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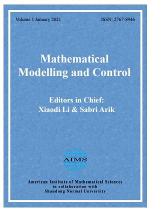

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Research article

On the fractional total domatic numbers of incidence graphs

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Abstract: For a hypergraph H with vertex set X and edge set Y, the incidence graph of hypergraph H is a bipartite graph $I(H) = (X, Y, E)$, where $xy \in E$ if and only if $x \in X$, $y \in Y$ and $x \in y$. A total dominating set of graph G is a vertex subset that intersects every open neighborhood of G. Let $\mathscr{M}$ be a family of (not necessarily distinct) total dominating sets of G and $r_{\mathscr{M}}$ be the maximum times that any vertex of G appears in $\mathscr{M}$. The fractional domatic number G is defined as $FTD(G) = \sup\limits_{\mathscr{M}} \frac{|\mathscr{M}|}{r_{\mathscr{M}}}$. In 2018, Goddard and Henning showed that the incidence graph of every complete k-uniform hypergraph H with order n has $FTD(I(H)) = \frac{n}{n-k+1}$ when $n \geq 2k \geq 4$. We extend the result to the range $n > k \geq 2$. More generally, we prove that every balanced n-partite complete k-uniform hypergraph H has $FTD(I(H)) = \frac{n}{n-k+1}$ when $n \geq k$ and $H \ncong K_n^{(n)}$, where $FTD(I(K_n^{(n)})) = 1$.

Keywords: fractional; total dominating set; hypergraph; incident graph

1. Introduction

Let $H = (V, \mathscr{E})$ be a *hypergraph*, where V is a finite vertex set and $\mathscr{E}$ is a finite edge set such that every edge $e \in \mathscr{E}$ is a subset of V. A vertex subset $T \subseteq V$ is called a *transversal* of H, if each edge of H contains at least one vertex of T. The *transversal number* of H is the minimum size among all transversals in H, and it is denoted by $\tau(H)$. The *disjoint transversal number* of a hypergraph H is the maximum number of disjoint transversals in H, and it is denoted by $disj_\tau(H)$. Goddard and Henning [1] studied the fractional disjoint transversal number. Given a hypergraph H and a family of (not necessarily distinct) transversals $\mathscr{F}$ of H, let $r_{\mathscr{F}}$ be the maximum times that any vertex appears in $\mathscr{F}$. The *fractional disjoint transversal number* is defined as

$$DT_f(H) = \sup_{\mathscr{F}} \frac{|\mathscr{F}|}{r_{\mathscr{F}}}.$$

Goddard and Henning [1] gave some bounds of $DT_f(H)$.

Lemma 1.1. *[1] For every isolate-free hypergraph H of order n,*

$$disj_\tau(H) \leq DT_f(H) \leq \frac{n}{\tau(H)}.$$

The minimum size of all edges of H is called the *anti-rank* of H and denoted by $r(H)$. Goddard and Henning [1] showed a lower bound on $DT_f(H)$ using the anti-rank of a hypergraph H.

Lemma 1.2. *[1] If a hypergraph H has order n and anti-rank k, then*

$$DT_f(H) \geq n/(n-k+1).$$

For $e \in \mathscr{E}(H)$, if edge e has size k, e is called a k-edge. When every edge of H is a k-edge, we call H k-*uniform*. A *complete* k-uniform hypergraph on n vertices, denoted by $K_n^{(k)}$, has all k-subsets of $\{1, \cdots, n\}$ as edges. If D is a minimum transversal of $K_n^{(k)}$, then $\tau(K_n^{(k)}) = |D| \geq n-k+1$ because each k-subset of $V(K_n^{(k)})$ contains at least one vertex in D. By Lemmas 1.1 and 1.2, $DT_f(K_n^{(k)}) = n/(n-k+1)$.

Goddard and Henning [1] discussed the fractional disjoint transversal number of the disjoint union of hypergraphs.

Lemma 1.3. *[1] If H is the disjoint union of isolate-free hypergraphs $H_1, H_2, \cdots, H_k$, then*

$$DT_f(H) = \min\{DT_f(H_1), DT_f(H_2), \cdots, DT_f(H_k)\}.$$

The *degree* of a vertex v in H, denoted by $d(v)$, is the number of edges containing v in H. Let $\delta(H)$ and $\Delta(H)$ be the minimum degree and the maximum degree of hypergraph H, respectively. If every vertex $v \in V(H)$ has degree k, we say that H is a *k-regular* hypergraph. For $k \geq 2$, let $\mathscr{H}_k$ denote the class of all k-regular k-uniform hypergraphs. Henning and Yeo [2] showed that if $H \in \mathscr{H}_3$, then $DT_f(H) \geq 2$. They also proved that for all $k \geq 3$, if $H \in \mathscr{H}_k$, then $DT_f(H) \geq \frac{k}{1+\ln k}$, and this bound is essentially the best possible.

A *polychromatic* (or *panchromatic*) *m-coloring* of hypergraph H is a mapping $f : V(H) \to \{1, 2, \cdots, m\}$ such that all m colors appear on each edge in H (see [3, 4]). In particular, when $m = 2$, it is also called a 2-coloring of H. Obviously, in a polychromatic m-coloring of H, each color class is a transversal of H. So, a hypergraph H has a polychromatic m-coloring if and only if H has m disjoint transversals. Let H be a hypergraph with maximum degree Δ and anti-rank r. The *hall ratio* of hypergraph H is defined as $h(H) = \min\{| \cup \mathscr{J}|/|\mathscr{J}| : \emptyset \neq \mathscr{J} \subseteq \mathscr{E}\}$, where $\cup\mathscr{J} = \cup_{J\in\mathscr{J}} J$. Kostochka and Woodall [4] showed that, if (i) $h(H) > r - 1$ or (ii) $r \geq 3$, $h(H) \geq r - 1$, then $disj_\tau(H) \geq r$. Bollobás et al. [3] proved that, for a family $\mathscr{H}$ of hypergraphs with maximum degree Δ and anti-rank r, each $H \in \mathscr{H}$ has $disj_\tau(H) \geq r/(\ln\Delta + O(\ln\ln\Delta))$; (ii) for all $\Delta \geq 2$, $r \geq 1$, $\min_{H\in\mathscr{H}}\{disj_\tau(H)\} \leq \max\{1, O(r/\ln\Delta)\}$; (iii) for each sequence Δ, $r \to \infty$ with $r = \omega(\ln\Delta)$, $\min_{H\in\mathscr{H}} disj_\tau(H) \leq (1 + o(1))r/\ln\Delta$. Li and Zhang [5] gave a lower bound $disj_\tau(H) \geq \lfloor r/\ln(c\Delta r^2)\rfloor$, where $0 < c = c(\Delta, r) < 1.5582$. Henning and Yeo [6] confirmed that every hypergraph $H \in \mathscr{H}_k$ contains m disjoint transversals for $k \geq 2$ and $2 \leq m \leq \frac{k}{3\ln k}$. Jiang, Yue and Zhang [7] showed that, for a fixed constant q, every hypergraph H with h edges and anti-rank r has a polychromatic m-coloring such that each color appears at least q times on every edge, where $m \geq r(1 - o(1))/(2\ln h)$.

Let G be a graph. A *total dominating set* of G is a vertex set that intersects the neighborhood of every vertex of graph G. The *total domatic number* of graph G is the maximum number of disjoint total dominating sets of G and denoted by $td(G)$. Construct the *open neighborhood hypergraph* $ON(G)$ of G as follows. The vertex set of $ON(G)$ is $V(G)$, and the edges of $ON(G)$ are the open neighborhoods of vertices in G. Clearly, a total dominating set of G is a transversal of $ON(G)$. Therefore, $td(G) = disj_\tau(ON(G))$. That is to say, graph G has m disjoint total dominating sets if and only if $ON(G)$ has m disjoint transversals. So, the disjoint total dominating set partition problem of graphs corresponds to the disjoint transversal partition problem of a special class of hypergraphs.

An m-vertex-coloring of graph G is called a *coupon coloring* (or *thoroughly dispersed coloring*) if every color appears in every open neighborhood of G (see [1, 8]). Obviously, in a coupon coloring of G, a color class is a total dominating set, and thus the maximum number m of colors in coupon colorings of G is equivalent to $td(G)$. Chen et al. [8] proved every k-regular graph G has $td(G) \geq (1 - o(1))k/\ln k$. Henning and Yeo [6] showed every k-regular graph G has $td(G) \geq 2$, provided $k \geq 4$. Goddard and Henning [1] confirmed that every planar graph has $td(G) \leq 4$, and the bound is tight.

Let $\mathscr{M}$ be a family of (not necessarily distinct) total dominating sets of G. Let $r_{\mathscr{M}}$ be the maximum times that any vertex of G appears in $\mathscr{M}$. Let

$$FTD(G) = \sup_{\mathscr{M}} \frac{|\mathscr{M}|}{r_{\mathscr{M}}}.$$

It is called the *fractional total domatic number* of G [1]. Let γ_t be the minimum size among all total dominating sets in G. Goddard and Henning [1] gave lower and upper bounds on the fractional total domatic number of a graph.

Lemma 1.4. *[1] If G is an isolate-free graph of order n, then*

$$td(G) \leq FTD(G) \leq \frac{n}{\gamma_t(G)}.$$

Goddard and Henning [1] established the relation between the fractional total domatic number of graph G and the fractional disjoint transversal number of its open neighborhood hypergraph $ON(G)$.

Lemma 1.5. *[1] For every isolate-free graph G, $FTD(G) = DT_f(ON(G))$.*

Goddard and Henning [1] showed that every claw-free graph G with $\delta(G) \geq 2$ has $FTD(G) \geq 3/2$. Also, they showed that every planar graph has $td(G) \leq 4$ and $FTD(G) \leq 5 - \frac{12}{n}$. Henning and Yeo [2] verified a conjecture in [1] that every connected cubic graph had $FTD(G) \geq 2$. For a k-regular graph G, its open neighborhood hypergraph $ON(G) \in \mathscr{H}_k$. By Lemma 1.5 and the conclusion that $DT_f(H) \geq \frac{k}{1+\ln k}$ for each $H \in \mathscr{H}_k$ due to Henning and Yeo [2], it follows that if G is a k-regular graph, then $FTD(G) \geq \frac{k}{1+\ln k}$, where $k \geq 3$. For more relevant results, see [9].

The *subdivision graph* of graph G, denoted by $S(G)$, is the graph obtained from G by subdividing every edge of G exactly once. So, $V(S(G)) = V(G) \bigcup E(G)$. Goddard and Henning [1] obtained the following result.

Theorem 1.1. *[1] For integer $n \geq 3$, $FTD(S(K_n)) = \frac{n}{n-1}$.*

The *incidence graph* of hypergraph H is a bipartite graph $I(H) = (X, Y, E)$, where $X = V(H)$, $Y = \mathscr{E}(H)$, $xy \in E$ if and only if $x \in X$, $y \in Y$ and $x \in y$. Note that $S(G)$ is exactly the incidence graph of G. Goddard and Henning [1] extended Theorem 1.1 to incidence graphs of complete k-uniform hypergraphs for any integer $k \geq 3$.

Theorem 1.2. *[1] Let $k \geq 3, n \geq 2k$ be two integers. Then,*

$$FTD(I(K_n^{(k)})) = \frac{n}{n-k+1}.$$

In this paper, we extend Theorem 1.2 by removal of the restriction that "$n \geq 2k$". Moreover, we generalize Theorem 1.2 to the incident graphs of h-balanced n-partite complete k-uniform hypergraphs and obtain a similar result. In Section 2, we discuss and determine the fractional disjoint transversal numbers for n-partite complete k-uniform hypergraphs. In Section 3, we determine the fractional total domatic numbers for the incidence graphs of balanced n-partite complete k-uniform hypergraphs.

2. The fractional disjoint transversal numbers of complete uniform hypergraphs

A hypergraph H is called an *n-partite complete k-uniform* hypergraph if H satisfies the following:
(1) H has a vertex set partition $\{V_1, V_2, \cdots, V_n\}$;
(2) $|V_i| = h_i \geq 1$, $1 \leq i \leq n$;
(3) edge set of H consists of all k-tuples, each of which exactly intersects k V_is.
We denote an n-partite complete k-uniform hypergraph by $K^{(k)}_{h_1,h_2,\cdots,h_n}$. The following result shows us that the fractional disjoint transversal number of $K^{(k)}_{h_1,h_2,\cdots,h_n}$ is irrelevant to the indices h_i, $1 \leq i \leq n$.

Theorem 2.1. *For positive integers h_i, n, k, where $n \geq k$ and $1 \leq i \leq n$, $DT_f(K^{(k)}_{h_1,h_2,\cdots,h_n}) = \frac{n}{n-k+1}$.*

Proof. Define $H = K^{(k)}_{h_1,h_2,\cdots,h_n}$. Let the vertex set partition of H be $V(H) = \{V_1, V_2, \cdots, V_n\}$. For any transversal F of H, $V(H) \setminus F$ intersects at most $k-1$ V_is. That is to say, F contains at least $n-k+1$ distinct V_is. Furthermore, for an arbitrary transversal family $\mathscr{F}$ of H, by the Pigeonhole Principle, we have that $r_{\mathscr{F}} \geq \frac{|\mathscr{F}|(n-k+1)}{n}$, implying that $\frac{|\mathscr{F}|}{r_{\mathscr{F}}} \leq \frac{n}{n-k+1}$. So, $DT_f(H) \leq \frac{n}{n-k+1}$. Consider a transversal family $\mathscr{F}$, which consists of all distinct unions of any $n-k+1$ different $V_i s$. Then, $|\mathscr{F}|=\binom{n}{n-k+1}$. Since every vertex of H is in $\binom{n-1}{n-k}$ transversals of $\mathscr{F}$, we have $\frac{|\mathscr{F}|}{r_{\mathscr{F}}}=\binom{n}{n-k+1}/\binom{n-1}{n-k} = \frac{n}{n-k+1}$. Consequently, $DT_f(H) = \frac{n}{n-k+1}$. □

When $h_i = h$ for each $1 \leq i \leq n$, $K^{(k)}_{h_1,h_2,\cdots,h_n}$ is called *h-balanced*, simply denoted by $K^{(k)}_{h\times n}$.

Corollary 2.1. *For positive integers n, k, h with $n \geq k$, $DT_f(K^{(k)}_{h\times n}) = \frac{n}{n-k+1}$.*

3. The fractional total domatic numbers of the incidence graphs of balanced complete uniform hypergraphs

Let $H = (V, \mathscr{E})$ be a hypergraph with $V = \{x_1, x_2, \cdots, x_n\}$ and $\mathscr{E} = (E_1, E_2, \cdots, E_s)$. The maximum size of all edges of H is called the *rank* of H and denoted by $R(H)$. An edge subset $S \subseteq \mathscr{E}$ is called a *cover* of H if S contains all vertices in $V(H)$. If the edge set of H can be partitioned into m disjoint covers, then H has a cover m-decomposition. The maximum number m is denoted by $cd(G)$. The *dual* of H is denoted by H^*, whose vertices $e_1, e_2, \cdots, e_s$ correspond to the edges of H and whose edges are $X_i = \{e_j | x_i \in E_j, 1 \leq j \leq s\}$, $i = 1, 2, \cdots, n$. Thus, $\delta(H) = r(H^*)$ and $\Delta(H) = R(H^*)$. Clearly, an m-transversal-partition of H corresponds to a cover m-decomposition of H^*, and $disj_\tau(H) = cd(H^*)$.

Goddard and Henning [1] referred to the following relationship between H and the open neighborhood hypergraph of $I(H)$. For completeness, we give a proof.

Lemma 3.1. *Let H be a hypergraph. Then, $ON(I(H))$ consists of two components, H and its dual H^*.*

Proof. Let $V(H)=\{u_1, u_2, \cdots, u_n\}$. By the definition of incidence graph, the vertex set of $I(H)$ is $V(H) \cup \mathscr{E}(H)$. Then, the $ON(I(H))$ consists of two components: H_1, which has vertex set $V(H)$, and H_2, which has vertex set $\mathscr{E}(H)$.

H_1 is formed by the open neighborhoods of the elements in $\mathscr{E}(H)$. Note that in $I(H)$, for each $e = \{u_1, u_2, \cdots, u_s\} \in \mathscr{E}(H)$, the open neighborhood of e is itself, i.e., $\{u_1, u_2, \cdots, u_s\}$. So, $H_1 \cong H$.

In $I(H)$, the open neighborhood of an element $u_i \in V(H)$ is the set of incident edges of u_i, $i = 1, 2, \cdots, n$, which forms an edge of H_2. For each $u_i \in V(H)$, we denote its corresponding edge in H_2 by U_i. For an element $e \in V(H_2) = \mathscr{E}(H)$ and an element $U_i \in E(H_2)$, e is incident with U_i in H_2 if and only if e is incident with u_i in H. This means that $H_2 \cong H^* \cong H_1^*$. □

Next, we improve Theorem 1.2 by removal of the restriction that "$n \geq 2k$". Let $P = (w_1, w_2, \cdots, w_p)$ be a permutation. We call $(w_j, w_{j+1}, \cdots, w_p, w_1, \cdots, w_{j-1})$ the *jth rotation* of P, $1 \leq j \leq p$. Clearly, the permutation P has p different rotations.

Theorem 3.1. *Let k, n be two integers and $n > k \geq 2$. Then,*

$$FTD(I(K_n^{(k)})) = \frac{n}{n-k+1}.$$

Proof. We define $H = K_n^{(k)}$. By Lemmas 1.3, 1.5 and 3.1, $FTD(I(H)) = DT_f(ON(I(H))) = \min\{DT_f(H), DT_f(H^*)\} \leq DT_f(H)$. Also, by Corollary 2.1, there is $FTD(I(H)) \leq \frac{n}{n-k+1}$.

Next, we prove that the bound is also a lower bound of $FTD(I(H))$ by constructing a special family of total dominating sets.

We know that $I(H)$ is a bipartite graph with $V(I(H)) = V(H) \cup E(H)$. Clearly, $|E(H)| = \binom{n}{k}$. Let $V(H) = \{v_1, v_2, \cdots, v_n\}$. Assume that $n = ak + b$, where $a \geq 1$, $0 \leq b < k$. Given a permutation $(v_1, v_2, \cdots, v_n)$, we pick a subset $F_1 \subset V(I(H))$ following three rules, as below:

(R1) $A_1 = \{v_1, v_2, \cdots, v_{n-k}, v_n\}$;
(R2) $B_1 = \{e_0^1, e_1^1, e_2^1, \cdots, e_a^1\}$,
where $e_0^1 = \{v_{n-k+1}, \cdots, v_n\}$ and
$e_j^1 = \{v_{kj-k+1}, \cdots, v_{kj}\}$, $1 \leq j \leq a$;
(R3) $F_1 = A_1 \cup B_1$.

Note that $e_0^1 = e_a^1$ if and only if $b = 0$. Since each element of $E(H)$ has k neighbors, and there are exactly $k - 1$ vertices of $V(H)$ not in F_1, each element of $E(H)$ is dominated by F_1 in $I(H)$. Also, by $\cup_{j=0}^{a} e_j^1 = V(H)$, each element of $V(H)$ is dominated by F_1. That is to say, F_1 is a total dominating set of $I(H)$.

Let $(u_1^i, u_2^i, \cdots, u_n^i) = (v_i, v_{i+1}, \cdots, v_n, v_1, \cdots, v_{i-1})$ denote the ith rotation of the sequence $(v_1, v_2, \cdots, v_n)$. Analogously, based on the sequence $(u_1^i, u_2^i, \cdots, u_n^i)$, we can obtain a total dominating set $F_i = A_i \cup B_i$ of $I(H)$ for each $1 \leq i \leq n$. It is easy to see that each element of $V(H)$ appears exactly $n - k + 1$ times in the family of total dominating sets $\mathscr{F} = \{F_1, F_2, \cdots, F_n\}$. When $b = 0$, for each $1 \leq i \leq n$, $e_1^i, e_2^i, \cdots, e_a^i$ are distinct, and each of them appears exactly a times in $\mathscr{F}$, i.e., $B_h = B_{k+h} = \cdots = B_{(a-1)k+h}$ for each $1 \leq h \leq k$. When $b > 0$, for each $1 \leq i \leq n$, $e_0^i, e_1^i, e_2^i, \cdots, e_a^i$ are distinct (See Figure 1). Note that, for any $1 \leq j \leq n$, the sequential k-tuple $\{v_j, v_{j+1}, \cdots, v_{j+k-1}\} = e_0^{k+j}$ (both the subscripts and $k + j$ are mod n.) This means that $\cup_{i=1}^{n}\{e_0^i, e_1^i, e_2^i, \cdots, e_a^i\} = \cup_{i=1}^{n}\{e_0^i\}$. It is easy to see that $e_0^1, e_0^2, \cdots, e_0^n$ are distinct. Next, we show e_0^i appears exactly $a + 1$ times for each $1 \leq i \leq n$.

- When $1 \leq i \leq k$, $e_0^i = e_a^{i+b} = e_{a-1}^{i+b+k} = \cdots = e_1^{i+b+(a-1)k}$;
- when $tk + 1 \leq i \leq tk + k$ and $1 \leq t \leq a - 1$, $e_t^{i-tk} = \cdots = e_1^{i-k} = e_0^i = e_a^{i+b} = e_{a-1}^{i+b+k} = \cdots = e_{t+1}^{i+b+(a-t-1)k}$;
- when $ak + 1 \leq i \leq n$, $e_0^i = e_a^{i-ak} = \cdots = e_2^{i-2k} = e_1^{i-k}$.

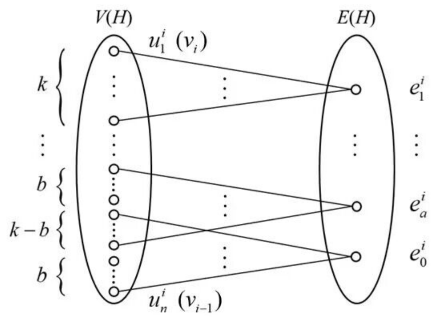

Figure 1. When $b > 0$, a demonstration for $B_i = \{e_1^i, \cdots, e_a^i, e_0^i\}$, $1 \leq i \leq n$.

We give a claim as follows.

Claim. $n - k + 1 \geq a + 1$.

Recall that $n > k \geq 2$. If $a = 1$, then $b \geq 1$. So, $n-k+1 = k+b-k+1 \geq 2 = a+1$. If $a \geq 2$, then $n-k+1 \geq ak-k+1 = a + a(k-1) - k + 1 = a + (a-1)(k-1) \geq a + 1$.

In short, $|\mathscr{F}| = n$, and $r_{\mathscr{F}} = n - k + 1$. By the definition of the fractional total domatic number, $FTD(I(H)) \geq \frac{n}{n-k+1}$.

□

Remark. According to the definition of $K_n^{(k)}$, the condition that $n \geq k$ is fundamental. The case that $n = k$ will be discussed in Theorem 3.2, and the case that $n > k = 1$ is contained in Theorem 3.3. In either of the cases, $FTD(I(K_n^{(k)})) = 1$.

The following Theorems 3.2, 3.3 determine the fractional total domatic number of $I(K_{h\times n}^{(k)})$ and generalize Theorem 1.2. Before that, we give a relation between a hypergraph and its sub-hypergraph on the fractional disjoint transversal number. For two hypergraphs H and $\widehat{H}$, we call $\widehat{H}$ a *sub-hypergraph* of H if $V(\widehat{H}) \subseteq V(H)$ and $\mathscr{E}(\widehat{H}) \subseteq \mathscr{E}(H)$.

Lemma 3.2. *Let $\widehat{H}$ be a sub-hypergraph of H. Then, $DT_f(\widehat{H}) \geq DT_f(H)$.*

Proof. Let $DT_f(H) = r$. By the definition of the fractional disjoint transversal number, there exists a transversal family $\mathscr{F}$ of H such that $\frac{|\mathscr{F}|}{r_{\mathscr{F}}} = r$. By $V(\widehat{H}) \subseteq V(H)$ and $\mathscr{E}(\widehat{H}) \subseteq \mathscr{E}(H)$, we can obtain a corresponding transversal family $\widehat{\mathscr{F}}$ of $\widehat{H}$ by removal of the vertices in $V(H) \setminus V(\widehat{H})$ from $\mathscr{F}$. Obviously, $|\widehat{\mathscr{F}}| = |\mathscr{F}|$, and $r_{\widehat{\mathscr{F}}} \leq r_{\mathscr{F}}$. So, $\frac{|\widehat{\mathscr{F}}|}{r_{\widehat{\mathscr{F}}}} \geq \frac{|\mathscr{F}|}{r_{\mathscr{F}}} = r$. Then, we have $DT_f(\widehat{H}) \geq \frac{|\widehat{\mathscr{F}}|}{r_{\widehat{\mathscr{F}}}} \geq r$. □

A *matching* in a hypergraph is a set of non-intersecting edges. The matching number $\mu(H)$ is the size of a largest matching of hypergraph H.

Theorem 3.2. *Let n, h be two positive integers. Then,*

$$FTD(I(K_{h\times n}^{(n)})) = \begin{cases} n, & h \geq 2; \\ 1, & h = 1. \end{cases}$$

Proof. When $n = 1$, $|V(I(K_{h\times 1}^{(1)}))| = 2h$. Clearly, $td(I(K_{h\times 1}^{(1)})) = 1$, and $\gamma_t(I(K_{h\times 1}^{(1)})) = 2h$. By Lemma 1.4, we have $FTD(I(K_{h\times 1}^{(1)})) = 1$. Next, we assume that $n \geq 2$.

When $h = 1$, $K_{1\times n}^{(n)} \cong K_n^{(n)}$ has only one edge, which contains n vertices. Clearly, $I(K_n^{(n)})$ is a star, and $|V(I(K_n^{(n)}))| = n + 1$. The edge e in $K_n^{(n)}$ corresponds to a vertex e in $I(K_n^{(n)})$. Every total dominating set must contain vertex e in $I(K_n^{(n)})$. Thus, for every family $\mathscr{M}$ of total dominating sets of $I(K_n^{(n)})$, vertex e appears $|\mathscr{M}|$ times in $\mathscr{M}$. So, $r_{\mathscr{M}} = |\mathscr{M}|$. By the definition of the fractional total domatic number, $FTD(I(K_n^{(n)})) = 1$.

Next, we focus on the cases that $n \geq 2$ and $h \geq 2$. Define $H = K_{h\times n}^{(n)}$. By Lemma 3.1, $ON(I(H))$ consists of two components H and H^*. Then, by Corollary 2.1, $DT_f(H) = n$. In the following, we will show that $DT_f(H^*) \geq n$. By Lemma 1.3, $DT_f(ON(I(H))) = \min\{DT_f(H), DT_f(H^*)\} = n$. Also, by Lemma 1.5, $FTD(I(H)) = DT_f(ON(I(H))) = n$.

Now, we discuss $DT_f(H^*)$. First, $|V(H)| = nh$, and $|\mathscr{E}(H)| = h^n$. Since each edge contributes n to the degree sum of H, $\sum_{v\in V(H)} d(v) = h^n n$. By H regularity, we have $d_H(v) = \frac{h^n n}{nh} = h^{n-1}$ for each $v \in V(H)$. Then, we know that H^* is h^{n-1}-uniform and has h^n vertices.

Here, we establish a claim.

Claim. $\mathscr{E}(H)$ can be partitioned into h^{n-1} disjoint maximum matchings.

Array all vertices of H into a matrix with h rows and n columns as follows, in such a way that the jth column consists of the jth partite vertices of $V(H)$, and denote the matrix by $A_{1,1,\cdots,1}$.

$$A_{1,1,\cdots,1} = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ v_{h1} & v_{h2} & \cdots & v_{hn} \end{bmatrix}$$

Every row of $A_{1,1,\cdots,1}$ corresponds to an edge in H. The set of all rows of $A_{1,1,\cdots,1}$ corresponds to a matching of H. Clearly, it is a maximum matching of H, and $\mu(H) = h$. We use $A_{1,i_2,i_3,\cdots,i_n}$ to denote the matrix obtained from $A_{1,1,\cdots,1}$ by replacing the jth column $(v_{1j}, v_{2j}, \cdots, v_{hj})^T$ with its i_jth rotation, where $2 \leq j \leq n$, $1 \leq i_j \leq h$. Then, we can obtain h^{n-1} distinct matrices, which correspond to h^{n-1} maximum matchings. We show that arbitrary two of such matchings are disjoint. Pick arbitrarily two distinct matrices $A = A_{1,i_2,\cdots,i_n}$ and $B = A_{1,i'_2,\cdots,i'_n}$. Without loss of generality, assume $i_j \neq i'_j$. For the sth row of A and the tth row of B, when $s \neq t$, the first elements are different; when $s = t$, the jth elements are different. That is to say, the corresponding matchings of A, B are disjoint. Recalling that $|\mathscr{E}(H)| = h^n$, we partition $\mathscr{E}(H)$ into h^{n-1} disjoint maximum matchings. The claim is proved.

Note that every maximum matching in H corresponds to a part of $V(H^*)$. Thus, H^* has h^{n-1} parts, each of which contains h vertices. It is easy to see that H^* is h^{n-1}-partite h^{n-1}-uniform and non-complete. We can extend H^* into a $K_{h\times h^{n-1}}^{(h^{n-1})}$ by adding the missing h^{n-1}-edges. By Corollary 2.1, we have $DT_f(K_{h\times h^{n-1}}^{(h^{n-1})}) = h^{n-1}$. By Lemma 3.2, $DT_f(H^*) \geq$

$DT_f(K^{(h^{n-1})}_{h\times h^{n-1}}) = h^{n-1}$. Noting that $h^{n-1} \geq n$ when $h \geq 2$, we have $DT_f(H^*) \geq n$ and then complete the proof. □

Theorem 3.3. *Let n, k, h be three positive integers and n > k. Then,* $FTD(I(K^{(k)}_{h\times n})) = \frac{n}{n-k+1}$.

Proof. Define $H = K^{(k)}_{h\times n}$. When $k = 1$, $|V(I(H))| = 2nh$. Clearly, $td(I(H)) = 1$ and $\gamma_t(I(H)) = 2nh$. By Lemma 1.4, we have $FTD(I(H)) = 1$. When $h = 1$, it is done by Theorem 3.1. In the following, we assume that $h \geq 2, k \geq 2$.

By Lemma 3.1, $ON(I(H))$ consists of two components, H and H^*. By Corollary 2.1, $DT_f(H) = \frac{n}{n-k+1}$. If $DT_f(H^*) \geq \frac{n}{n-k+1}$, then $FTD(I(K^{(k)}_{h\times n})) = DT_f(ON(I(H))) = \min\{DT_f(H), DT_f(H^*)\} = \frac{n}{n-k+1}$. By $FTD(I(K^{(k)}_n)) = \frac{n}{n-k+1}$, we know that $DT_f((K^{(k)}_n)^*) \geq \frac{n}{n-k+1}$. Next, we show that $DT_f(H^*) \geq \frac{n}{n-k+1}$ for $h \geq 2$ and $k \geq 2$. Before that, we need some properties of $K^{(k)}_n$ and its dual.

Let $V(K^{(k)}_n) = \{v_1, v_2, \cdots, v_n\}$ and $E(K^{(k)}_n) = \{E : E \subset V(K^{(k)}_n), |E| = k\} = \{E_i : 1 \leq i \leq \binom{n}{k}\}$. By the duality, we know that $V((K^{(k)}_n)^*) = \{e_i : 1 \leq i \leq \binom{n}{k}\}$, $E((K^{(k)}_n)^*) = \{V_1, V_2, \cdots, V_n\}$, and e_i is incident with V_j if and only if E_i is incident with v_j, i.e., $e_i \in V_j$ if and only if $v_j \in E_i$ for all $1 \leq i \leq \binom{n}{k}$, $1 \leq j \leq n$. Pick a family of (not necessarily distinct) transversals $\mathscr{F}$ of $(K^{(k)}_n)^*$ such that $DT_f((K^{(k)}_n)^*) = \frac{|\mathscr{F}|}{r_{\mathscr{F}}}$. According to the definition of the dual of a hypergraph, we can establish the following observation.

Observation. Let $F = \{e_1, e_2, \cdots, e_s\} \subseteq V((K^{(k)}_n)^*)$ and $f = \{E_1, E_2, \cdots, E_s\} \subseteq E(K^{(k)}_n)$. Then, the following statements are equivalent.

(1) F is a transversal of $(K^{(k)}_n)^*$;
(2) $F \cap V_j \neq \emptyset$ for each element $V_j \in E((K^{(k)}_n)^*)$;
(3) for each element $V_j \in E((K^{(k)}_n)^*)$, there exists an element $e_{i_j} \in F$ such that $e_{i_j} \in V_j$;
(4) for each element $v_j \in V(K^{(k)}_n)$, there exists an element $E_{i_j} \in f$ such that $v_j \in E_{i_j}$;
(5) $\cup_{i=1}^{s} E_i = V(K^{(k)}_n) = \{v_1, v_2, \cdots, v_n\}$;
(6) f is a cover of $K^{(k)}_n$.

Assume that the n parts of $K^{(k)}_{h\times n}$ are $X_1, X_2, \cdots, X_n$, where $X_p = \{v^1_p, v^2_p, \cdots, v^h_p\}$ for each $1 \leq p \leq n$. We can partition $V(K^{(k)}_{h\times n})$ into h subsets $Y_1, Y_2, \cdots, Y_h$ such that $Y_q = \{v^q_1, v^q_2, \cdots, v^q_n\}$ for each $1 \leq q \leq h$. We next give a family of transversals of $(K^{(k)}_{h\times n})^*$ based on $\mathscr{F}$. For each $F \in \mathscr{F}$, we construct a corresponding transversal $F(h)$ of $(K^{(k)}_{h\times n})^*$ as follows. Without loss of generality, assume that $F = \{e_1, e_2, \cdots, e_s\}$. Then, $f = \{E_1, E_2, \cdots, E_s\}$ is a cover of $K^{(k)}_n$, i.e., $\cup_{i=1}^{s} E_i = \{v_1, v_2, \cdots, v_n\}$. Noting that $E_i \in E(K^{(k)}_n)$, we may assume that $E_i = \{v_{i_1}, \cdots, v_{i_k}\}$ for each $1 \leq i \leq s$. Set $f^q = \{E^q_1, E^q_2, \cdots, E^q_s\}$, where $E^q_i = \{v^q_{i_1}, \cdots, v^q_{i_k}\}$, $1 \leq i \leq s$, $1 \leq q \leq h$. Then, $\cup_{i=1}^{s} E^q_i = \{v^q_1, v^q_2, \cdots, v^q_n\} = Y_q$. Define

$$f(h) = \cup_{q=1}^{h} f^q = \{E^q_i : 1 \leq i \leq s, 1 \leq q \leq h\}.$$

It follows that $f(h)$ is a cover of $K^{(k)}_{h\times n}$ because $\cup_{E\in f(h)} E = \cup_{q=1}^{h} \cup_{i=1}^{s} E^q_i = \cup_{q=1}^{h} Y_q = V(K^{(k)}_{h\times n})$. By the duality, we know that

$$F(h) = \{e^q_i : 1 \leq i \leq s, 1 \leq q \leq h\}$$

is a transversal of $(K^{(k)}_{h\times n})^*$. Let $\mathscr{F} = \{F_1, F_2, \cdots, F_{|\mathscr{F}|}\}$. Then, $\mathscr{F}(h) = \{F_1(h), F_2(h), \cdots, F_{|\mathscr{F}|}(h)\}$ is a family of transversals of $(K^{(k)}_{h\times n})^*$. Obviously, $r_{\mathscr{F}(h)} = r_{\mathscr{F}}$. Hence, we have

$$DT_f((K^{(k)}_{h\times n})^*) \geq \frac{|\mathscr{F}(h)|}{r_{\mathscr{F}(h)}} = \frac{|\mathscr{F}|}{r_{\mathscr{F}}} \geq \frac{n}{n-k+1}$$

for $h \geq 2$ and $k \geq 2$.

□

4. Concluding remarks

By Theorems 3.2 and 3.3, we have completely determined the fractional total domatic number on the incident graph of $K^{(k)}_{h\times n}$ for all positive integers n, k, h.

Theorem 4.1. *Let n, k, h be positive integers, $n \geq k$. Then,*

$$FTD(I(K^{(k)}_{h\times n})) = \begin{cases} 1, & n = k \text{ and } h = 1; \\ \frac{n}{n-k+1}, & \text{otherwise}. \end{cases}$$

When $k = 2$, we simply denote $K^{(k)}_{h\times n}$ by $K_{h\times n}$. Recall that the incidence graph of a graph G is exactly the subdivision graph $S(G)$. Then, we have the following result, which extends Theorem 1.1.

Theorem 4.2. *For integers $n \geq 3$, $h \geq 1$,*

$$FTD(S(K_{h\times n})) = \frac{n}{n-1}.$$

As discussed in Lemma 3.1, for an arbitrary hypergraph H, the open neighborhood hypergraph $ON(I(H))$ of its incident graph $I(H)$ consists of two components: H and its dual hypergraph H^*. By Lemmas 1.3 and 1.5, there is $FTD(I(H)) = DT_f(ON(I(H))) = \min\{DT_f(H), DT_f(H^*)\} \leq DT_f(H)$. In this paper, we have proved that $FTD(I(H)) = DT_f(H)$ when H is an h-balanced n-partite complete k-uniform hypergraph for any positive integers h, n, k ($n \geq k$). It is interesting to determine the class of hypergraphs H with $FTD(I(H)) = DT_f(H)$.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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