Mathematical Modelling and Control

http://www.aimspress.com/journal/mmc

MMC, 3(2): 127–138
DOI:10.3934/mmc.2023012
Received: 18 November 2022
Revised: 14 March 2023
Accepted: 19 March 2023
Published: 08 June 2023

Research article

Existence theory of fractional order three-dimensional differential system at resonance

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Abstract: This paper deals with three-dimensional differential system of nonlinear fractional order problem

$$\begin{aligned} D^{\alpha}_{0^+}\upsilon(\varrho) &= f(\varrho,\omega(\varrho),\omega'(\varrho),\omega''(\varrho),...,\omega^{(n-1)}(\varrho)),\ \varrho\in(0,1),\\ D^{\beta}_{0^+}\nu(\varrho) &= g(\varrho,\upsilon(\varrho),\upsilon'(\varrho),\upsilon''(\varrho),...,\upsilon^{(n-1)}(\varrho)),\ \varrho\in(0,1),\\ D^{\gamma}_{0^+}\omega(\varrho) &= h(\varrho,\nu(\varrho),\nu'(\varrho),\nu''(\varrho),...,\nu^{(n-1)}(\varrho)),\ \varrho\in(0,1), \end{aligned}$$

with the boundary conditions,

$$\begin{aligned} \upsilon(0)=\upsilon'(0)=...=\upsilon^{(n-2)}(0)=0,&\ \upsilon^{(n-1)}(0)=\upsilon^{(n-1)}(1),\\ \nu(0)=\nu'(0)=...=\nu^{(n-2)}(0)=0,&\ \nu^{(n-1)}(0)=\nu^{(n-1)}(1),\\ \omega(0)=\omega'(0)=...=\omega^{(n-2)}(0)=0,&\ \omega^{(n-1)}(0)=\omega^{(n-1)}(1), \end{aligned}$$

where $D^{\alpha}_{0^+}, D^{\beta}_{0^+}, D^{\gamma}_{0^+}$ are the standard Caputo fractional derivative, $n-1<\alpha,\beta,\gamma\le n$, $n\ge 2$ and we derive sufficient conditions for the existence of solutions to the fraction order three-dimensional differential system with boundary value problems via Mawhin's coincidence degree theory, and some new existence results are obtained. Finally, an illustrative example is presented.

Keywords: fractional differential equation; coincidence degree theory; resonance

1. Introduction

In the recent years, the glorious developments have been envisaged in the field of fractional differential equations due to their applications being used in various fields such as blood flow phenomena, electro Chemistry of corrosion, industrial robotics, probability and Statistics and so on, refer [1–7]. In particular, the fractional derivative has been used in lot of physical applications such as propagation of fractional diffusive waves in viscoelastic solids [8], charge transmit-time dispersion amorphous semi-conductor [9] and a non-Markovian diffusion process with memory [10].

Although fixed point theorems like the Banach contraction principle and the Schauder fixed point theorem are used to establish the existence of solutions, stronger conditions on the nonlinear functions involved limit their application to a limited number of problems. We employ Mawhin's topological degree theory method to include additional types of boundary value problems (BVP's) and apply fewer restricted conditions.

In the field of fractional systems, many results have been obtained through assured extensions of existing results given only to integer systems. Despite the enormous amount of published work on fractional differential systems, there are still many difficult open problems. Indeed, the theory and applications of these systems are still very active areas of research.

Recently, two-point BVP's for fractional differential equations have been studied in some papers (see [11, 12]). The existence of solutions to coupled systems of fractional differential equations has been given in papers [13–16]. Moreover, some authors discussed the existence of solutions for nonlinear fractional multi-point BVP's; for instance, refer [17–21], and the references cited therein. There are few papers which deals with the BVP's for fractional differential equations at nonresonance. Meanwhile, fractional BVP's at resonance have been intensively explored, as shown by references to several recent works on the subject [22–27].

Hu and Zhang [28] investigated the existence, uniqueness of solutions to integer higher-order nonlinear coupled fractional differential equations at resonance by the coincidence degree theory. Hu [29] discussed the solution of a higher-order coupled system of nonlinear fractional differential equations with infinite-point boundary conditions by coincidence degree theory.

Motivated by the results mentioned above, the two point BVP's of system of higher-order fractional differential equations have been studied by some authors, to the best of our knowledge, no work has been done on the BVP of system involving three-dimensional differential system higher-order fractional differential equations with Caputo fractional derivative. Inspired by the aforementioned studies, in this manuscript, we establish sufficient conditions for the existence of solutions to the nonlinear fractional order three-dimensional differential system with BVP's of the form.

$$D^{\alpha}_{0^+}\upsilon(\varrho) = f(\varrho, \omega(\varrho), \omega'(\varrho), \omega''(\varrho), \cdots, \omega^{(n-1)}(\varrho)), \quad (1.1)$$

$$D^{\beta}_{0^+}\nu(\varrho) = g(\varrho, \upsilon(\varrho), \upsilon'(\varrho), \upsilon''(\varrho), \cdots, \upsilon^{(n-1)}(\varrho)), \quad (1.2)$$

$$D^{\gamma}_{0^+}\omega(\varrho) = h(\varrho, \nu(\varrho), \nu'(\varrho), \nu''(\varrho), \cdots, \nu^{(n-1)}(\varrho)), \quad (1.3)$$

$\varrho \in (0, 1)$, with the boundary conditions,

$$\upsilon(0) = \upsilon'(0) = \cdots = \upsilon^{(n-2)}(0) = 0, \ \upsilon^{(n-1)}(0) = \upsilon^{(n-1)}(1), \quad (1.4)$$

$$\nu(0) = \nu'(0) = \cdots = \nu^{(n-2)}(0) = 0, \ \nu^{(n-1)}(0) = \nu^{(n-1)}(1), \quad (1.5)$$

$$\omega(0) = \omega'(0) = \cdots = \omega^{(n-2)}(0) = 0, \ \omega^{(n-1)}(0) = \omega^{(n-1)}(1), \quad (1.6)$$

where $D^{\alpha}_{0^+}, D^{\beta}_{0^+}$ and $D^{\gamma}_{0^+}$ denote the standard Caputo fractional derivative, $n-1 < \alpha, \beta, \gamma \leq n$, $n \geq 2$. Boundary value problems being at resonance means that the associated linear homogeneous equation $D^{\alpha}_{0^+}\upsilon(\varrho) = 0$ has a nontrivial solution $\upsilon(\varrho) = ct^{n-1}$, where $0 < \varrho < 1, c \in R$.

Our main aim of this paper is to establish some new criteria for the existence of solutions of (1.1) and (1.4). By using Mawhin's coincidence degree theory, some new existence results are obtained. This paper presents a new existence result which is a generalization of some known results in the existing literature.

This paper is organized in the following fashion: In Section 2, we shall present some notations, definitions and some properties of the fractional calculus. In Section 3, we investigate the existence of solutions of equation (1.1) and (1.4) by the Mawhin's coincidence degree theory [30]. In Section 4, we illustrate the main result further by providing an example.

2. Preliminaries

This section starts with a quick review of the fractional calculus concepts that will be used in this work. So let's start with the Riemann–Liouville fractional integrals and derivatives definitions.

Definition 2.1. *[15] The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : R_+ \to R$ on the half-axis R_+ is given by*

$$(I^{\alpha}_{0^+}f)(\varrho) := \frac{1}{\Gamma(\alpha)}\int_0^{\varrho}(\varrho - \nu)^{\alpha-1}f(\nu)d\nu \quad \text{for } \varrho > 0$$

provided the right hand side is pointwise defined on R_+.

Definition 2.2. *[15] The Riemann-Liouville fractional derivative of order $\alpha > 0$ on continuous function $f : R_+ \to R$*

is given by

$$(D_{0^+}^{\alpha} f)(\varrho) := \frac{1}{\Gamma(n-\alpha)} \int_0^{\varrho} (\varrho - \nu)^{n-\alpha-1} f^n(\nu) d\nu \quad for \quad \varrho > 0, \tag{2.1}$$

where $n-1 < \alpha \leq n$ and Γ is the gamma function, such that the integral is pointwise defined on R_+.

Definition 2.3. *[12] Assume that f is $(n-1)$-times absolutely continuous function, the Caputo fractional derivative of order $\alpha > 0$ of f is given by*

$$\begin{aligned}(D_{0^+}^{\alpha} f)(\varrho) &:= I_{0^+}^{n-\alpha} \frac{d^n f(\varrho)}{d\varrho^n} \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^{\varrho} (\varrho - \nu)^{n-\alpha-1} f^n(\nu) d\nu \quad for \quad \varrho > 0,\end{aligned} \tag{2.2}$$

where n is the smallest integer greater than or equal to α, provided that the right side integral is pointwise defined on $(0, +\infty)$.

Lemma 2.4. *[12] Assume that $n-1 < \alpha \leq n$, $\widetilde{x} \in C(0,1) \cap L^1(0,1)$, then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} \widetilde{x}(\varrho) := \widetilde{x}(\varrho) + c_0 + c_1\varrho + c_2\varrho^2 + ... + c_{n-1}\varrho^{n-1}$$

where $c_i = -\frac{\widetilde{x}^{(i)}(0)}{i!} \in R$, $(i = 0, 1, 2, \cdots, n-1)$ and $n \geq \alpha$, n is the smallest integer.

Lemma 2.5. *[12] Let $\beta > 0$, $\alpha + \beta > 0$, then*

$$I_{0^+}^{\alpha} I_{0^+}^{\beta} f(\widetilde{x}) := I_{0^+}^{\alpha+\beta} f(\widetilde{x})$$

is satisfied for continuous function f.

Lemma 2.6. *[20] Let $L : domL \subset X \to Z$ be a Fredholm operator with index zero and N be L-compact on $\overline{\Omega}$. Assume that the following relations hold.*

1. *$Lx \neq \lambda Nx$ for every $(\widetilde{x}, \lambda) \in [(domL \setminus KerL) \cap \partial\Omega] \times (0,1)$;*
2. *$Nx \notin ImL$ for every $\widetilde{x} \in KerL \cap \partial\Omega$;*
3. *For some isomorphism $J : ImQ \to KerL$, we have $deg(JQN|_{KerL}, KerL \cap \Omega, 0) \neq 0$, where $Q : Z \to Z$ is a continuous projection such that ImL=KerQ. Then the operator equation $Lx = Nx$ has at least one solution in $domL \cap \overline{\Omega}$.*

3. Main results

Our main result is as follows.

Let $X = C^{n-1}[0,1]$ with the norm $\|X\|_X = \max\left\{\|X\|_\infty, \left\|X'\right\|_\infty, \cdots, \left\|X^{n-1}\right\|_\infty\right\}$ and $Z = C[0,1]$ with the norm $\|\widetilde{z}\|_Z = \|\widetilde{z}\|_\infty$, where $\|X\|_\infty = \max_{\varrho\in[0,1]} |X(\varrho)|$ and $\|\widetilde{z}\|_\infty = \max_{\varrho\in[0,1]} |\widetilde{z}(\varrho)|$. Then we indicate $\overline{X} = X\times X\times X$ with the norm $\|(\upsilon, \nu, \omega)\|_{\overline{X}} = \max\{\|\upsilon\|_X, \|\nu\|_X, \|\omega\|_X\}$ and $\overline{Z} = Z\times Z\times Z$ with the norm $\|(\widetilde{x}, \widetilde{y}, \widetilde{z})\|_{\overline{Z}} = \max\{\|\widetilde{x}\|_Z, \|\widetilde{y}\|_Z, \|\widetilde{z}\|_Z\}$. Obviously, $\overline{X}$ and $\overline{Z}$ are Banach spaces.

Define $L_i : domL \subset X \to Z$, (i=1,2,3) by

$$L_1\upsilon = D_{0^+}^{\alpha}\upsilon,\ L_2\nu = D_{0^+}^{\beta}\nu \text{ and } L_3\omega = D_{0^+}^{\gamma}\omega,$$

where

$$\begin{aligned}domL_1 &= \Big\{\upsilon \in X |\ D_{0^+}^{\alpha}\upsilon(\varrho) \in Z, \upsilon^{(j)}(0) = 0, \\ &\qquad \upsilon^{(n-1)}(0) = \upsilon^{(n-1)}(1), j = 0, 1, ..., n-2\Big\}, \\ domL_2 &= \Big\{\nu \in X |\ D_{0^+}^{\beta}\nu(\varrho) \in Z, \nu^{(j)}(0) = 0, \\ &\qquad \nu^{(n-1)}(0) = \nu^{(n-1)}(1), j = 0, 1, ..., n-2\Big\}, \\ domL_3 &= \Big\{\omega \in X |\ D_{0^+}^{\gamma}\omega(\varrho) \in Z, \omega^{(j)}(0) = 0, \\ &\qquad \omega^{(n-1)}(0) = \omega^{(n-1)}(1), j = 0, 1, ..., n-2\Big\}.\end{aligned}$$

Define $L : domL \subset \overline{X} \to \overline{Z}$ as

$$L(\upsilon, \nu, \omega) = (L_1\upsilon, L_2\nu, L_3\omega), \tag{3.1}$$

where

$$domL = \Big\{(\upsilon, \nu, \omega) \in \overline{X} |\ \upsilon \in domL_1, \nu \in domL_2, \omega \in domL_3\Big\}.$$

Define the operator (Nemytski) $N : \overline{X} \to \overline{Z}$ as

$$N(\upsilon, \nu, \omega) = (N_1\omega, N_2\upsilon, N_3\nu),$$

where $N_i : X \to Z$, (i=1,2,3) as follows:

$$\begin{aligned}N_1\omega(\varrho) &= f(\varrho, \omega(\varrho), \omega'(\varrho), ..., \omega^{(n-1)}(\varrho)), \\ N_2\upsilon(\varrho) &= g(\varrho, \upsilon(\varrho), \upsilon'(\varrho), ..., \upsilon^{(n-1)}(\varrho)), \\ N_3\nu(\varrho) &= h(\varrho, \nu(\varrho), \nu'(\varrho), ..., \nu^{(n-1)}(\varrho)).\end{aligned}$$

The operator equation is then equivalent to the BVP's (1.1) and (1.4).

$$L(\upsilon, \nu, \omega) = N(\upsilon, \nu, \omega),\ (\upsilon, \nu, \omega) \in domL.$$

Lemma 3.1. *Let the operator L be defined by* (3.1). *Then*

$$\begin{aligned} kerL &= (KerL_1, KerL_2, KerL_3) \\ &= \Big\{(\upsilon,\nu,\omega) \in \overline{x} | (\upsilon,\nu,\omega) \\ &= (m_1\varrho^{n-1}, m_2\varrho^{n-1}, m_3\varrho^{n-1}), m_1, m_2, m_3 \in R\Big\}, \end{aligned} \quad (3.2)$$

and

$$\begin{aligned} ImL &= (ImL_1, ImL_2, ImL_3) \\ &= \Big\{(\widetilde{x},\widetilde{y},\widetilde{z}) \in \overline{Z} | \int_0^1 (1-\kappa)^{\alpha-n}\widetilde{x}(\kappa)d\kappa = 0, \\ &\int_0^1 (1-\kappa)^{\beta-n}\widetilde{y}(\kappa)d\kappa = 0, \ \int_0^1 (1-\kappa)^{\gamma-n}\widetilde{z}(\kappa)d\kappa = 0\Big\}. \end{aligned} \quad (3.3)$$

Proof. By Lemma 2.1, $L_1\upsilon = D^{\alpha}_{0^+}\upsilon(\varrho) = 0$ has the solution

$$\upsilon(\varrho) = c_0 + c_1\varrho + ... + c_{n-1}\varrho^{n-1}.$$

Using the boundary condition, we have

$$KerL_1 = \Big\{\upsilon \in X | \upsilon = m_1\varrho^{n-1}, m_1 \in R\Big\}.$$

For $(\widetilde{x},\widetilde{y},\widetilde{z}) \in ImL$, there exists $(\upsilon,\nu,\omega) \in domL$ such that $(\widetilde{x},\widetilde{y},\widetilde{z}) = L(\upsilon,\nu,\omega)$. Again, by Lemma 2.1, we get

$$\begin{aligned} \upsilon(\varrho) &= I^{\alpha}_{0^+}\widetilde{x}(\varrho) + a_0 + a_1\varrho + ... + a_{n-1}\varrho^{n-1}, \\ \nu(\varrho) &= I^{\beta}_{0^+}\widetilde{y}(\varrho) + b_0 + b_1\varrho + ... + b_{n-1}\varrho^{n-1}, \\ \omega(\varrho) &= I^{\gamma}_{0^+}\widetilde{z}(\varrho) + c_0 + c_1\varrho + ... + c_{n-1}\varrho^{n-1}. \end{aligned}$$

By definition of $domL$, we have $a_j = b_j = c_j = 0$, $j = 0, 1, 2, \cdots, n-2$. We can get

$$\begin{aligned} \upsilon(\varrho) &= I^{\alpha}_{0^+}\widetilde{x}(\varrho) + a_{n-1}\varrho^{n-1}, \\ \nu(\varrho) &= I^{\beta}_{0^+}\widetilde{y}(\varrho) + b_{n-1}\varrho^{n-1}, \\ \omega(\varrho) &= I^{\gamma}_{0^+}\widetilde{z}(\varrho) + c_{n-1}\varrho^{n-1}. \end{aligned}$$

From Lemma 2.2, we have

$$\upsilon^{(n-1)}(\varrho) = I^{\alpha-n+1}_{0^+}\widetilde{x}(\varrho) + a_{n-1}(n-1)!,$$

$$\nu^{(n-1)}(\varrho) = I^{\beta-n+1}_{0^+}\widetilde{y}(\varrho) + b_{n-1}(n-1)!,$$

$$\omega^{(n-1)}(\varrho) = I^{\gamma-n+1}_{0^+}\widetilde{z}(\varrho) + c_{n-1}(n-1)!.$$

By using the boundary conditions, we obtain

$$\begin{cases} \int_0^1 (1-\kappa)^{\alpha-n}\widetilde{x}(\kappa)d\kappa = 0, \\ \int_0^1 (1-\kappa)^{\beta-n}\widetilde{y}(\kappa)d\kappa = 0, \\ \int_0^1 (1-\kappa)^{\gamma-n}\widetilde{z}(\kappa)d\kappa = 0. \end{cases}$$

Further, suppose $(\widetilde{x},\widetilde{y},\widetilde{z}) \in Z)$ and satisfies above conditions.

Let $\upsilon(\varrho) = I^{\alpha}_{0^+}\widetilde{x}(\varrho)$, $\nu(\varrho) = I^{\beta}_{0^+}\widetilde{y}(\varrho)$, $\omega(\varrho) = I^{\gamma}_{0^+}\widetilde{z}(\varrho)$ then $(\upsilon,\nu,\omega) \in domL$ and $D^{\alpha}_{0^+}\upsilon(\varrho) = \widetilde{x}(\varrho)$, $D^{\beta}_{0^+}\nu(\varrho) = \widetilde{y}(\varrho)$, $D^{\gamma}_{0^+}\omega(\varrho) = \widetilde{z}(\varrho)$. Hence, $(\widetilde{x},\widetilde{y},\widetilde{z}) \in ImL$. Then we get

$$ImL_1 = \left\{\widetilde{x} \in Z | \int_0^1 (1-\kappa)^{\alpha-n}\widetilde{x}(\kappa)d\kappa = 0\right\}.$$

Similarly, we get that

$$\begin{aligned} KerL_2 &= \Big\{\nu \in X | \nu = m_2\varrho^{n-1}, m_2 \in R\Big\}, \\ ImL_2 &= \left\{\widetilde{y} \in Z | \int_0^1 (1-\kappa)^{\beta-n}\widetilde{y}(\kappa)d\kappa = 0\right\}, \\ KerL_3 &= \Big\{\omega \in X | \omega = m_3\varrho^{n-1}, m_3 \in R\Big\}, \\ ImL_3 &= \left\{\widetilde{z} \in Z | \int_0^1 (1-\kappa)^{\gamma-n}\widetilde{z}(\kappa)d\kappa = 0\right\}. \end{aligned}$$

□

Lemma 3.2. *Let L be defined by* $L(\upsilon,\nu,\omega) = (L_1\upsilon, L_2\nu, L_3\omega)$. *Then L is a Fredholm operator of index zero, the linear continuous projector operators* $P : \overline{X} \to \overline{X}$ *and* $Q : \overline{Z} \to \overline{Z}$ *can be defined as*

$$P(\upsilon,\nu,\omega) = (P_1\upsilon, P_2\nu, P_3\omega),$$

where

$$\begin{cases} P_1\upsilon = \dfrac{\upsilon^{(n-1)}(0)}{(n-1)!}\varrho^{n-1}, \\ P_2\nu = \dfrac{\nu^{(n-1)}(0)}{(n-1)!}\varrho^{n-1}, \\ P_3\omega = \dfrac{\omega^{(n-1)}(0)}{(n-1)!}\varrho^{n-1}. \end{cases} \quad (3.4)$$

and

$$Q(\widetilde{x},\widetilde{y},\widetilde{z}) = (Q_1\widetilde{x}, Q_2\widetilde{y}, Q_3\widetilde{z}),$$

where

$$\begin{cases} Q_1\widetilde{x}(\varrho) = (\alpha-n+1)\int_0^1 (1-\kappa)^{\alpha-n}\widetilde{x}(\kappa)d\kappa, \\ Q_2\widetilde{y}(\varrho) = (\beta-n+1)\int_0^1 (1-\kappa)^{\beta-n}\widetilde{y}(\kappa)d\kappa, \\ Q_3\widetilde{z}(\varrho) = (\gamma-n+1)\int_0^1 (1-\kappa)^{\gamma-n}\widetilde{z}(\kappa)d\kappa. \end{cases} \quad (3.5)$$

Furthermore, the operator $K_P : ImL \to domL \cap KerP$ can be written by $K_P(\widetilde{x},\widetilde{y},\widetilde{z}) = (I^{\alpha}_{0^+}\widetilde{x}, I^{\beta}_{0^+}\widetilde{y}, I^{\gamma}_{0^+}\widetilde{z})$, that is, $K_P = (L\,|_{domL\cap KerP})^{-1}$.

Proof. Define $P_i : X \to X$, $(i = 1,2,3)$ and $P : (\upsilon,\nu,\omega) \to (P_1\upsilon, P_2\nu, P_3\omega)$, from (3.4) we get

$$\begin{aligned} P_1^2\upsilon(\varrho) &= \frac{(P_1\upsilon)^{(n-1)}(0)}{(n-1)!}\varrho^{n-1} \\ &= \frac{1}{(n-1)!}\left(\frac{d^{n-1}}{dt^{n-1}}\frac{\upsilon^{(n-1)}(0)}{(n-1)!}\varrho^{n-1}\right)|_{\varrho=0}\varrho^{n-1} \\ &= \frac{a_{n-1}}{(n-1)!}\left(\frac{d^{n-1}}{dt^{n-1}}\varrho^{n-1}\right)|_{\varrho=0}\varrho^{n-1} = a_{n-1}\varrho^{n-1} = P_1\upsilon(\varrho). \end{aligned}$$

Clearly, $P_2^2 = P_2$, $P_3^2 = P_3$.

Obviously, $ImP = KerL$ and $P^2(\upsilon,\nu,\omega) = P(\upsilon,\nu,\omega)$. Note that $KerP = \left\{(\upsilon,\nu,\omega)|\upsilon^{(n-1)}(0) = 0, \nu^{(n-1)}(0) = 0, \omega^{(n-1)}(0) = 0\right\}$. Since $(\upsilon,\nu,\omega) = ((\upsilon,\nu,\omega) - P(\upsilon,\nu,\omega)) + P(\upsilon,\nu,\omega)$. It is clear that $\overline{X} = KerP + KerL$. Furthermore, by the definition of KerP, we can get that $KerL \cap KerP = \{(0,0,0)\}$. Then we get $\overline{X} = KerP \oplus KerL$.

Clearly, we have $Q(\widetilde{x},\widetilde{y},\widetilde{z}) = (Q_1\widetilde{x}, Q_2\widetilde{y}, Q_3\widetilde{z})$. By the definition of Q_1, we have

$$\begin{aligned} Q_1^2\widetilde{x}(\varrho) &= Q_1(Q_1\widetilde{x}(\varrho)) \\ &= Q_1\widetilde{x}(\varrho).(\alpha-n+1)\int_0^1(1-\kappa)^{\alpha-n}d\kappa = Q_1\widetilde{x}(\varrho). \end{aligned}$$

Similarly, we can show that $Q_2^2\widetilde{y} = Q_2\widetilde{y}$ and $Q_3^2\widetilde{z} = Q_3\widetilde{z}$. This gives $Q^2(\widetilde{x},\widetilde{y},\widetilde{z}) = Q(\widetilde{x},\widetilde{y},\widetilde{z})$. It follows from

$$(\widetilde{x},\widetilde{y},\widetilde{z}) = ((\widetilde{x},\widetilde{y},\widetilde{z}) - Q(\widetilde{x},\widetilde{y},\widetilde{z})) + Q(\widetilde{x},\widetilde{y},\widetilde{z}),$$

where $(\widetilde{x},\widetilde{y},\widetilde{z}) - Q(\widetilde{x},\widetilde{y},\widetilde{z}) \in KerQ$, $Q(\widetilde{x},\widetilde{y},\widetilde{z}) \in ImQ$ that $\overline{Z} = ImL + ImQ$. Let $\widetilde{x} \in ImQ_1 \cap ImL_1$ and set $\widetilde{x}(\varrho) = m_1\varrho^{n-1}$ to obtain that

$$\begin{aligned} 0 &= (\alpha-n+1)\int_0^1(1-\kappa)^{\alpha-n}\widetilde{x}(\kappa)d\kappa \\ &= (\alpha-n+1)m_1\int_0^1(1-\kappa)^{\alpha-n}\kappa^{n-1}d\kappa \\ &= \frac{\Gamma(n)\Gamma(\alpha-n+2)}{\Gamma(\alpha+1)}m_1 \end{aligned}$$

which implies that $m_1 = 0$. Similarly, $\widetilde{y} \in ImQ_2 \cap ImL_2$ implies that $m_2 = 0$, also $\widetilde{z} \in ImQ_3 \cap ImL_3$ implies that $m_3 = 0$. Moreover, by $KerQ = ImL$ and $Q^2(\widetilde{x},\widetilde{y},\widetilde{z}) = Q(\widetilde{x},\widetilde{y},\widetilde{z})$, we obtain $ImQ \cap ImL = \{(0,0,0)\}$. Hence, we get

$$\overline{Z} = ImL \oplus ImQ.$$

Thus

$$dimKerL = dimImQ = codimImL.$$

This shows that L is a Fredholm operator of index zero.

In fact, $(\widetilde{x},\widetilde{y},\widetilde{z}) \in ImL$, we have

$$LK_P(\widetilde{x},\widetilde{y},\widetilde{z}) = (D^{\alpha}_{0^+}(I^{\alpha}_{0^+}\widetilde{x}), D^{\beta}_{0^+}(I^{\beta}_{0^+}\widetilde{y}), D^{\gamma}_{0^+}(I^{\gamma}_{0^+}\widetilde{z})) = (\widetilde{x},\widetilde{y},\widetilde{z}). \tag{3.6}$$

On the other hand, for $(\upsilon,\nu,\omega) \in domL \cap KerP$, we have

$$\begin{aligned} I^{\alpha}_{0^+}L_1\upsilon(\varrho) &= I^{\alpha}_{0^+}D^{\alpha}_{0^+}\upsilon(\varrho) = \upsilon(\varrho) + a_0 + a_1\varrho + ... + a_{n-1}\varrho^{n-1}, \\ I^{\beta}_{0^+}L_2\nu(\varrho) &= I^{\beta}_{0^+}D^{\beta}_{0^+}\nu(\varrho) = \nu(\varrho) + b_0 + b_1\varrho + ... + b_{n-1}\varrho^{n-1}, \\ I^{\gamma}_{0^+}L_3\omega(\varrho) &= I^{\gamma}_{0^+}D^{\gamma}_{0^+}\omega(\varrho) = \omega(\varrho) + c_0 + c_1\varrho + ... + c_{n-1}\varrho^{n-1}. \end{aligned}$$

By the definitions of domL and KerP, we have $\upsilon^{(j)}(0) = \nu^{(j)}(0) = \omega^{(j)}(0) = 0$, $j = 0,1,2,...,n-1$ in the above expressions are all coefficients equal to zero. Thus, we obtain

$$\begin{aligned} K_PL(\widetilde{x},\widetilde{y},\widetilde{z}) &= (I^{\alpha}_{0^+}D^{\alpha}_{0^+}\widetilde{x}(\varrho), I^{\beta}_{0^+}D^{\beta}_{0^+}\widetilde{y}(\varrho), I^{\gamma}_{0^+}D^{\gamma}_{0^+}\widetilde{z}(\varrho)) \\ &= (\widetilde{x},\widetilde{y},\widetilde{z}). \end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7), we get $K_P = (L|_{domL\cap KerP})^{-1}$. □

Setting $d_1 = \dfrac{1}{\Gamma(\alpha-n+2)}$, $d_2 = \dfrac{1}{\Gamma(\beta-n+2)}$, $d_3 = \dfrac{1}{\Gamma(\gamma-n+2)}$.

Again, for every $(\widetilde{x},\widetilde{y},\widetilde{z}) \in ImL$,

$$\begin{aligned} \|K_P(\widetilde{x},\widetilde{y},\widetilde{z})\|_{\overline{X}} &= \|(I^{\alpha}_{0^+}X, I^{\beta}_{0^+}\widetilde{y}, I^{\gamma}_{0^+}\widetilde{z})\|_{\overline{X}} \\ &= \max\left\{\left\|I^{\alpha}_{0^+}\widetilde{x}\right\|_X, \|I^{\beta}_{0^+}\widetilde{y}\|_X, \|I^{\gamma}_{0^+}\widetilde{z}\|_X\right\} \\ &= \max\left\{\frac{1}{\Gamma(\alpha-n+2)}\|\widetilde{x}\|_{\infty}, \frac{1}{\Gamma(\beta-n+2)}\|\widetilde{y}\|_{\infty}, \frac{1}{\Gamma(\gamma-n+2)}\|\widetilde{z}\|_{\infty}\right\} \\ &\leq \max\{d_1\|\widetilde{x}\|_{\infty}; d_2\|\widetilde{y}\|_{\infty}; d_3\|\widetilde{z}\|_{\infty}\}. \end{aligned} \tag{3.8}$$

Lemma 3.3. *Assume $\Omega \subset \overline{X}$ is an open bounded subset such that $domL \cap \overline{\Omega} \neq \phi$, then N is L-compact on $\overline{\Omega}$.*

Proof. Since the functions f, g and h are continuous, we get $QN(\overline{\Omega})$ is bounded and by the definition of operators Q and K_P on the interval [0,1], we can get that $K_P(I-Q)N(\overline{\Omega})$ is bounded. On the other hand, there exist constants $r_i > 0,\ i = 1, 2, 3$, such that for all $(\upsilon, \nu, \omega) \in \overline{\Omega},\ \varrho \in [0, 1]$, then

$$|(I - Q_1)N_1\omega| \le r_1,\ \ |(I - Q_2)N_2\upsilon| \le r_2,\ \ |(I - Q_3)N_3\nu| \le r_3.$$

Next, denote $K_{P,Q} = K_P(I - Q)N$ and for $0 \le \varrho_1 < \varrho_2 \le 1$, we get

$$\begin{aligned}
&\Big|K_{P,Q}(\upsilon, \nu, \omega)(\varrho_2) - K_{P,Q}(\upsilon, \nu, \omega)(\varrho_1)\Big|\\
&= \Big|K_P(I - Q)(N_1\omega(\varrho_2), N_2\upsilon(\varrho_2), N_3\nu(\varrho_2))\\
&\qquad - K_P(I - Q)(N_1\omega(\varrho_1), N_2\upsilon(\varrho_1), N_3\nu(\varrho_1))\Big|\\
&= \Big|I^{\alpha}_{0^+}(I - Q_1)N_1\omega(\varrho_2) - I^{\alpha}_{0^+}(I - Q_1)N_1\omega(\varrho_1),\\
&\qquad I^{\beta}_{0^+}(I - Q_2)N_2\upsilon(\varrho_2) - I^{\beta}_{0^+}(I - Q_2)N_2\upsilon(\varrho_1),\\
&\qquad I^{\gamma}_{0^+}(I - Q_3)N_3\nu(\varrho_2) - I^{\gamma}_{0^+}(I - Q_3)N_3\nu(\varrho_1)\Big|.
\end{aligned}$$

Here,

$$\begin{aligned}
&\Big|I^{\alpha}_{0^+}(I - Q_1)N_1\omega(\varrho_2) - I^{\alpha}_{0^+}(I - Q_1)N_1\omega(\varrho_1)\Big|\\
&\le \frac{1}{\Gamma(\alpha)}\Big|\int_0^{\varrho_2}(\varrho_2 - \kappa)^{\alpha-1}(I - Q_1)N_1\omega(\kappa)d\kappa\\
&\qquad - \int_0^{\varrho_1}(\varrho_1 - \kappa)^{\alpha-1}(I - Q_1)N_1\omega(\kappa)d\kappa\Big|\\
&\le \frac{r_1}{\Gamma(\alpha)}\left[\int_0^{\varrho_1}((\varrho_2 - \kappa)^{\alpha-1} - (\varrho_1 - \kappa)^{\alpha-1})d\kappa + \int_{\varrho_1}^{\varrho_2}(\varrho_2 - \kappa)^{\alpha-1}d\kappa\right]\\
&= \frac{r_1}{\Gamma(\alpha + 1)}(\varrho_2^{\alpha} - \varrho_1^{\alpha}).
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&\Big|(I^{\alpha}_{0^+}(I - Q_1)N_1\omega)^{(j)}(\varrho_2) - (I^{\alpha}_{0^+}(I - Q_1)N_1\omega)^{(j)}(\varrho_1)\Big|\\
&\le \frac{1}{\Gamma(\alpha - j)}\Big|\int_0^{\varrho_2}(\varrho_2 - \kappa)^{\alpha-j-1}(I - Q_1)N_1\omega(\kappa)d\kappa\\
&\qquad - \int_0^{\varrho_1}(\varrho_1 - \kappa)^{\alpha-j-1}(I - Q_1)N_1\omega(\kappa)d\kappa\Big|\\
&\le \frac{r_1}{\Gamma(\alpha - j)}\Big[\int_0^{\varrho_1}((\varrho_2 - \kappa)^{\alpha-j-1} - (\varrho_1 - \kappa)^{\alpha-j-1})d\kappa\\
&\qquad + \int_{\varrho_1}^{\varrho_2}(\varrho_2 - \kappa)^{\alpha-j-1}d\kappa\Big]\\
&= \frac{r_1}{\Gamma(\alpha - j + 1)}(\varrho_2^{\alpha-j} - \varrho_1^{\alpha-j}),
\end{aligned}$$

where $j = 0, 1, 2, ..., n - 1$. Thus,

$$\Big|(I^{\alpha}_{0^+}(I - Q_1)N_1\omega)^{(j)}(\varrho_2) - (I^{\alpha}_{0^+}(I - Q_1)N_1\omega)^{(j)}(\varrho_1)\Big| \to 0$$

uniformly as $\varrho_2 \to \varrho_1$. Similarly, we can show that

$$\begin{aligned}
&\Big|I^{\beta}_{0^+}(I - Q_2)N_2\upsilon(\varrho_2) - I^{\beta}_{0^+}(I - Q_2)N_2\upsilon(\varrho_1)\Big|\\
&\qquad \le \frac{r_2}{\Gamma(\beta + 1)}(\varrho_2^{\beta} - \varrho_1^{\beta}),\\
&\Big|(I^{\beta}_{0^+}(I - Q_2)N_2\upsilon)^{j}(\varrho_2) - (I^{\beta}_{0^+}(I - Q_2)N_2\upsilon)^{j}(\varrho_1)\Big|\\
&\qquad \le \frac{r_2}{\Gamma(\beta - j + 1)}(\varrho_2^{\beta-j} - \varrho_1^{\beta-j}),\\
&\Big|I^{\gamma}_{0^+}(I - Q_3)N_3\nu(\varrho_2) - I^{\gamma}_{0^+}(I - Q_3)N_3\nu(\varrho_1)\Big|\\
&\qquad \le \frac{r_3}{\Gamma(\gamma + 1)}(\varrho_2^{\gamma} - \varrho_1^{\gamma}),\\
&\Big|(I^{\gamma}_{0^+}(I - Q_3)N_3\nu)'(\varrho_2) - (I^{\gamma}_{0^+}(I - Q_3)N_3\nu)'(\varrho_1)\Big|\\
&\qquad \le \frac{r_3}{\Gamma(\gamma - j + 1)}(\varrho_2^{\gamma-j} - \varrho_1^{\gamma-j}).
\end{aligned}$$

Since ϱ^{α}, $\varrho^{\alpha-j}$, ϱ^{β}, $\varrho^{\beta-j}$, ϱ^{γ} and $\varrho^{\gamma-j}$ are uniformly continuous on [0,1], we can get that $K_{P,Q}(\overline{\Omega}) \subset C[0, 1]$, $K^{(j)}_{P,Q}(\overline{\Omega}) \subset C[0, 1]$, $j = 1, 2, ..., n - 1$ are equicontinuous. By the Arzela-Ascoli theorem, we can obtain $K_P(I - Q)N$ is completely continuous. Hence N is L-compact on $\overline{\Omega}$. □

Theorem 3.4. *Let $f, g, h : [0, 1]\times R^{(n-1)} \to R$ be continuous. Assume that*

(B_1) There exist positive constants $\delta_i, \rho_i, \tau_i \in [0, 1]$, $i = 1, 2, ...n$, such that for all $(\widetilde{y}_1, \widetilde{y}_2, ..., \widetilde{y}_n) \in R^n$ and $\varrho \in [0, 1]$,

$$\begin{aligned}
|f(\varrho, \widetilde{y}_1, \widetilde{y}_2, ..., \widetilde{y}_n)| &\le \rho_0 + \rho_1|\widetilde{y}_1| + \rho_2|\widetilde{y}_2| + ... + \rho_n|\widetilde{y}_n|,\\
|g(\varrho, \widetilde{y}_1, \widetilde{y}_2, ..., \widetilde{y}_n)| &\le \delta_0 + \delta_1|\widetilde{y}_1| + \delta_2|\widetilde{y}_2| + ... + \delta_n|\widetilde{y}_n|,\\
|h(\varrho, \widetilde{y}_1, \widetilde{y}_2, ..., \widetilde{y}_n)| &\le \tau_0 + \tau_1|\widetilde{y}_1| + \tau_2|\widetilde{y}_2| + ... + \tau_n|\widetilde{y}_n|.
\end{aligned}$$

(B_2) There exists a positive constant D such that for any $m_1, m_2, m_3 \in R$, if $\min\{|m_1|, |m_2|, |m_3|\} > D$, one has either

$$m_1N_1(m_2\varrho^{n-1}) > 0,\ m_2N_2(m_3\varrho^{n-1}) > 0,\ m_3N_3(m_1\varrho^{n-1}) > 0$$

or

$$m_1N_1(m_2\varrho^{n-1}) < 0,\ m_2N_2(m_3\varrho^{n-1}) < 0,\ m_3N_3(m_1\varrho^{n-1}) < 0.$$

(B_3) $\max\Big\{2d_1\sum_{j=1}^n\rho_j, 2d_2\sum_{j=1}^n\delta_j, 2d_3\sum_{j=1}^n\tau_j, d_1\sum_{j=1}^n\rho_j + d_2\sum_{j=1}^n\delta_j, d_2\sum_{j=1}^n\delta_j + d_3\sum_{j=1}^n\tau_j, d_1\sum_{j=1}^n\rho_j + d_3\sum_{j=1}^n\tau_j\Big\} < 1.$

Then the system (1.1) and (1.4) has at least one solution.

Lemma 3.5. *Assume that $(B_1) - (B_3)$ hold, then the set*

$$\Omega_1 = \{(\upsilon, \nu, \omega) \in domL \setminus KerL \mid L(\upsilon, \nu, \omega) = \lambda N(\upsilon, \nu, \omega), \lambda \in (0, 1)\}$$

is bounded.

Proof. For $(\upsilon,\nu,\omega) \in \Omega_1, \lambda \neq 0$, then $L(\upsilon,\nu,\omega) = \lambda N(\upsilon,\nu,\omega) \in ImL = KerQ$, that is, $QN(\upsilon,\nu,\omega) = 0$. By (3.3), we have

$$\lambda(\alpha-n+1)\int_0^1 (1-\kappa)^{\alpha-n} f(\kappa,\omega(\kappa),\omega'(\kappa),...,\omega^{(n-1)}(\kappa))d\kappa = 0,$$
$$\lambda(\beta-n+1)\int_0^1 (1-\kappa)^{\beta-n} g(\kappa,\upsilon(\kappa),\upsilon'(\kappa),...,\upsilon^{(n-1)}(\kappa))d\kappa = 0,$$
$$\lambda(\gamma-n+1)\int_0^1 (1-\kappa)^{\gamma-n} h(\kappa,\nu(\kappa),\nu'(\kappa),...,\nu^{(n-1)}(\kappa))d\kappa = 0.$$

Applying integral mean value theorem, there exist constants $\varrho_0,\varrho_1,\varrho_2 \in [0,1]$ such that

$$f(\varrho_0,\omega(\varrho_0),\omega'(\varrho_0),...,\omega^{(n-1)}(\varrho_0)) = 0,$$
$$g(\varrho_1,\upsilon(\varrho_1),\upsilon'(\varrho_1),...,\upsilon^{(n-1)}(\varrho_1)) = 0,$$
$$h(\varrho_2,\nu(\varrho_2),\nu'(\varrho_2),...,\nu^{(n-1)}(\varrho_2)) = 0.$$

From (B_2), we can get $|\nu^{(n-1)}(\varrho_2)| \leq K$, $|\upsilon^{(n-1)}(\varrho_1)| \leq K$ and $|\omega^{(n-1)}(\varrho_0)| \leq K$.
By $L_2\nu = \lambda N_2\upsilon$, we have

$$\nu(\varrho) = \frac{1}{\Gamma(\beta)}\int_0^{\varrho} (\varrho-\kappa)^{\beta-1} g(\kappa,\upsilon(\kappa),...,\upsilon^{(n-1)}(\kappa))\,d\kappa - \nu(0) - \nu'(0) - ... - \frac{\nu^{(n-1)}(0)}{(n-1)!}\varrho^{n-1}.$$

Furthermore, we have that,

$$\nu^{(n-1)}(\varrho) = \frac{1}{\Gamma(\beta-n+1)}\int_0^{\varrho} (\varrho-\kappa)^{\beta-n} g(\kappa,\upsilon(\kappa),...,\upsilon^{(n-1)}(\kappa))d\kappa - \nu^{(n-1)}(0).$$

Substituting $\varrho = \varrho_2$ in the above equation, we can get

$$\nu^{(n-1)}(\varrho_2) = \frac{1}{\Gamma(\beta-n+1)} \times \int_0^{\varrho_2} (\varrho_2-\kappa)^{\beta-n} g(\kappa,\upsilon(\kappa),...,\upsilon^{(n-1)}(\kappa))d\kappa - \nu^{(n-1)}(0).$$

Together with $|\nu^{(n-1)}(\varrho_2)| \leq K$, we have

$$\begin{aligned}
|\nu^{(n-1)}(0)| &\leq \Big|\frac{1}{\Gamma(\beta-n+1)} \times \int_0^{\varrho_2} (\varrho_2-\kappa)^{\beta-n} g(\kappa,\upsilon(\kappa),...,\upsilon^{(n-1)}(\kappa))d\kappa\Big| + |\nu^{(n-1)}(\varrho_2)| \\
&\leq K + \frac{1}{\Gamma(\beta-n+1)} \times \int_0^{\varrho_2} (\varrho_2-\kappa)^{\beta-n} |g(\kappa,\upsilon(\kappa),...,\upsilon^{(n-1)}(\kappa))|d\kappa \\
&\leq K + \frac{1}{\Gamma(\beta-n+1)} \times \int_0^{\varrho_2} (\varrho_2-\kappa)^{\beta-n} \left(\delta_0 + \sum_{j=1}^{n} \delta_j \left|\upsilon_j\right|\right) d\kappa \\
&\leq K + \frac{1}{\Gamma(\beta-n+1)} \left(\delta_0 + \sum_{j=1}^{n} \delta_j \left\|\upsilon_j\right\|_\infty\right) \times \int_0^{\varrho_2} (\varrho_2-\kappa)^{\beta-n} d\kappa \\
&\leq K + d_2\delta_0 + d_2 \sum_{j=1}^{n} \delta_j \left\|\upsilon_j\right\|_\infty.
\end{aligned} \tag{3.9}$$

Using similar argument, we get

$$|\upsilon^{(n-1)}(0)| \leq K + d_1\rho_0 + d_1 \sum_{j=1}^{n} \rho_j \left\|\omega_j\right\|_\infty, \tag{3.10}$$

$$|\omega^{(n-1)}(0)| \leq K + d_3\tau_0 + d_3 \sum_{j=1}^{n} \tau_j \left\|\nu_j\right\|_\infty. \tag{3.11}$$

For every $(\upsilon,\nu,\omega) \in \overline{X}$,

$$\begin{aligned}
||P(\upsilon,\nu,\omega)||_{\overline{X}} &= ||(P_1\upsilon, P_2\nu, P_3\omega)||_{\overline{X}} \\
&= \max\{||P_1\upsilon||_X, ||P_2\nu||_X, ||P_3\omega||_X\} \\
&= \max\left\{\frac{|\upsilon^{(n-1)}(0)|}{(n-1)!}\left\|\varrho^{n-1}\right\|_X, \frac{|\nu^{(n-1)}(0)|}{(n-1)!}\left\|\varrho^{n-1}\right\|_X, \frac{|\omega^{(n-1)}(0)|}{(n-1)!}\left\|\varrho^{n-1}\right\|_X\right\} \\
&\leq \max\left\{|\upsilon^{(n-1)}(0)|; |\nu^{(n-1)}(0)|; |\omega^{(n-1)}(0)|\right\}.
\end{aligned} \tag{3.12}$$

Again, for $(\upsilon,\nu,\omega) \in \Omega_1$, $(\upsilon,\nu,\omega) \in domL \setminus KerL$, then $(I-P)(\upsilon,\nu,\omega) \in domL \cap KerP$ and $LP(\upsilon,\nu,\omega) = (0,0,0)$. Thus, from (3.5), we have

$$\begin{aligned}
||(I-P)(\upsilon,\nu,\omega)||_{\overline{X}} &= ||K_P L(I-P)(\upsilon,\nu,\omega)||_{\overline{X}} \\
&= ||K_P(L_1\upsilon, L_2\nu, L_3\omega)||_{\overline{X}} \\
&\leq \max\{d_1\,||N_1\omega||_\infty; d_2\,||N_2\upsilon||_\infty; d_3||N_3\nu||_\infty\}.
\end{aligned} \tag{3.13}$$

From (3.12) and (3.13), we get

$$\begin{aligned}
||(\upsilon,\nu,\omega)||_{\overline{X}} &= ||P(\upsilon,\nu,\omega) + (I-P)(\upsilon,\nu,\omega)||_{\overline{X}} \\
&\leq ||P(\upsilon,\nu,\omega)||_{\overline{X}} + ||(I-P)(\upsilon,\nu,\omega)||_{\overline{X}} \\
&\leq \max\left\{|\upsilon^{(n-1)}(0)|; |\nu^{(n-1)}(0)|; |\omega^{(n-1)}(0)|\right\}
\end{aligned}$$

$$+\max\{d_1\|N_1\omega\|_\infty ; d_2\|N_2\upsilon\|_\infty ; d_3\|N_3\nu\|_\infty\}$$
$$\le \max\Big\{|\upsilon^{(n-1)}(0)| + d_1\|N_1\omega\|_\infty, |\upsilon^{(n-1)}(0)| + d_2\|N_2\upsilon\|_\infty,$$
$$|\upsilon^{(n-1)}(0)| + d_3\|N_3\nu\|_\infty, |\nu^{(n-1)}(0)| + d_1\|N_1\omega\|_\infty,$$
$$|\nu^{(n-1)}(0)| + d_2\|N_2\upsilon\|_\infty, |\nu^{(n-1)}(0)| + d_3\|N_3\nu\|_\infty,$$
$$|\omega^{(n-1)}(0)| + d_1\|N_1\omega\|_\infty, |\omega^{(n-1)}(0)| + d_2\|N_2\upsilon\|_\infty,$$
$$|\omega^{(n-1)}(0)| + d_3\|N_3\nu\|_\infty\Big\}. \quad (3.14)$$

The proof is divided into nine cases as follows.

Case 1. $\|(\upsilon,\nu,\omega)\|_{\overline{X}} \le |\upsilon^{(n-1)}(0)| + d_1\|N_1\omega\|_\infty$.

By (3.10), and (B_1), we have

$$\|(\upsilon,\nu,\omega)\|_{\overline{X}} \le K + d_1\rho_0 + d_1\sum_{j=1}^{n}\rho_j\left\|\omega_j\right\|_\infty + d_1\rho_0 + d_1\sum_{j=1}^{n}\rho_j\left\|\omega_j\right\|_\infty \le K + 2d_1\rho_0 + 2d_1\sum_{j=1}^{n}\rho_j\left\|\omega_j\right\|_\infty \quad (3.15)$$

According to (B_3) and the definition of $\|(\upsilon,\nu,\omega)\|_{\overline{X}}$, we can get $\|\omega\|_X$ are bounded. Therefore Ω_1 is bounded.

Case 2. $\|(\upsilon,\nu,\omega)\|_{\overline{X}} \le |\upsilon^{(n-1)}(0)| + d_2\|N_2\upsilon\|_\infty$. By (3.10), and (B_1), we have

$$\|(\upsilon,\nu,\omega)\|_{\overline{X}} \le K + d_1\rho_0 + d_1\sum_{j=1}^{n}\rho_j\left\|\upsilon_j\right\|_\infty + d_2\delta_0 + d_2\sum_{j=1}^{n}\delta_j\left\|\upsilon_j\right\|_\infty \quad (3.16)$$

By (B_3), $\|(\upsilon,\nu,\omega)\|_{\overline{X}}$ is bounded. Therefore Ω_1 is bounded.

Case 3. $\|(\upsilon,\nu,\omega)\|_{\overline{X}} \le |\upsilon^{(n-1)}(0)| + d_3\|N_3\nu\|_\infty$. By (3.11), and (B_1), we have

$$\|(\upsilon,\nu,\omega)\|_{\overline{X}} \le K + d_1\rho_0 + d_1\sum_{j=1}^{n}\rho_j\left\|\nu_j\right\|_\infty + d_3\tau_0 + d_3\sum_{j=1}^{n}\tau_j\left\|\nu_j\right\|_\infty \quad (3.17)$$

By (B_3), $\|(\upsilon,\nu,\omega)\|_{\overline{X}}$ is bounded. Therefore Ω_1 is bounded.

Case 4. $\|(\upsilon,\nu,\omega)\|_{\overline{X}} \le |\nu^{(n-1)}(0)| + d_1\|N_1\omega\|_\infty$. The proof is similar to that of Case 2, hence the details are omitted.

Case 5. $\|(\upsilon,\nu,\omega)\|_{\overline{X}} \le |\nu^{(n-1)}(0)| + d_2\|N_2\upsilon\|_\infty$. The proof is similar to Case 1, hence the details are omitted.

Case 6. $\|(\upsilon,\nu,\omega)\|_{\overline{X}} \le |\nu^{(n-1)}(0)| + d_3\|N_3\nu\|_\infty$.By (3.9), and (B_1), we have

$$\|(\upsilon,\nu,\omega)\|_{\overline{X}} \le K + d_2\delta_0 + d_2\sum_{j=1}^{n}\delta_j\left\|\upsilon_j\right\|_\infty + d_3\tau_0 + d_3\sum_{j=1}^{n}\tau_j\left\|\nu_j\right\|_\infty \quad (3.18)$$

By (B_3), $\|(\upsilon,\nu,\omega)\|_{\overline{X}}$ is bounded. Therefore Ω_1 is bounded.

Case 7. $\|(\upsilon,\nu,\omega)\|_{\overline{X}} \le |\omega^{(n-1)}(0)| + d_1\|N_1\omega\|_\infty$. The proof is similar to Case 3, hence the details are omitted.

Case 8. $\|(\upsilon,\nu,\omega)\|_{\overline{X}} \le |\omega^{(n-1)}(0)| + d_2\|N_2\upsilon\|_\infty$. The proof is similar to Case 6, hence the details are omitted.

Case 9. $\|(\upsilon,\nu,\omega)\|_{\overline{X}} \le |\omega^{(n-1)}(0)| + d_3\|N_3\nu\|_\infty$. The proof is similar to Case 5, hence the details are omitted.

Ω_1 is bounded, according to the preceding arguments. □

Lemma 3.6.

$$\Omega_2 = \left\{(\upsilon,\nu,\omega)|\ (\upsilon,\nu,\omega) \in KerL, N(\upsilon,\nu,\omega) \in ImL\right\}$$

is bounded.

Proof. For $(\upsilon,\nu,\omega) \in \Omega_2$, so we have $(\upsilon,\nu,\omega) = (m_1\varrho^{n-1}, m_2\varrho^{n-1}, m_3\varrho^{n-1})$, $m_1, m_2,$ $m_3 \in R$. Then from $N(\upsilon,\nu,\omega) \in ImL = KerQ$, we have $Q_1(N_1\omega) = 0, Q_2(N_2\upsilon) = 0, Q_3(N_3\nu) = 0$, that is,

$$\int_0^1 (1-\kappa)^{\alpha-n} f(\varrho, m_3\varrho^{n-1}, (n-1)m_3\varrho^{n-2}, ..., m_3(n-1)!)dt = 0,$$
$$\int_0^1 (1-\kappa)^{\beta-n} g(\varrho, m_1\varrho^{n-1}, (n-1)m_1\varrho^{n-2}, ..., m_1(n-1)!)dt = 0,$$
$$\int_0^1 (1-\kappa)^{\gamma-n} h(\varrho, m_2\varrho^{n-1}, (n-1)m_2\varrho^{n-2}, ..., m_2(n-1)!)dt = 0.$$

By integral mean value theorem, there exist constants $\varrho_0, \varrho_1, \varrho_2 \in [0,1]$ such that

$$f(\varrho_0, m_3\varrho_0^{n-1}, (n-1)m_3\varrho_0^{n-2}, ..., m_3(n-1)!) = 0,$$
$$g(\varrho_1, m_1\varrho_1^{n-1}, (n-1)m_1\varrho_1^{n-2}, ..., m_1(n-1)!) = 0,$$
$$h(\varrho_2, m_2\varrho_2^{n-1}, (n-1)m_2\varrho_2^{n-2}, ..., m_2(n-1)!) = 0.$$

By (B_2) imply that $|m_1|, |m_2|, |m_3| \le \dfrac{D}{(n-1)!}$. Therefore Ω_2 is bounded. □

Lemma 3.7.

$$\Omega_3 = \Big\{(\upsilon,\nu,\omega)\in KerL \,|\, \lambda(\upsilon,\nu,\omega)+(1-\lambda)QN(\upsilon,\nu,\omega)=(0,0,0), \lambda\in(0,1)\Big\}$$

is bounded.

Proof. For $(\upsilon,\nu,\omega)\in\Omega_3$, so we have

$$(\upsilon,\nu,\omega) = (m_1\varrho^{n-1}, m_2\varrho^{n-1}, m_3\varrho^{n-1}),$$

$m_1, m_2, m_3 \in R$ and

$$\lambda m_1\varrho^{n-1} + (1-\lambda)(\alpha-n+1) \times \int_0^1 (1-\kappa)^{\alpha-n} f(\varrho, m_3\varrho^{n-1}, ..., m_3(n-1)!)dt = 0, \tag{3.19}$$

$$\lambda m_2\varrho^{n-1} + (1-\lambda)(\beta-n+1) \times \int_0^1 (1-\kappa)^{\beta-n} g(\varrho, m_1\varrho^{n-1}, ..., m_1(n-1)!)dt = 0, \tag{3.20}$$

$$\lambda m_3\varrho^{n-1} + (1-\lambda)(\gamma-n+1) \times \int_0^1 (1-\kappa)^{\gamma-n} h(\varrho, m_2\varrho^{n-1}, ..., m_2(n-1)!)dt = 0. \tag{3.21}$$

If $\lambda = 0$, then by (B_2), we get $|m_1|, |m_2|, |m_3| \le \frac{D}{(n-1)!}$. For $\lambda \in (0,1]$, we obtain $|m_1|, |m_2|, |m_3| \le \frac{D}{(n-1)!}$. Otherwise, if $|m_i| > \frac{D}{(n-1)!}$, $i = 1,2,3$, from (B_2), one has

$$\lambda m_1^2\varrho^{n-1} + (1-\lambda)(\alpha-n+1) \times \int_0^1 (1-\kappa)^{\alpha-n} m_1 f(\varrho, m_3\varrho^{n-1}, ..., m_3(n-1)!)dt > 0,$$

$$\lambda m_2^2\varrho^{n-1} + (1-\lambda)(\beta-n+1) \times \int_0^1 (1-\kappa)^{\beta-n} m_2 g(\varrho, m_1\varrho^{n-1}, ..., m_1(n-1)!)dt > 0,$$

$$\lambda m_3^2\varrho^{n-1} + (1-\lambda)(\gamma-n+1) \times \int_0^1 (1-\kappa)^{\gamma-n} m_3 h(\varrho, m_2\varrho^{n-1}, ..., m_2(n-1)!)dt > 0.$$

which contradict to (3.19) or (3.20) or (3.21). Hence, Ω_3 is bounded. □

Remark 3.8. *Suppose the second part of* (H_3) *holds, then the set*

$$\Omega_3' = \Big\{(\upsilon,\nu,\omega)\in KerL \,|\, -\lambda(\upsilon,\nu,\omega) + (1-\lambda)QN(\upsilon,\nu,\omega)=(0,0,0), \lambda\in(0,1)\Big\}$$

is bounded.

Proof of the Theorem 3.1: Suppose $\cup_{i=1}^{3}\overline{\Omega} \subset \Omega$ be a bounded open subset of X. From the Lemma 3.2 and Lemma 3.3, we get L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$. By Lemma 3.4 and Lemma 3.5, we get
(1) $L(\upsilon,\nu,\omega) \neq \lambda N(\upsilon,\nu,\omega)$ for every $((\upsilon,\nu,\omega),\lambda) \in [(domL \backslash KerL) \cap \partial\Omega] \times (0,1)$;
(2) $Nx \notin ImL$ for every $(\upsilon,\nu,\omega) \in KerL \cap \partial\Omega$. Choose

$$H((\upsilon,\nu,\omega),\lambda) = \pm\lambda(\upsilon,\nu,\omega) + (1-\lambda)QN(\upsilon,\nu,\omega).$$

By Lemma 3.6 (or Remark 3.1), we get $H((\upsilon,\nu,\omega),\lambda) \neq 0$ for $(\upsilon,\nu,\omega) \in KerL \cap \partial\Omega$. By the homotopic property of degree, we have

$$\begin{aligned} deg(JQN|_{KerL}, KerL\cap\Omega, 0) &= deg(H(.,0), KerL\cap\Omega, 0) \\ &= deg(H(.,1), KerL\cap\Omega, 0) \\ &= deg(\pm I, KerL\cap\Omega, 0) \neq 0. \end{aligned}$$

Thus, the condition (3) of Lemma 3.3 is satisfied. By Lemma 3.3, we obtain $L(\upsilon,\nu,\omega) = N(\upsilon,\nu,\omega)$ has at least one solution in $domL\cap\overline{\Omega}$. Hence BVP (1.1) and (1.4) has at least one solution. This completes the proof.

4. Example

Consider the BVP of fractional differential equation of the form

$$\begin{cases} D_{0^+}^{2.25}\upsilon(\varrho) &= \frac{\varrho}{8} + \frac{\varrho^3}{4}e^{\sin\omega(\varrho)} + \frac{1}{4}(1+\omega'(\varrho))^{-\frac{1}{3}} \\ & \quad +\varrho^5\sec(\omega''(\varrho)),\ \varrho\in(0,1), \\ D_{0^+}^{2.5}\nu(\varrho) &= \frac{\varrho^4}{5} + \frac{1}{5}e^{\upsilon(\varrho)}\cos\upsilon(\varrho) + \frac{\varrho}{10}\log(1+\upsilon'(\varrho)) \\ & \quad +\frac{1}{10}\arctan\upsilon''(\varrho),\ \varrho\in(0,1), \\ D_{0^+}^{2.75}\omega(\varrho) &= \frac{\varrho^2}{7} + \varrho^5\cos\nu(\varrho) + \frac{\varrho^8}{7}e^{-|\nu'(\varrho)|} \\ & \quad +\frac{1}{14}(1+\nu''(\varrho))^{-2},\ \varrho\in(0,1) \end{cases} \tag{4.1}$$

and

$$\begin{cases} \upsilon(0) = \upsilon'(0) = 0, \upsilon''(0) = \upsilon''(1), \\ \nu(0) = \nu'(0) = 0, \nu''(0) = \nu''(1), \\ \omega(0) = \omega'(0) = 0, \omega''(0) = \omega''(1), \end{cases} \tag{4.2}$$

Here $\alpha = 2.25, \beta = 2.5, \gamma = 2.75, n = 3$. Moreover,

$$f(\varrho, \omega(\varrho), \omega^{'}(\varrho)) = \frac{\varrho}{8} + \frac{\varrho^3}{4} e^{\sin\omega(\varrho)} + \frac{1}{4}(1 + \omega^{'}(\varrho))^{-\frac{1}{3}} + \varrho^5 sec(\omega^{''}(\varrho)),$$

$$g(\varrho, \upsilon(\varrho), \upsilon^{'}(\varrho)) = \frac{\varrho^4}{5} + \frac{1}{5} e^{\upsilon(\varrho)} \cos\upsilon(\varrho) + \frac{\varrho}{10}\log(1 + \upsilon^{'}(\varrho)) + \frac{1}{10}\arctan\upsilon^{''}(\varrho),$$

$$h(\varrho, \nu(\varrho), \nu^{'}(\varrho)) = \frac{\varrho^2}{7} + \varrho^5\cos\nu(\varrho) + \frac{\varrho^8}{7} e^{-|\nu^{'}(\varrho)|} + \frac{1}{14}(1 + \nu^{''}(\varrho))^{-2}.$$

Now let us compute $\rho_0,\ \rho_1,\ \rho_2,\ \rho_3$ from $f(\varrho, \omega(\varrho), \omega^{'}(\varrho))$.

$$f(\varrho, \omega(\varrho), \omega^{'}(\varrho)) = \frac{\varrho}{8} + \frac{\varrho^3}{4}(1 + \omega(\varrho) + \frac{\omega^2(\varrho)}{2} - ...) + \frac{1}{4}(1 - \frac{1}{3}\omega^{'}(\varrho) + \frac{4}{18}\omega^{'2} - ...) + \varrho^5(1 + \frac{\omega^{''2}(\varrho)}{2} + ...)$$

$$\left|f(\varrho, \omega(\varrho), \omega^{'}(\varrho))\right| \leq \frac{13}{8} + \frac{1}{4}|\omega(\varrho)| + \frac{1}{12}\left|\omega^{'}(\varrho)\right|$$

From the above inequality, we get $\rho_0 = \frac{13}{8}, \rho_1 = \frac{1}{4}, \rho_2 = \frac{1}{12}$, $\rho_3 = 0$. Also,

$$g(\varrho, \upsilon(\varrho), \upsilon^{'}(\varrho)) = \frac{\varrho^4}{5} + \frac{1}{5}(1 + \upsilon(\varrho) - \frac{\upsilon^3(\varrho)}{3} + ...) + \frac{\varrho}{10}(\upsilon^{'}(\varrho) - \frac{\upsilon^{'2}(\varrho)}{2} + ...) + \frac{1}{10}(\upsilon^{''}(\varrho) - \frac{\upsilon^{''3}(\varrho)}{3} + ...)$$

$$\left|g(\varrho, \upsilon(\varrho), \upsilon^{'}(\varrho))\right| \leq \frac{2}{5} + \frac{1}{5}|\upsilon(\varrho)| + \frac{1}{10}\left|\upsilon^{'}(\varrho)\right| + \frac{1}{10}\left|\upsilon^{''}(\varrho)\right|.$$

Here, $\delta_0 = \frac{2}{5},\ \delta_1 = \frac{1}{5},\ \delta_2 = \frac{1}{10},\ \delta_3 = \frac{1}{10}$. Similarly,

$$h(\varrho, \nu(\varrho), \nu^{'}(\varrho)) = \frac{\varrho^2}{7} + \varrho^5\left(1 - \frac{\nu^2(\varrho)}{2!} + ...\right)\nu(\varrho) + \frac{\varrho^8}{7}\left(1 - |\nu^{'}(\varrho)| + \frac{|\nu^{'}(\varrho)|^2}{2!} - ...\right) + \frac{1}{14}\left(1 - 2\nu^{''}(\varrho) + 3\nu^{''2}(\varrho) - ...\right)$$

$$\left|h(\varrho, \nu(\varrho), \nu^{'}(\varrho))\right| \leq \frac{19}{14} + \frac{1}{7}\left|\nu^{'}(\varrho)\right| + \frac{2}{14}\nu^{''}(\varrho).$$

Here, $\tau_0 = \frac{19}{14},\ \tau_1 = 0,\ \tau_2 = \frac{1}{7},\ \tau_3 = \frac{1}{7}$.
We get, $d_1 = \frac{1}{\Gamma(\alpha-n+2)} \approx 1.1033$, $d_2 = \frac{1}{\Gamma(\beta-n+2)} \approx 1.1284$, $d_3 = \frac{1}{\Gamma(\gamma-n+2)} \approx 1.0881$. Also, to compute $\sum_{j=1}^{3}\rho_j = \frac{1}{3}, \sum_{j=1}^{3}\delta_j = \frac{2}{5}, \sum_{j=1}^{3}\tau_j = \frac{2}{7}$

$$\max\left\{2d_1\sum_{j=1}^{n}\rho_j, 2d_2\sum_{j=1}^{n}\delta_j, 2d_3\sum_{j=1}^{n}\tau_j, d_1\sum_{j=1}^{n}\rho_j + d_2\sum_{j=1}^{n}\delta_j, d_2\sum_{j=1}^{n}\delta_j + d_3\sum_{j=1}^{n}\tau_j, d_1\sum_{j=1}^{n}\rho_j + d_3\sum_{j=1}^{n}\tau_j\right\}$$
$$\approx \max\left\{0.7355, 0.9027, 0.6218, 0.8191, 0.7622, 0.6786\right\} < 1.$$

Hence all the conditions of Theorem 3.1 are satisfied. Therefore, BVP's (4.1), (4.2) has atleast one solution.

5. Conclusions

To provide sufficient conditions for the existence of solutions to the fraction order three-dimensional differential system with boundary value problems in order to ensure that the existence of solutions for the BVP's of fractional differential equation of the form (1.1) and (1.4). By using Mawhin's coincidence degree method we proved that the problem has atleast one solution. This paper provides an example to further illustrate the main result.

Conflict of interest

The authors declare that there is no conflicts of interest in this paper.

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