_\mu^T\right)_1^7, \left(X_\mu^T\right)_2^{15}\right] = \left[\begin{array}{cc} [0.1, 0.5] & [0.2, 0.2] \\ [0, 0] & [0.5, 0.7] \end{array}\right],$$

$$\left(\widetilde{Q}_\mu\right)_2 = \left[\left(X_\mu^T\right)_1^{10}, \left(X_\mu^T\right)_2^{15}\right] = \left[\begin{array}{cc} [0, 0.5] & [0.2, 0.2] \\ [0.1, 0.1] & [0.5, 0.7] \end{array}\right].$$

(4) Based on the maximum and minimal solutions of the primary fuzzy matrix equation, we can work out all the parameter set solutions of the primary fuzzy matrix equation.

$$\left(\widetilde{X}_\mu^T\right)_1 = \left[\begin{array}{cc} [0.1, 0.5] & [0.2, 0.2 \le \overline{x}_{12} \le 0.5] \\ [0 \le \underline{x}_{21} \le 0.1, 0 \le \overline{x}_{21} \le 0.3] & [0.5 \le \underline{x}_{22} \le 1, 0.7 \le \overline{x}_{22} \le 1] \end{array}\right],$$

$$\left(\widetilde{X}_\mu^T\right)_2 = \left[\begin{array}{cc} [0 \le \underline{x}_{11} \le 0.1, 0.5] & [0.2, 0.2 \le \overline{x}_{12} \le 0.5] \\ [0.1, 0.1 \le \overline{x}_{21} \le 0.3] & [0.5 \le \underline{x}_{22} \le 1, 0.7 \le \overline{x}_{22} \le 1] \end{array}\right].$$

(5) The solution set of IT2 FRE is

$$\left(\widetilde{X}_\mu^T\right)_1 = \left[\begin{array}{cc} \frac{1}{[0.1, 0.5]} & \frac{1}{[0 \le \underline{x}_{21} \le 0.1, 0 \le \overline{x}_{21} \le 0.3]} \\ \frac{1}{[0.2, 0.2 \le \overline{x}_{12} \le 0.5]} & \frac{1}{[0.5 \le \underline{x}_{22} \le 1, 0.7 \le \overline{x}_{22} \le 1]} \end{array}\right],$$

$$\left(\widetilde{X}_\mu^T\right)_2 = \left[\begin{array}{cc} \frac{1}{[0 \le \underline{x}_{11} \le 0.1, 0.5]} & \frac{1}{[0.1, 0.1 \le \overline{x}_{21} \le 0.3]} \\ \frac{1}{[0.2, 0.2 \le \overline{x}_{12} \le 0.5]} & \frac{1}{[0.5 \le \underline{x}_{22} \le 1, 0.7 \le \overline{x}_{22} \le 1]} \end{array}\right].$$

6. Conclusions

This paper focused on the solution of IT2 FRE $\widetilde{A} \circ \widetilde{X} = \widetilde{B}$. First, the STP of interval matrices and its properties were introduced, and the matrix representation of the interval-valued logic was given. Then, the IT2 FRE was considered as the primary fuzzy matrix equation and secondary fuzzy matrix equation. The solution of secondary fuzzy matrix is known, so only the primary fuzzy matrix equation needs to be solved. Moreover, the solvability of the primary matrix equation was studied, and a specific algorithm for solving IT2 FREs based on the STP of interval matrices was given. Finally, a numerical example was given to verify the effectiveness of the proposed method.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that there are no conflicts of interest in this paper.

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