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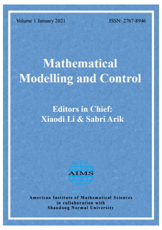

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Research article

Existence of solutions for a class of fractional dynamical systems with two damping terms in Banach space

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Abstract: This paper studies the existence of solutions for fractional dynamical systems with two damping terms in Banach space. First, we generalize the well-known Gronwall inequality. Next, according to fixed-point theorems and inequalities, the existence results for the considered system are obtained. At last, an example is used to support the main results.

Keywords: Gronwall inequality; fixed point theorems; fractional differential equation; two damping terms

1. Introduction

In recent years, fractional calculus has gained much attention. Compared with the integer derivative, the fractional derivative can lead to better results in many practical problems as it serves as a powerful tool to describe the memory and genetic properties of various materials and processes. Fractional differential equations (FDEs) appear in plenty of scientific and engineering disciplines because they describe many events and processes in the domains of aerodynamics, chemistry, physics, the rheology of polymers, etc. For more details, see [1–3] and the references therein.

In addition, fractional damping systems based on velocity history have attracted extensive attention and been the focus of much research in the last few decades (see [4–7]). For example, problems of non-viscous damping with hysteresis have been studied in the application of magnetorheological fluids (see [8]). Similarly, this concept was used to simulate damping in a vehicle tire and plates made of composite materials (see [9, 10]). The authors of [11] considered a mechanical system with viscoelastic damping, mass and a rapid jump by producing the FDEs based on Caputo fractional derivatives. For more exciting results about damped dynamic systems, see [4–8].

The existence of solutions for fractional dynamical systems has been widely investigated because it is a fundamental problem of fractional dynamical systems as well as a necessary condition to consider other properties such as controllability and stability (see [12–14]). For instance, the authors of [15] investigated the existence of solutions for fractional neutral differential equations with infinite delay. The authors of [16] obtained existence results for an impulsive fractional integro-differential equation with state-dependent delay. In addition, many authors have studied multi-term fractional systems as they have been successfully used in gas dynamics, mechanical systems, etc. (see [17–21]). These systems are more complicated and interesting than one-term fractional systems. For example, Sheng and Jiang discussed the following system in [22]:

$$\begin{cases} {}^{C}D_{0^+}^{\alpha}x(t) - A\,{}^{C}D_{0^+}^{\beta}x(t) = f(t, x(t)),\ t \in J = [0, T]; \\ x(0) = x_0,\ x'(0) = x'_0, \end{cases}$$

where $0 < \beta \leq 1 < \alpha \leq 2$, $x \in R^n$, A is an $n \times n$ matrix and $f : J \times R^n \to R^n$ is continuous. The existence results were obtained by utilizing fixed-point theorems.

It is well known that many significant results have been achieved in the study of finite-dimensional dynamical systems. However, the research on dynamical systems is not limited to the finite dimension. From the perspective of practical problems in physics, many important dynamical system problems, such as turbulence in fluid mechanics, discrete attractors and small dissipative dynamics (see [23]), are studied in infinite dimensions.

Inspired by the above, we investigate the following fractional dynamical system with two damping terms in Banach space $\mathbb{H}$:

$$\begin{cases} {}^C D_{0^+}^{\alpha}\xi(t) - \mathcal{A}{}^C D_{0^+}^{\beta}\xi(t) - \mathcal{B}{}^C D_{0^+}^{\gamma}\xi(t) = \hbar(t,\xi(t)), \\ t \in \mathfrak{T} = [0,\iota]; \\ \xi(0) = \xi_0,\ \xi'(0) = y_0,\ \xi''(0) = z_0, \end{cases} \tag{1.1}$$

where $0 < \gamma \leq 1 < \beta \leq 2 < \alpha \leq 3$, ξ_0, y_0, $z_0 \in \mathbb{H}$, $\mathcal{A}$, $\mathcal{B} \in L(\mathbb{H})$ and $\hbar : \mathfrak{T} \times \mathbb{H} \to \mathbb{H}$ is a given function.

To the best of our knowledge, few people have studied this type of system. Only Zhang and Xu [24] has studied (1.1) in a finite-dimensional space. The existence and uniqueness of solutions for (1.1) have been obtained by using the Banach fixed point theorem. It is remarkable that $\hbar(t,\xi)$ meets the Lipschitz condition, which is difficult to satisfy in practical problems. Compared with [24], this article has the following distinctive features. First, we consider (1.1) in abstract space. Second, on the basis of [22], we generalize the Gronwall inequality, which is crucial for the proof of our results. Third, $\hbar(t,\xi)$ here is no longer required to satisfy the Lipschitz conditions.

The remainder of this article is organized as follows. In Section 2, we introduce some fundamental concepts and lemmas, which will be used throughout the paper. In Section 3, first, the Gronwall inequality is extended. Second, the main results are presented and proved. Last, we give an illustrative instance to support the main results. In Section 4, the conclusion of the full text is given.

2. Preliminaries

In this section, we introduce some definitions and lemmas used to prove the conclusion. Throughout this paper, let $\mathcal{PC}(\mathfrak{T},\mathbb{H})$ be the Banach space of all continuous functions from $\mathfrak{T} = [0,\iota]$ to $\mathbb{H}$ with the norm $\|\xi\|_c = \sup\{\|\xi(t)\| : t \in \mathfrak{T}\}$ for $\xi \in \mathcal{PC}(\mathfrak{T},\mathbb{H})$, where $\|\cdot\|$ is the norm of the Banach space $\mathbb{H}$. In addition, $\mathcal{X}(\cdot)$ and $\mathcal{X}_c(\cdot)$ represent the Hausdorff measure of noncompactness of a bounded set in $\mathbb{H}$ and $\mathcal{PC}(\mathfrak{T},\mathbb{H})$ respectively.

Definition 2.1 [1] The Riemann-Liouville fractional integral of order $\alpha \in R^+$ of a function $f : [0, +\infty) \to \mathbb{H}$ with the lower zero is defined as follows:

$$(I_{0^+}^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_{0^+}^{t} (t-s)^{\alpha-1} f(s)ds,$$

where $t > 0$, $\alpha > 0$ and $\Gamma(\cdot)$ is a gamma function.

Definition 2.2 [1] The Riemann-Liouville fractional derivative of order $\alpha \in R^+$ of a function $f : [0,+\infty) \to \mathbb{H}$ with the lower zero is defined as follows:

$$(D_{0^+}^{\alpha} f)(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_{0^+}^{t} (t-s)^{n-\alpha-1} f(s)ds,$$

where $n = [\alpha] + 1$, $t > 0$ and $\alpha > 0$.

Definition 2.3 [1] The Caputo fractional derivative of order $\alpha \in R^+$ of a function $f : [0,+\infty) \to \mathbb{H}$ with the lower zero is defined as follows:

$$({}^C D_{0^+}^{\alpha} f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_{0^+}^{t} (t-s)^{n-\alpha-1} f^{(n)}(s)ds.$$

From the definition of fractional integrals and Caputo derivatives, we have the following results.

Lemma 2.4 [1]

$$I_{0^+}^{\alpha}\ ({}^C D_{0^+}^{\alpha} f(t)) = f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0),\ \ t > 0,\ n-1 < \alpha < n.$$

Lemma 2.5 [25] Let $0 < \gamma \leq 1 < \beta \leq 2 < \alpha \leq 3$; then,

$$I_{0^+}^{\alpha}\ ({}^C D_{0^+}^{\beta} f(t)) = I_{0^+}^{\alpha-\beta} f(t) - \frac{f(0)t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \frac{f'(0)t^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)},$$

$$I_{0^+}^{\alpha}\ ({}^C D_{0^+}^{\gamma} f(t)) = I_{0^+}^{\alpha-\gamma} f(t) - \frac{f(0)t^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}.$$

Next, for $\zeta \in L^p(\mathfrak{T},R)$, define the norm

$$\|\zeta\|_{L^p(\mathfrak{T})} = \begin{cases} (\int_{\mathfrak{T}} \|\zeta(t)\|^p dt)^{\frac{1}{p}}, & 1 \leq p < \infty; \\ \inf\limits_{\mu(\bar{\mathfrak{T}})=0} \{ \sup\limits_{t\in\mathfrak{T}-\bar{\mathfrak{T}}} \|\zeta(t)\|\}, & p = \infty, \end{cases}$$

where $\mu(\overline{\mathfrak{T}})$ is the Lebesgue measure on $\overline{\mathfrak{T}}$. Then, $L^P(\mathfrak{T}, R)$ is a Banach space composed of all Lebesgue measurable functions $\zeta : \mathfrak{T} \to R$ with $\|\zeta\|_{L^P(\mathfrak{T})} < \infty$.

Lemma 2.6 [26] Suppose that $\mathbb{H}$ is a Banach space. Let D be a closed and convex subset of $\mathbb{H}$ and $x_0 \in D$. Assume that the continuous operator $Q : D \to D$ satisfies the following:

$C \subset D$ countable and $C \subset \overline{co}(\{x_0\} \cup Q(C)) \to C$ is relatively compact.

Then, Q has a fixed point in D.

Definition 2.7 [27] Let $\mathbb{H}$ be a Banach space and $Q : \mathbb{H} \to \mathbb{H}$ be continuous and bounded. We say that Q is condensing if $\mathcal{X}(Q(\Omega)) \le \mathcal{X}(\Omega)$ for any bounded and not relatively compact set $\Omega \subset \mathbb{H}$.

Lemma 2.8 [28] If S is bounded, then for each $\epsilon > 0$, there exists a sequence $\{v_n\}_{n=1}^{\infty} \subseteq S$ such that

$$\mathcal{X}(S) \le 2\mathcal{X}(\{v_n\}_{n=1}^{\infty}) + \epsilon.$$

Lemma 2.9 [29] Let $\mathbb{H}$ be a Banach space and $Q : \mathbb{H} \to \mathbb{H}$ be a condensing operator. If the set

$$E(Q) = \{\xi \in \mathbb{H} : \xi = \lambda Q\xi,\ \lambda \in [0, 1]\}$$

is bounded, then Q has at least one fixed point.

3. Main results

3.1. Generalization of Gronwall inequality

In this section, we extend the Gronwall inequality.

Theorem 3.1. *Suppose that α, β, $\gamma > 0$ and $z(t)$ is a nonnegative and integrable function on $[0, \iota]$. Let $\widetilde{h}(t)$, $\overline{h}(t)$ and $h_1(t)$ be nondecreasing, nonnegative and continuous functions defined on $[0, \iota]$. Assume that $\xi(t)$ is nonnegative and integrable on $[0, \iota]$ with*

$$\begin{aligned}\xi(t) \le{}& z(t) \\ &+ \widetilde{h}(t)\int_0^t (t-s)^{\alpha-1}\xi(s)ds + \overline{h}(t)\int_0^t (t-s)^{\beta-1}\xi(s)ds \\ &+ h_1(t)\int_0^t (t-s)^{\gamma-1}\xi(s)ds,\ t \in [0, \iota].\end{aligned}$$

Then,

$$\begin{aligned}&\xi(t) \\ &\le z(t) + \int_0^t \sum_{r=1}^{\infty}[h(t)]^r \sum_{p=0}^{r}\sum_{q=0}^{r-p} \frac{C_r^p C_{r-p}^q [\Gamma(\alpha)]^{r-p-q}[\Gamma(\beta)]^p[\Gamma(\gamma)]^q}{\Gamma[(r-p-q)\alpha + p\beta + q\gamma]} \\ &\times (t-s)^{(r-p-q)\alpha+p\beta+q\gamma-1} z(s)ds,\end{aligned} \tag{3.1}$$

where $h(t) = \widetilde{h}(t) + \overline{h}(t) + h_1(t)$ and $C_r^p = \frac{r(r-1)(r-2)\cdots(r-p+1)}{p!}$.

Proof. Let $h(t) = \widetilde{h}(t) + \overline{h}(t) + h_1(t)$. Then,

$$\xi(t) \le z(t) + h(t)\int_0^t [(t-s)^{\alpha-1} + (t-s)^{\beta-1} + (t-s)^{\gamma-1}]\xi(s)ds.$$

Define

$$\mathcal{B}\xi(t) = h(t)\int_0^t [(t-s)^{\alpha-1} + (t-s)^{\beta-1} + (t-s)^{\gamma-1}]\xi(s)ds.$$

Then,

$$\xi(t) \le z(t) + \mathcal{B}\xi(t), \qquad t \in [0, \iota],$$

which implies that

$$\xi(t) \le \sum_{i=0}^{r-1} \mathcal{B}^i z(t) + \mathcal{B}^r\xi(t). \tag{3.2}$$

Now, we prove that

$$\begin{aligned}\mathcal{B}^r\xi(t) \le{}&[h(t)]^r \int_0^t \sum_{p=0}^{r}\sum_{q=0}^{r-p} \frac{C_r^p C_{r-p}^q [\Gamma(\alpha)]^{r-p-q}[\Gamma(\beta)]^p[\Gamma(\gamma)]^q}{\Gamma[(r-p-q)\alpha + p\beta + q\gamma]} \\ &\times (t-s)^{(r-p-q)\alpha+p\beta+q\gamma-1}\xi(s)ds,\end{aligned} \tag{3.3}$$

and that $\mathcal{B}^r\xi(t) \to 0$ as $r \to +\infty$ for each $t \in [0, \iota]$.

When $r = 1$, the inequality (3.3) holds obviously. Suppose that it holds for $r = k$.

Let $r = k + 1$; then,

$$\begin{aligned}&\mathcal{B}^{k+1}\xi(t) \\ =&\mathcal{B}(\mathcal{B}^k\xi(t)) \\ \le& h(t)\int_0^t [(t-s)^{\alpha-1} + (t-s)^{\beta-1} + (t-s)^{\gamma-1}] \\ &\times\left[\int_0^s [h(s)]^k \sum_{p=0}^{k}\sum_{q=0}^{k-p} \frac{C_k^p C_{k-p}^q [\Gamma(\alpha)]^{k-p-q}[\Gamma(\beta)]^p[\Gamma(\gamma)]^q}{\Gamma[(k-p-q)\alpha + p\beta + q\gamma]} \right. \\ &\left.\times(s-\tau)^{(k-p-q)\alpha+p\beta+q\gamma-1}\xi(\tau)d\tau\right] ds.\end{aligned} \tag{3.4}$$

Since $h(t)$ is nondecreasing,

$$\begin{aligned}
&\mathscr{B}^{k+1}\xi(t)\\
\leq&[h(t)]^{k+1}\int_0^t[(t-s)^{\alpha-1}+(t-s)^{\beta-1}+(t-s)^{\gamma-1}]\\
&\times\Bigg[\int_0^s\sum_{p=0}^{k}\sum_{q=0}^{k-p}\frac{C_k^pC_{k-p}^q[\Gamma(\alpha)]^{k-p-q}[\Gamma(\beta)]^p[\Gamma(\gamma)]^q}{\Gamma[(k-p-q)\alpha+p\beta+q\gamma]}\\
&\times(s-\tau)^{(k-p-q)\alpha+p\beta+q\gamma-1}\xi(\tau)d\tau\Bigg]ds.
\end{aligned}\tag{3.5}$$

We exchange the integral order. Then,

$$\begin{aligned}
&B^{k+1}\xi(t)\\
\leq&[h(t)]^{k+1}\\
&\times\int_0^t\Bigg[\int_\tau^t\Bigg(\sum_{p=0}^{k}\sum_{q=0}^{k-p}\frac{C_k^pC_{k-p}^q[\Gamma(\alpha)]^{k-p-q}[\Gamma(\beta)]^p[\Gamma(\gamma)]^q}{\Gamma[(k-p-q)\alpha+p\beta+q\gamma]}\\
&\times(t-s)^{\alpha-1}(s-\tau)^{(k-p-q)\alpha+p\beta+q\gamma-1}\\
&+\sum_{p=0}^{k}\sum_{q=0}^{k-p}\frac{C_k^pC_{k-p}^q[\Gamma(\alpha)]^{k-p-q}[\Gamma(\beta)]^p[\Gamma(\gamma)]^q}{\Gamma[(k-p-q)\alpha+p\beta+q\gamma]}\\
&\times(t-s)^{\beta-1}(s-\tau)^{(k-p-q)\alpha+p\beta+q\gamma-1}\\
&+\sum_{p=0}^{k}\sum_{q=0}^{k-p}\frac{C_k^pC_{k-p}^q[\Gamma(\alpha)]^{k-p-q}[\Gamma(\beta)]^p[\Gamma(\gamma)]^q}{\Gamma[(k-p-q)\alpha+p\beta+q\gamma]}\\
&\times(t-s)^{\gamma-1}(s-\tau)^{(k-p-q)\alpha+p\beta+q\gamma-1}\Bigg)ds\Bigg]\xi(\tau)d\tau\\
=&[h(t)]^{k+1}\\
&\times\int_0^t\Bigg[\sum_{p=0}^{k}\sum_{q=0}^{k-p}\frac{C_k^pC_{k-p}^q[\Gamma(\alpha)]^{k+1-p-q}[\Gamma(\beta)]^p[\Gamma(\gamma)]^q}{\Gamma[(k+1-p-q)\alpha+p\beta+q\gamma]}\\
&\times(t-s)^{(k+1-p-q)\alpha+p\beta+q\gamma-1}\\
&+\sum_{p=0}^{k}\sum_{q=0}^{k-p}\frac{C_k^pC_{k-p}^q[\Gamma(\alpha)]^{k-p-q}[\Gamma(\beta)]^{(p+1)}[\Gamma(\gamma)]^q}{\Gamma[(k-p-q)\alpha+(p+1)\beta+q\gamma]}\\
&\times(t-s)^{(k-p-q)\alpha+(p+1)\beta+q\gamma-1}\\
&+\sum_{p=0}^{k}\sum_{q=0}^{k-p}\frac{C_k^pC_{k-p}^q[\Gamma(\alpha)]^{k-p-q}[\Gamma(\beta)]^p[\Gamma(\gamma)]^{(q+1)}}{\Gamma[(k-p-q)\alpha+p\beta+(q+1)\gamma]}\\
&\times(t-s)^{(k-p-q)\alpha+p\beta+(q+1)\gamma-1}\Bigg]\xi(s)ds\\
\leq&[h(t)]^{k+1}\\
&\times\int_0^t\Bigg[\sum_{p=0}^{k}\sum_{q=0}^{k-p}\frac{C_k^pC_{k-p}^q[\Gamma(\alpha)]^{k+1-p-q}[\Gamma(\beta)]^p[\Gamma(\gamma)]^q}{\Gamma[(k+1-p-q)\alpha+p\beta+q\gamma]}\\
&\times(t-s)^{(k+1-p-q)\alpha+p\beta+q\gamma-1}\\
&+\sum_{p=1}^{k+1}\sum_{q=0}^{k-p+1}\frac{C_k^{p-1}C_{k-p+1}^q[\Gamma(\alpha)]^{k+1-p-q}[\Gamma(\beta)]^p[\Gamma(\gamma)]^q}{\Gamma[(k+1-p-q)\alpha+p\beta+q\gamma]}\\
&\times(t-s)^{(k+1-p-q)\alpha+p\beta+q\gamma-1}\\
&+\sum_{p=0}^{k}\sum_{q=1}^{k-p+1}\frac{C_k^pC_{k-p}^{q-1}[\Gamma(\alpha)]^{k+1-p-q}[\Gamma(\beta)]^p[\Gamma(\gamma)]^q}{\Gamma[(k+1-p-q)\alpha+p\beta+q\gamma]}\\
&\times(t-s)^{(k+1-p-q)\alpha+p\beta+q\gamma-1}\Bigg]\xi(s)ds.
\end{aligned}$$

So, we have

$$\begin{aligned}
&B^{k+1}\xi(t)\\
\leq&[h(t)]^{k+1}\\
&\times\int_0^t\Bigg[\sum_{p=1}^{k}\sum_{q=1}^{k-p}\frac{(C_k^pC_{k-p}^q+C_k^{p-1}C_{k-p+1}^q+C_k^pC_{k-p}^{q-1})}{\Gamma[(k+1-p-q)\alpha+p\beta+q\gamma]}\\
&\times[\Gamma(\alpha)]^{k+1-p-q}[\Gamma(\beta)]^p[\Gamma(\gamma)]^q(t-s)^{(k+1-p-q)\alpha+p\beta+q\gamma-1}\\
&+\sum_{q=0}^{k}\Bigg|_{p=0}\frac{C_k^pC_{k-p}^q[\Gamma(\alpha)]^{k+1-p-q}[\Gamma(\beta)]^p[\Gamma(\gamma)]^q}{\Gamma[(k+1-p-q)\alpha+p\beta+q\gamma]}\\
&\times(t-s)^{(k+1-p-q)\alpha+p\beta+q\gamma-1}\\
&+\sum_{p=0}^{k}\Bigg|_{q=0}\frac{C_k^pC_{k-p}^q[\Gamma(\alpha)]^{k+1-p-q}[\Gamma(\beta)]^p[\Gamma(\gamma)]^q}{\Gamma[(k+1-p-q)\alpha+p\beta+q\gamma]}\\
&\times(t-s)^{(k+1-p-q)\alpha+p\beta+q\gamma-1}\\
&+\Bigg|_{\substack{p=k+1\\q=0}}\frac{C_k^{p-1}C_{k-p+1}^q[\Gamma(\alpha)]^{k+1-p-q}[\Gamma(\beta)]^p[\Gamma(\gamma)]^q}{\Gamma[(k+1-p-q)\alpha+p\beta+q\gamma]}\\
&\times(t-s)^{(k+1-p-q)\alpha+p\beta+q\gamma-1}\\
&+\sum_{p=1}^{k+1}\Bigg|_{q=0}\frac{C_k^{p-1}C_{k-p+1}^q[\Gamma(\alpha)]^{k+1-p-q}[\Gamma(\beta)]^p[\Gamma(\gamma)]^q}{\Gamma[(k+1-p-q)\alpha+p\beta+q\gamma]}\\
&\times(t-s)^{(k+1-p-q)\alpha+p\beta+q\gamma-1}\\
&+\sum_{p=1}^{k+1}\Bigg|_{p=k-p-1}\frac{C_k^{p-1}C_{k-p+1}^q[\Gamma(\alpha)]^{k+1-p-q}[\Gamma(\beta)]^p[\Gamma(\gamma)]^q}{\Gamma[(k+1-p-q)\alpha+p\beta+q\gamma]}\\
&\times(t-s)^{(k+1-p-q)\alpha+p\beta+q\gamma-1}\\
&+\sum_{q=1}^{k+1}\Bigg|_{p=0}\frac{C_k^pC_{k-p}^{q-1}[\Gamma(\alpha)]^{k+1-p-q}[\Gamma(\beta)]^p[\Gamma(\gamma)]^q}{\Gamma[(k+1-p-q)\alpha+p\beta+q\gamma]}\\
&\times(t-s)^{(k+1-p-q)\alpha+p\beta+q\gamma-1}\\
&+\sum_{p=0}^{k}\Bigg|_{\substack{p=\\k-p-1}}\frac{C_k^pC_{k-p}^{q-1}[\Gamma(\alpha)]^{k+1-p-q}[\Gamma(\beta)]^p[\Gamma(\gamma)]^q}{\Gamma[(k+1-p-q)\alpha+p\beta+q\gamma]}\\
&\times(t-s)^{(k+1-p-q)\alpha+p\beta+q\gamma-1}\Bigg]\xi(s)ds.
\end{aligned}\tag{3.6}$$

Notice that

$$\sum_{q=0}^{k}\Big|_{p=0}\frac{C_k^pC_{k-p}^q[\Gamma(\alpha)]^{k+1-p-q}[\Gamma(\beta)]^p[\Gamma(\gamma)]^q}{\Gamma[(k+1-p-q)\alpha+p\beta+q\gamma]}\times(t-s)^{(k+1-p-q)\alpha+p\beta+q\gamma-1}$$
$$=\Big|_{\substack{p=0\\q=0}}\frac{[\Gamma(\alpha)]^{k+1}}{\Gamma[(k+1)\alpha]}(t-s)^{(k+1)\alpha-1}+\sum_{q=1}^{k}\Big|_{p=0}\frac{C_k^q[\Gamma(\alpha)]^{k+1-q}[\Gamma(\gamma)]^q}{\Gamma[(k+1-q)\alpha+q\gamma]}(t-s)^{(k+1-q)\alpha+q\gamma-1}. \tag{3.7}$$

$$\sum_{q=1}^{k+1}\Big|_{p=0}\frac{C_k^pC_{k-p}^{q-1}[\Gamma(\alpha)]^{k+1-p-q}[\Gamma(\beta)]^p[\Gamma(\gamma)]^q}{\Gamma[(k+1-p-q)\alpha+p\beta+q\gamma]}\times(t-s)^{(k+1-p-q)\alpha+p\beta+q\gamma-1}$$
$$=\Big|_{\substack{p=0\\q=k+1}}\frac{[[\Gamma(\gamma)]^{k+1}}{\Gamma[(k+1)\gamma]}(t-s)^{(k+1)\gamma-1}+\sum_{q=1}^{k}\Big|_{p=0}\frac{C_k^{q-1}[\Gamma(\alpha)]^{k+1-q}[\Gamma(\gamma)]^q}{\Gamma[(k+1-q)\alpha+q\gamma]}(t-s)^{(k+1-q)\alpha+q\gamma-1}. \tag{3.8}$$

Combining (3.7) and (3.8), we obtain

$$\sum_{q=0}^{k}\Big|_{p=0}\frac{C_k^pC_{k-p}^q[\Gamma(\alpha)]^{k+1-p-q}[\Gamma(\beta)]^p[\Gamma(\gamma)]^q}{\Gamma[(k+1-p-q)\alpha+p\beta+q\gamma]}\times(t-s)^{(k+1-p-q)\alpha+p\beta+q\gamma-1}$$
$$+\sum_{q=1}^{k+1}\Big|_{p=0}\frac{C_k^pC_{k-p}^{q-1}[\Gamma(\alpha)]^{k+1-p-q}[\Gamma(\beta)]^p[\Gamma(\gamma)]^q}{\Gamma[(k+1-p-q)\alpha+p\beta+q\gamma]}\times(t-s)^{(k+1-p-q)\alpha+p\beta+q\gamma-1}$$
$$=\Big|_{\substack{p=0\\q=0}}\frac{[\Gamma(\alpha)]^{k+1}}{\Gamma[(k+1)\alpha]}(t-s)^{(k+1)\alpha-1}+\Big|_{\substack{p=0\\q=k+1}}\frac{[\Gamma(\gamma)]^{k+1}}{\Gamma[(k+1)\gamma]}(t-s)^{(k+1)\gamma-1}+\sum_{q=1}^{k}\Big|_{p=0}\frac{C_{k+1}^q[\Gamma(\alpha)]^{k+1-q}[\Gamma(\gamma)]^q}{\Gamma[(k+1-q)\alpha+q\gamma]}(t-s)^{(k+1-q)\alpha+q\gamma-1}. \tag{3.9}$$

Similarly,

$$\sum_{p=0}^{k}\Big|_{q=0}\frac{C_k^pC_{k-p}^q[\Gamma(\alpha)]^{k+1-p-q}[\Gamma(\beta)]^p[\Gamma(\gamma)]^q}{\Gamma[(k+1-p-q)\alpha+p\beta+q\gamma]}\times(t-s)^{(k+1-p-q)\alpha+p\beta+q\gamma-1}$$
$$+\sum_{p=1}^{k+1}\Big|_{q=0}\frac{C_k^{p-1}C_{k-p+1}^q[\Gamma(\alpha)]^{k+1-p-q}[\Gamma(\beta)]^p[\Gamma(\gamma)]^q}{\Gamma[(k+1-p-q)\alpha+p\beta+q\gamma]}\times(t-s)^{(k+1-p-q)\alpha+p\beta+q\gamma-1}$$
$$=\Big|_{\substack{p=0\\q=0}}\frac{[\Gamma(\alpha)]^{k+1}}{\Gamma[(k+1)\alpha]}(t-s)^{(k+1)\alpha-1}+\Big|_{\substack{p=k+1\\q=0}}\frac{[\Gamma(\beta)]^{k+1}}{\Gamma[(k+1)\beta]}(t-s)^{(k+1)\beta-1}$$
$$+\sum_{p=1}^{k}\Big|_{q=0}\frac{C_{k+1}^p[\Gamma(\alpha)]^{k+1-p-q}[\Gamma(\beta)]^p[\Gamma(\gamma)]^q}{\Gamma[(k+1-p-q)\alpha+p\beta+q\gamma]}\times(t-s)^{(k+1-p-q)\alpha+p\beta+q\gamma-1}, \tag{3.10}$$

and

$$\sum_{p=1}^{k+1}\Big|_{q=k-p+1}\frac{C_k^{p-1}C_{k-p+1}^q[\Gamma(\alpha)]^{k+1-p-q}[\Gamma(\beta)]^p[\Gamma(\gamma)]^q}{\Gamma[(k+1-p-q)\alpha+p\beta+q\gamma]}\times(t-s)^{(k+1-p-q)\alpha+p\beta+q\gamma-1}$$
$$+\sum_{p=0}^{k}\Big|_{q=k-p+1}\frac{C_k^pC_{k-p}^{q-1}[\Gamma(\alpha)]^{k+1-p-q}[\Gamma(\beta)]^p[\Gamma(\gamma)]^q}{\Gamma[(k+1-p-q)\alpha+p\beta+q\gamma]}\times(t-s)^{(k+1-p-q)\alpha+p\beta+q\gamma-1}$$
$$=\Big|_{\substack{p=k+1\\q=k-p+1}}\frac{[\Gamma(\alpha)]^{p-k-1}[\Gamma(\beta)]^{k+1}[\Gamma(\gamma)]^{k-p+1}}{\Gamma[(p-k-1)\alpha+(k+1)\beta+(k-p+1)\gamma]}\times(t-s)^{(p-k-1)\alpha+(k+1)\beta+(k-p+1)\gamma-1}$$
$$+\Big|_{\substack{p=0\\q=k-p+1}}\frac{[\Gamma(\gamma)]^{k+1}}{\Gamma[(k+1)\gamma]}\times(t-s)^{(k+1)\gamma-1}$$
$$+\sum_{p=1}^{k}\Big|_{q=k-p+1}\frac{C_{k+1}^p[\Gamma(\alpha)]^{k+1-p-q}[\Gamma(\beta)]^p[\Gamma(\gamma)]^q}{\Gamma[(k+1-p-q)\alpha+p\beta+q\gamma]}\times(t-s)^{(k+1-p-q)\alpha+p\beta+q\gamma-1}. \tag{3.11}$$

By calculation, we can obtain

$$C_k^pC_{k-p}^q+C_k^{p-1}C_{k-p+1}^q+C_k^pC_{k-p}^{q-1}=C_{k+1}^pC_{k+1-p}^q. \tag{3.12}$$

From (3.9), (3.10), (3.11) and (3.12), the inequality (3.6) can be rewritten as

$$\mathcal{B}^{k+1}\xi(t) \le [h(t)]^{k+1}$$
$$\times \int_0^t \sum_{p=0}^{k+1}\sum_{q=0}^{k+1-p} \frac{C_{k+1}^p C_{k+1-p}^q [\Gamma(\alpha)]^{k+1-p-q}[\Gamma(\beta)]^p[\Gamma(\gamma)]^q}{\Gamma[(k+1-p-q)\alpha+p\beta+q\gamma]}$$
$$\times (t-s)^{(k+1-p-q)\alpha+p\beta+q\gamma-1}\xi(s)ds.$$

As a result, the inequality (3.3) is proved.

Next, since $\widetilde{h}(t)$, $\overline{h}(t)$ and $h_1(t)$ are continuous on $[0,\iota]$, there exists $\mathcal{H} > 0$ such that $h(t) \le \mathcal{H}$. From (3.3),

$$\mathcal{B}^r\xi(t) \le \int_0^t \sum_{p=0}^{r}\sum_{q=0}^{r-p} \frac{\mathcal{H}^r C_r^p C_{r-p}^q [\Gamma(\alpha)]^{r-p-q}[\Gamma(\beta)]^p[\Gamma(\gamma)]^q}{\Gamma[(r-p-q)\alpha+p\beta+q\gamma]}$$
$$\times (t-s)^{(r-p-q)\alpha+p\beta+q\gamma-1}\xi(s)ds.$$

Notice that

$$(a+b+c)^r = \sum_{p=0}^{r}\sum_{q=0}^{r-p} C_r^p C_{r-p}^q a^{r-p-q}b^p c^q.$$

Using Stirling's formula, $\Gamma(\sigma+1) \sim \sqrt{2\pi\sigma}(\frac{\sigma}{e})^{\sigma}$, one can get that

$$\lim_{r\to\infty}\mathcal{B}^r\xi(t)$$
$$\le \lim_{r\to\infty}\ell \sum_{p=0}^{r}\sum_{q=0}^{r-p} \frac{\mathcal{H}^r C_r^p C_{r-p}^q [\Gamma(\alpha)]^{r-p-q}[\Gamma(\beta)]^p[\Gamma(\gamma)]^q}{\Gamma[(r-p-q)\alpha+p\beta+q\gamma]}$$
$$\times \int_0^t (t-s)^{(r-p-q)\alpha+p\beta+q\gamma-1}ds$$
$$\le \lim_{r\to\infty}\ell \sum_{p=0}^{r}\sum_{q=0}^{r-p} \frac{\mathcal{H}^r C_r^p C_{r-p}^q [\Gamma(\alpha)\iota^\alpha]^{r-p-q}[\Gamma(\beta)\iota^\beta]^p[\Gamma(\gamma)\iota^\gamma]^q}{\Gamma[(r-p-q)\alpha+p\beta+q\gamma+1]}$$
$$= \lim_{r\to\infty}\ell \sum_{p=0}^{r}\sum_{q=0}^{r-p} \frac{\mathcal{H}^r C_r^p C_{r-p}^q}{\sqrt{2\pi[(r-p-q)\alpha+p\beta+q\gamma]}}$$
$$\times \left(\frac{\Gamma(\alpha)\iota^\alpha}{(\frac{(r-p-q)\alpha+p\beta+q\gamma}{e})^\alpha}\right)^{r-p-q}$$
$$\times \left(\frac{\Gamma(\beta)\iota^\beta}{(\frac{(r-p-q)\alpha+p\beta+q\gamma}{e})^\beta}\right)^{p}$$
$$\times \left(\frac{\Gamma(\gamma)\iota^\gamma}{(\frac{(r-p-q)\alpha+p\beta+q\gamma}{e})^\gamma}\right)^{q}$$
$$\le \lim_{r\to\infty}\ell \frac{[\mathcal{H}(C_1+C_2+C_3)]^r}{\sqrt{2\pi r\delta}},$$

where the symbol $\sim$ means that $\Gamma(\sigma+1)$ is equivalent infinity of $\sqrt{2\pi\sigma}(\frac{\sigma}{e})^{\sigma}$, $\ell := \sup_{t\in[0,\iota]}|\xi(t)|$, $\delta = \min\{\alpha,\beta,\gamma\}$,

$$C_1 = \frac{\Gamma(\alpha)\iota^\alpha}{(\frac{(r-p-q)\alpha+p\beta+q\gamma}{e})^\alpha},$$
$$C_2 = \frac{\Gamma(\beta)\iota^\beta}{(\frac{(r-p-q)\alpha+p\beta+q\gamma}{e})^\beta},$$
$$C_3 = \frac{\Gamma(\gamma)\iota^\gamma}{(\frac{(r-p-q)\alpha+p\beta+q\gamma}{e})^\gamma}.$$

Notice that $C_1,\ C_2,\ C_3 \to 0$ as $r\to\infty$. So, $\mathcal{H}(C_1+C_2+C_3) < 1$ if r is large enough, which implies that $[\mathcal{H}(C_1+C_2+C_3)]^r \to 0$ as $r\to\infty$. Then, we can obtain that $\mathcal{B}^r\xi(t)\to 0$ as $r\to\infty$.

Therefore, from the inequality (3.2), it is easy to get (3.1). □

Corollary 3.1. *Assume that the assumptions of Theorem 3.1.1 hold. Moreover, suppose that* $z(t)$ *is nondecreasing on* $[0,\iota]$. *Then,*

$$\xi(t) \le z(t)E_\delta[h(t)(\Gamma(\alpha)t^\alpha + \Gamma(\beta)t^\beta) + \Gamma(\gamma)t^\gamma],$$

where $\delta = \min\{\alpha,\ \beta,\ \gamma\}$ *and* E_δ *is the Mittag-Leffler function defined by*

$$E_\delta(\sigma) = \sum_{k=0}^{\infty}\frac{\sigma^k}{\Gamma(k\delta+1)},\ \sigma\in C,\ Re(\delta) > 0.$$

Proof. Since $z(t)$ is nondecreasing on $[0,\iota]$, from (3.1), we have

$$\xi(t) \le z(t)\Bigg[1 + \int_0^t \sum_{r=1}^{\infty}[h(t)]^r \sum_{p=0}^{r}\sum_{q=0}^{r-p}$$
$$\times \frac{C_r^p C_{r-p}^q [\Gamma(\alpha)]^{r-p-q}[\Gamma(\beta)]^p[\Gamma(\gamma)]^q}{\Gamma[(r-p-q)\alpha+p\beta+q\gamma]}$$
$$\times (t-s)^{(r-p-q)\alpha+p\beta+q\gamma-1}ds\Bigg]$$

$$
\begin{aligned}
=&z(t)\left[1+\sum_{r=1}^{\infty}[h(t)]^r\sum_{p=0}^{r}\sum_{q=0}^{r-p}\right.\\
&\times\frac{C_r^pC_{r-p}^q[\Gamma(\alpha)]^{r-p-q}[\Gamma(\beta)]^p[\Gamma(\gamma)]^q}{\Gamma[(r-p-q)\alpha+p\beta+q\gamma+1]}\\
&\left.\times(t-s)^{(r-p-q)\alpha+p\beta+q\gamma}\right]\\
\le&z(t)\left[1+\sum_{r=1}^{\infty}[h(t)]^r\sum_{p=0}^{r}\sum_{q=0}^{r-p}\right.\\
&\left.\times\frac{C_r^pC_{r-p}^q[\Gamma(\alpha)t^\alpha]^{r-p-q}[\Gamma(\beta)t^\beta]^p[\Gamma(\gamma)t^\gamma]^q}{\Gamma(r\delta+1)}\right]\\
=&z(t)\left[1+\sum_{r=1}^{\infty}\frac{[h(t)]^r[\Gamma(\alpha)t^\alpha+\Gamma(\beta)t^\beta+\Gamma(\gamma)t^\gamma]^r}{\Gamma(r\delta+1)}\right]\\
=&z(t)E_\delta[h(t)(\Gamma(\alpha)t^\alpha+\Gamma(\beta)t^\beta)+\Gamma(\gamma)t^\gamma].
\end{aligned}
$$

□

3.2. Existence of solutions

The existence results of (1.1) are given in this section. Before that, we list the following hypotheses.

($\mathcal{H}1$) $\hbar:\mathfrak{T}\times\mathbb{H}\to\mathbb{H}$ satisfies Carathéodory type conditions.
($\mathcal{H}2$) There exist $k_1,\ k_2\in(0,1)$ and real functions $l_1\in L^{\frac{1}{k_1}}(\mathfrak{T},\mathbb{H})$, $l_2\in L^{\frac{1}{k_2}}(\mathfrak{T},\mathbb{H})$ such that

$$\|\hbar(t,\xi)\|\le l_1(t)\|\xi\|+l_2(t) \text{ for all } \xi\in\mathbb{H} \text{ and a.e. } t\in\mathfrak{T}.$$

($\mathcal{H}2'$) There exist real functions $l_1,\ l_2\in L^\infty(\mathfrak{T},\mathbb{H})$ such that

$$\|\hbar(t,\xi)\|\le l_1(t)\|\xi\|+l_2(t) \text{ for all } \xi\in\mathbb{H} \text{ and a.e. } t\in\mathfrak{T}.$$

($\mathcal{H}3$) There exist $k_3\in(0,1)$ and a real function $\omega\in L^{\frac{1}{k_3}}(\mathfrak{T},\mathbb{H})$ such that

$$\mathcal{X}(\hbar(t,D))\le\omega(t)\mathcal{X}(D) \text{ for any bounded } D\subset\mathbb{H} \text{ and a.e. } t\in\mathfrak{T}.$$

For convenience, denote

$$\aleph:=\frac{\alpha-1}{1-k_1},\quad \mathfrak{R}:=\frac{\alpha-1}{1-k_2},\quad \mathfrak{I}:=\frac{\alpha-1}{1-k_3}.$$

Now, we introduce the following primary results.

Theorem 3.2. *Assume that ($\mathcal{H}1$), ($\mathcal{H}2$) and ($\mathcal{H}3$) hold. In addition, suppose that*

$$
\begin{aligned}
d:=&\frac{2\|\mathcal{A}\|\iota^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+\frac{2\|\mathcal{B}\|\iota^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}+\frac{2\|\omega\|_{L^{\frac{1}{k_3}}(\mathfrak{T})}\iota^{(1+\mathfrak{I})(1-k_3)}}{\Gamma(\alpha)(1+\mathfrak{I})^{1-k_3}}\\
<&1,
\end{aligned}
\tag{3.13}
$$

$$
\begin{aligned}
\kappa:=&\frac{\|\mathcal{A}\|\iota^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+\frac{\|\mathcal{B}\|\iota^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}+\frac{\|l_1\|_{L^{\frac{1}{k_1}}(\mathfrak{T})}\iota^{(1+\aleph)(1-k_1)}}{\Gamma(\alpha)(1+\aleph)^{1-k_1}}\\
<&1.
\end{aligned}
\tag{3.14}
$$

Then, (1.1) has at least one solution on $\mathfrak{T}$.

Proof. Suppose that $\xi(t)$ is the solution of (1.1). Taking the integral of order α on both sides of (1.1), by Lemma 2.4 and Lemma 2.5, we can get that

$$
\begin{aligned}
\xi(t)=&\xi_0+y_0t+z_0\frac{t^2}{2}+\frac{\mathcal{A}}{\Gamma(\alpha-\beta)}\int_0^t(t-s)^{\alpha-\beta-1}\xi(s)ds\\
&-\mathcal{A}\frac{\xi_0t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}-\mathcal{A}\frac{y_0t^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}\\
&-\mathcal{B}\frac{\xi_0t^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}+\frac{\mathcal{B}}{\Gamma(\alpha-\gamma)}\int_0^t(t-s)^{\alpha-\gamma-1}\xi(s)ds\\
&+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\hbar(s,\xi(s))ds.
\end{aligned}
$$

Define the operator Q on $\mathcal{PC}(\mathfrak{T},\mathbb{H})$.

$$
\begin{aligned}
(Q\xi)(t)=&\xi_0+y_0t+z_0\frac{t^2}{2}+\frac{\mathcal{A}}{\Gamma(\alpha-\beta)}\int_0^t(t-s)^{\alpha-\beta-1}\xi(s)ds\\
&-\mathcal{A}\frac{\xi_0t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}-\mathcal{A}\frac{y_0t^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}-\mathcal{B}\frac{\xi_0t^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}\\
&+\frac{\mathcal{B}}{\Gamma(\alpha-\gamma)}\int_0^t(t-s)^{\alpha-\gamma-1}\xi(s)ds\\
&+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\hbar(s,\xi(s))ds.
\end{aligned}
\tag{3.15}
$$

Obviously, we only need to certify the existence of fixed points of Q.

Step 1. We state that Q is continuous.

First, let $\{\xi_n\}$ be a sequence such that $\xi_n\to\xi$ as $n\to\infty$ in $\mathcal{PC}(\mathfrak{T},\mathbb{H})$. For arbitrary $t_1,\ t_2\in\mathfrak{T}$, $t_2>t_1$, we have

$$\begin{aligned}
&\|(Q\xi_n)(t_2)-(Q\xi_n)(t_1)\| \\
\leq&\|y_0\|(t_2-t_1)+\frac{\|z_0\|}{2}(t_2^2-t_1^2)\frac{\|\mathcal{A}\|\|\xi_0\|}{\Gamma(\alpha-\beta+1)}(t_2^{\alpha-\beta}-t_1^{\alpha-\beta}) \\
&+\frac{\|\mathcal{A}\|\|y_0\|}{\Gamma(\alpha-\beta+2)}(t_2^{\alpha-\beta+1}-t_1^{\alpha-\beta+1}) \\
&+\frac{\|\mathcal{B}\|\|\xi_0\|}{\Gamma(\alpha-\gamma+1)}(t_2^{\alpha-\gamma}-t_1^{\alpha-\gamma}) \\
&+\frac{\|\mathcal{A}\|}{\Gamma(\alpha-\beta)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-\beta-1}\|\xi_n(s)\|ds
\end{aligned}$$

$$\begin{aligned}
&+\frac{\|\mathcal{A}\|}{\Gamma(\alpha-\beta)}\int_0^{t_1}\left[(t_2-s)^{\alpha-\beta-1}-(t_1-s)^{\alpha-\beta-1}\right]\|\xi_n(s)\|ds \\
&+\frac{\|\mathcal{B}\|}{\Gamma(\alpha-\gamma)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-\gamma-1}\|\xi_n(s)\|ds \\
&+\frac{\|\mathcal{B}\|}{\Gamma(\alpha-\gamma)}\int_0^{t_1}\left[(t_2-s)^{\alpha-\gamma-1}-(t_1-s)^{\alpha-\gamma-1}\right]\|\xi_n(s)\|ds \\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\|\hbar(s,\xi_n(s))\|ds \\
&+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]\|\hbar(s,\xi_n(s))\|ds \\
\leq&\|y_0\|(t_2-t_1)+\frac{\|z_0\|}{2}(t_2^2-t_1^2) \\
&+\frac{\|\mathcal{A}\|\|\xi_0\|}{\Gamma(\alpha-\beta+1)}(t_2^{\alpha-\beta}-t_1^{\alpha-\beta}) \\
&+\frac{\|\mathcal{A}\|\|y_0\|}{\Gamma(\alpha-\beta+2)}(t_2^{\alpha-\beta+1}-t_1^{\alpha-\beta+1}) \\
&+\frac{\|\mathcal{B}\|\|\xi_0\|}{\Gamma(\alpha-\gamma+1)}(t_2^{\alpha-\gamma}-t_1^{\alpha-\gamma})+\frac{\|\mathcal{A}\|\|\xi_n\|_c}{\Gamma(\alpha-\beta+1)}(t_2^{\alpha-\beta}-t_1^{\alpha-\beta}) \\
&+\frac{\|\mathcal{B}\|\|\xi_n\|_c}{\Gamma(\alpha-\gamma+1)}(t_2^{\alpha-\gamma}-t_1^{\alpha-\gamma})+\frac{W}{\Gamma(\alpha+1)}(t_2^{\alpha}-t_1^{\alpha}) \\
\leq&\|y_0\|(t_2-t_1)+\frac{\|z_0\|}{2}(t_2^2-t_1^2) \\
&+\frac{\|\mathcal{A}\|\|y_0\|}{\Gamma(\alpha-\beta+2)}(t_2^{\alpha-\beta+1}-t_1^{\alpha-\beta+1}) \\
&+\frac{2\|\mathcal{A}\|\|\xi_n\|_c}{\Gamma(\alpha-\beta+1)}(t_2^{\alpha-\beta}-t_1^{\alpha-\beta}) \\
&+\frac{2\|\mathcal{B}\|\|\xi_n\|_c}{\Gamma(\alpha-\gamma+1)}(t_2^{\alpha-\gamma}-t_1^{\alpha-\gamma})+\frac{W}{\Gamma(\alpha+1)}(t_2^{\alpha}-t_1^{\alpha}),
\end{aligned}$$

where $W := \sup\{\|\hbar(t,\xi_n(t))\| : t \in \mathfrak{T},\ \xi_n \in \mathcal{PC}(\mathfrak{T},\mathbb{H})\} < \infty$. It is easy to get that $\|(Q\xi_n)(t_2)-(Q\xi_n)(t_1)\| \to 0$ as $(t_2-t_1)\to 0$. Therefore, $\{Q\xi_n\}$ is equicontinuous.

On the other side,

$$\begin{aligned}
&\|(Q\xi_n)(t)-(Q\xi)(t)\| \\
\leq&\frac{\|\mathcal{A}\|}{\Gamma(\alpha-\beta)}\int_0^t(t-s)^{\alpha-\beta-1}\|\xi_n(s)-\xi(s)\|ds \\
&+\frac{\|\mathcal{B}\|}{\Gamma(\alpha-\gamma)}\int_0^t(t-s)^{\alpha-\gamma-1}\|\xi_n(s)-\xi(s)\|ds \\
&+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\|\hbar(s,\xi_n(s))-\hbar(s,\xi(s))\|ds.
\end{aligned}$$

Thus,

$$\|Q\xi_n(t)-Q\xi(t)\|\to 0 \text{ as } n\to\infty, \quad \forall t\in\mathfrak{T}. \tag{3.16}$$

By using the Arzela-Ascoli theorem, we conclude that $\{Q\xi_n\}$ is relatively compact on $\mathfrak{T}$.

We now claim that

$$\|Q\xi_n-Q\xi\|_c\to 0 \text{ as } n\to\infty. \tag{3.17}$$

If (3.17) is not true, there exist $\varepsilon_1>0$ and $\{\xi_{n_i}\}$ of $\{\xi_n\}$ such that

$$\|Q\xi_{n_i}-Q\xi\|_c\geq\varepsilon_1, \quad (n=1,2,3\cdots). \tag{3.18}$$

Because $\{Q\xi_n\}$ is relatively compact, $\{Q\xi_{n_i}\}$ contains a subsequence. It converges to $z\in\mathcal{PC}(\mathfrak{T},\mathbb{H})$. We assume that $\{Q\xi_{n_i}\}$ itself converges to z:

$$\|Q\xi_{n_i}-z\|_c\to 0 \text{ as } n\to\infty. \tag{3.19}$$

By (3.16) and (3.19), we obtain that $z=Q\xi$. So (3.19) contradicts with (3.18). Thus, (3.17) is true.

Step 2. Choose the constant ϱ which satisfies the following inequality:

$$\varrho\geq\frac{\mu}{1-\kappa}, \tag{3.20}$$

where

$$\begin{aligned}
\mu :=&\|\xi_0\|+\iota\|y_0\|+\frac{\iota^2}{2}\|z_0\|+\frac{\|\xi_0\|\|\mathcal{A}\|\iota^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+\frac{\|y_0\|\|\mathcal{A}\|\iota^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)} \\
&+\frac{\|\xi_0\|\|\mathcal{B}\|\iota^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)})+\frac{\|l_2\|_{L^{\frac{1}{k_2}}(\mathfrak{T})}\iota^{(1+\mathfrak{R})(1-k_2)}}{\Gamma(\alpha)(1+\mathfrak{R})^{1-k_2}}.
\end{aligned}$$

We now claim that

$$Q: D_\varrho\to D_\varrho, \tag{3.21}$$

where $D_\varrho := \{\xi \in \mathcal{PC}(\mathfrak{T}, \mathbb{H}) : \|\xi\|_c \le \varrho\}$.

In fact, for arbitrary $\xi \in D_\varrho$, using $(\mathcal{H}2)$, we have

$$\begin{aligned}
&\|Q(\xi)\| \\
\le&\|\xi_0\| + \iota\|y_0\| + \frac{\iota^2}{2}\|z_0\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\|\xi(s)\|ds \\
&+\frac{|\xi_0\||\mathcal{A}\|\iota^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{|y_0\||\mathcal{A}\|\iota^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)} \\
&+\frac{\|\mathcal{B}\|}{\Gamma(\alpha-\gamma)}\int_0^t (t-s)^{\alpha-\gamma-1}\|\xi(s)\|ds \\
&+\frac{\|\xi_0\|\|\mathcal{B}\|\iota^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|\hbar(s,\xi(s))\|ds \\
\le&\|\xi_0\| + \iota\|y_0\| + \frac{\iota^2}{2}\|z_0\| + \frac{\|\mathcal{A}\|\varrho\iota^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \\
&+\frac{\|\xi_0\|\|\mathcal{A}\|\iota^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{\|y_0\|\|\mathcal{A}\|\iota^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)} \\
&+\frac{\|\mathcal{B}\|\varrho\iota^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + \frac{\|\xi_0\|\|\mathcal{B}\|\iota^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} \\
&+\frac{\varrho}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}l_1(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}l_2(s)ds.
\end{aligned} \tag{3.22}$$

Notice that

$$\int_0^t (t-s)^{\alpha-1}l_1(s)ds \le \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-k_1}}ds\right)^{1-k_1}\left(\int_0^t l_1(s)^{\frac{1}{k_1}}ds\right)^{k_1},$$

$$\int_0^t (t-s)^{\alpha-1}l_2(s)ds \le \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-k_2}}ds\right)^{1-k_2}\left(\int_0^t l_2(s)^{\frac{1}{k_2}}ds\right)^{k_2}.$$

These, together with (3.22), guarantee that

$$\begin{aligned}
&\|Q(\xi)\|_c \\
\le&\|\xi_0\| + \iota\|y_0\| + \frac{\iota^2}{2}\|z_0\| + \frac{\|\xi_0\|\|\mathcal{A}\|\iota^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{\|y_0\|\|\mathcal{A}\|\iota^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)} \\
&+\frac{\|\xi_0\|\|\mathcal{B}\|\iota^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + \frac{\|l_2\|_{L^{\frac{1}{k_2}}(\mathfrak{T})}\iota^{(1+\mathfrak{R})(1-k_2)}}{\Gamma(\alpha)(1+\mathfrak{R})^{1-k_2}} \\
&+\frac{\|\mathcal{A}\|\iota^{\alpha-\beta}\varrho}{\Gamma(\alpha-\beta+1)} + \frac{\|\mathcal{B}\|\iota^{\alpha-\gamma}\varrho}{\Gamma(\alpha-\gamma+1)} + \frac{\|l_1\|_{L^{\frac{1}{k_1}}(\mathfrak{T})}\iota^{(1+\aleph)(1-k_1)}\varrho}{\Gamma(\alpha)(1+\aleph)^{1-k_1}}.
\end{aligned}$$

From (3.20), we have that $\|Q(\xi)\|_c \le \varrho$ if $\|\xi\|_c \le \varrho$.

Step 3. We demonstrate that S is relatively compact if $S \subset D_\varrho$ is countable and

$$S \subset \overline{co}(\{\xi_0\} \cup Q(S)) \tag{3.23}$$

for some $\xi_0 \in D_\varrho$.

Let $S = \{\xi_n\}_{n=1}^\infty \subset D_\varrho$. By $(\mathcal{H}3)$, one can get that

$$\begin{aligned}
&\mathcal{X}(\{(Q\xi_n)(t)\}_{n=1}^\infty) \\
\le&\mathcal{X}(\{\frac{\mathcal{A}}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\xi_n(s)ds\}_{n=1}^\infty) \\
&+\mathcal{X}(\{\frac{\mathcal{B}}{\Gamma(\alpha-\gamma)}\int_0^t (t-s)^{\alpha-\gamma-1}\xi_n(s)ds\}_{n=1}^\infty) \\
&+\mathcal{X}(\{\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\hbar(s,\xi_n(s))ds\}_{n=1}^\infty) \\
\le&\frac{2\|\mathcal{A}\|}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\mathcal{X}(\{\xi_n(s)\}_{n=1}^\infty)ds \\
&+\frac{2\|\mathcal{B}\|}{\Gamma(\alpha-\gamma)}\int_0^t (t-s)^{\alpha-\gamma-1}\mathcal{X}(\{\xi_n(s)\}_{n=1}^\infty)ds \\
&+\frac{2}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathcal{X}(\{\hbar(s,\xi_n(s))\}_{n=1}^\infty)ds \\
\le&\Big(\frac{2\|\mathcal{A}\|}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}ds \\
&+\frac{2\|\mathcal{B}\|}{\Gamma(\alpha-\gamma)}\int_0^t (t-s)^{\alpha-\gamma-1}ds \\
&+\frac{2}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\omega(s)ds\Big)\mathcal{X}_c(\{\xi_n\}_{n=1}^\infty) \\
\le&\Big(\frac{2\|\mathcal{A}\|\iota^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{2\|\mathcal{B}\|\iota^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + \frac{2\|\omega\|_{L^{\frac{1}{k_3}}(\mathfrak{T})}\iota^{(1+\mathfrak{I})(1-k_3)}}{\Gamma(\alpha)(1+\mathfrak{I})^{1-k_3}}\Big) \\
&\times\mathcal{X}_c(\{\xi_n\}_{n=1}^\infty)
\end{aligned} \tag{3.24}$$

for each $t \in \mathfrak{T}$.

Note that $\{Q\xi_n\}_{n=1}^\infty$ is equicontinuous by *Step 1*. From a well-known result on the measure of noncompactness, we have

$$\mathcal{X}_c(\{Q\xi_n\}_{n=1}^\infty) = \sup_{t\in\mathfrak{T}}\mathcal{X}(\{(Q\xi_n)(t)\}_{n=1}^\infty).$$

This together with (3.13), (3.23) and (3.24) guarantees that

$$\mathcal{X}_c(\{\xi_n\}_{n=1}^\infty) \le \mathcal{X}_c(\{Q\xi_n\}_{n=1}^\infty) \le d\mathcal{X}_c(\{\xi_n\}_{n=1}^\infty),$$

which implies that $S = \{\xi_n\}_{n=1}^\infty$ is relatively compact.

From Lemma 2.6, Q has a fixed point in D_ϱ. It is a solution of (1.1). □

Theorem 3.3. *Assuming that $(\mathcal{H}1)$, $(\mathcal{H}2')$ and $(\mathcal{H}3)$ hold. In addition, suppose that*

$$\begin{aligned}
l :=&\frac{4\|\mathcal{A}\|\iota^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{4\|\mathcal{B}\|\iota^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + \frac{4\|\omega\|_{L^{\frac{1}{k_3}}(\mathfrak{T})}\iota^{(1+\mathfrak{I})(1-k_3)}}{\Gamma(\alpha)(1+\mathfrak{I})^{1-k_3}} \\
&<1.
\end{aligned} \tag{3.25}$$

Then (1.1) has at least one solution on $\mathfrak{T}$.

Proof. First, it is easy to get that $Q : \mathcal{PC}(\mathfrak{T}, \mathbb{H}) \to \mathcal{PC}(\mathfrak{T}, \mathbb{H})$ defined as (3.15) is bounded and continuous.

Then, from Lemma 2.8, for any $\varepsilon > 0$ and bounded $C \subset \mathcal{PC}(\mathfrak{T}, \mathbb{H})$, there exists $\{\nu_n\}_{n=1}^{\infty} \subseteq C$ satisfying $\mathcal{X}(C) \le 2\mathcal{X}(\{\nu_n\}_{n=1}^{\infty}) + \varepsilon$ and

$$
\begin{aligned}
&\mathcal{X}((QC)(t)) \\
\le&\mathcal{X}(\{\frac{\mathcal{A}}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\xi(s)ds : \xi \in C\}) \\
&+ \mathcal{X}(\{\frac{\mathcal{B}}{\Gamma(\alpha-\gamma)}\int_0^t (t-s)^{\alpha-\gamma-1}\xi(s)ds : \xi \in C\}) \\
&+ \mathcal{X}(\{\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\hbar(s,\xi(s))ds : \xi \in C\}) \\
\le&2\mathcal{X}(\{\frac{\mathcal{A}}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\{\nu_n(s)\}_{n=1}^{\infty}ds\}) \\
&+ 2\mathcal{X}(\{\frac{\mathcal{B}}{\Gamma(\alpha-\gamma)}\int_0^t (t-s)^{\alpha-\gamma-1}\{\nu_n(s)\}_{n=1}^{\infty}ds\}) \\
&+ 2\mathcal{X}(\{\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\{\hbar(s,\nu_n(s))\}_{n=1}^{\infty}ds\}) + 3\varepsilon \\
\le&\frac{4\|\mathcal{A}\|}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\mathcal{X}(\{\nu_n(s)\}_{n=1}^{\infty})ds \\
&+ \frac{4\|\mathcal{B}\|}{\Gamma(\alpha-\gamma)}\int_0^t (t-s)^{\alpha-\gamma-1}\mathcal{X}(\{\nu_n(s)\}_{n=1}^{\infty})ds \\
&+ \frac{4}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathcal{X}(\{\hbar(s,\nu_n(s))\}_{n=1}^{\infty})ds + 3\varepsilon \\
\le&\Big(\frac{4\|\mathcal{A}\|\iota^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{4\|\mathcal{B}\|\iota^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + \frac{4\|\omega\|_{L^{\frac{1}{k_3}}(\mathfrak{T})}\iota^{(1+\mathfrak{I})(1-k_3)}}{\Gamma(\alpha)(1+\mathfrak{I})^{1-k_3}}\Big) \\
&\times \mathcal{X}_c(\{\nu_n\}_{n=1}^{\infty}) + 3\varepsilon
\end{aligned}
$$

for arbitrary $t \in \mathfrak{T}$. Notice that

$$\mathcal{X}_c(\{\nu_n\}_{n=1}^{\infty}) \le \mathcal{X}_c(C),\ \mathcal{X}_c((QC)) = \sup_{t\in\mathfrak{T}} \mathcal{X}((QC(t))).$$

Let $\varepsilon \to 0$. We obtain

$$\mathcal{X}_c((QC)) \le l\mathcal{X}_c(C),$$

which implies that $Q : \mathcal{PC}(\mathfrak{T}, \mathbb{H}) \to \mathcal{PC}(\mathfrak{T}, \mathbb{H})$ is a condensing operator by (3.25).

At last, define the set

$$E(Q) = \{\xi \in \mathcal{PC}(\mathfrak{T}, \mathbb{H}) : \xi = \lambda Q\xi, \lambda \in [0,1]\}.$$

We declare that $E(Q)$ is bounded.

From $(\mathcal{H}2')$, we choose $K_1 = \|l_1\|_\infty$ and $K_2 = \|l_2\|_\infty$. Then, $\|\hbar(t,\xi)\| \le K_1\|\xi\| + K_2$.

For each $\xi \in E(Q)$ and $t \in \mathfrak{T}$, we have

$$
\begin{aligned}
&\|\xi(t)\| \\
\le&\|\xi_0\| + \iota\|y_0\| + \frac{\iota^2}{2}\|z_0\| + \frac{\|\xi_0\|\|\mathcal{A}\|\iota^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \\
&+ \frac{\|y_0\|\|\mathcal{A}\|\iota^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)} + \frac{\|\xi_0\|\|\mathcal{B}\|\iota^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} \\
&+ \frac{\|\mathcal{A}\|}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\|\xi(s)\|ds \\
&+ \frac{\|\mathcal{B}\|}{\Gamma(\alpha-\gamma)}\int_0^t (t-s)^{\alpha-\gamma-1}\|\xi(s)\|ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|\hbar(s,\xi(s))\|ds \\
\le&\|\xi_0\| + \iota\|y_0\| + \frac{\iota^2}{2}\|z_0\| + \frac{\|\xi_0\|\|\mathcal{A}\|\iota^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \\
&+ \frac{\|y_0\|\|\mathcal{A}\|\iota^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)} + \frac{\|\xi_0\|\|\mathcal{B}\|\iota^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + \frac{K_2\iota^{\alpha}}{\Gamma(\alpha+1)} \\
&+ \frac{\|\mathcal{A}\|}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\|\xi(s)\|ds \\
&+ \frac{\|\mathcal{B}\|}{\Gamma(\alpha-\gamma)}\int_0^t (t-s)^{\alpha-\gamma-1}\|\xi(s)\|ds \\
&+ \frac{K_1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|\xi(s)\|ds.
\end{aligned}
$$

For convenience, define

$$
\begin{aligned}
l :=&\|\xi_0\| + \iota\|y_0\| + \frac{\iota^2}{2}\|z_0\| + \frac{\|\xi_0\|\|\mathcal{A}\|\iota^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \\
&+ \frac{\|y_0\|\|\mathcal{A}\|\iota^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)} + \frac{\|\xi_0\|\|\mathcal{B}\|\iota^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + \frac{K_2\iota^{\alpha}}{\Gamma(\alpha+1)},
\end{aligned}
$$

$$\chi_1 := \frac{\|\mathcal{A}\|}{\Gamma(\alpha-\beta)}, \quad \chi_2 := \frac{\|\mathcal{B}\|}{\Gamma(\alpha-\gamma)}, \quad \chi_3 := \frac{K_1}{\Gamma(\alpha)}.$$

By Corollary 3.1.2, we can obtain

$$
\begin{aligned}
&\|\xi(t)\| \\
\le&l + \chi_1\int_0^t (t-s)^{\alpha-\beta-1}\|\xi(s)\|ds \\
&+ \chi_2\int_0^t (t-s)^{\alpha-\gamma-1}\|\xi(s)\|ds + \chi_3\int_0^t (t-s)^{\alpha-1}\|\xi(s)\|ds \\
\le&lE_{\alpha-\beta}\big[(\chi_1+\chi_2+\chi_3)(\Gamma(\alpha)t^{\alpha} + \Gamma(\alpha-\beta)t^{\alpha-\beta} \\
&+ \Gamma(\alpha-\gamma)t^{\alpha-\gamma})\big] \\
:=& M^*.
\end{aligned}
$$

This means that $\|\xi(t)\| \le M^*$ for each $\xi \in E(Q)$. Thus, the set $E(Q)$ is bounded.

From Lemma 2.9, (1.1) has at least one solution. □

Remark 3.1. *Notice that $l_1 \in L^{\infty}(\mathfrak{T}, \mathbb{H})$, which implies that $l_1(t)$ is no longer required to satisfy (3.14).*

3.3. Examples

In this section, an example is used to illustrate the effectiveness of the obtained results.

Example Consider the following fractional dynamical system with two damping terms in Banach space c_0:

$$\begin{cases} \frac{\partial^{\frac{5}{2}}}{\partial t^{\frac{5}{2}}}\xi_n(t) - \varepsilon_1 \frac{\partial^{\frac{3}{2}}}{\partial t^{\frac{3}{2}}}\xi_n(t) - \varepsilon_2 \frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}\xi_n(t) = \hbar_n(t, \xi_1, \xi_2, \cdots), \\ t \in \mathfrak{T} = [0,1]; \\ \xi_n(0) = 0,\ \xi_n'(0) = 0,\ \xi_n''(0) = 0, \qquad (n = 1, 2, 3, \cdots), \end{cases} \tag{3.26}$$

where ε_i $(i = 1, 2)$ are positive numbers, and

$$\hbar_n(t, \xi_1, \xi_2, \cdots) = \tfrac{1}{t^2+1}\left(\xi_n + \tfrac{t}{n^2}\right) \text{ for } t \in \mathfrak{T},\ (\xi_1, \xi_2, \cdots) \in c_0,\ n = 1, 2, 3, \cdots;$$

c_0 represents the space where all sequences converge to zero, which is a Banach space with respect to the norm $\|\xi\|_\infty = \sup_n |\xi_n|$.

Conclusion: System (3.26) has at least one solution on $\mathfrak{T}$ if $2\varepsilon_1 + \varepsilon_2 < 1 - \frac{1}{\sqrt{\pi}}$.

Proof System (3.26) can be considered as a system of the style of (1.1), where

$$\alpha = \frac{5}{2},\ \beta = \frac{3}{2},\ \gamma = \frac{1}{2},\ \xi_0 = y_0 = z_0 = (0, \cdots, 0, \cdots),$$

$$\hbar(t, \xi) = (\hbar_1(t, \xi), \cdots, \hbar_n(t, \xi), \cdots) = \frac{1}{t^2+1}\left\{\xi_n + \frac{t}{n^2}\right\}_{n=1}^{\infty}.$$

In addition, $A = \varepsilon_1 \mathcal{E}$ and $B = \varepsilon_2 \mathcal{E}$, where $\mathcal{E}$ is the identity operator.

From Theorem 3.2.1, we only need to show that the conditions $(\mathcal{H}1)$, $(\mathcal{H}2)$, $(\mathcal{H}3)$, (3.13) and (3.14) are satisfied.

We can easily conclude that the function $\hbar$ satisfies $(\mathcal{H}1)$. To verify condition $(\mathcal{H}2)$, let $\xi = \{\xi_j\}_{j=1}^{\infty} \in c_0$. Then

$$\|\hbar(t, \xi)\|_\infty = \frac{1}{t^2+1}\|\{\xi_j + \frac{t}{j^2}\}_{j=1}^{\infty}\|_\infty \le \frac{1}{t^2+1}(\|\xi\|_\infty + t)$$

for $t \in \mathfrak{T}$. So we choose $l_1(t) = \frac{1}{t^2+1}$, $l_2(t) = \frac{t}{t^2+1}$. This shows that $(\mathcal{H}2)$ holds. To prove $(\mathcal{H}3)$, we review the Hausdorff measure of noncompactness $\mathcal{X}$ in c_0:

$$\mathcal{X}(D) = \lim_{n\to\infty} \sup_{\xi \in D} \|(I - T_n)\xi\|_\infty,$$

where D is a bounded subset in c_0 and T_n is the projection onto the linear span of the first n vectors in the standard basis [27].

Let $u = \{u_j\}_{j=1}^{\infty} \in D \subseteq c_0$; we have

$$\begin{aligned} \mathcal{X}(\hbar(t, D)) &= \frac{1}{t^2+1} \lim_{n\to\infty} \sup_{u\in D} \|(I - T_n)\{u_j + \frac{t}{j^2}\}_{j=1}^{\infty})\|_\infty \\ &\le \frac{1}{t^2+1} \lim_{n\to\infty} \sup_{u\in D} \|(I - T_n)\{u_j\}_{j=1}^{\infty})\|_\infty \\ &= \frac{1}{t^2+1}\mathcal{X}(D) \end{aligned}$$

for $t \in \mathfrak{T}$. So we can choose $\omega(t) = \frac{1}{t^2+1}$. Hence, $(\mathcal{H}3)$ is satisfied. Obviously, $\|\omega\|_{L^2} \le 1$ and $\mathfrak{I} = \frac{\alpha-1}{1-k_3} = 3$. So,

$$\begin{aligned} &\frac{2\|\mathcal{A}\|\iota^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{2\|\mathcal{B}\|\iota^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + \frac{2\|\omega\|_{L^{\frac{1}{p_3}}(\mathfrak{T})}\iota^{(1+\mathfrak{I})(1-k_3)}}{\Gamma(\alpha)(1+\mathfrak{I})^{1-k_3}} \\ &\le \frac{2\varepsilon_1}{\Gamma(2)} + \frac{2\varepsilon_2}{\Gamma(3)} + \frac{2}{\Gamma(\frac{5}{2})} < 1, \end{aligned}$$

$$\frac{\|\mathcal{A}\|\iota^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{\|\mathcal{B}\|\iota^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + \frac{\|l_1\|_{L^{\frac{1}{k_1}}(\mathfrak{T})}\iota^{(1+\aleph)(1-k_1)}}{\Gamma(\alpha)(1+\aleph)^{1-k_1}} < 1.$$

Thus, (3.26) has at least one solution on $\mathfrak{T} = [0, 1]$. □

4. Conclusions

This paper studies the existence of solutions for fractional dynamical systems with two damping terms in abstract space. The desired results have been obtained by using the non-compact measurement method and the fixed point theorem. For the first time, we consider systems with damping in an infinite-dimensional space. The conclusions of this paper are important for systems with two damping terms. In the future, the controllability of similar systems can be considered on this basis.

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Conflict of interest

The authors declare that there is no conflict of interest that may influence the publication of this paper.

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