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Research article

A novel numerical approach for solving delay differential equations arising in population dynamics

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Abstract: In this paper, the initial-value problem for a class of first order delay differential equations, which emerges as a model for population dynamics, is considered. To solve this problem numerically, using the finite difference method including interpolating quadrature rules with the basis functions, we construct a fitted difference scheme on a uniform mesh. Although this scheme has the same rate of convergence, it has more efficiency and accuracy compared to the classical Euler scheme. The different models, Nicolson's blowfly and Mackey–Glass models, in population dynamics are solved by using the proposed method and the classical Euler method. The numerical results obtained from here show that the proposed method is reliable, efficient, and accurate.

Keywords: delay differential equation; finite difference method; convergence

1. Introduction

Delay differential equations (DDE) arise in as model problems in many fields of science and engineering such as dynamical diseases, biosciences, population ecology, economics and control theory [1–8]. For example, population dynamics, neural networks, time lag cell growth, signal transmission and climate systems are modeled by DDEs [9–22]. Especially, single-delayed population models such as Nicholson and Mackey-Glass have been very well worked by many authors (see, e.g., survey articles [23, 24] and references therein).

In particular, in population dynamics, the famous Nicholson's sheep blowfly model with time delay can be described as follows:

$$u'(t) + \delta u(t) = \beta u(t-\tau)e^{-\gamma u(t-\tau)},$$

$$u(t) = \varphi(t),\ t \in [-\tau, 0],\ \varphi > 0,$$

where β, γ, δ are positive parameters, where δ is per capita daily adult mortality rate, β is the maximum per capita daily egg production rate, $1/\gamma$ is the size at which the population reproduces at its maximum rate, $\tau > 0$ is the generation time (for details see [13]).

Another model, as called Mackey-Glass, can be defined as follows:

$$u'(t) + \gamma u(t) = \frac{\beta\theta^n u(t-\tau)}{\theta^n + u^n(t-\tau)},$$

$$u(t) = \varphi(t),\ t \in [-\tau, 0],\ \varphi > 0,$$

where β, γ, θ are positive parameters, where γ is the destruction rate, θ is the shape parameter, τ is the maturation delay between the start of the production of immature cells in the bone marrow and their release into circulation and $n \geq 1$ (for details, see [17, 25]).

Motivated by the above works (especially the works of [13, 17]), we consider the following delay differential problem in the interval $\bar{I} = [0, T]$:

$$Lu := u'(t) + a(t)u(t) = f(t, u(t-r)),\ t \in I, \tag{1.1}$$

$$u(t) = \varphi(t),\ t \in I_0, \tag{1.2}$$

where $I = (0, T] = \cup_{p=1}^{m} I_p$, $I_p = \{t : r_{p-1} < t \leq r_p\}$, $1 \leq p \leq m$ and $r_k = kr$, $0 \leq k \leq m$ and, $I_0 = [-r, 0]$ (for simplicity we suppose that T/r is integer; i.e., $T = mr$). In addition, r is a constant large delay.

The functions $a(t) \in C(\bar{I})$, $f(t, v) \in C(\bar{I} \times \mathbb{R})$, $\frac{\partial f}{\partial v}(t, v) \in C^1(\bar{I} \times \mathbb{R})$ and $\varphi(t) \in C(I_0)$ are assumed to be continuous at specified intervals and such that (1.1) has a unique solution $u(t) \in C^1(\bar{I})$ satisfying the given condition (1.2) (see [26, 27]). Furthermore, there exist an α such that

$$a(t) \geq \alpha > 0,\ \left|\frac{\partial f}{\partial v}\right| \leq M.$$

Besides, studies on the existence and uniqueness of solutions of DDEs can be found in [1, 26, 28–31, 39] and references therein.

Because of this problem is semi-linear, it may not always be possible to find the exact solution. Therefore, it is important to develop effective numerical methods to solve these problems. In the past two decades, several numerical techniques have been put forward to solve DDEs problems, such as the Runge-Kutta methods [28, 32], the neural network approximation [33], the multistep method [34], the Legendre-Gauss spectral collocation method [35,40] and the collocation method with interpolation [36].

The aim of this paper is discretizing (1.1)-(1.2) using a numerical method, which is composed of an implicit fitted finite difference scheme on uniform mesh. This method is based on integral identities using interpolated quadrature rules which remainder terms in integral form. As a result, a local truncation error occurs, which includes only the first derivative of the exact solution, thus facilitating the examination of convergence. This method of approximation has the advantage that the scheme can also be more effective than a classical scheme such as Euler in the case when the continuous problem is considered under certain restrictions.

The structure of this paper is as follows: In Section 2, we analyze some properties of the continuous problem. In Section 3, for the approximate solution of the problem, we construct the finite difference discretization. We present the stability and convergence analysis of the discretized problem in Section 4. In Section 5, we apply our approach in some familiar models in population dynamics and compare our results with the classical Euler method. Finally, we summarize findings from this study in Section 6.

Notation. For any continuous function $g(t)$, we use $\|g\|_\infty = \max_{0 \leq t \leq T} |g(t)|$ for the continuous maximum norm on the corresponding interval and $\|g\|_{\infty,p} = \max_{r_{p-1} \leq t \leq r_p} |g(t)|$. Throughout the paper, C denotes a generic positive constant. Some specific, fixed constants of this kind are indicated by subscripting C and D.

2. The continuous problem

In this section, we present a priori estimates for the solution of (1.1)-(1.2), which are needed in later sections for the analysis of appropriate numerical solutions.

Lemma 2.1. *Assume that $a \in C(\bar{I})$, $f \in C(\bar{I} \times \mathbb{R})$, $\frac{\partial f}{\partial v} \in C^1(\bar{I} \times \mathbb{R})$, $\varphi \in C(I_0)$ and u be the solution of the problem (1.1)-(1.2), then it satisfies the bounds:*

$$\|u\|_{\infty,p} \leq C_p,\ 1 \leq p \leq m, \tag{2.1}$$

$$\left\|u'\right\|_{\infty,p} \leq D_p,\ 1 \leq p \leq m, \tag{2.2}$$

where

$$C_1 = |\varphi(0)| + \alpha^{-1}(\|F\|_{\infty,1} + M\|\varphi\|_{\infty,0}),$$

$$C_p = C_{p-1} + \alpha^{-1}(\|F\|_{\infty,p} + MC_{p-1}),\ 2 \leq p \leq m,$$

$$D_1 = \|F\|_{\infty,1} + \|a\|_{\infty,1} C_1 + M\|\varphi\|_{\infty,0},$$

$$D_p = \|F\|_{\infty,p} + \|a\|_{\infty,p} C_p + MC_{p-1},\ 2 \leq p \leq m,$$

$$F(t) = f(t, 0).$$

Proof. The proof is by induction in k. We can rewrite the problem (1.1) in the form

$$u'(t) + a(t)u(t) - b(t)u(t-r) = F(t), \tag{2.3}$$

where

$$b(t) = \frac{\partial f}{\partial v}(t, \tilde{v}),\ \tilde{v} = \gamma u(t-r),\ (0 < \gamma < 1)-\text{intermediate values.}$$

For $t \in I_p$, from (2.3), we have

$$u(t) = u(r_{p-1})e^{-\int_{r_{p-1}}^{t} a(\tau)d\tau} + \int_{r_{p-1}}^{t} [F(s) + b(s)u(s-r)]e^{-\int_{s}^{t} a(\tau)d\tau} ds$$

and here

$$|u(t)| \leq \left|u(r_{p-1})\right| e^{-\alpha(t-r_{p-1})}+\int_{r_{p-1}}^{t} [|F(s)|+|b(s)|\,|u(s-r)|]e^{-\alpha(t-s)}ds. \quad (2.4)$$

So, for $t \in I_1$, we get

$$\begin{aligned}|u(t)| &\leq |\varphi(0)|\, e^{-\alpha t} + (\|F\|_{\infty,1} + M\,\|\varphi\|_{\infty,0}) \int_0^t e^{-\alpha(t-s)}ds \\ &\leq |\varphi(0)|\, e^{-\alpha t} + \alpha^{-1}(\|F\|_{\infty,1} + M\,\|\varphi\|_{\infty,0})(1-e^{-\alpha t}) \\ &\leq |\varphi(0)| + \alpha^{-1}(\|F\|_{\infty,1} + M\,\|\varphi\|_{\infty,0}).\end{aligned}$$

Now, for $t \in I_2$, we have

$$\begin{aligned}|u(t)| &\leq |u(r)|\, e^{-\alpha(t-r)} + \int_r^t [|F(s)| + |b(s)|\,|u(s-r)|]e^{-\alpha(t-s)}ds \\ &\leq |u(r)|\, e^{-\alpha(t-r)} + (\|F\|_{\infty,2} + M\,\|u\|_{\infty,1}) \int_r^t e^{-\alpha(t-s)}ds \\ &\leq |u(r)| + \alpha^{-1}(\|F\|_{\infty,2} + M\,\|u\|_{\infty,1})(1-e^{-\alpha(t-r)}) \\ &\leq C_1 + \alpha^{-1}(\|F\|_{\infty,2} + MC_1) \equiv C_2.\end{aligned}$$

Thus, the inequality (2.1) is valid for $p = 1, 2$. Let the inequality (2.1) be true for $p = k$. That is

$$\|u\|_{\infty,k} \leq C_{k-1} + \alpha^{-1}(\|F\|_{\infty,k} + MC_{k-1}) \equiv C_k,\ 2 \leq k \leq m.$$

For $t \in I_{k+1}$, because of (2.4) we get

$$\begin{aligned}|u(t)| &\leq |u(r_k)|\, e^{-\alpha(t-r_k)} + \int_{r_k}^t [|F(s)|+|b(s)|\,|u(s-r)|]e^{-\alpha(t-s)}ds \\ &\leq |u(r_k)|\, e^{-\alpha(t-r_k)} + (\|F\|_{\infty,k+1} + M\,\|u\|_{\infty,k}) \int_{r_k}^t e^{-\alpha(t-s)}ds \\ &\leq |u(r_k)|\, e^{-\alpha(t-r_k)} + \alpha^{-1}(\|F\|_{\infty,k+1} + M\,\|u\|_{\infty,k})(1-e^{-\alpha(t-r_k)}) \\ &\leq C_k + \alpha^{-1}(\|F\|_{\infty,k+1} + MC_k).\end{aligned}$$

and hence the inequality (2.1) is valid for $p = k + 1$.

From (1.1), we can write

$$u'(t) = F(t) - a(t)u(t) - b(t)u(t-r),$$

$$\left|u'(t)\right| \leq |F(t)| + |a(t)|\,|u(t)| + |b(t)|\,|u(t-r)|. \quad (2.5)$$

For $t \in I_1$, from (2.5) we get

$$\begin{aligned}\left|u'(t)\right| &\leq \|F\|_{\infty,1} + \|a\|_{\infty,1}\,\|u\|_{\infty,1} + M\,\|\varphi\|_{\infty,0} \\ &\leq \|F\|_{\infty,1} + \|a\|_{\infty,1}\,C_1 + M\,\|\varphi\|_{\infty,0}.\end{aligned}$$

Also, for $t \in I_2$, from (2.5) we have

$$\begin{aligned}\left|u'(t)\right| &\leq \|F\|_{\infty,2} + \|a\|_{\infty,2}\,\|u\|_{\infty,2} + M\,\|u\|_{\infty,1} \\ &\leq \|F\|_{\infty,2} + \|a\|_{\infty,1}\,C_2 + MC_1.\end{aligned}$$

So, the inequality (2.2) be valid for $p = 1, 2$. Let the inequality (2.2) is valid for $p = k$. That is

$$\left\|u'\right\|_{\infty,k} \leq \|F\|_{\infty,k} + \|a\|_{\infty,k}\,C_k + MC_{k-1}.$$

For $t \in I_{k+1}$, because of (2.5) we obtain

$$\begin{aligned}\left|u'(t)\right| &\leq \|F\|_{\infty,k+1} + \|a\|_{\infty,k+1}\,\|u\|_{\infty,k+1} + M\,\|u\|_{\infty,k} \\ &\leq \|F\|_{\infty,k+1} + \|a\|_{\infty,k+1}\,C_{k+1} + MC_k,\end{aligned}$$

hence the inequality (2.2) is valid for $p = k + 1$. Thus the proof is complete. □

3. The difference scheme and mesh

ω_{N_0} is a uniform mesh on I defined by

$$\omega_{N_0} = \{t_i = ih,\ i = 1, 2, ..., N_0,\ h = T/N_0 = r/N\},$$

which contains by N mesh point at each subinterval I_p ($1 \leq p \leq m$):

$$\omega_{N_p} = \{t_i : (p-1)N + 1 \leq i \leq pN\},\ 1 \leq p \leq m.$$

As a result, we describe

$$\omega_{N_0} = \cup_{p=1}^{m} \omega_{N_p},\ \overline{\omega}_{N_0} = \omega_{N_0} \cup \{t_0 = 0\}.$$

Also, for any mesh function $g(t)$, we denote $g_i = g(t_i)$ and y_i denotes an approximation of $u(t)$ at t_i and

$$g_{\bar{t},i} = (g_i - g_{i-1})/h,\ \|g\|_{\infty,p} = \|g\|_{\infty,\omega_{N_p}} := \max_{(p-1)N+1 \leq i \leq pN} |g_i|.$$

We use the following identity for the difference approximation of the problem (1.1):

$$h^{-1} \int_{t_{i-1}}^{t_i} Lu(t)\phi_i(t)dt = h^{-1} \int_{t_{i-1}}^{t_i} f(t, u(t-r))\phi_i(t)dt,\ 1 \leq i \leq N_0, \quad (3.1)$$

with basis function

$$\phi_i(t) = e^{-\int_t^{t_i} a(s)ds},\ t_{i-1} \leq t \leq t_i.$$

which is the solution of the following problem

$$-\phi_i(t) + a(t)\phi_i(t) = 0, t_{i-1} < t \le t_i,\ \phi_i(t_i) = 1. \quad (3.2)$$

We can rewrite (3.1) as

$$h^{-1}\int_{t_{i-1}}^{t_i} u'(t)\phi_i(t)dt + h^{-1}\int_{t_{i-1}}^{t_i} a(t)u(t)\phi_i(t)dt$$

$$= h^{-1}\int_{t_{i-1}}^{t_i} f(t, u(t-r))\phi_i(t)dt.$$

To obtain the difference scheme, we will use the following formulas (2.1) and (2.2) from [37], respectively,

$$\int_a^b p(x)g(x)dx = [\sigma g(b) + (1-\sigma)g(a)]\int_a^b p(x)dx$$

$$+g(a;b)\int_a^b (x - x^{(\sigma)})p(x)dx + R(g),$$

$$R(g) = \int_a^b dxp(x)\int_a^b g'(s)K_0(x,s)ds,$$

and

$$\int_a^b p(x)g'(x)dx = g(a;b)\int_a^b p(x)dx + \overline{R}(g),$$

$$\overline{R}(g) = -\int_a^b dxp'(x)\int_a^b g'(s)K_0(x,s)ds,$$

$$K_0(x,s) = T_0(x-s) - (b-a)^{-1}(x-a),$$

$$x^{(\sigma)} = \sigma b + (1-\sigma)a,$$

$$T_0(\lambda) = 1, \lambda > 0; T_0(\lambda) = 0, \lambda < 0.$$

Using these formulas on each interval (t_{i-1}, t_i) (where $g = u, p = \phi$ and $\sigma = 1$) taking into account (3.2) we have the following precise relation

$$\ell u_i \equiv A_i u_{\bar{t},i} + B_i u_i$$

$$= C_i f(t_i, u_{i-N}) + D_i f(t_i, u_{i-N-1}) + R_i,\ 1 \le i \le N_0, \quad (3.3)$$

where

$$A_i = h^{-1}\int_{t_{i-1}}^{t_i} [\phi_i(t) + (t-t_i)a(t)\phi_i(t)]dt,$$

$$B_i = h^{-1}\int_{t_{i-1}}^{t_i} a(t)\phi_i(t)dt,$$

$$C_i = h^{-1}\int_{t_{i-1}}^{t_i} \phi_i(t)dt + h^{-2}\int_{t_{i-1}}^{t_i} (t-t_i)\phi_i(t)dt,$$

$$D_i = -h^{-2}\int_{t_{i-1}}^{t_i} (t-t_i)\phi_i(t)dt,$$

$$R_i = h^{-1}\int_{t_{i-1}}^{t_i} dt\phi_i(t)\int_{t_{i-1}}^{t_i} \frac{d}{ds}f(s, u(s-r))[T_0(t-s) - h^{-1}(t-t_{i-1})]ds, \quad (3.4)$$

$$T_0(t) = 1,\ t \ge 0;\ T_0(t) = 0,\ t < 0.$$

By virtue of (3.3), for the approximate solution of the problem (1.1)-(1.2), we put forward the following difference scheme:

$$\ell y_i \equiv A_i y_{\bar{t},i} + B_i y_i = C_i f(t_i, y_{i-N}) + D_i f(t_i, y_{i-N-1}),\ 1 \le i \le N_0, \quad (3.5)$$

$$y_i = \varphi_i,\ -N \le i \le 0. \quad (3.6)$$

On the other hand, using the implicit Euler method, we easily present an alternative difference scheme for approximating (1.1)-(1.2) [27]:

$$y_{\bar{t},i} + a_i y_i = f(t_i, y_{i-N}),\ 1 \le i \le N_0, \quad (3.7)$$

$$y_i = \varphi_i,\ -N \le i \le 0. \quad (3.8)$$

4. Error analysis and convergence

To investigate the convergence of the present method, note that the error function $z_i = y_i - u_i$, $0 \le i \le N_0$ is the solution of the following discrete problem

$$\ell z_i \equiv A_i z_{\bar{t},i} + B_i z_i = C_i[f(t_i, y_{i-N}) - f(t_i, u_{i-N})]$$

$$+D_i[f(t_i, y_{i-N-1}) - f(t_i, u_{i-N-1})] - R_i,\ 1 \le i \le N_0, \quad (4.1)$$

$$z_i = 0,\ -N \le i \le 0, \quad (4.2)$$

where the truncation error R_i is given by (3.4).

Lemma 4.1. *Under the assumptions of Lemma 2.1, for the truncation error R_i we have*

$$\|R\|_{\infty,p} \le CN^{-1},\ 1 \le p \le m. \quad (4.3)$$

Proof. From (3.4), we get

$$|R_i| \le h^{-1}\int_{t_{i-1}}^{t_i} dt\phi_i(t)\int_{t_{i-1}}^{t_i}\left[\left|\frac{\partial f}{\partial s}\right| + \left|\frac{\partial f}{\partial v}\right|\left|u'(s-r)\right|\right]ds.$$

Owing to Lemma 2.1, $\left|\frac{\partial f}{\partial v}\right| \le M$,

$$\le Ch^{-1}\int_{t_{i-1}}^{t_i} dt\phi_i(t)\int_{t_{i-1}}^{t_i}(1 + |u'(s-r)|)d\xi,$$

and $0 < \phi_i(t) \le 1$, we obtain

$$|R_i| \le Ch,$$

which leads to the inequality (4.3). □

Lemma 4.2. *[38] Let $|G_i| \le \overline{G}_i$ and $\overline{G}_i$ be a nondecreasing function, we consider the following problem*

$$L^h v_i \equiv A_i v_{\bar{t},i} + B_i v_i = G_i,\ 1 \le i \le N,$$
$$v_0 = \beta.$$

Then, the solution of this difference problem holds:

$$|v_i| \le |\beta| + \alpha^{-1}\overline{G}_i.$$

Lemma 4.3. *Let z_i be the solution of the problem (4.1)-(4.2). Then, the following inequality holds:*

$$\|z\|_{\infty,p} \le \alpha^{-1}\sum_{j=1}^{p}(1 + 2\alpha^{-1}M)^{p-j}\,\|R\|_{\infty,p}\,,\ 1 \le p \le m. \quad (4.4)$$

Proof. From (4.1), we have

$$A_i z_{\bar{t},i} + B_i z_i = C_i\widetilde{C}_i z_{i-N} + D_i\widetilde{D}_i z_{i-N-1} - R_i,$$

where

$$\widetilde{C}_i = \frac{\partial f}{\partial v}(t_i, u_{i-N} + \delta z_{i-N}),\ \widetilde{D}_i = \frac{\partial f}{\partial v}(t_i, u_{i-N-1} + \delta z_{i-N-1}),$$
$$(0 < \delta < 1) - \text{intermediate values.}$$

For estimate C_i, we can write

$$|C_i| \le h^{-1}\int_{t_{i-1}}^{t_i}\phi_i(t)dt + h^{-2}\int_{t_{i-1}}^{t_i}|t - t_i|\,\phi_i(t)dt,$$

owing to $0 < \phi_i(t) \le 1$, we obtain

$$|C_i| \le h^{-1}\int_{t_{i-1}}^{t_i}dt + h^{-2}\int_{t_{i-1}}^{t_i}|t - t_i|\,dt \le 2.$$

For estimate D_i, in a similar way, we have

$$|D_i| \le h^{-2}\int_{t_{i-1}}^{t_i}|t - t_i|\,\phi_i(t)dt \le 1.$$

Because of Lemma 4.2, applying the discrete maximum principle for difference operator $L^h z_i := A_i z_{\bar{t},i} + B_i z_i$, we have

$$\begin{aligned}\|z\|_{\infty,p} &\le \left|z_{p-1}\right| + \alpha^{-1}(2M\,\|z\|_{\infty,p-1} + \|R\|_{\infty,p})\\ &\le (1 + 2\alpha^{-1}M)\,\|z\|_{\infty,p-1} + \alpha^{-1}\,\|R\|_{\infty,p}\,.\end{aligned}$$

By solving this difference inequality, we arrive at (4.4). □

Now we give the theorem, which is the main result of this paper.

Theorem 4.4. *Let u and y be the solution of problems (1.1)-(1.2) and (3.1)-(3.2), respectively. Then they satisfy the following bound, in which first order convergence in the discrete maximum norm is obtained:*

$$\|y - u\|_{\infty,\overline{\omega}_{N_0}} \le CN^{-1}.$$

Proof. This follows immediately by combining the previous lemmas. □

5. Numerical illustrations

In this section, we present numerical results obtained by applying the numerical methods (3.5)-(3.6) and (3.7)-(3.8) to particular problems.

Example 5.1. *We consider the test problem:*

$$u'(t) + 2u(t) = u^2(t-1),\ 0 < t \le 2,$$

subject to the interval condition,

$$u(t) = e^t,\ -1 \le t \le 0,$$

whose exact solution is given by

$$u(t) = \begin{cases} e^{-2t} + \frac{1}{4}e^{2(t-1)} - \frac{1}{4}e^{-2(t+1)}, & t \in (0,1],\\ \frac{1}{4}e^{-2} - \frac{1}{16}e^{-4} + (\frac{1}{2} - \frac{1}{6}e^{-2} + \frac{3}{4}e^{2})e^{-2t} & \\ +(-\frac{1}{2} + \frac{1}{4}e^{-2} - \frac{1}{32}e^{-4})e^{-4(t-1)} + \frac{1}{96}e^{4(t-2)}, & t \in (1,2].\end{cases}$$

Example 5.2. *As another test problem, we consider a special problem in population dynamics introduced in Sect. 1, called the Nicolson's model:*

$$u'(t) + 2u(t) = 0.01u(t-1)e^{-10u(t-1)} = 0,\ 0 < t \le 2,$$

subject to the interval condition,

$$u(t) = 1,\ -1 \le t \le 0,$$

whose exact solution is unknown. But, we have found the following approximate solution to this problem via Mathematica:

$$u(t) = \begin{cases} (1 - \frac{e^{-10}}{200})e^{-2t} + \frac{e^{-10}}{200}, & t \in (0,1],\\ 1.135 \times 10^{-9}e^{g(t)} & \\ +(1 - 0.0369451 \times ExpIntegralEi[g(t)])e^{-2t}, & t \in (1,2],\\ g(t) = -73.8905e^{-2t}. & \end{cases}$$

Example 5.3. *Another test problem is the Mackey–Glass model:*

$$u'(t) + 2u(t) = \frac{2u(t-1)}{1+u^3(t-1)}, \ 0 < t \le 2,$$

subject to the interval condition,

$$u(t) = 1, \ -1 \le t \le 0,$$

whose exact solution is

$$u(t) = \begin{cases} \frac{1}{2}(1+e^{-2t}), & t \in (0,1], \\ \frac{4}{9} + \frac{1}{2}e^{-2t} + \frac{1}{54}e^{-2(t-1)}\{3+4\sqrt{3}\pi \\ -12\sqrt{3}\arctan(\sqrt{3}h(t)) + 4\ln\frac{1+3h(t)}{4} + 6\ln(\frac{1+3h^2(t)}{4})\}, & t \in (1,2], \\ h(t) = e^{2(t-1)}. \end{cases}$$

We define the exact error E_i^N and the computed maximum pointwise error E^N for any N as follows:

$$E_i^N = |y_i - u_i|, \ E^N = \max_{0 \le i \le N} E_i^N$$

where y_i is the numerical approximation to exact value u_i for the nodes t_i. The computational results of the test problems obtained by using both methods (Euler method (EM) and proposed method (PM)) are presented in the Tables 1-9 and Figures 1-6.

Table 1. The numerical results on $(0,2]$ (EM) for Example 5.1

t_i	u_i	$y_i(N=64)$	E_i^{64}	$y_i(N=128)$	E_i^{128}
0.125	0.7958945	0.7989118	$3.017E-3$	0.7974169	$1.522E-3$
0.250	0.6417920	0.6465677	$4.776E-3$	0.6441991	$2.407E-3$
0.375	0.5280108	0.5337125	$5.702E-3$	0.5308818	$2.871E-3$
0.500	0.4474025	0.4535031	$6.101E-3$	0.4504713	$3.069E-3$
0.625	0.3949029	0.4010935	$6.191E-3$	0.3980138	$3.110E-3$
0.750	0.3672135	0.3733419	$6.128E-3$	0.3702901	$3.077E-3$
0.875	0.3625947	0.3686229	$6.028E-3$	0.3656182	$3.024E-3$
1.000	0.3807564	0.3867324	$5.976E-3$	0.3837513	$2.995E-3$
1.125	0.3843352	0.3868855	$2.550E-3$	0.3855884	$1.253E-3$
1.250	0.3555932	0.3571401	$1.547E-3$	0.3563410	$7.479E-4$
1.375	0.3141976	0.3157584	$1.561E-3$	0.3149574	$7.598E-4$
1.500	0.2706128	0.2725290	$1.916E-3$	0.2715566	$9.439E-4$
1.625	0.2300922	0.2324090	$2.317E-3$	0.2312417	$1.149E-3$
1.750	0.1950488	0.1977069	$2.658E-3$	0.1963726	$1.324E-3$
1.875	0.1664717	0.1694018	$2.930E-3$	0.1679335	$1.462E-3$
2.000	0.1448033	0.1479719	$3.169E-3$	0.1463851	$1.582E-3$

Table 2. The numerical results on $(0,2]$ (PM) for Example 5.1

t_i	u_i	$y_i(N=64)$	E_i^{64}	$y_i(N=128)$	E_i^{128}
0.125	0.7958945	0.7958959	$1.391E-6$	0.7958948	$3.478E-7$
0.250	0.6417920	0.6417948	$2.869E-6$	0.6417927	$7.174E-7$
0.375	0.5280108	0.5280153	$4.528E-6$	0.5280119	$1.132E-6$
0.500	0.4474025	0.4474090	$6.471E-6$	0.4474042	$1.618E-6$
0.625	0.3949029	0.3949117	$8.821E-6$	0.3949051	$2.205E-6$
0.750	0.3672135	0.3672252	$1.172E-5$	0.3672164	$2.931E-6$
0.875	0.3625947	0.3626101	$1.536E-5$	0.3625985	$3.841E-6$
1.000	0.3807564	0.3807763	$1.997E-5$	0.3807614	$4.993E-6$
1.125	0.3843352	0.3843772	$4.192E-5$	0.3843457	$1.048E-5$
1.250	0.3555932	0.3556421	$4.895E-5$	0.3556054	$1.224E-5$
1.375	0.3141976	0.3142460	$4.837E-5$	0.3142097	$1.209E-5$
1.500	0.2706128	0.2706571	$4.434E-5$	0.2706239	$1.108E-5$
1.625	0.2300922	0.2301314	$3.918E-5$	0.2301020	$9.794E-6$
1.750	0.1950488	0.1950830	$3.418E-5$	0.1950574	$8.544E-6$
1.875	0.1664717	0.1665018	$3.009E-5$	0.1664792	$7.523E-6$
2.000	0.1448033	0.1448308	$2.746E-5$	0.1448102	$6.866E-6$

Table 3. Comparison of E^N both methods on $(0,2]$ for Example 5.1

N	$E^N(EM)$	$E^N(PM)$	N	$E^N(EM)$	$E^N(PM)$
32	$1.226E-2$	$1.972E-4$	256	$1.560E-3$	$3.084E-6$
64	$6.191E-3$	$4.934E-5$	512	$7.808E-4$	$7.710E-7$
128	$3.111E-3$	$1.234E-5$	1024	$3.907E-4$	$1.928E-7$

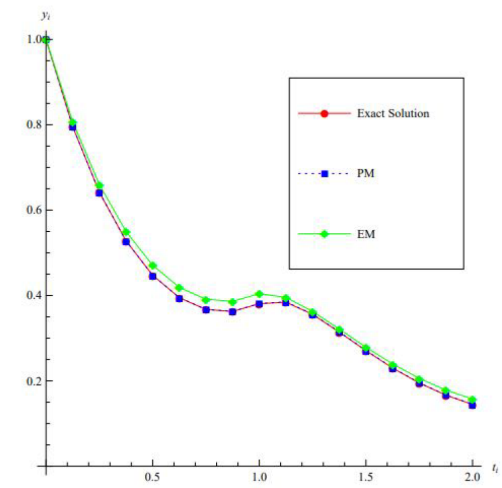

Figure 1. Numerical results of Example 5.1 for $N = 16$.

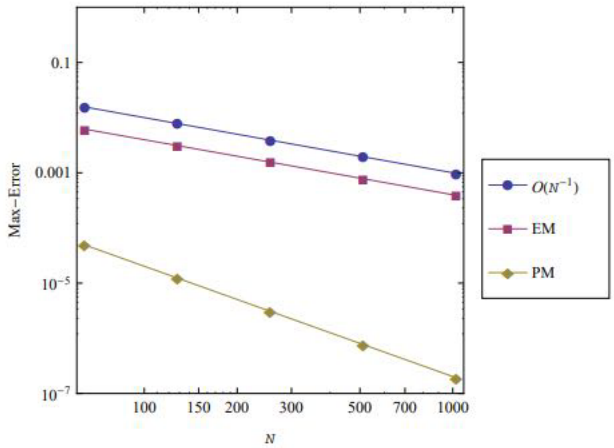

Figure 2. Maximum errors comparison between EM and PM for Example 5.1.

Table 4. The numerical results on (0, 2] (EM) for Example 5.2

t_i	u_i	$y_i(N = 64)$	E_i^{64}	$y_i(N = 128)$	E_i^{128}
0.125	0.7788008	0.7817868	$2.986E-3$	0.7803077	$1.507E-3$
0.250	0.6065307	0.6111906	$4.660E-3$	0.6088802	$2.349E-3$
0.375	0.4723667	0.4778208	$5.454E-3$	0.4751139	$2.747E-3$
0.500	0.3678796	0.3735540	$5.674E-3$	0.3707351	$2.855E-3$
0.625	0.2865050	0.2920396	$5.535E-3$	0.2892875	$2.783E-3$
0.750	0.2231303	0.2283128	$5.182E-3$	0.2257333	$2.603E-3$
0.875	0.1737741	0.1784919	$4.718E-3$	0.1761415	$2.367E-3$
1.000	0.1353355	0.1395427	$4.207E-3$	0.1374446	$2.109E-3$
1.125	0.1053995	0.1090928	$3.693E-3$	0.1072492	$1.850E-3$
1.250	0.0820861	0.0852882	$3.202E-3$	0.0836883	$1.602E-3$
1.375	0.0639317	0.0666802	$2.748E-3$	0.0653056	$1.374E-3$
1.500	0.0497974	0.0521370	$2.340E-3$	0.0509658	$1.168E-3$
1.625	0.0387965	0.0407739	$1.977E-3$	0.0397831	$9.866E-4$
1.750	0.0302372	0.0318985	$1.661E-3$	0.0310653	$8.281E-4$
1.875	0.0235792	0.0249676	$1.388E-3$	0.0242706	$6.914E-4$
2.000	0.0184001	0.0195552	$1.156E-3$	0.0189747	$5.746E-4$

Table 5. The numerical results on (0, 2] (PM) for Example 5.2

t_i	u_i	$y_i(N = 64)$	E_i^{64}	$y_i(N = 128)$	E_i^{128}
0.125	0.7788008	0.7788008	$2.220E-16$	0.7788008	$1.221E-15$
0.250	0.6065307	0.6065307	$3.331E-16$	0.6065307	$2.109E-15$
0.375	0.4723667	0.4723667	$4.441E-16$	0.4723667	$2.554E-15$
0.500	0.3678796	0.3678796	$3.886E-16$	0.3678796	$2.109E-15$
0.625	0.2865050	0.2865050	$2.220E-16$	0.2865050	$1.887E-15$
0.750	0.2231303	0.2231303	$8.327E-17$	0.2231303	$1.554E-15$
0.875	0.1737741	0.1737741	$5.551E-17$	0.1737741	$1.193E-15$
1.000	0.1353355	0.1353355	$1.943E-16$	0.1353355	$9.714E-16$
1.125	0.1053995	0.1053995	$6.589E-10$	0.1053995	$1.648E-10$
1.250	0.0820861	0.0820861	$2.321E-9$	0.0820861	$5.803E-10$
1.375	0.0639317	0.0639317	$4.884E-9$	0.0639317	$1.221E-9$
1.500	0.0497974	0.0497974	$7.128E-9$	0.0497974	$1.782E-9$
1.625	0.0387965	0.0387965	$7.512E-9$	0.0387965	$1.878E-9$
1.750	0.0302372	0.0302372	$5.460E-9$	0.0302372	$1.365E-9$
1.875	0.0235792	0.0235792	$1.713E-9$	0.0235792	$4.281E-10$
2.000	0.0184001	0.0184001	$2.349E-9$	0.0184001	$5.874E-10$

Table 6. Comparison of E^N both methods on (0, 2] for Example 5.2

N	$E^N(EM)$	$E^N(PM)$	N	$E^N(EM)$	$E^N(PM)$
32	$1.121E-2$	$3.059E-8$	256	$1.432E-3$	$4.784E-10$
64	$5.674E-3$	$7.653E-9$	512	$7.173E-4$	$1.196E-10$
128	$2.855E-3$	$1.914E-9$	1024	$3.590E-4$	$2.990E-11$

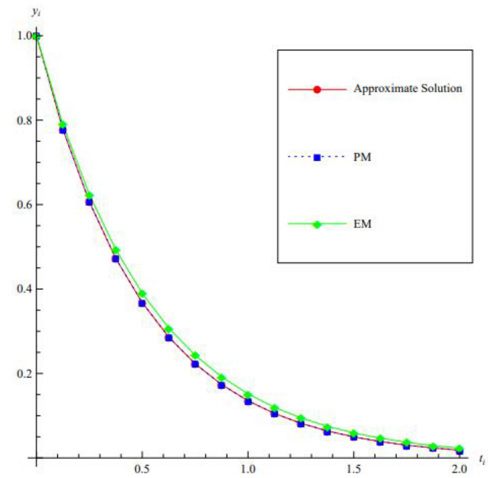

Figure 3. Numerical results of Example 5.2 for $N = 16$.

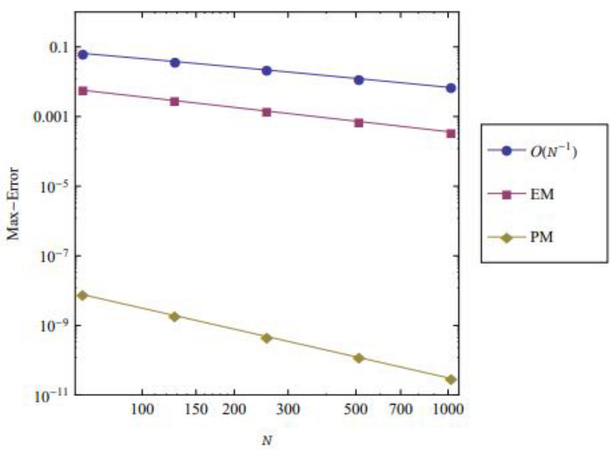

Figure 4. Maximum errors comparison between EM and PM for Example 5.2.

Table 7. The numerical results on $(0, 2]$ (EM) for Example 5.3

t_i	u_i	$y_i(N = 64)$	E_i^{64}	$y_i(N = 128)$	E_i^{128}
0.125	0.8894004	0.8908934	$1.493E-3$	0.8901538	$7.534E-4$
0.250	0.8032653	0.8055953	$2.330E-3$	0.8044400	$1.175E-3$
0.375	0.7361833	0.7389103	$2.727E-3$	0.7375569	$1.374E-3$
0.500	0.6839397	0.6867769	$2.837E-3$	0.6853675	$1.428E-3$
0.625	0.6432524	0.6460197	$2.767E-3$	0.6446437	$1.391E-3$
0.750	0.6115651	0.6141563	$2.591E-3$	0.6128666	$1.301E-3$
0.875	0.5868870	0.5892459	$2.359E-3$	0.5880706	$1.184E-3$
1.000	0.5676676	0.5697712	$2.104E-3$	0.5687222	$1.055E-3$
1.125	0.5555687	0.5576293	$2.061E-3$	0.5566033	$1.035E-3$
1.250	0.5491945	0.5509498	$1.755E-3$	0.5500749	$8.804E-4$
1.375	0.5445477	0.5459707	$1.423E-3$	0.5452601	$7.124E-4$
1.500	0.5396283	0.5407961	$1.168E-3$	0.5402120	$5.837E-4$
1.625	0.5337302	0.5347415	$1.011E-3$	0.5342354	$5.051E-4$
1.750	0.5268499	0.5277863	$9.364E-4$	0.5273179	$4.679E-4$
1.875	0.5192898	0.5202043	$9.145E-4$	0.5197473	$4.575E-4$
2.000	0.5114309	0.5123498	$9.190E-4$	0.5118911	$4.603E-4$

Table 8. The numerical results on $(0, 2]$ (PM) for Example 5.3

t_i	u_i	$y_i(N = 64)$	E_i^{64}	$y_i(N = 128)$	E_i^{128}
0.125	0.8894004	0.8894004	0.0	0.8894004	0.0
0.250	0.8032653	0.8032653	0.0	0.8032653	0.0
0.375	0.7361833	0.7361833	0.0	0.7361833	0.0
0.500	0.6839397	0.6839397	0.0	0.6839397	0.0
0.625	0.6432524	0.6432524	0.0	0.6432524	0.0
0.750	0.6115651	0.6115651	0.0	0.6115651	0.0
0.875	0.5868870	0.5868870	0.0	0.5868870	0.0
1.000	0.5676676	0.5676676	0.0	0.5676676	0.0
1.125	0.5555687	0.5555637	$5.042E-6$	0.5555674	$1.260E-6$
1.250	0.5491945	0.5491870	$7.480E-6$	0.5491926	$1.870E-6$
1.375	0.5445477	0.5445398	$7.900E-6$	0.5445457	$1.975E-6$
1.500	0.5396283	0.5396212	$7.116E-6$	0.5396265	$1.779E-6$
1.625	0.5337302	0.5337244	$5.803E-6$	0.5337288	$1.451E-6$
1.750	0.5268499	0.5268455	$4.391E-6$	0.5268488	$1.098E-6$
1.875	0.5192898	0.5192867	$3.110E-6$	0.5192891	$7.773E-7$
2.000	0.5114309	0.5114288	$2.050E-6$	0.5114303	$5.126E-7$

Table 9. Comparison of E^N both methods on $(0, 2]$ for Example 5.3

N	$E^N(EM)$	$E^N(PM)$	N	$E^N(EM)$	$E^N(PM)$
32	$5.603E-3$	$3.177E-5$	256	$7.162E-4$	$4.962E-7$
64	$2.837E-3$	$7.939E-6$	512	$3.587E-4$	$1.240E-7$
128	$1.428E-3$	$1.985E-6$	1024	$1.795E-4$	$3.101E-8$

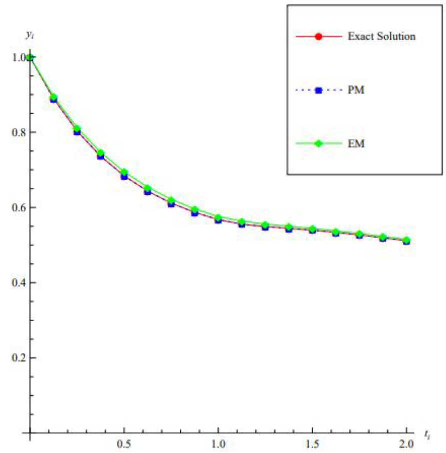

Figure 5. Numerical results of Example 5.3 for $N = 16$.

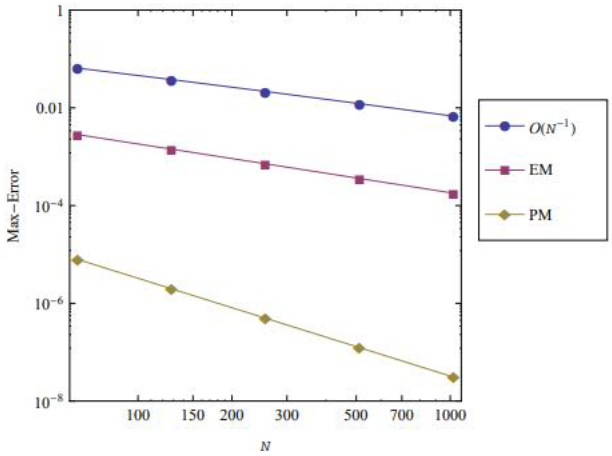

Figure 6. Maximum errors comparison between EM and PM for Example 5.3.

6. Conclusions

In this study, we have proposed a novel difference scheme for solving a class of first order DDEs by using the finite difference method. This method was based on an exponentially fitted difference scheme on an equidistant mesh on each time subinterval. Therefore, the proposed method was useful when applying DDE to various model examples because the method was faster and more effective than the classical Euler method. The examples were solved using both the proposed method and classical Euler method and the computational results for various values of N were displayed in Tables 1-9 and were plotted in Figures 1-6.

In particular, the numerical results in Tables 3, 6, 9 and Figures 2, 4, 6 revealed that the proposed method gave more impressive and remarkable results than the classical method. Theoretical results represents an ongoing work within a further research, such as other nonlinear delay or neutral delay problems involving biological models.

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Conflict of interest

All authors declare that there are no conflicts of interest in this paper.

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