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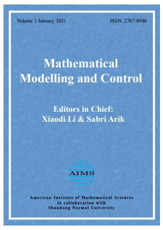

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Review

Recent advances of finite-field networks

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Abstract: Finite-field networks (FFNs) are a class of multi-agent systems over finite fields with sensing, computing, and communication capabilities. FFNs have been investigated extensively to save computing and communication resources. This paper summarizes the current research results to provide a direction for future research. First, different models of FFNs are reviewed, including FFNs with time-delays, switching topology, and leader-following structures. Then, the consensus and synchronization problems of multi-agent systems over finite fields are analyzed, and the necessary and sufficient conditions for consensus and synchronization of some autonomous systems have been derived in recent research. Finally, the distributed control of multi-agent systems over finite fields has been developed by many scholars based on various approaches.

Keywords: multi-agent system; finite-field network; controllability; consensus; semi-tensor product (STP) of matrices

1. Introduction

A multi-agent system consists of a group of agents or nodes who communicate with each other based on local information and aims to achieve some purpose under contro [1, 2]. Due to its broad applications, many critical problems are investigated in multi-agent systems, such as consensus [3, 4], controllability [5, 6] and stabilizability [7, 8]. Among these practical research directions of multi-agent systems, the consensus problem is one of the most important problems, which requires members of a network to reach an agreement on certain information of interest [9]. Scholars have also studied the consensus problem with structures such as time-delay and switching signal [3, 10]. It is worth mentioning that research on consensus has been applied in many practical areas, including robotics, drones, robotic arm collaboration, and other directions. The controllability of multi-agent systems with different structures [11–13] has been studied which was proposed by Tanner in 2004 [14]. Then the structural controllability submitted by Lin [15] in the control system was introduced into the multi-agent systems [16, 17].

Due to the limitation of communication bandwidth, memory constraints, and information safety, many scholars employ finite fields rather than the fields of real numbers to model multi-agent systems [18]. It means the system takes values from finite sets, and operations are performed according to modular arithmetic. In 2013, Shreyas and Christoforos [19] investigated the conditions for structural controllability and observability of linear systems over finite fields. Subsequently, Lu et al. [20] extended the results of [19] to higher dimensions and studied the theory of structural controllability of general linear dynamics and switching topology over finite fields. Pasqualetti [18] provided the necessary and sufficient conditions for the consensus of networks over finite fields based on graph theory and the characteristic polynomial in 2014. The results of finite-field consensus [18] were then expanded by Li et al. to the case with time delays and switching topologies [21, 22]. Meng [23] developed the synchronization problem of finite-field networks (FFNs) and gave some sufficient conditions for synchronization based on graph theory and characteristic polynomials of network matrices. In 2022, Zhu [24] studied the synchronization problem of FFNs with time-delays, strengthened some conclusions obtained by Li et al. [21, 22]. Xu and Hong [25] investigated the leader-following consensus problem of multi-agent systems with dynamics of high dimensions over finite fields, requiring that the interaction graph of the FFNs was a directed acyclic graph. They provided consensusability conditions for fixed and switching topologies. Then the controllability of multi-agent systems with switching topology over finite fields was developed in [26].

In addition to the above study, Li et al. [27–29] proposed a new approach to study the consensus problem and the controllability of FFNs via the semi-tensor product (STP) of matrices and obtained the necessary and sufficient conditions of consensus with switching topology and controllability of multi-agent systems. With the help of STP, the leader-follower consensus of multi-agent systems with time-delays was researched [30], and some consensus criteria were presented based on set stability. Then the set stability was applied to study switched delayed logical networks [31], consensus of FFNs with stochastic time-delays [32] and containment problem of FFNs with fixed and switching topologies [33].

The rest of this paper is organized as follows. In Section 2, some necessary preliminaries are presented. Section 3-5 introduce the latest results in FFNs research, including models, analysis, and control of FFNs. Section 6 gives a brief summary and prospect of this paper.

2. Preliminaries

2.1. Preliminaries on STP

For a matrix A, the i-th column and j-th row are denoted by $Col_i(A)$ and $Row_j(A)$, respectively. $\mathcal{D}_k := \{0, \dots, k-1\}$, $\mathcal{D}_k^n := \mathcal{D}_k \times \cdots \times \mathcal{D}_k$. $\Delta_n := \{Col_k(I_n), k = 1, \dots, n\}$, where I_n is the n-dimensional identity matrix. $\mathcal{L}_{n\times t}$:= set of $n \times t$ logical matrices. $\mathbf{1_n} := [\underbrace{1, \dots, 1}_{\mathbf{n}}]^{\mathrm{T}}$.

Definition 2.1. *[34] Let $A = [a_{ij}] \in \mathbb{R}^{m\times n}$, $B = [b_{ij}] \in \mathbb{R}^{p\times q}$, the Kronecker product of matrices A and B is*

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{21}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix} \in \mathcal{M}_{mp\times nq}. \tag{2.1}$$

Definition 2.2. *[35] Let $A \in \mathbb{R}^{m\times r}, B \in \mathbb{R}^{n\times r}$, the Khatri-Rao product of matrices A and B is denoted by*

$$A * B = [Col_1(A) \ltimes Col_1(B), \dots, Col_r(A) \ltimes Col_r(B)]. \tag{2.2}$$

Definition 2.3. *[36] The STP of matrices A and B is defined as*

$$A \ltimes B = (A \otimes I_{t/n})(B \otimes I_{t/p}) \in \mathcal{M}_{(mt/n)\times(qt/p)},\ t = lcm(n, p). \tag{2.3}$$

Lemma 2.1. *[36] Let $f(x_1, x_2, \dots, x_n) : \mathcal{D}_k^n \mapsto \mathcal{D}_k$ be a k-valued logical function. Then, there exists a unique matrix $M_f \in \mathcal{L}_{k\times k^n}$, called the structural matrix of f, such that*

$$f(x_1, x_2, \dots, x_n) = M_f \ltimes_{i=1}^n x_i,\ x_i \in \Delta_k, \tag{2.4}$$

where $\ltimes_{i=1}^n x_i = x_1 \ltimes x_2 \ltimes \cdots \ltimes x_n$.

2.2. Preliminaries on finite fields and graph theory

The definition and properties of finite fields are given as follows. A finite field $\mathbb{F}$ is a set of elements with addition "+" and multiplication "·" satisfying the following axioms:

• Closure under addition and multiplication. For $\forall u, v \in \mathbb{F}$, $u + v \in \mathbb{F}$ and $u \cdot v \in \mathbb{F}$ hold;

• Associativity of addition and multiplication. For $\forall u$, v, $w \in \mathbb{F}$, $u + (v + w) = (u + v) + w$ and $u \cdot (v \cdot w) = (u \cdot v) \cdot w$ hold;

• Commutativity of addition and multiplication. For $\forall u$, $v \in \mathbb{F}$, it holds $u + v = v + u$, $u \cdot v = v \cdot u$;

• Distributivity of multiplication over addition. For $\forall u$, v, $w \in \mathbb{F}$, it holds $u \cdot (v + w) = u \cdot v + u \cdot w$;

• Existence of additive and multiplicative identity elements. For $\forall u \in \mathbb{F}$, $\exists$ elements $0, 1 \in \mathbb{F}$, such that $u+0 = u$ and $u \cdot 1 = u$;

• Existence of additive and multiplicative inverse elements. For $\forall u \in \mathbb{F}$, $\exists -u, u^{-1} \in \mathbb{F}$, such that $u + (-u) = 0$ and $u \cdot u^{-1} = 1$, with $u \neq 0$.

The field $\mathbb{F}$ is finite if and only if the number of elements in the field is finite. In this study, the finite field is considered as a prime field, i.e., the number of elements in the finite field is prime. In $\mathbb{F}_p$, $\mathbb{F}_p = \{0, 1, ..., p-1\}$, p is a prime number, with addition operator "$+_p$" and the multiplication operator "$\times_p$" defined as in modular arithmetic.

(i) The structural matrix of "$+_p$" is

$$M_{+,p} = \delta_p[U_1, U_2, ..., U_p], \tag{2.5}$$

where $U_1 = (1, ..., p)$, $U_s = (s, ..., p, 1, ..., s-1)$.

(ii) The structural matrix of "$\times_p$" is

$$M_{\times,p} = \delta_p[V_1, V_2, ..., V_p], \tag{2.6}$$

where $V_s = ((0 \times s)mod(p) + 1, (1 \times s)mod(p) + 1, ..., ((p-1) \times s)mod(p) + 1))$, $s = 1, ..., p$.

Finally, we recall some standard definitions in graph theory. A directed graph $\mathcal{G} = (\mathcal{V}, \varepsilon)$, consists of a set of vertices $\mathcal{V}$ and a set of edges $\varepsilon \subseteq \mathcal{V} \times \mathcal{V}$. An edge $(v_i, v_j) \in \varepsilon$ is directed from vertex v_j to vertex v_i. For a vertex $v_i \in \mathcal{V}$, the set of neighbors of v_i is defined as $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_i, v_j) \in \varepsilon\}$. The adjacency matrix of $\mathcal{G}$ is defined as $A = (a_{ij}) \in \mathbb{F}_p^{n \times n}$: if $v_i \in \mathcal{N}_i$, $a_{ij} \neq 0$, $a_{ij} = 0$, otherwise. A path in $\mathcal{G}$ is a subgraph $P = (\{v_1, ..., v_{k+1}\}, \{e_1, ..., e_k\})$ such that $v_i \neq v_j$ for all $i \neq j$, and $e_i = (v_{i+1}, v_i)$ for each $i \in \{1, ..., k\}$. A cycle is a path in which the first and last vertex in the sequence are the same.

3. Different models of the networks over finite fields

Due to the different dynamical behaviors of multi-agent systems and the uncertainty of communication topology, there are many different dynamics of multi-agent systems in finite fields. This section will introduce several models of FFNs.

3.1. Models of multi-agent systems over finite fields

Consider a network with $n \in \mathbb{N}$ agents over the finite field $\mathbb{F}_p$ [18], where $\mathbb{F}_p$ is defined in last section. The communication topology between agents is described by the directed graph $\mathcal{G} = (\mathcal{V}, \varepsilon)$ and requires each agent to manipulate and transmit values over $\mathbb{F}_p$ according to a distributed protocol. Let $x_i(t) \in \mathbb{F}_p$ denotes the state of the i-th agent at time t. Then, the evolution of the network state $x(t) = [x_1(t), ..., x_n(t)]^{\mathrm{T}}$ can be described by the network:

$$x(t+1) = Ax(t). \tag{3.1}$$

3.2. Models of multi-agent systems with switching topology and time-delays over finite fields

The deferred response between multi-agents and sensors sometimes leads to delays in systems. A form of FFNs with time-delays was proposed in [21], and its dynamics is

$$x_i(t+1) = a_{ii}x_i(t) + \sum_{j \in \mathcal{N}_i} a_{ij}x_j(t - \tau_{ij}), i = 1, 2, \ldots, N. \tag{3.2}$$

If all time-delays are constant and equal, let $x(t) = [x_1(t), ..., x_n(t)]^{\mathrm{T}}$, the dynamics of the network can be rewritten as

$$x(t+1) = Bx(t) + Cx(t - \tau), \tag{3.3}$$

where $B = diag(A), C = A - B$. If $\tau_{ij}(t)$ is independent of time t for $i \neq j$, and denoted by τ_{ij}, the dynamics of the network can be rewritten as

$$x(t+1) = B_0x(t) + C_1x(t-1) + C_2x(t-2) + \cdots + C_{\tau_0}x(t - \tau_0), \tag{3.4}$$

where $B_0 = diag(A)$, and $C_k, k = 1, 2, \ldots, \tau_0$, respectively, represent an interaction matrix of the agents that send information by time-delay k. So, one of c_{ij}^k in matrix $C_k, k = 1, 2, \ldots, \tau_0$ equals a_{ij} in matrix A. Thus, $A = B_0 + \sum_{k=1}^{\tau_0} C_k$. The dynamics of a network with time-delays was also described in the synchronization problem [24].

Both [27] and [22] proposed switched networks over finite fields, the evolution of the FFN with switching topology and linear protocols can be described as

$$x(t+1) = A_{\sigma(t)}x(t), \tag{3.5}$$

where $\sigma : \mathbb{N} \mapsto \{1, \ldots, w\}$ is the switching signal.

In addition, [22] also studied FFNs with switching topology and time-delays:

$$x_i(t+1) = \sum_{j \in \mathcal{N}_i \bigcup \{i\}} a_{ij}^{\sigma(t)}x_j(t - \tau_{ij}), i = 1, 2, \ldots, N. \tag{3.6}$$

In [22], the network can be rewritten as

$$\begin{aligned} x(t+1) =& C_{0,\sigma(t)}x(t) + C_{1,\sigma(t)}x(t-1) + C_{2,\sigma(t)}x(t-2) + \ldots \\ &+ C_{\tau_0,\sigma(t)}x(t - \tau_0), \end{aligned} \tag{3.7}$$

where $C_{k,\sigma(t)}, k = 0, 1, \ldots, \tau$ is determined by time-delay k that is experienced by information transmission on the link received at time t. It always holds

$$\sum_{k=0}^{\tau} C_{k,s(t)} = A_{s(t)}. \tag{3.8}$$

Then [32] analyzed FFNs with two kinds of stochastic time-delays.

$$x_i(t+1) = \sum_{j \in \mathcal{N}_i^{in} \bigcup i} a_{ij} x_j(t - d_t). \tag{3.9}$$

where stochastic time-delays is given as follows:

(i) Probabilistic time-delay: $\mathbb{P}\{d(t) = l\} = p_l \geq 0$ satisfying $\sum_{l=0}^{\tau} p_l = 1, \forall l \in \{0, \ldots, \tau\}$.

(ii) Markov jump time-delay: d(t) is a Markov chain. $\mathbb{P}\{d(t+1) = s | d(t) = l\} = p_{s,l} \geq 0, s, l \in \{0, \ldots, \tau\}$, and $\sum_{l=0}^{\tau} p_{s,l} = 1, \forall l \in \{0, \ldots, \tau\}$.

3.3. Models of leader-follower multi-agent systems over finite fields

In [25], consider a leader-follower multi-agent system with one leader and N followers. For the leader, the system is autonomous. For each follower, the system can obtain local information input from itself and its neighbors. The dynamics of the leader is described by a autonomous system:

$$x_0(t+1) = Ax_0(t). \tag{3.10}$$

The dynamics of the i-th follower is described by a linear control system:

$$x_i(t+1) = Ax_i(t) + bu_i(k), \tag{3.11}$$

where $x_i(t) \in \mathbb{F}_p^n$, b is a column vector and $u_i(k)$ is the input.

A multi-agent system consists of M leaders and $N - M$ followers in [33]. The dynamics of the leader can be given as the following form:

$$x_l(t+1) = Ax_l(t). \tag{3.12}$$

The dynamics of the f-th follower can be given as the following form:

$$x_f(t+1) = \sum_{j \in \mathcal{N}_i \bigcup \{f\}} a_{fj} x_j(t). \tag{3.13}$$

Lu et al. [20] proposed the dynamics of a system which is different from (3.12) and (3.13), distinguishing leader and follower through external control. For each follower, there is a following linear dynamical system:

$$x_i(t+1) = Ax_i(t) + Bu_i(t). \tag{3.14}$$

For each leader, the linear dynamical system is given by

$$x_i(t+1) = Ax_i(t) + Bu_i(t) + u_i^{ext}(t), \tag{3.15}$$

where $u_i(t) \in \mathbb{F}_p^m$ and $u_i^{ext}(t) \in \mathbb{F}_p^n$ are the control input of i-th agent and the external control input of i-th agent, respectively.

Lu et al. [26] developed a multi-agent system over finite fields which consists of $\mathcal{N}$ agents. For the given multi-agent system, an agent is said to be a leader if external control inputs actuate the agent; an agent is said to be a follower if the agent obeys linear distributed protocol. It holds that $\mathcal{N} = \mathcal{N}_f \bigcup \mathcal{N}_l$ and $\mathcal{N}_f \bigcap \mathcal{N}_l = \emptyset$. The dynamics of the leader-follower multi-agent system is given as follows:

$$x_i(t+1) = a_{ii} x_i(t) + \sum_{j \in \mathcal{N}_i} a_{ij} x_j(t), i \in \mathcal{N}_f, \tag{3.16}$$

$$x_i(t+1) = a_{ii} x_i(t) + \sum_{j \in \mathcal{N}_i} a_{ij} x_j(t) + u_i(t), i \in \mathcal{N}_l, \tag{3.17}$$

where $x_i(t) \in \mathbb{F}_p$.

In [19, 26], for system model (3.16) and (3.17), it can be written into a compact form:

$$x(t+1) = Ax(t) + Bu(k). \tag{3.18}$$

Due to link failure or creation, the communication topology of expressing the information flow among agents may vary at times. Consider the following switched multi-agent system:

$$x(t+1) = A_{\sigma(t)} x(t) + B_{\sigma(t)} u(k). \tag{3.19}$$

Similarly, there are time-delays in leader-follower multi-agent systems. A leader-follower multi-agent system with one leader and N followers was considered in [28, 30]. The dynamics of the leader is given as follows:

$$x_0(t+1) = A_0 x_0(t - \tau_0), \tag{3.20}$$

where $x_0(t) = (x_0^1(t), \dots, x_0^n(t))^T \in \mathbb{F}_p^n$. The dynamics of the i-th follower ($i \in \{1, \dots, N\}$) is given as follows:

$$x_i(t+1) = A_i x_i(t - \tau_{ij}), \tag{3.21}$$

or

$$x_i(t+1) = a_{ii} x_i(t) + \sum_{j \in \mathcal{N}_i \bigcup \{i\}} a_{ij} x_j(t - \tau_{ij}), \tag{3.22}$$

or

$$x_i(t+1) = a_{ii} x_i(t) + \sum_{j \in \mathcal{N}_i \bigcup \{i\}} a_{ij}^{\sigma(t)} x_j(t - \tau_{ij}), \tag{3.23}$$

where $x_i(t) = (x_i^1(t), \dots, x_i^n(t))^T \in \mathbb{F}_p^n$.

In [28, 30], the follower has several dynamics. The i-th follower in (3.21) updates the state of the agent according to the initial condition of the states and dynamic matrix of the follower's autonomous system. The i-th follower in (3.22) updates the state according to the initial condition of the states and weighted adjacency matrix associated with the directed graph $\mathcal{G}$. In addition, the dynamics of the follower in [30] adds a switching signal. Note that each agent in those mentioned above leader-following multi-agent systems except (3.16) and (3.17) is an n-dimensional vector over finite field $\mathbb{F}_p$. Each agent in (3.16), (3.17), and other models without leader-following structure is a 1-dimensional vector, so these systems are $|\mathcal{N}|$-dimensional. Therefore, using various methods, scholars have developed the research of multi-agent systems over finite field $\mathbb{F}_p$ based on the model's differences in terms of communication topology as well as state dimensionality. These progress can be summarized in two aspects: analysis and control of FFNs. The following sections will present these methods and the results obtained in the analysis and control of FFNs.

4. Several analysis results of the networks over finite fields

In the research related to FFNs, consensus is the most fundamental and important research direction, and many complete and exceptional outcomes have been obtained. Graph theory, characteristic polynomials and matrix STP are the key methodologies utilized for the consensus analysis of different models. Most of the finite-field networks in these studies are autonomous systems. The consensus problem is developed by analyzing communication topology and dynamic matrix of FFNs.

The definition of consensus of FFNs is defined as follows:

Definition 4.1. *[18] The network (3.1) over $\mathbb{F}_p$ achieves (finite-time) consensus if for all initial states in $\mathbb{F}_p^n$, there exist a finite time $T \in \mathbb{N}$ and some constant $\eta \in \mathbb{F}_p$ such that $x(T+k) = x(T) = \eta \mathbf{1_n}$ for all $k \in \mathbb{N}$.*

4.1. Analysis of FFNs by algebraic and graphical methods

Consider the analysis of FFNs under algebraic and graphical methods. [18] proposed model (3.1) to start the research related to FFNs. First, the preconditions for consensus of networks over finite fields were proposed.

Lemma 4.1. *[18] Consider the network (3.1) over finite field $\mathbb{F}_p$. If consensus is achieved, then A is either nilpotent or row-stochastic.*

When A is a nilpotent matrix, it is obvious that the system will achieve consensus after finite step iterations. Hence, the analysis follows presupposes that the network matrix A is row-stochastic. [18] derived a set of consensus equivalence requirements on this basis. Considering the state transition graph of matrix A, they proposed the following conclusion.

Theorem 4.1. *[18] The network (3.1) over a finite field $\mathbb{F}_p$ achieves consensus with row-stochastic matrix A if and only if the transition graph $\mathcal{G}_A = (\mathcal{V}_A, \varepsilon_A)$ contains exactly p cycles, corresponding to the unit cycles around the vertices $\eta \mathbf{1}$, $\eta \in \mathbb{F}_p$.*

The above theorem is a necessary and sufficient condition for achieving the finite-field network consensus based on the state transition graph. When the number of agents rises, the size of the related state transition graph grows exponentially, making it harder to verify the consensus of the network. Consider the following inverse recursion in [18]:

$$\delta_\alpha^{k+1} = \widehat{A}^{-1}(\delta_\alpha^k), \tag{4.1}$$

where $\delta_\alpha^k \in \mathbb{F}_p^{\omega r}$ for all time $k \in \mathbb{N}$ and $\delta_\alpha^0 = \{\alpha \mathbf{1}\}, \alpha \in \mathbb{F}_p^{\omega r}$. Then, the consensus of network (3.1) can be verified by inverse recursion (4.1) without analyzing the state transition graph. The two previously mentioned approaches are not explicit enough for finite-field consensus. Thus, [18] presented an additional equivalence requirement based on the characteristic polynomial of the network matrix.

Theorem 4.2. *[18] The network (3.1) over a finite field $\mathbb{F}_p$ achieves consensus with row-stochastic matrix A if and only if the characteristic polynomial*

$$P_A(\lambda) = \lambda^{n-1}(\lambda - 1). \tag{4.2}$$

Theorem 4.2 enables the design of finite-field consensus matrices. Finite-field consensus time and value were obtained by a theorem. It indicated that consensus time depended on the dimension of the largest Jordan block associated with the eigenvalue 0. Subsequently, they gave the relevant results for average consensus. These conclusions in [18] were extended to consensus of networks with switching topology and time-delays over finite fields by [21, 22].

For networks (3.3), (3.4), and (3.7), taking $y(k) = [x^T(t+\tau), x^T(t+\tau-1), ..., x^T(t)]^T \in \mathbb{F}_p^{n(\tau+1)}$, there is the following equivalent form:

$$y(k+1) = D_{s(t)}y(k), \tag{4.3}$$

where

$$D_{s(t)} = \begin{bmatrix} C_{0,s(t+\tau)} & C_{1,s(t+\tau)} & \cdots & C_{\tau-1,s(t+\tau)} & C_{\tau,s(t+\tau)} \\ I & 0 & \dots & 0 & 0 \\ 0 & I & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & I & 0 \end{bmatrix}. \tag{4.4}$$

Similar to the form of (4.3), networks (3.3) and (3.4) can be equivalently transformed into a network without time-delays. Then they can obtain a series of theorems on consensus based on graph theory, characteristic polynomials, and other methods. The above definition of consensus requires that the state in the network reach a common value and stay at the value forever. In some practical systems, it requires the state of the network to be equal to each other but not remain at a fixed value. Consequently, [23] defined network synchronization over finite fields and provided some results related to synchronization. The research model of [23] is still (3.1).

Definition 4.2. *[23] The network (3.1) over $\mathbb{F}_p$ achieves synchronization if for all initial states in $\mathbb{F}_p^n$, there exist a finite time $K \in \mathbb{N}$ such that $x_1(t) = x_2(t) = \cdots = x_n(t)$ for all $t \geq K$.*

When network (3.1) achieves synchronization, the state trajectory of the network converges to $\Omega = \{\alpha\mathbf{1_n} | \alpha \in \mathbb{F}_\mathbf{p}\}$. Synchronization of FFNs requires the network matrix to satisfy preconditions.

Lemma 4.2. *[23] If synchronization of network (3.1) is achieved, then either A is a nilpotent matrix or the row sums of A are the same and nonzero.*

For synchronization of FFNs, there exists a class of initial state $x(0) \in \{e_{i,n} | i = 1, \dots, n\}$, where $e_{i,n}$ is an n-dimensional vector with the i-th element being 1 and others 0. After a finite time, it can achieve synchronization regardless of the form of the network matrix. In [18], the transition graph can be used to verify consensus of network (3.1). The synchronization problem has similar results as theorem 4.1. The difference is that the transition graph of consensus FFNs contains p unit cycles, while the transition graph of synchronization FFNs has r same length cycles except for the unit cycle around $\mathbf{0_n}$, and the vertex sets of $r+1$ cycles constitute a partition of Ω. [24] can verify the consensus problem based on the characteristic polynomial, and derive a theorem that is similar to (4.2) as a necessary and sufficient condition for synchronization of network (3.1). Assume $A\mathbf{1_n} = \alpha\mathbf{1_n}$, network (3.1) over $\mathbb{F}_p$ achieves synchronization if and only if the characteristic polynomial of A, is $P_A(\lambda) = \lambda^{n-1}(\lambda - \alpha)$. Besides, time and cycles of finite-field synchronization can be obtained.

In [24], synchronization of FFNs with time-delays was investigated from a perspective of linear recursion theory. It extended results in [21, 22], and derived a sufficient condition for synchronization.

4.2. Analysis of FFNs via STP

Since many previous methods for studying the consensus of multi-agent systems over the field of real numbers are difficult to apply to finite fields, the aforementioned papers have derived several algebra-theoretic and graph-theoretic conditions for FFNs. Studies for linear FFNs with time delays and switching topology are currently insufficient, and it is not easy to verify consensus by some existing conclusions. Many researchers attempted to use the STP of matrices to study the consensus problem of FFNs, gave more explicit and concise results for some proposed models, and further explored some unsolved problems. This is due to the excellent performance of STP in Boolean networks and multi-valued logical networks. [27, 28] first proposed to use STP to study the consensus problem of FFNs, converted switched FFN (3.5) and leader-follower multi-agent system (3.20,3.21) into algebraic forms via STP to give a preliminary analysis. Subsequently, these results were advanced by [29–33].

For switched FFN (3.5), by Lemma 1, there exists a structural matrix such that $x_i = S_i^r x(t), i = 1, \dots, n$, where $S_i^r = (M_{+,p})^{n-1} \ltimes_{k=1}^{n} [I_{p^{k-1}} \otimes (M_{\times,p} \ltimes a_{ik}^r)] \in \mathcal{L}_{p \times p^n}$, $M_{+,p}$ and $M_{\times,p}$ are given to represent the addition operator and the multiplication operator over the finite field $\mathbb{F}_p$.

Then, it can be converted into the form:

$$x(t+1) = L_r x(t), \tag{4.5}$$

where $L_r = S_1^r * S_2^r * \cdots * S_n^r \in \mathcal{L}_{p^n \times p^n}$. So the algebraic form of (3.5) can be obtained as follows:

$$x(t+1) = L_{\sigma(t)} x(t). \tag{4.6}$$

After the above model transformation, the network can be analyzed. First, the definition of switching point reachability and its equivalent conditions can be given. Then, a necessary and sufficient condition can be presented for consensus of network (3.5).

Theorem 4.3. *[27] For each A_r, suppose that conditions of Theorem 1 holds. Then, the network (3.5) achieves consensus under arbitrary switching signal, if and only if there exists a positive integer $\tau \le p^n$ such that*

$$Row_i(M^\tau)\mathbf{1}_{\mathbf{p^n}} = \mathbf{0} \tag{4.7}$$

holds for any $i \in \{1, \dots, p^n\}/\{c(\alpha) : \alpha \in \mathbb{F}_p\}$, where $c(\alpha) = \alpha\frac{p^n-1}{p-1} + 1$.

Li et al. [22] and [27] both analyzed the finite-field network with switching topology, and provided the equivalence conditions of consensus. The former primarily used methods like graph theory and characteristic polynomials, while the latter employed STP to transform network (3.5) into a new algebraic form, with the results can be confirmed by straightforward calculations.

Li et al. [30] studied leader-follower consensus of multi-agent systems with time-delays (3.20,3.22) over $\mathbb{F}_p$. Note that the definition of consensus in [30] is the same as the definition of synchronization in [23] (Definition 4.2), which requires that every component in state vector of an agent equal to corresponding component in state vector of other agents and can change over time. By Lemma 1, there are $x_0(t+1) = L_0 Z(t)$ and $x_i(t+1) = L_i z(t), i = 1, \dots, N$. Then, multiplying the $N+1$ equations of a leader and followers by the Khatri-Rao matrix product, there is $x(t+1) = LZ(t)$. Using the pseudo-commutative law and the dummy matrices, one can obtain the equivalent algebraic form:

$$z(t+1) = LZ(t). \tag{4.8}$$

Therefore, a new system without time-delays is established by dimensionality expansion, and the subsequent analysis can be performed. [30] first presented the concept of set stability for system (4.8) over $\mathbb{F}_p$.

Definition 4.3. *[30] Given a nonempty set $\Omega \subseteq \Delta_{p^{n(N+1)(\tau+1)}}$. System (4.8) is said to be stable at Ω, if there exists a positive integer μ such that*

$$z(t; z_0) \in \Omega \tag{4.9}$$

holds for any integer $t \ge \mu$ and any initial state $z_0 \in \Delta_{p^{n(N+1)(\tau+1)}}$.

Defining the set Λ,

$$\begin{aligned} \Lambda = \{&\delta^j_{p^{n(N+1)(\tau+1)}} = \ltimes_{k=1}^{\tau+1}\delta^{j_k}_{p^{n(N+1)}} : j_k \in \{i_1, \dots, i_{p^n}\} \\ &k = 1, \dots, \tau+1\} \\ := \{&\delta^{l_1}_{p^{n(N+1)(\tau+1)}}, \dots, \delta^{l_{p^{n(\tau+1)}}}_{p^{n(N+1)(\tau+1)}}\} \end{aligned} \tag{4.10}$$

where $i_1 < i_2 < \cdots < i_{p^n}$ and $l_1 < l_2 < \cdots < l_{p^{n(\tau+1)}}$. Then one can obtain the following theorem.

Theorem 4.4. *[30] The follower (3.20) achieves (finite-time) consensus with the leader (3.22), if and only if system (4.8) is stable at Λ.*

The consensus problem of leader-follower multi-agent systems with time-delays over finite fields was converted into the problem of set stability. It only needs to investigate requirements for the set stability of system (4.8), there is the following conclusion.

Theorem 4.5. *[30] System (4.8) is stable at Λ, if and only if there exists a positive integer $\mu \le p^{n(N+1)(\tau+1)}$ such that*

$$\sum_{c\in\Gamma} Row_c(\hat{L}^{\mu}) = 0^T_{p^{n(N+1)(\tau+1)}}, \tag{4.11}$$

where $\Gamma = \{1, \dots, p^{n(N+1)(\tau+1)}\} \setminus \{l_1, \dots, l_{n(\tau+1)}\}$.

Based on Theorems 4.4 and 4.5, a criterion for the leader-follower consensus of system (3.22) and (3.20) can be presented. The follower (3.22) achieves (finite-time) consensus with the leader (3.20), if and only if there exists a positive integer $\mu \le p^{n(N+1)(\tau+1)}$ such that (4.11) holds.

Li et al. [30] also discussed the case of a follower with a switching signal (3.23). Using STP, the system was converted into an algebraic system without time-delays by expanding dimensions. The definition of system consensus and set stability under any switching signal was provided. The definition of switching point reachability was proposed, and the relevant criterion for the reachability of switching point was given. Finally, they gave the necessary and sufficient conditions for leader-follower consensus of the system with switching topology based on the obtained criterion .

Switched delayed logical networks were studied by set stability which was applied to finite-field consensus in [31]. They converted switched delayed logical networks into an equivalent algebraic form, and proved that the set stability of switched delayed logical networks is equivalent to the set stability of the algebraic form with respect to trajectory. Based on the algebraic form and the switching point reachability, a necessary and sufficient condition for the set stability of switched delayed logical networks can be obtained. They applied the above results to the consensus of FFNs with switching topology and time-delays, and showed the effectiveness of the new results.

The set stability was used to investigate FFNs with two kinds of stochastic time-delays in [32]. By STP, they converted systems with stochastic time-delays into the corresponding linear discrete-time stochastic systems. Then, they revealed the relation between the finite-time consensus of FFNs with stochastic time-delays and the finite-time set stability of the obtained stochastic systems and proposed two new criteria for the finite-time consensus problem.

Liu et al. [33] extended the concept of containment to FFNs which was a multi-agent system with M leaders and $N-M$ followers.

Definition 4.4. *[33] The followers (3.13) achieve containment with the leaders (3.12) in $\mathbb{F}_p$, if there exists $\rho \in \mathbb{Z}_+$, which satisfies the condition that*

$$x_f(t) \in \{x_1(t), \dots, x_M(t)\}, f = M+1, \dots, N, \tag{4.12}$$

holds for any initial state and any integer $t \ge \rho$.

The idea of this research is similar to the previous articles. [33] used the STP method to obtain the corresponding algebraic form of the system. They studied the consensus problem under fixed and switching topologies through set stability and set stability under arbitrary switching signal, respectively.

5. Some control problems of the networks over finite fields

In this section, controllability and consensus protocols of multi-agent control systems over finite fields are investigated, structural controllability of FFNs is derived, and controllability of FFNs are researched via STP.

5.1. Consensus and controllability of FFNs

The leader-follower consensus problem of multi-agent systems over finite fields was considered in [25]. For system models (3.10) and (3.11), the input is a distributed control protocol that has been used and intensively investigated for the consensus problem of real-valued multi-agent systems. The control protocol has the following form:

$$u_i(t) = K \sum_{j=0}^{N} a_{ij}(x_j(t) - x_i(t)). \tag{5.1}$$

where $K \in \mathbb{F}_p^{1\times n}$ is the feedback gain matrix. Actually, the consensus problem is similar to synchronization of FFNs in [23] (Definition 4.2), that is, $x_i(t) = x_0(t), i = 1, \dots, N$. Let $\delta_i(t) = x_i(t) - x_0(t) \in \mathbb{F}_p^n$, the consensus problem is equivalent to the existence of T such that, $t \ge T$,

$$\delta_i(t) = 0, i = 1, \dots, N. \tag{5.2}$$

The interaction graph describing the information transmission among the $N + 1$ agents is denoted by $\mathcal{G} = (\mathcal{V}, \varepsilon)$. The subgraph induced by the N followers is denoted by $\hat{\mathcal{G}}$. Note that $\mathcal{A} = (a_{ij}) \in \mathbb{F}_p^{(N+1)\times(N+1)}$ and $\mathcal{D} \in \mathbb{F}_p^{(N+1)\times(N+1)}$ are the weighted adjacency matrix and degree matrix of $\mathcal{G}$, $\hat{\mathcal{A}}$ and $\hat{\mathcal{D}}$ is the induced adjacency submatrix and degree submatrix corresponding to $\hat{\mathcal{G}}$. This paper assumed the induced subgraph $\hat{\mathcal{G}}$ is a directed acyclic graph (DAG), which has been used in some studies of consensus problems. If A is nilpotent, then consensus can be easily achieved by just letting $K = 0$. So the theorem in [25] assumed matrix A is not nilpotent. Then the necessary and sufficient conditions for consensus were provided for fixed and switching topologies.

For the multi-agent system with switching topology over finite fields, researchers studied the controllability problem of model (3.19) in [26]. First, several graphical conditions for controllability of multi-agent systems over finite fields were established. It was proved that a switched multi-agent system is controllable over $\mathbb{F}_p$ if each graph of the subsystem is a spanning forest. The conclusion can be obtained that a multi-agent system with switching topology can be controllable over $\mathbb{F}_p$ even if each of its subsystems is not controllable. Besides, this paper showed that the switched system is controllable if the union of graphs is a path graph or a star graph.

5.2. Structural controllability of FFNs

When solving many control problems of systems, system matrices are usually prescribed. However, in some cases, it needs to analyze systems whose parameters are not exactly known. In order to deal with these problems, scholars have developed a characterization of system properties based on the structure of the system. Then, a system of the form (3.18) is said to be structured if every entry in the system matrices is either zero or a free independent parameter. [19, 20] extended this concept to the study of multi-agent systems over finite fields.

Note that (3.16) and (3.17) can be compacted into (3.18) in [19]. With $y(t) = Cx(t)$, it can be written as a form of a linear system. But unlike the general linear system, the matrix B in the system actually is an $n \times |\mathcal{N}_l|$ matrix of the form $B = \begin{bmatrix} e_{i_1,N} & e_{i_2,N} & \cdots & e_{i_m,N} \end{bmatrix}$. Although a form of a linear system is used to represent multi-agent systems over finite fields, some traditional methods for linear systems over field of real numbers may not be applicable over $\mathbb{F}_p$. [19] developed a characterization of structural controllability over finite fields, and the definition of structural controllability is as follows.

Definition 5.1. *[19] The system (3.18) is said to be structurally controllable if one can fix all free parameter entries of (A, B) at some particular values from $\mathbb{F}_p$ such that system (3.18) is controllable over finite fields in the classical sense.*

Then they proved that a linear system will be structurally controllable (or observable) over $\mathbb{F}_p$ if the graph of the system satisfies specific properties, and the size of the field is sufficiently large.

Lu [20] researched the structural controllability of multi-agent systems of the form (3.14) and (3.15). For the control input, there is the following protocol:

$$u_i(t + 1) = K \sum_{j\in N_i^{in} \bigcup\{i\}} a_{ij}x_j(t), \tag{5.3}$$

where $K \in \mathbb{F}_p^{m\times n}$ is the feedback gain matrix. Under the given protocol (5.3), the system can be written into a compact form:

$$x(t + 1) = \Phi x(t) + \Gamma U^{ext}(t), \tag{5.4}$$

where $\Phi = I_n \otimes A + \hat{A} \otimes BK$, $\Gamma = D \otimes I_N$, and D is the same as matrix B of (3.18). The multi-agent systems with switching topology are given by

$$x(t + 1) = \Phi x_{\sigma(t)}(t) + \Gamma_{\sigma(t)} U^{ext}(t), \tag{5.5}$$

where $\Phi_{\sigma(t)} = I_n \otimes A + \hat{A}_{\sigma(t)} \otimes BK$, $\Gamma_{\sigma(t)} = D_{\sigma(t)} \otimes I_N$. Then they proved that a multi-agent system is structurally controllable over a finite field if the graph has a spanning forest, and a switched multi-agent system is structurally controllable if each switching network has a spanning forest.

5.3. Controllability of FFNs via STP

The previous section used the STP of matrices to analyze the finite-field networks. Consider leader-follower FFN

(3.16) and (3.17), by Lemma 1, can be converted into an equivalent algebraic form:

$$x(t+1) = Lu(t)x(t), \tag{5.6}$$

where $L \in \mathcal{L}_{p^n \times p^{2n}}$. The definition of reachability and controllability for the leader-follower FFN was given in [29]. By using the algebraic form, they proposed the controllability matrix for the FFN, which can be used to verify reachability and controllability of the multi-agent systems.

Consider system (5.6), L can be split into p^n blocks: $L = \begin{bmatrix} L_1 & L_2 & \dots & L_{p^n} \end{bmatrix}$, where $L_i \in \mathcal{L}_{p^n \times p^n}$. Let $M = \sum_{i=1}^{p^n} L_i$, the controllability matrix of system (5.6) is defined as follows:

$$C = \sum_{s=1}^{p^n + |N_l|} M^s. \tag{5.7}$$

Then, there is the corresponding theorem.

Theorem 5.1. *Consider system (5.6).*
(i) $x_d = \delta_{p^n}^{\beta}$ *is reachable from* $x_0 = \delta_{p^n}^{\alpha}$ *at the s-th step, if and only if*

$$(M^s)_{\beta,\alpha} > 0. \tag{5.8}$$

(ii) $x_d = \delta_{p^n}^{\beta}$ *is reachable from* $x_0 = \delta_{p^n}^{\alpha}$*, if and only if*

$$(C)_{\beta,\alpha} > 0. \tag{5.9}$$

(iii) System (5.6) is controllable, if and only if

$$C > 0. \tag{5.10}$$

Finally, an algorithm to find the minimal number of leaders which can make the multi-agent systems over finite fields be controllable was proposed.

6. Conclusions

This paper has presented recent research advances around networks over finite fields, divided into three main aspects: The first part introduced the models of multi-agent systems over finite fields; The second part analyzed the consensus and synchronization problems of FFNs through graph theory, characteristic polynomial, and the STP of matrics; The third part investigated multi-agent control systems over finite fields and proposed relevant conclusions on consensus and controllability. There are many models on FFNs in recent research that focuses on linear iterative strategies, but the research on nonlinear systems is still insufficient. The models on finite fields are mainly studied for the constant systems with time-delays and switching topology structures. The time-varying systems over finite fields have yet to be involved. Currently, many results have been obtained by STP for FFNs, and this method can be used to analyze and control nonlinear multi-agent systems over finite fields. Sometimes, excessive computational complexity exists due to the large matrix dimension of STP when dealing with complex systems. Other theoretical methods urgently need to be introduced into the research of FFNs, which requires further exploration by scholars.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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