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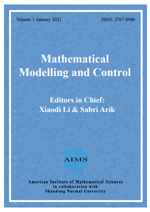


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***Research article***

# Exploring complicated behaviors of a delay differential equation

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**Abstract:** Complicated behaviors of a delay differential equation are explored through the Euler discretization method. It rigorously shows that the corresponding discrete equation can be chaotic under some conditions, which reflects that there exist complicated behaviors in the original delay differential equation.

**Keywords:** chaos; delay; snap-back repeller

## 1. Introduction

As we know, there are very complex dynamical behaviors in nonlinear differential equations [1–5]. In particular, when it concerns a delay or some delays in those nonlinear differential equations, there will appear extremely complicated dynamical behaviors. As we can see from the monograph [1], even through the study of stability of nonlinear systems with a single delay, it is very difficult to obtain the stability criterion. The delay usually represents a feedback in a dynamical system, which gives great influence to the system. This will effect the original dynamical systems to lose their stability and occur very complicated dynamical behaviors, that is, to appear unstable dynamical behaviors, for example, chaos, bifurcation and oscillation, etc.

However, it is very difficult to directly solve a nonlinear differential equation and use the precise solutions to study those complicated behaviors as mentioned above, especially, when there exist one delay or some delays. Then, there appear a lot of numerical analysis methods to obtain the approximate solutions and use them to study some properties of the original nonlinear systems. There are more than four frequently used methods, for example, the methods of boundary element, finite difference, finite element, and collocation, we refer to the monograph [3] for more details. Among these usually used methods, the barycentric interpolation collocation method becomes a very useful and active method in studying a variety of nonlinear problems. This method exhibits enormous potential to solve nonlinear dynamical systems, which can obtain the approximate solutions and make them to have excellent error precision. For instance, the paper [4] used this method to obtain the approximate solution of one kind of differential equations and showed the errors can be very small with high accuracy. The paper [5] used this method to study hyperchaotic problems in several systems and find some attractors by numerical simulations, which shows that this method is also very useful and powerful in chaos study.

Among those unstable behaviors, chaos is a particular dynamical behavior which seems a random-like behavior in nonlinear deterministic dynamical systems without stochastic terms. The most important characteristic of a chaotic system is that its evolution has highly sensitive dependence on initial conditions. So, the future behaviors of a chaotic system are unpredictable from long-term perspective. With the development of numerical computation methods and computer, many results have been published to study chaos by using the difference formats. The essence of numerical methods mentioned above is to obtain the discrete formats from the space variables or time variable in the corresponding nonlinear differential equations. When it is concerned with a time variable, the difference method is usually employed. For instance, many scholars had used the simplest Euler discretization method, including the forward difference or backward difference, to exhibit complicated behaviors of nonlinear dynamical systems and rigorously prove the existence of chaos in such nonlinear systems, one can see [6, 7] and references therein. In recent years, some scholars had researched some special cases of the system below

$$\dot{v}(y) = -av(y) + rv(y-\tau)q(v(y-\tau)), \tag{1.1}$$

in which $\tau$ represents the delay, $a > 0$ is a constant, $r$ is an undetermined real number, and $q(\cdot)$ is a map. Equation (1.1) has the form of the well-known Mackey-Glass equation, which has been used in many practical applications, such as population models [8] and economics [9]. Many scholars had studied different kinds of the Mackey-Glass equation. For example, when $q(v) = 1 - v$, equation (1.1) becomes a very simple delay differential equation, which is the well-known delay logistic equation studied by [10–12]. Adhikari et al. studied the periodic solutions of (1.1) in [10]. Jiang et al. studied the Hopf bifurcation of (1.1) and found the chaotic phenomena by computer simulations in [11] and [12]. When $q(v) = 1 - v^2$, Kaplan and Yorke [13], and Dormayer [14], studied the stability and the periodic solutions of (1.1). However, there are rarely rigorously mathematical results for proving chaos in the differential or discrete forms of them. Recently, we succeeded in proving chaos in the discretization of equation (1.1) when $q(v) = 1 - v^2$ in [7]. This inspires us to study chaos for the discretization of (1.1) for more general forms of $q(v)$ . That is, the delay difference equation below

$$v(m+1) = \delta v(m) + \mu v(m-n)q(v(m-n)),\ m \geq 0, \tag{1.2}$$

where $\delta, \mu$ are real numbers, $n$ is a positive integer, and $q(\cdot)$ is a map.

In the rest of the paper, we first give some preliminaries in Section 2 which will be employed in the following section. Next, we will show our main result about chaos in Section 3 and give two computer simulations to show the validity of our theorem. In the end, we give a conclusion in Section 4.

## 2. Preliminaries

Shi and Chen [15] established some chaos criteria on Banach spaces in 2004. As a special case, that is, when the Banach space becomes the Euclidean space $\mathbf{R}^n$, they gave a criterion of chaos for maps on $\mathbf{R}^n$, which can be viewed a modifying of the famous Marotto's Theorem, we refer to [15] and the references therein. One year later, Marotto [16] also published a short paper on redefining the snap-back repeller and gave a small correction on his former famous result which was first proposed by him in 1978. Actually, the results on chaos in $\mathbf{R}^n$ in the two papers are consistent. Here, we give a brief statement of Theorem 4.4 in [15], which is slightly modified for convenience of application in this paper, for more details on the reason of this modification, we refer to Theorem 4.4 of [15] and Lemma 2.4 of [17].

**Lemma 2.1.** *Assume that a map $G : \mathbf{R}^m \to \mathbf{R}^m$ is continuously differentiable on $\mathbf{R}^m$ which satisfies $G(w^*) = w^*$, $w^* \in \mathbf{R}^m$, and the following conditions*

(I) *any eigenvalue $\lambda$ of $DG(w^*)$ satisfies $|\lambda| > 1$;*
(II) *there are a point $y_0 \in \mathbf{R}^m$ in some neighborhood of $w^*$ with $y_0 \neq w^*$, and an integer $n > 0$, such that $G^n(y_0) = w^*$, and $\det DG(y_i) \neq 0$ for $0 \leq i \leq n-1$, where $y_i = G(y_{i-1})$ for $1 \leq i \leq n-1$.*

*Then G has Devaney chaos and Li-Yorke chaos.*

## 3. Main result

Set

$$w_i(m) := v(m+i-n-1),\ 1 \leq i \leq n+1,\ m \geq 0.$$

Thus, system (1.2) becomes the following discrete system on $\mathbf{R}^{n+1}$

$$w(m+1) = G(w(m)), \tag{3.1}$$

where $G(w) = (w_2, \cdots, w_{n+1}, \delta w_{n+1} + \mu w_1 q(w_1))^T \in \mathbf{R}^{n+1}$, $w = (w_1, w_2, \cdots, w_{n+1})^T \in \mathbf{R}^{n+1}$.

Since equation (1.2) is transformed into equation (3.1), we say that equation (1.2) is chaotic when equation (3.1) is chaotic. The definition of chaos on the equivalent transformation systems is inspired by Definitions 5.1 and 5.2 in [18].

Now, a main theorem about chaos corresponding to system (3.1) is established.

**Theorem 3.1.** *Assume that* $q : I \subset \mathbf{R} \to \mathbf{R}$ *is a continuously differentiable map with*

$$q(w^*) = 0,\ q'(w^*) \neq 0,\ q(0) \neq 0, \tag{3.2}$$

*where* $w^*(w^* \neq 0)$ *and* $0$ *are two interior points of the interval* $I$*, then we can obtain a positive number* $\mu_0$ *to make system (3.1), that is, system (1.2), be chaotic when* $|\mu| > \mu_0$.

*Proof.* Obviously, the point $P := (0, \cdots, 0)^T \in I^{n+1}$ is always a fixed point of system (3.1) and any other possible fixed point is $Q := (v_0, \cdots, v_0)^T \in I^{n+1}$, which satisfies the following equation

$$\mu q(v_0) = 1 - \delta.$$

Inspired from the idea in [7], Lemma 2.1 is employed to show this theorem. For convenience, we will show that the map $G$ with the fixed point $P$ satisfies the two conditions of Lemma 2.1.

First, we prove that the map $G$ meets the condition (I). From assumption of Theorem 3.1 and equation (3.1), we gain $G$ is continuously differentiable in $\mathbf{R}^{n+1}$ with

$$DG(P) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ .. & .. & .. & \cdots & .. \\ 0 & 0 & 0 & \cdots & 1 \\ \mu q(0) & 0 & 0 & \cdots & \delta \end{pmatrix}_{(n+1)\times(n+1).}$$

Its characteristic equation is

$$\lambda^{n+1} - \delta\lambda^n - \mu q(0) = 0. \tag{3.3}$$

Set $\mu_1 := \frac{1+|\delta|}{|q(0)|}$. It follows from (3.3) that each eigenvalue $\lambda$ of $DG(P)$ satisfies $|\lambda| > 1$ for $|\mu| > \mu_1$. In fact, if there is an eigenvalue $\lambda_0$ of $DG(P)$ satisfying $|\lambda_0| \leq 1$, then there is a contradiction

$$1 + |\delta| \geq |\lambda_0^{n+1}| + |\delta\lambda_0^n| \geq |\lambda_0^{n+1} - \delta\lambda_0^n| = |\mu q(0)| > 1 + |\delta|.$$

So, condition (I) of Lemma 2.1 holds. Sequentially, we gain a number $r > 0$ and some norm in $\mathbf{R}^{n+1}$ to make $G$ expand in $\bar{B}_r(P)$ in this norm, where $\bar{B}_r(P) \subset I^{n+1}$ is a closed ball of radius $r$ centered at $P$, one can see the detailed description in Theorem 4.4 of [15].

Next, we will show the condition (II) of Lemma 2.1 is also met. Let $U \subset B_r(P)$ be any neighborhood of $P$ in $\mathbf{R}^{n+1}$. We can take a small interval $V \subset I$ which contains 0 and satisfies $\underbrace{V \times V \times \cdots \times V}_{n+1} \subset U$. The rest of the proof is divided into two steps.

**Step 1.** It is to show that we can gain a point $P_0 \in U$ with $P_0 \neq P$, and

$$G^{n+2}(P_0) = P.$$

Set $p(v) := vq(v)$. Then, the function $p(v)$ is continuously differentiable. It follows from (3.2) that $p(v)$ satisfies

$$p(0) = 0,\ \ p'(0) = q(0) \neq 0, \tag{3.4}$$

which implies that $p(v)$ is rigorously monotonous near $0 \in I$.

The case $n = 1$. We can take $-\frac{\delta}{|\mu|}w^*$ to lie in a small neighborhood of 0 for sufficiently large $|\mu|$. Then, by the monotonicity of $p(v)$ near 0, the equation $p(v) = -\frac{\delta}{|\mu|}w^*$, that is,

$$\mu v q(v) = -\delta w^* \tag{3.5}$$

has a solution $v_2 \in V$ for sufficiently large $|\mu|$. Similarly, the following equation

$$\mu v q(v) = w^* - \delta v_2 \tag{3.6}$$

also has a solution $v_1 \in V$ for sufficiently large $|\mu|$. Therefore, we can take a common positive constant $\mu_2'$ such that $v_1, v_2 \in V$ and satisfy equations (3.5) and (3.6) for each $\mu$ with $|\mu| > \mu_2'$. Take $P_0 = (v_1, v_2)^T$. Then, it is easy to obtain that the point $P_0 \in V \times V \subset U$, $P_0 \neq P$, $G(P_0) = (v_2, w^*)^T$, $G^2(P_0) = (w^*, 0)^T$, and $G^3(P_0) = P$.

The case $n > 1$. With a similar discussion as above, we can get a number $\mu_2^* > 0$, for each $|\mu| > \mu_2^*$, there exist two points $v_1, v_2 \in V$ satisfying the equations below

$$\begin{cases} \mu v_1 q(v_1) = w^*, \\ \mu v_2 q(v_2) = -\delta w^*. \end{cases} \tag{3.7}$$

Take $P_0 = (v_1, v_2, 0, \cdots, 0)^T$. Then, we gain $P_0 \in \underbrace{V \times V \times \cdots \times V}_{n+1} \subset U$, $P_0 \neq P$, $G(P_0) = (v_2, 0, \cdots, 0, w^*)^T$,

$G^i(P_0) = (0, \cdots, 0, \underbrace{w^*, 0, \cdots, 0}_{i})^T$ for $2 \le i \le n+1$, and $G^{n+2}(P_0) = P$.

In a summary, setting $\mu_0 := \max\{\mu_1, \mu_2', \mu_2^*\}$, we can obtain a point $P_0 \in U$ satisfying $P_0 \neq P$ and $G^{n+2}(P_0) = P$ for each each $\mu$ with $|\mu| > \mu_0$, in the two cases.

**Step 2.** It is to show that for any $\mu$ satisfying $|\mu| > \mu_0$,

$$\det DG(P_i) \neq 0, \quad 0 \le i \le n+1,$$

where $P_i := G(P_{i-1})$ for $1 \le i \le n+1$.

From the second relation of (3.4), we get an interval $V_1 \subset V$ which contains 0 and satisfies $p'(v) \neq 0$ for any $v \in V_1$. We can take $\mu_0$ sufficiently large such that $v_1$ and $v_2$ lie in $V_1$ for any $\mu$ satisfying $|\mu| > \mu_0$, where $v_1$ and $v_2$ satisfy equations (3.5) and (3.6) or equations (3.7) in the above. Consequently, $p'(v_1) \neq 0$ and $p'(v_2) \neq 0$. It is easy to prove that for any $w = (w_1, \cdots, w_{n+1})^T \in I^{n+1}$,

$$\det DG(w) = (-1)^n p'(w_1).$$

For the case $n = 1$, it is clear that $P_0 = (v_1, v_2)^T$, $P_1 = (v_2, w^*)^T$, $P_2 = (w^*, 0)^T$. Then for any $\mu$ satisfying $|\mu| > \mu_0$,

$$\det DG(P_i) = -p'(v_{i+1}) \neq 0, \quad \text{for } i = 0, 1,$$

$$\det DG(P_2) = -p'(w^*) = -w^* q'(w^*) \neq 0.$$

For the case $n > 1$, it is also clear that $P_0 = (v_1, v_2, 0, \cdots, 0)^T$, $P_1 = (v_2, 0, \cdots, 0, w^*)^T$, and $P_i = (0, \cdots, 0, \underbrace{w^*, 0, \cdots, 0}_{i})^T$ for $2 \le i \le n+1$. Hence, for any $\mu$ satisfying $|\mu| > \mu_0$,

$$\det DG(P_i) = (-1)^n p'(v_{i+1}) \neq 0, \quad \text{for } i = 0, 1,$$

$$\det DG(P_i) = (-1)^n p'(0) \neq 0, \quad \text{for } 2 \le i \le n,$$

$$\det DG(P_{n+1}) = (-1)^n p'(w^*) = (-1)^n w^* q'(w^*) \neq 0.$$

Thus, the condition (II) in Lemma 2.1 is also met. Therefore, equation (3.1), i.e., equation (1.2) is chaotic. □

**Remark 3.1.** *It is not easy to determine the concrete parameter $\mu_0$ in Theorem 3.1. In practice, we can take $\mu_0$ such that $|\mu_0|$ is sufficiently large to satisfy all the conditions in the proof of Theorem 3.1. The method of determining the concrete parameter is going to be studied later.*

Here, we provide one example to show this theorem. Take $q(v) = 1 - v$ in system (1.2). Obviously, conditions of Theorem 3.1 hold for $q(v)$ with $w^* = 1$. Therefore, we can obtain a positive number $\mu_0$ to make system (3.1), that is, system (1.2), be chaotic when $|\mu| > \mu_0$.

For convenience, we give two computer simulations to show the complex behaviors of system (3.1). The parameters are taken as $\delta = 0.1$, $n = 1, 2$. Then, each eigenvalue $\lambda$ of $DG(P)$ satisfies $|\lambda| > 1$ when $\mu$ satisfying $|\mu| > \mu_1 := \frac{1+|\delta|}{|q(0)|} = 1.1$. Thus, we can gain a number $\mu_0 > 1.1$ to make equation (3.1) be chaotic for $\mu$ satisfying $|\mu| > \mu_0$. Here, it takes $\mu = -1.8$ for computer simulations. See Figures 1 and 2.

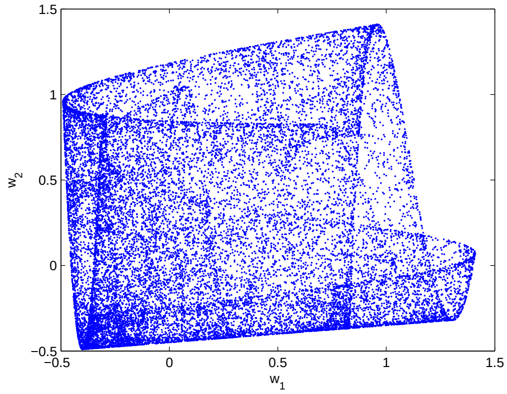


**Figure 1.** Complicated behaviors of equation (3.1) when $\delta = 0.1$, $\mu = -1.8$, $n = 1$, $w(0) = (0.01, 0.01)^T$.

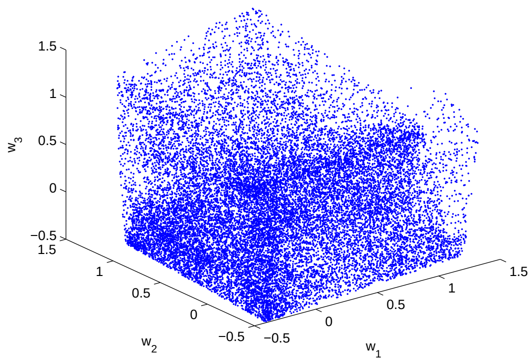


**Figure 2.** Complicated behaviors of equation (3.1) when $\delta = 0.1$, $\mu = -1.8$, $n = 2$, $w(0) = (0.01, 0.01, 0.01)^T$.

## 4. Conclusions

We explore the complicated behaviors of a delay differential equation through its difference form. We rigorously show the corresponding discrete equation can be chaotic under some conditions. This reflects that the original delay differential equation has complex dynamical behaviors. We will explore the chaotic behavior of more general delay differential equations in a later day.

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## Conflict of interest

The author declares that there are no conflicts of interest.

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