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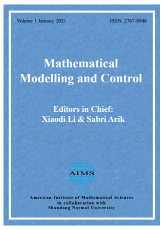

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Research article

On the solutions of the dual matrix equation $A^{\top}XA = B$

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Abstract: Let $\mathbb{D}^{m\times n} = \{A = A_1 + \varepsilon A_2 | A_1, A_2 \in \mathbb{R}^{m\times n}\}$ be the set of all $m \times n$ real dual matrices. In this paper, the following problems are considered. **Problem I:** Given dual matrices $A = A_1 + \varepsilon A_2 \in \mathbb{D}^{m\times n}$ and $B = B_1 + \varepsilon B_2 \in \mathbb{D}^{n\times n}$, find $X \in S$ such that the dual matrix equation $A^{\top}XA = B$ is satisfied, where $S = \{X \in \mathbb{D}^{m\times m} | CX = D, C, D \in \mathbb{D}^{p\times m}\}$. **Problem II:** Given dual matrices $A = A_1 + \varepsilon A_2 \in \mathbb{D}^{m\times n}, B = B_1 + \varepsilon B_2 \in \mathbb{D}^{n\times n}$ and $\tilde{X} = \tilde{X}_1 + \varepsilon \tilde{X}_2 \in \mathbb{D}^{m\times m}$, with $B_i = B_i^{\top}, i = 1, 2$, find $\hat{X} \in T$ such that $\|\hat{X} - \tilde{X}\|_{\mathrm{D}} = \min_{X\in T} \|X - \tilde{X}\|_{\mathrm{D}} = \min_{X\in T} \sqrt{\|X_1 - \tilde{X}_1\|^2 + \|X_2 - \tilde{X}_2\|^2}$, where $T = \{X = X_1 + \varepsilon X_2 \in \mathbb{D}^{m\times m} | A^{\top}XA = B \text{ s. t. } X_i = X_i^{\top}, i = 1, 2\}$. We derive the solvability conditions and the representation of the general solution of Problem I using the Moore-Penrose inverse. Also, we deduce the solvability conditions and the explicit formula of T and the unique approximation solution $\hat{X}$ of Problem II by applying the Moore-Penrose inverse and Kronecker product of matrices. Finally, we give a numerical example to show the correctness of our result.

Keywords: dual matrix equation; optimal approximation; linear manifold; Kronecker product

1. Introduction

We will adopt the following terminology. $\mathbb{R}^{m\times n}$ denotes the set of all $m \times n$ real matrices. I_n denotes the identity matrix of size n. $A^{\top}, A^{\dagger}, \mathrm{tr}(A)$ and $\|A\|$ represent the transpose, the Moore-Penrose inverse, the trace and the Frobenius norm of the matrix A, respectively. Given two matrices $A = [a_{ij}] \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$, the Kronecker product of A and B is defined by $A \otimes B = [a_{ij}B] \in \mathbb{R}^{mp\times nq}$. Also, for a matrix $A = (a_1, a_2, \cdots, a_n) \in \mathbb{R}^{m\times n}, a_i \in \mathbb{R}^m, i = 1, 2, \cdots, n$, the stretch function $\mathrm{vec}(A)$ is defined as $\mathrm{vec}(A) = (a_1^{\top}, a_2^{\top}, \cdots, a_n^{\top})^{\top}$. Further, the symbols E_A and F_A stand for two orthogonal projectors $E_A = I_m - AA^{\dagger},\ F_A = I_n - A^{\dagger}A$ induced by $A \in \mathbb{R}^{m\times n}$.

Many scholars considered the following matrix equation

$$A^{\top}XA = B \tag{1.1}$$

in real and complex matrix spaces. For example, Dai and Lancaster [1] considered symmetric, positive semi-definite, and positive definite solutions of the matrix equation (1.1) with the help of the singular value decomposition. Peng et al. [2] provided the necessary and sufficient conditions and the expression of the symmetric ortho-symmetric solutions of the matrix equation (1.1) by applying the generalized singular value decomposition. Li [3] gave the necessary and sufficient conditions and the expressions for the D-symmetric solutions of the matrix equation (1.1) on a linear manifold using the generalized singular value decomposition.

In 1873, Clifford [4] introduced dual numbers. Subsequently, the dual algebra develops rapidly and has been widely applied to kinematic analysis [5], robotics [6], screw motion [7] and rigid body motion analysis [8, 9]. The set of the dual numbers is usually denoted by

$$\mathbb{D} = \{a = a_1 + \varepsilon a_2 | a_1, a_2 \in \mathbb{R}\}.$$

The real unit ε is subjected to the rules: $\varepsilon \neq 0,\ 0\varepsilon = \varepsilon 0 = 0,\ 1\varepsilon = \varepsilon 1 = \varepsilon,\ \varepsilon^2 = 0$. For the operation rules about the dual numbers, the readers can see Ref. [5]. A matrix whose elements are dual numbers is called a dual matrix, namely, the set of all $m \times n$ real dual matrices is

$$\mathbb{D}^{m\times n} = \{A = A_1 + \varepsilon A_2 | A_1, A_2 \in \mathbb{R}^{m\times n}\}.$$

The operational rules for dual matrices are similar to those of dual numbers. Dual matrices have important applications in kinematic analysis [5, 10], spatial kinematics [11, 12] and robotics [6, 13]. The solutions of linear dual equations are widely used in kinematic analysis and sensor calibration problems. For instance, Angeles [10] applied the dual algebra to compute the parameters of the serew of a rigid body between two finitely-separated positions and of the instant screw. Condurache and Burlacu [14] solved the $AX = XB$ sensor calibration problem by means of the orthogonal dual tensor method. Condurache and Ciureanu [15] explored the $AX = YB$ sensor calibration problem using dual algebra.

Furthermore, many authors considered the solutions of the dual matrix equation $Ax = b$. Udwadia [16] considered this equation using the dual generalized inverses. Zhong and Zhang [17] introduced the dual group-inverse solution of $Ax = b$. Pennestrì and Valentini [18] proposed to solve this dual equation by applying the QR-decomposition.

We observe that the solutions of the dual matrix equation seems to be rarely considered. Therefore, in this paper, we will consider two problems of the dual matrix equation (1.1), that is :

Problem I. Given dual matrices $A = A_1 + \varepsilon A_2 \in \mathbb{D}^{m\times n}$ and $B = B_1 + \varepsilon B_2 \in \mathbb{D}^{n\times n}$, find $X \in S$ such that the dual matrix equation (1.1) is satisfied, where $S = \{X \in \mathbb{D}^{m\times m} | CX = D, C, D \in \mathbb{D}^{p\times m}\}$.

Problem II. Given dual matrices $A = A_1 + \varepsilon A_2 \in \mathbb{D}^{m\times n}$, $B = B_1 + \varepsilon B_2 \in \mathbb{D}^{n\times n}$ and $\tilde{X} = \tilde{X}_1 + \varepsilon \tilde{X}_2 \in \mathbb{D}^{m\times m}$, with $B_i = B_i^\top, i = 1, 2$, find $\hat{X} \in T$ such that $\|\hat{X} - \tilde{X}\|_{\mathrm{D}} = \min_{X\in T} \|X - \tilde{X}\|_{\mathrm{D}} = \min_{X\in T} \sqrt{\|X_1 - \tilde{X}_1\|^2 + \|X_2 - \tilde{X}_2\|^2}$, where $T = \{X = X_1 + \varepsilon X_2 \in \mathbb{D}^{m\times m} | A^\top XA = B \text{ s. t. } X_i = X_i^\top, i = 1, 2\}$.

The outline of the rest of this paper is as follows. In Section 2, we introduce some lemmas. In Section 3, the solvability conditions and the representation of the general solution of Problem I are derived by applying the Moore-Penrose inverse. In Section 4, by utilizing the Moore-Penrose inverse and Kronecker product of matrices, we obtain the unique approximation solution $\hat{X}$ of Problem II. In Section 5, a numerical algorithm to solve Problem II and a numerical example are provided. Some concluding remarks are given in Section 6.

2. Preliminaries

First, we should point out that $\|P\|_{\mathrm{D}} = \sqrt{\|P_1\|^2 + \|P_2\|^2}$ is indeed a matrix norm for the the dual matrix $P = P_1 + \varepsilon P_2$. In fact, for all $k \in \mathbb{R}$ and for all the m-by-p dual matrices $P = P_1 + \varepsilon P_2$ and $Q = Q_1 + \varepsilon Q_2$, where $P_i,\ Q_i \in \mathbb{R}^{m\times p}$ $(i = 1, 2)$, we have • $\|P\|_{\mathrm{D}} = \sqrt{\|P_1\|^2 + \|P_2\|^2} \geq 0$ and $\|P\|_{\mathrm{D}} = 0 \Leftrightarrow P_1 = 0, P_2 = 0$;

- $\|kP\|_{\mathrm{D}} = \sqrt{\|kP_1\|^2 + \|kP_2\|^2} = \sqrt{k^2(\|P_1\|^2 + \|P_2\|^2)} = |k| \sqrt{\|P_1\|^2 + \|P_2\|^2} = |k| \cdot \|P\|_{\mathrm{D}}$;
- Since

$$\begin{aligned}
\|P + Q\|_{\mathrm{D}}^2 =& \|P_1 + P_2\|^2 + \|Q_1 + Q_2\|^2 \\
\leq& (\|P_1\| + \|P_2\|)^2 + (\|Q_1\| + \|Q_2\|)^2 \\
=& \|P_1\|^2 + \|P_2\|^2 + \|Q_1\|^2 + \|Q_2\|^2 \\
&+ 2(\|P_1\| \cdot \|P_2\| + \|Q_1\| \cdot \|Q_2\|), \\
(\|P\|_{\mathrm{D}} + \|Q\|_{\mathrm{D}})^2 =& \|P_1\|^2 + \|P_2\|^2 + \|Q_1\|^2 + \|Q_2\|^2 \\
&+ 2\sqrt{\|P_1\|^2 + \|P_2\|^2} \cdot \sqrt{\|Q_1\|^2 + \|Q_2\|^2},
\end{aligned}$$

and

$$\|P_1\| \cdot \|P_2\| + \|Q_1\| \cdot \|Q_2\| \leq \sqrt{\|P_1\|^2 + \|P_2\|^2} \cdot \sqrt{\|Q_1\|^2 + \|Q_2\|^2}.$$

Thus, the inequality $\|P + Q\|_{\mathrm{D}} \leq \|P\|_{\mathrm{D}} + \|Q\|_{\mathrm{D}}$ follows.

Next, in order to solve Problems I and II, we introduce the following lemmas.

Lemma 2.1. *[19] If $A \in \mathbb{R}^{m\times p}, B \in \mathbb{R}^{q\times n}$ and $D \in \mathbb{R}^{m\times n}$. Then the matrix equation $AXB = D$ has a solution $X \in \mathbb{R}^{p\times q}$ if and only if $AA^\dagger DB^\dagger B = D$. In this case, the general solution is $X = A^\dagger DB^\dagger + F_A V_1 + V_2 E_B$, where V_1, V_2 are arbitrary matrices.*

Lemma 2.2. *[20] Let $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{p\times q}, C \in \mathbb{R}^{m\times r}, D \in \mathbb{R}^{s\times q}$ and $E \in \mathbb{R}^{m\times q}$. Then the linear matrix equation $AXB + CYD = E$ is consistent if and only if*

$$E_G E_A E = 0, E_A E F_D = 0, E_C E F_B = 0, E F_B F_H = 0,$$

where $G = E_A C,\ H = DF_B$. In this case, the general solution is

$$\begin{aligned}
Y =& G^\dagger E_A E D^\dagger + (F_G C^\dagger + F_C G^\dagger E_A) E F_B H^\dagger + W \\
&- C^\dagger C F_G W H H^\dagger - G^\dagger G W D D^\dagger, \\
X =& A^\dagger (E - CYD) B^\dagger + Z - A^\dagger A Z B B^\dagger,
\end{aligned}$$

where $W,\ Z$ are arbitrary matrices.

Lemma 2.3. *[21] If $A \in \mathbb{R}^{m\times n}, D \in \mathbb{R}^{m\times m}$. Then the matrix equation $AXA^\top = D$ has a symmetric solution if and only if $D = D^\top, E_A D = 0$, in this case, the general symmetric solution is $X = A^\dagger D(A^\dagger)^\top + F_A V + V^\top F_A$, where V is an arbitrary matrix.*

Lemma 2.4. *[22] Suppose that A, B are two real matrices, and X is an unknown variable matrix. Then*

$$\frac{\partial \mathrm{tr}(BX)}{\partial X} = B^\top, \ \frac{\partial \mathrm{tr}(X^\top B^\top)}{\partial X} = B^\top,$$
$$\frac{\partial \mathrm{tr}(AXBX)}{\partial X} = (BXA + AXB)^\top,$$
$$\frac{\partial \mathrm{tr}(AX^\top BX^\top)}{\partial X} = BX^\top A + AX^\top B,$$
$$\frac{\partial \mathrm{tr}(AXBX^\top)}{\partial X} = AXB + A^\top XB^\top.$$

Lemma 2.5. *[23] Let $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{n\times l}, C \in \mathbb{R}^{l\times s}$. Then*

$$\mathrm{vec}(ABC) = \left(C^\top \otimes A\right)\mathrm{vec}(B).$$

Lemma 2.6. *[24] Let $V \in \mathbb{R}^{m\times n}$, then $\mathrm{vec}(V^\top) = T_{mn}\mathrm{vec}(V)$, where*

$$T_{mn} = \begin{bmatrix} J_{11}^\top & J_{12}^\top & \cdots & J_{1n}^\top \\ J_{21}^\top & J_{22}^\top & \cdots & J_{2n}^\top \\ \vdots & \vdots & \ddots & \vdots \\ J_{m1}^\top & J_{m2}^\top & \cdots & J_{mn}^\top \end{bmatrix} \in \mathbb{R}^{mn\times mn}$$

with $J_{ij}, i = 1, \cdots, m, j = 1, \cdots, n$ is an $m \times n$ matrix with the element at position (i, j) is 1 and the others are 0, T_{mn} can be uniquely determined by m and n.

3. Solving Problem I

Theorem 3.1. *Given dual matrices $A = A_1 + \varepsilon A_2 \in \mathbb{D}^{m\times n}$, $B = B_1 + \varepsilon B_2 \in \mathbb{D}^{n\times n}$, $C = C_1 + \varepsilon C_2 \in \mathbb{D}^{p\times m}$ and $D = D_1 + \varepsilon D_2 \in \mathbb{D}^{p\times m}, i = 1, 2$, if write*

$$\begin{aligned}
&G_1 = E_{C_1} C_2 F_{C_1},\ G_2 = A_1^\top F_{C_1} F_{G_1},\\
&G_3 = A_2^\top F_{C_1} F_{G_1} F_{G_2} - A_1^\top C_1^\dagger C_2 F_{C_1} F_{G_1} F_{G_2},\\
&J_1 = C_1^\dagger D_1 + F_{C_1} G_1^\dagger E_{C_1}(D_2 - C_2 C_1^\dagger D_1),\\
&J_2 = (C_1^\dagger - C_1^\dagger C_2 F_{C_1} G_1^\dagger E_{C_1})(D_2 - C_2 C_1^\dagger D_1),\\
&J_3 = J_1 + F_{C_1} F_{G_1} G_2^\dagger (B_1 - A_1^\top J_1 A_1) A_1^\dagger,\\
&J_4 = J_2 - C_1^\dagger C_2 F_{C_1} F_{G_1} G_2^\dagger (B_1 - A_1^\top J_1 A_1) A_1^\dagger,\\
&J_5 = B_2 - A_2^\top J_3 A_1 - A_1^\top J_4 A_1 - A_1^\top J_3 A_2,
\end{aligned}$$

$$\begin{aligned}
&M = [G_3\ \ A_1^\top F_{C_1}],\ M^\dagger = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix},\\
&N = E_{A_1} A_2,\ K = E_M G_2,\ H = N F_{A_1},\\
&J_6 = J_3 + F_{C_1} F_{G_1} F_{G_2} M_1 J_5 A_1^\dagger\\
&\quad - F_{C_1} F_{G_1} F_{G_2} M_1 G_2 K^\dagger E_M J_5 N^\dagger N A_1^\dagger\\
&\quad - F_{C_1} F_{G_1} F_{G_2} M_1 G_2 F_K G_2^\dagger J_5 F_{A_1} H^\dagger N A_1^\dagger\\
&\quad + F_{C_1} F_{G_1} K^\dagger E_M J_5 N^\dagger E_{A_1}\\
&\quad + F_{C_1} F_{G_1} (F_K G_2^\dagger + F_{G_2} K^\dagger E_M) J_5 F_{A_1} H^\dagger E_{A_1},\\
&J_7 = J_4 - C_1^\dagger C_2 F_{C_1} F_{G_1} F_{G_2} M_1 J_5 A_1^\dagger\\
&\quad + C_1^\dagger C_2 F_{C_1} F_{G_1} F_{G_2} M_1 G_2 K^\dagger E_M J_5 N^\dagger N A_1^\dagger\\
&\quad + C_1^\dagger C_2 F_{C_1} F_{G_1} F_{G_2} M_1 G_2 F_K G_2^\dagger J_5 F_{A_1} H^\dagger N A_1^\dagger\\
&\quad - C_1^\dagger C_2 F_{C_1} F_{G_1} K^\dagger E_M J_5 N^\dagger E_{A_1}\\
&\quad - C_1^\dagger C_2 F_{C_1} F_{G_1} (F_K G_2^\dagger + F_{G_2} K^\dagger E_M) J_5 F_{A_1} H^\dagger E_{A_1}\\
&\quad + F_{C_1} M_2 J_5 A_1^\dagger - F_{C_1} M_2 G_2 K^\dagger E_M J_5 N^\dagger N A_1^\dagger\\
&\quad - F_{C_1} M_2 G_2 F_K G_2^\dagger J_5 F_{A_1} H^\dagger N A_1^\dagger.
\end{aligned}$$

Then Problem I is solvable if and only if

$$E_{C_1} D_1 = 0,\ E_{G_1} E_{C_1}(D_2 - C_2 C_1^\dagger D_1) = 0, \tag{3.1}$$
$$G_2 G_2^\dagger B_1 A_1^\dagger A_1 + E_{G_2} A_1^\top J_1 A_1 = B_1, \tag{3.2}$$
$$E_K E_M J_5 = 0,\ E_M J_5 F_N = 0,\ E_{G_2} J_5 F_{A_1} = 0,\ J_5 F_{A_1} F_H = 0. \tag{3.3}$$

In this case, the general solution of Problem I can be expressed as $X = X_1 + \varepsilon X_2$, where

$$\begin{aligned}
X_1 =& J_6 - F_{C_1} F_{G_1} F_{G_2} M_1 G_2 F_K W_6 E_H N A_1^\dagger + F_{C_1} F_{G_1} F_{G_2} W_{71}\\
&- F_{C_1} F_{G_1} F_{G_2} M_1 G_3 W_{71} A_1 A_1^\dagger - F_{C_1} F_{G_1} K^\dagger K W_6 N N^\dagger E_{A_1}\\
&- F_{C_1} F_{G_1} F_{G_2} M_1 A_1^\top F_{C_1} W_{72} A_1 A_1^\dagger\\
&+ F_{C_1} F_{G_1} W_6 E_{A_1} - F_{C_1} F_{G_1} G_2^\dagger G_2 F_K W_6 H H^\dagger E_{A_1},
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
X_2 =& J_7 + C_1^\dagger C_2 F_{C_1} F_{G_1} F_{G_2} M_1 G_2 F_K W_6 E_H N A_1^\dagger\\
&- C_1^\dagger C_2 F_{C_1} F_{G_1} F_{G_2} W_{71} + F_{C_1} W_{72}\\
&+ C_1^\dagger C_2 F_{C_1} F_{G_1} F_{G_2} M_1 G_3 W_{71} A_1 A_1^\dagger\\
&+ C_1^\dagger C_2 F_{C_1} F_{G_1} F_{G_2} M_1 A_1^\top F_{C_1} W_{72} A_1 A_1^\dagger\\
&- F_{C_1} M_2 G_3 W_{71} A_1 A_1^\dagger - F_{C_1} M_2 A_1^\top F_{C_1} W_{72} A_1 A_1^\dagger\\
&+ C_1^\dagger C_2 F_{C_1} F_{G_1} G_2^\dagger G_2 F_K W_6 H H^\dagger E_{A_1}\\
&+ C_1^\dagger C_2 F_{C_1} F_{G_1} K^\dagger K W_6 N N^\dagger E_{A_1}\\
&- F_{C_1} M_2 G_2 F_K W_6 E_H N A_1^\dagger - C_1^\dagger C_2 F_{C_1} F_{G_1} W_6 E_{A_1},
\end{aligned} \tag{3.5}$$

and W_6, W_{71}, W_{72} are arbitrary matrices.

Proof. By separating the dual matrix equations $CX = D$ and (1.1) into the real part and the dual part leads to the following four equations:

$$C_1X_1 = D_1, \tag{3.6}$$
$$C_2X_1 + C_1X_2 = D_2, \tag{3.7}$$
$$A_1^\top X_1A_1 = B_1, \tag{3.8}$$
$$A_1^\top X_2A_1 + A_2^\top X_1A_1 + A_1^\top X_1A_2 = B_2. \tag{3.9}$$

By using Lemma 2.1, Eq. (3.6) is solvable if and only if the first condition of (3.1) is satisfied, and the general solution is

$$X_1 = C_1^\dagger D_1 + F_{C_1}W_1, \tag{3.10}$$

where W_1 is an arbitrary matrix. Plugging (3.10) into (3.7), we have

$$C_1X_2 = D_2 - C_2C_1^\dagger D_1 - C_2F_{C_1}W_1. \tag{3.11}$$

By Lemma 2.1, Eq. (3.11) with respect to X_2 is solvable if and only if

$$G_1W_1 = E_{C_1}(D_2 - C_2C_1^\dagger D_1), \tag{3.12}$$

In this case, the general solution is

$$X_2 = C_1^\dagger(D_2 - C_2C_1^\dagger D_1) - C_1^\dagger C_2F_{C_1}W_1 + F_{C_1}W_2, \tag{3.13}$$

where W_2 is an arbitrary matrix. By applying Lemma 2.1, Eq. (3.12) is solvable if and only if the second condition of (3.1) is satisfied, and the general solution is

$$W_1 = G_1^\dagger E_{C_1}(D_2 - C_2C_1^\dagger D_1) + F_{G_1}W_3, \tag{3.14}$$

where W_3 is an arbitrary matrix. Substituting (3.14) into (3.10) and (3.13) yields

$$X_1 = J_1 + F_{C_1}F_{G_1}W_3, \tag{3.15}$$
$$X_2 = J_2 - C_1^\dagger C_2F_{C_1}F_{G_1}W_3 + F_{C_1}W_2. \tag{3.16}$$

Inserting (3.15) into (3.8) yields

$$G_2W_3A_1 = B_1 - A_1^\top J_1A_1. \tag{3.17}$$

Using Lemma 2.1 again, Eq. (3.17) with respect to W_3 is solvable if and only if (3.2) is satisfied, the general solution is

$$W_3 = G_2^\dagger(B_1 - A_1^\top J_1A_1)A_1^\dagger + F_{G_2}W_4 + W_5E_{A_1}, \tag{3.18}$$

where W_4 and W_5 are arbitrary matrices. Plugging (3.18) into (3.15) and (3,16) leads to

$$X_1 = J_3 + F_{C_1}F_{G_1}F_{G_2}W_4 + F_{C_1}F_{G_1}W_5E_{A_1}, \tag{3.19}$$

$$X_2 = J_4 - C_1^\dagger C_2F_{C_1}F_{G_1}F_{G_2}W_4 - C_1^\dagger C_2F_{C_1}F_{G_1}W_5E_{A_1} + F_{C_1}W_2. \tag{3.20}$$

Then, by substituting (3.19) and (3.20) into (3.9), we can get

$$MLA_1 + G_2W_5N = J_5, \tag{3.21}$$

where $L = \begin{bmatrix} W_4 \\ W_2 \end{bmatrix}$, by Lemma 2.2, Eq. (3.21) with respects to L and W_5 is solvable if and only if the conditions of (3.3) holds, and the general solution is

$$\begin{aligned} W_5 =& K^\dagger E_MJ_5N^\dagger + (F_KG_2^\dagger + F_{G_2}K^\dagger E_M)J_5F_{A_1}H^\dagger + W_6 \\ &- G_2^\dagger G_2F_KW_6HH^\dagger - K^\dagger KW_6NN^\dagger, \end{aligned} \tag{3.22}$$
$$L = M^\dagger(J_5 - G_2W_5N)A_1^\dagger + W_7 - M^\dagger MW_7A_1A_1^\dagger, \tag{3.23}$$

where W_6 and W_7 are arbitrary matrices. Then

$$\begin{aligned} W_4 =& M_1(J_5 - G_2W_5N)A_1^\dagger + W_{71} - M_1(G_3W_{71} \\ &+ A_1^\top F_{C_1}W_{72})A_1A_1^\dagger, \end{aligned} \tag{3.24}$$
$$\begin{aligned} W_2 =& M_2(J_5 - G_2W_5N)A_1^\dagger + W_{72} - M_2(G_3W_{71} \\ &+ A_1^\top F_{C_1}W_{72})A_1A_1^\dagger, \end{aligned} \tag{3.25}$$

where $W_7 = \begin{bmatrix} W_{71} \\ W_{72} \end{bmatrix}$ with $W_{71} \in \mathbb{R}^{m\times n}$. Inserting (3.22), (3.24) and (3.25) into (3.19) and (3.20), we can easily obtain the expressions (3.4) and (3.5). □

4. Solving Problem II

Theorem 4.1. *Given dual matrices $A = A_1 + \varepsilon A_2 \in \mathbb{D}^{m\times n}$, $B = B_1 + \varepsilon B_2 \in \mathbb{D}^{n\times n}$ and $\tilde{X} = \tilde{X}_1 + \varepsilon\tilde{X}_2 \in \mathbb{D}^{m\times m}$ with $B_i = B_i^\top, i = 1, 2$, if write*

$$P = B_2 - A_2^\top(A_1^\top)^\dagger B_1A_1^\dagger A_1 - A_1^\dagger A_1B_1A_1^\dagger A_2,$$

$$
\begin{aligned}
Q =& F_{A_1}A_2^\top E_{A_1}, \Theta = F_Q E_{A_1} A_2 A_1^\dagger, \\
V_1 =& (A_1^\top)^\dagger B_1 A_1^\dagger + E_{A_1} Q^\dagger F_{A_1} P A_1^\dagger + (A_1^\dagger)^\top P^\top F_{A_1} (Q^\dagger)^\top E_{A_1}, \\
V_2 =& (A_1^\top)^\dagger P A_1^\dagger - (A_1^\dagger)^\top A_2^\top E_{A_1} Q^\dagger F_{A_1} P A_1^\dagger \\
& - (A_1^\dagger)^\top P^\top F_{A_1} (Q^\dagger)^\top E_{A_1} A_2 A_1^\dagger, \\
R_1 =& \frac{1}{2} E_{A_1} \left((\tilde{X}_2 + \tilde{X}_2^\top) - (V_2^\top + V_2) \right), \\
R_2 =& \frac{1}{2} F_Q E_{A_1} \left((\tilde{X}_1 + \tilde{X}_1^\top) - (V_1^\top + V_1) \right) \\
& - \frac{1}{2} \Theta \left((\tilde{X}_2 + \tilde{X}_2^\top) - (V_2^\top + V_2) \right) A_1 A_1^\dagger, \\
R_3 =& \frac{1}{2} E_{A_1} \left((\tilde{X}_1 + \tilde{X}_1^\top) - (V_1^\top + V_1) \right) E_{A_1}.
\end{aligned}
$$

Then dual matrix equation (1.1) *has a symmetric solution if and only if*

$$F_{A_1} B_1 = 0, QQ^\dagger F_{A_1} P A_1^\dagger A_1 = F_{A_1} P. \tag{4.1}$$

and the general symmetric solution set of dual matrix equation (1.1) *can be expressed as*

$$T = \{X = X_1 + \varepsilon X_2 \in \mathbb{D}^{m\times m} | A^\top X A = B, X_i = X_i^\top, i = 1, 2\}, \tag{4.2}$$

where

$$X_1 = V_1 + E_{A_1}(F_Q U_3 + U_4 E_{A_1}) + (F_Q U_3 + U_4 E_{A_1})^\top E_{A_1}, \tag{4.3}$$

$$X_2 = V_2 - \Theta^\top U_3 A_1 A_1^\dagger - A_1 A_1^\dagger U_3^\top \Theta + E_{A_1} U_2 + U_2^\top E_{A_1}, \tag{4.4}$$

with U_2, U_3, U_4 are arbitrary matrices. In this case, Problem II has the unique solution $\hat{X}$, and $\hat{X}$ admits the following representation:

$$\hat{X} = \hat{X}_1 + \varepsilon \hat{X}_2, \tag{4.5}$$

where

$$\hat{X}_1 = V_1 + E_{A_1}(F_Q U_3 + U_4 E_{A_1}) + (F_Q U_3 + U_4 E_{A_1})^\top E_{A_1}, \tag{4.6}$$

$$\hat{X}_2 = V_2 - \Theta^\top U_3 A_1 A_1^\dagger - A_1 A_1^\dagger U_3^\top \Theta + E_{A_1} U_2 + U_2^\top E_{A_1}, \tag{4.7}$$

and U_2, U_3 and U_4 are determined by solving the unique solution of the equation

$$\Delta \begin{bmatrix} \mathrm{vec}(U_2) \\ \mathrm{vec}(U_3) \\ \mathrm{vec}(U_4) \end{bmatrix} = R, \tag{4.8}$$

with Δ and R being defined as in (4.25).

Proof. In the first step, we need to find the general symmetric solution of the dual matrix equation (1.1).

The dual matrix equation (1.1) is equivalent to Equations (3.8)–(3.9). Using Lemma 2.3, Eq. (3.8) has a symmetric solution if and only if the first condition of (4.1) is satisfied, and the general symmetric solution is

$$X_1 = (A_1^\top)^\dagger B_1 A_1^\dagger + E_{A_1} U_1 + U_1^\top E_{A_1}, \tag{4.9}$$

where U_1 is an arbitrary matrix. Inserting (4.9) into (3.9) yields

$$A_1^\top X_2 A_1 = P - A_2^\top F_{A_1} U_1^\top A_1 - A_1^\top U_1 F_{A_1} A_2. \tag{4.10}$$

Using Lemma 2.3 again, Eq. (4.10) has a symmetric solution if and only if

$$Q U_1 A_1 = F_{A_1} P, \tag{4.11}$$

the general symmetric solution is

$$
\begin{aligned}
X_2 =& (A_1^\top)^\dagger P A_1^\dagger - (A_1^\top)^\dagger A_2^\top E_{A_1} U_1 A_1 A_1^\dagger - A_1 A_1^\dagger U_1^\top E_{A_1} A_2 A_1^\dagger \\
& + E_{A_1} U_2^\top + U_2 E_{A_1},
\end{aligned} \tag{4.12}
$$

where U_2 is an arbitrary matrix. By Lemma 2.1, Eq. (4.11) with unknown matrix U_1 has a solution if and only if the second condition of (4.1) is satisfied, the general solution is

$$U_1 = Q^\dagger F_{A_1} P A_1^\dagger + F_Q U_3 + U_4 E_{A_1}, \tag{4.13}$$

where U_3, U_4 are arbitrary matrices. By substituting (4.13) into (4.9) and (4.12), we can get (4.3) and (4.4).

In the second step, we need to solve the minimization problem. For the given dual matrix $\tilde{X} \in \mathbb{D}^{m\times m}$ and any matrix $X \in T$ in (4.2), we have

$$
\begin{aligned}
& f(U_2, U_3, U_4) \\
=& \|X - \tilde{X}\|_{\mathbb{D}}^2 \\
=& \|X_1 - \tilde{X}_1\|^2 + \|X_2 - \tilde{X}_2\|^2 \\
=& \|V_1 + E_{A_1}(F_Q U_3 + U_4 E_{A_1}) + (F_Q U_3 + U_4 E_{A_1})^\top E_{A_1} - \tilde{X}_1\| \\
& + \|V_2 - \Theta^\top U_3 A_1 A_1^\dagger - A_1 A_1^\dagger U_3^\top \Theta + E_{A_1} U_2 + U_2^\top E_{A_1} - \tilde{X}_2\| \\
=& \mathrm{tr}(V_1^\top V_1 + U_3^\top F_Q E_{A_1} F_Q U_3 + E_{A_1} U_4^\top E_{A_1} U_4 E_{A_1} \\
& + E_{A_1} F_Q U_3 U_3^\top F_Q E_{A_1} + E_{A_1} U_4 E_{A_1} U_4^\top E_{A_1} + \tilde{X}_1^\top \tilde{X}_1 + V_2^\top V_2 \\
& + A_1 A_1^\dagger U_3^\top \Theta \Theta^\top U_3 A_1 A_1^\dagger + \Theta^\top U_3 A_1 A_1^\dagger U_3^\top \Theta + U_2^\top E_{A_1} U_2
\end{aligned}
$$

$$
\begin{aligned}
&+ E_{A_1}U_2U_2^\top E_{A_1} + \tilde{X}_2^\top \tilde{X}_2 + 2V_1^\top E_{A_1}F_QU_3 + 2V_1^\top E_{A_1}U_4E_{A_1}\\
&+ 2V_1^\top U_3^\top F_QE_{A_1} + 2V_1^\top E_{A_1}U_4^\top E_{A_1} - 2V_1^\top \tilde{X}_1\\
&+ 2U_3^\top F_QE_{A_1}U_4E_{A_1} + 2U_3^\top F_QE_{A_1}U_3^\top F_QE_{A_1}\\
&+ 2U_3^\top F_QE_{A_1}U_4^\top E_{A_1} - 2U_3^\top F_QE_{A_1}\tilde{X}_1 + 2U_2^\top E_{A_1}U_2^\top E_{A_1}\\
&+ 2E_{A_1}U_4^\top E_{A_1}U_3^\top F_QE_{A_1} + 2E_{A_1}U_4^\top E_{A_1}U_4^\top E_{A_1}\\
&- 2E_{A_1}U_4^\top E_{A_1}\tilde{X}_1 + 2E_{A_1}F_QU_3E_{A_1}U_4^\top E_{A_1} - 2E_{A_1}F_QU_3\tilde{X}_1\\
&- 2E_{A_1}U_4E_{A_1}\tilde{X}_1 - 2V_2^\top \Theta^\top U_3A_1A_1^\dagger - 2V_2^\top A_1A_1^\dagger U_3^\top \Theta\\
&+ 2V_2^\top E_{A_1}U_2 + 2V_2^\top U_2^\top E_{A_1} - 2U_2^\top E_{A_1}\tilde{X}_2\\
&+ 2U_3^\top \Theta U_3^\top \Theta - 2V_2^\top \tilde{X}_2 - 2E_{A_1}U_2\tilde{X}_2 + 2\Theta^\top U_3A_1A_1^\dagger \tilde{X}_2\\
&+ 2A_1A_1^\dagger U_3^\top \Theta \tilde{X}_2).
\end{aligned}
$$

Therefore, $f(U_2, U_3, U_4)$ is minimized if and only if $\frac{\partial f(U_2,U_3,U_4)}{\partial U_2} = 0, \frac{\partial f(U_2,U_3,U_4)}{\partial U_3} = 0, \frac{\partial f(U_2,U_3,U_4)}{\partial U_4} = 0$, which implies that

$$E_{A_1}U_2 + E_{A_1}U_2^\top E_{A_1} = R_1, \tag{4.14}$$

$$
\begin{aligned}
&F_QE_{A_1}F_QU_3 + F_QE_{A_1}U_3^\top F_QE_{A_1} + \Theta\Theta^\top U_3A_1A_1^\dagger + \Theta U_3^\top \Theta\\
&+ F_QE_{A_1}U_4E_{A_1} + F_QE_{A_1}U_4^\top E_{A_1} = R_2,
\end{aligned} \tag{4.15}
$$

$$E_{A_1}F_QU_3E_{A_1} + E_{A_1}U_3^\top F_QE_{A_1} + E_{A_1}U_4E_{A_1} + E_{A_1}U_4^\top E_{A_1} = R_3. \tag{4.16}$$

By applying the Kronecker product and stretching function, (4.14)–(4.16) can be equivalently written as

$$\Delta_{11}\mathrm{vec}(U_2) = \mathrm{vec}(R_1), \tag{4.17}$$

$$\Delta_{22}\mathrm{vec}(U_3) + \Delta_{23}\mathrm{vec}(U_4) = \mathrm{vec}(R_2), \tag{4.18}$$

$$\Delta_{32}\mathrm{vec}(U_3) + \Delta_{33}\mathrm{vec}(U_4) = \mathrm{vec}(R_3), \tag{4.19}$$

where

$$\Delta_{11} = I_m \otimes E_{A_1} + T_{m^2}(E_{A_1} \otimes E_{A_1}), \tag{4.20}$$

$$
\begin{aligned}
\Delta_{22} = {} & I_m \otimes (F_QE_{A_1}F_Q) + (A_1A_1^\dagger) \otimes (\Theta\Theta^\top) + T_{m^2}\Big(\Theta \otimes \Theta^\top\\
&+ (F_QE_{A_1}) \otimes (E_{A_1}F_Q)\Big),
\end{aligned} \tag{4.21}
$$

$$\Delta_{23} = E_{A_1} \otimes (F_QE_{A_1}) + T_{m^2}((F_QE_{A_1}) \otimes E_{A_1}), \tag{4.22}$$

$$\Delta_{32} = E_{A_1} \otimes (E_{A_1}F_Q) + T_{m^2}\left(E_{A_1} \otimes (E_{A_1}F_Q)\right), \tag{4.23}$$

$$\Delta_{33} = E_{A_1} \otimes E_{A_1} + T_{m^2}(E_{A_1} \otimes E_{A_1}), \tag{4.24}$$

with T_{m^2} is the $m^2 \times m^2$ commutation matrix which is defined by Lemma 2.6. Let

$$\Delta = \begin{bmatrix} \Delta_{11} & 0 & 0 \\ 0 & \Delta_{22} & \Delta_{23} \\ 0 & \Delta_{32} & \Delta_{33} \end{bmatrix}, \ R = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}. \tag{4.25}$$

Then, (4.17)–(4.19) can be expressed as the equation of (4.8). The proof is complete. □

5. Numerical algorithm and numerical example

Based on Theorem 4.1, we can formulate the following algorithm to solve Problem II.

Algorithm 1

1) Input matrices A_i, B_i and $\tilde{X}_i, i = 1, 2$.

2) Calculate $P, Q, \Theta, V_1, V_2, R_1, R_2$ and R_3 according to Theorem 4.1.

3) If the conditions (4.1) are satisfied, then continue, otherwise, Problem II has no solution, and stop.

4) Compute the matrices $\Delta_{11}, \Delta_{22}, \Delta_{23}, \Delta_{32}$ and Δ_{33} by (4.20)–(4.24).

5) Compute the matrices Δ and R by (4.25).

6) Solving Eq. (4.8), we obtain $U_2 = \mathrm{reshape}(\mathrm{vec}(U_2))$, $U_3 = \mathrm{reshape}(\mathrm{vec}(U_3))$ and $U_4 = \mathrm{reshape}(\mathrm{vec}(U_4))$.

7) Compute the matrices $\hat{X}_1, \hat{X}_2$ on the basis of (4.6)–(4.7).

8) Calculate the unique approximation solution $\hat{X} = \hat{X}_1 + \varepsilon\hat{X}_2$.

Example 5.1. Let $m = 6, n = 6$, and the matrices $A_1, A_2, B_1, B_2, \tilde{X}_1, \tilde{X}_2$ be given by

$$A_1 = \begin{bmatrix} 1.5712 & 0.5686 & 1.5930 & 1.2840 & 0.9838 & 0.6992 \\ 0.8688 & 0.5996 & 0.9163 & 0.7342 & 0.7349 & 0.6207 \\ 1.0319 & 0.1179 & 1.0144 & 0.8216 & 0.4750 & 0.2494 \\ 1.6420 & 0.6331 & 1.6696 & 1.3452 & 1.0542 & 0.7626 \\ 0.9667 & 0.4320 & 0.9903 & 0.7970 & 0.6603 & 0.4976 \\ 0.8451 & 0.0925 & 0.8303 & 0.6726 & 0.3863 & 0.2009 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.2920 & 0.3395 & 0.4177 & 0.1280 & 0.4607 & 0.1206 \\ 0.4317 & 0.9516 & 0.9831 & 0.9991 & 0.9816 & 0.5895 \\ 0.0155 & 0.9203 & 0.3015 & 0.1711 & 0.1564 & 0.2262 \\ 0.9841 & 0.0527 & 0.7011 & 0.0326 & 0.8555 & 0.3846 \\ 0.1672 & 0.7379 & 0.6663 & 0.5612 & 0.6448 & 0.5830 \\ 0.1062 & 0.2691 & 0.5391 & 0.8819 & 0.3763 & 0.2518 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 53.5564 & 18.5678 & 54.1964 & 43.6984 & 32.9892 & 23.1646 \\ 18.5678 & 6.5279 & 18.8010 & 15.1578 & 11.4979 & 8.1054 \\ 54.1964 & 18.8010 & 54.8454 & 44.2215 & 33.3910 & 23.4507 \\ 43.6984 & 15.1578 & 44.2215 & 35.6556 & 26.9221 & 18.9071 \\ 32.9892 & 11.4979 & 33.3910 & 26.9221 & 20.3610 & 14.3185 \\ 23.1646 & 8.1054 & 23.4507 & 18.9071 & 14.3185 & 10.0803 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 82.7260 & 46.3330 & 94.2048 & 73.4880 & 66.5812 & 44.0999 \\ 46.3330 & 22.3895 & 50.6798 & 40.1518 & 34.1939 & 23.0895 \\ 94.2048 & 50.6798 & 105.9817 & 82.9702 & 73.9555 & 49.2920 \\ 73.4880 & 40.1518 & 82.9702 & 64.8924 & 58.2046 & 38.7950 \\ 66.5812 & 34.1939 & 73.9555 & 58.2046 & 50.8615 & 34.1143 \\ 44.0999 & 23.0895 & 49.2920 & 38.7950 & 34.1143 & 22.8048 \end{bmatrix},$$

$$\tilde{X}_1 = \begin{bmatrix} 0.4243 & 0.4735 & 0.7655 & 0.6476 & 0.4501 & 0.2564 \\ 0.4609 & 0.1527 & 0.1887 & 0.6790 & 0.4587 & 0.6135 \\ 0.7702 & 0.3411 & 0.2875 & 0.6358 & 0.6619 & 0.5822 \\ 0.3225 & 0.6074 & 0.0911 & 0.9452 & 0.7703 & 0.5407 \\ 0.7847 & 0.1917 & 0.5762 & 0.2089 & 0.3502 & 0.8699 \\ 0.4714 & 0.7384 & 0.6834 & 0.7093 & 0.6620 & 0.2648 \end{bmatrix},$$

$$\tilde{X}_2 = \begin{bmatrix} 0.6074 & 0.0911 & 0.9452 & 0.7703 & 0.5407 & 0.5447 \\ 0.1917 & 0.5762 & 0.2089 & 0.3502 & 0.8699 & 0.6473 \\ 0.7384 & 0.6834 & 0.7093 & 0.6620 & 0.2648 & 0.5439 \\ 0.2428 & 0.5466 & 0.2362 & 0.4162 & 0.3181 & 0.7210 \\ 0.9174 & 0.4257 & 0.1194 & 0.8419 & 0.1192 & 0.5225 \\ 0.2691 & 0.6444 & 0.6073 & 0.8329 & 0.9398 & 0.9937 \end{bmatrix}.$$

It is easy to verity that the conditions (4.1) hold:

$$\|F_{A_1}B_1\| = 2.2693 \times 10^{-14},$$
$$\|QQ^{\dagger}F_{A_1}PA_1^{\dagger}A_1 - F_{A_1}P\| = 2.6989 \times 10^{-14}.$$

By using Algorithm 1, we can obtain the unique approximation solution $\hat{X} = \hat{X}_1 + \varepsilon\hat{X}_2$ of Problem II as follows:

$$\hat{X}_1 = \begin{bmatrix} 1.4250 & 1.2127 & 1.4706 & 1.2908 & 1.2738 & 0.7912 \\ 1.2127 & 1.1359 & 0.3465 & 0.8545 & 0.8276 & 0.8239 \\ 1.4706 & 0.3465 & 1.1821 & 1.3277 & 1.0543 & 1.1221 \\ 1.2908 & 0.8545 & 1.3277 & 1.4092 & 0.8873 & 1.2891 \\ 1.2738 & 0.8276 & 1.0543 & 0.8873 & 0.7528 & 1.0798 \\ 0.7912 & 0.8239 & 1.1221 & 1.2891 & 1.0798 & 0.4615 \end{bmatrix},$$

$$\hat{X}_2 = \begin{bmatrix} 1.7453 & 0.5624 & 1.3277 & 1.7978 & 1.0231 & 0.9802 \\ 0.5624 & 0.4391 & 0.5530 & 0.6539 & 0.6188 & 0.3080 \\ 1.3277 & 0.5530 & 1.1629 & 1.0739 & 0.9511 & 1.1491 \\ 1.7978 & 0.6539 & 1.0739 & 2.1033 & 0.5943 & 1.2673 \\ 1.0231 & 0.6188 & 0.9511 & 0.5943 & 0.7730 & 0.6016 \\ 0.9802 & 0.3080 & 1.1491 & 1.2673 & 0.6016 & 0.5225 \end{bmatrix}.$$

The absolute errors are estimated by

$$\|A_1^{\top}X_1A_1 - B_1\| = 2.9543 \times 10^{-12},$$
$$\|A_1^{\top}X_2A_1 + A_2^{\top}X_1A_1 + A_1^{\top}X_1A_2 - B_2\| = 1.2922 \times 10^{-12},$$

which implies that $\hat{X}$ is the unique approximation solution of Problem II.

6. Conclusions

Solving dual matrix equations is often required in kinematic analysis and sensor calibration. In this paper, the solvability conditions and explicit solutions of Problem I are obtained using the Moore-Penrose inverse (see Theorem 3.1). Further, by applying the Moore-Penrose inverse and Kronecker product of matrices, we obtain the unique approximation solution of Problem II (see Theorem 4.1).

Conflict of interest

All authors declare that they have no conflicts of interest.

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