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Research article

Complicate dynamics of a discrete predator-prey model with double Allee effect

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Abstract: In this paper, a discrete predator-prey model with double Allee effect is discussed. We first simplify the corresponding continuous predator-prey model, and use the semidiscretization method to obtain a new discrete model. Next, the existence and local stability of nonnegative fixed points of the new discrete model are studied by using a key lemma. Then, by using the center manifold theorem and bifurcation theory, the sufficient conditions for the occurrences of transcritical bifurcation and Neimark-Sacker bifurcation and the stability of closed orbit bifurcated are obtained. Finally, the numerical simulations are presented, which not only verify the existence of Neimark-Sacker bifurcation but also reveal some new dynamic phenomena of this model.

Keywords: discrete predator-prey system; double Allee effect; transcritical bifurcation; Neimark-Sacker bifurcation

1. Introduction

The dynamical analysis of complex ecosystem, such as food chain, is based on the interaction among species or between two species, especially the dynamical relationship between predator and prey [1–4]. The current theory of predator-prey dynamics must depend on the study of nonlinear mathematical model [5]. With the continuous improvement of ecological knowledge like theoretical research, empirical research and observational research, etc., there are more and more basic elements of predation to be considered. Therefore, modelers add some complexities to their abstraction in order to obtain authenticity from the emergence of the far-reaching Lotka-Volterra model [6] and the modifications introduced by Volterra [5], taking into account the self-interference of prey populations.

Allee effect, which affects the number of prey, is one of these factors. It changes the qualitative stability and quantitative aspects of predator-prey model dynamics. Because the interaction between predator and prey is naturally prone to vibration, it is obvious to study this phenomenon as a potential mechanism for the generation of population cycles. Lots of researches about predator-prey models are done with the Allee effect [7–9].

The most popular framework for modeling expert predator-prey interaction has the following structure:

$$\begin{cases} \frac{dx}{dt} = xg(x,k) - yh(x), \\ \frac{dy}{dt} = (ph(x) - c)y, \end{cases} \tag{1.1}$$

where $x(t)$ and $y(t)$ are the prey and predator population sizes in the time t, respectively, $p, c > 0$ indicate the birth rate and background mortality rate, respectively, $g(x,k)$ describes the specific growth rate of the prey in the absence of predator, and $h(x)$ describes the predator functional response.

Any mechanism leading to a positive correlation between the components of individual fitness and the number or density of similar individuals can be regarded as Allee effect; it describes a scenario in which low population size is affected by the positive correlation between population growth rate and density, increasing the possibility of their extinction.

Recent ecological studies have shown that two or more Allee effects can lead to mechanisms acting on a population at the same time. The combined effects of some of these phenomena are called multiple (double) Allee effects. The author's analysis in [10] showed that the results of strong and weak Allee effects on the dynamics of Volterra predator-prey model are similar, which originate from the limit cycle of the model.

In this paper, we continue to consider the following predator-prey model with double Allee effect functional response raised by [10].

$$\begin{cases} \frac{dX}{dT} = rX(1-\frac{X}{K})\frac{X-M}{X+N} - qXY, \\ \frac{dY}{dT} = (pX - C)Y, \end{cases} \tag{1.2}$$

where r scales the prey growth rate, K is the intrinsic carrying capacity for the environment to the prey in the absence of predator, M is the Allee threshold, and the auxiliary parameter N satisfies $N > 0$, q is the prey captured rate by the predator, $p, C > 0$ indicate the birth rate and background mortality rate, respectively.

One can see that the first equation in the model (1.2) includes double Allee effects, expressed by the first factor $m(X) = X - M$, and a second term $r(X) = \frac{rX}{X+N}$, This can be interpreted as an approximation of population dynamics, in which the difference between fertile and non-fertile are not clearly modeled.

In order to simplify the analysis of system (1.2), we make a topologically equivalent change of variables and time rescaling as in [11–14], defining the function ϕ, such that $\phi(x, y) = (Kx, \frac{r}{q}y) = (X, Y)$, $\frac{r}{x+n}dT = dt$. Then, system (1.2) is transformed into

$$\begin{cases} \frac{dx}{dt} = ((1-x)(x-m) - (x+n)y)x, \\ \frac{dy}{dt} = b(x-c)(x+n)y, \end{cases} \tag{1.3}$$

where $b = \frac{pK}{r}$, $\frac{C}{pK}$, $m = \frac{M}{K}$, $n = \frac{N}{K}$, and $K > X > M$ is obtained from equation (1.2), so $1 > m > 0$.

We now use the semidiscretization method, which has been applied in many studies [15–18], to study the discrete model of system (1.3). For this, suppose that $[t]$ denotes the greatest integer not exceeding t. Consider the following semidiscretization version of (1.3).

$$\begin{cases} \frac{1}{x(t)}\frac{dx(t)}{dt} = (1 - x([t]))(x([t]) - m) - (x([t]) + n)y([t]), \\ \frac{1}{y(t)}\frac{dy(t)}{dt} = b(x([t]) - c)(x([t]) + n). \end{cases} \tag{1.4}$$

It is easy to see that the system (1.4) has piecewise constant arguments, and that a solution $(x(t), y(t))$ of the system (1.4) for $t \in [0, +\infty)$ possesses the following natures:
1. on the interval $[0, +\infty)$, $x(t)$ and $y(t)$ are continuous;
2. when $t \in [0, +\infty)$ except for the points $t \in \{0, 1, 2, 3, \cdots\}$, $\frac{dx(t)}{dt}$ and $\frac{dy(t)}{dt}$ exist everywhere.

The following system can be obtained by integrating (1.4) over the interval $[n, t]$ for any $t \in [n, n+1)$ and $n = 0, 1, 2, \cdots$

$$\begin{cases} x(t) = x_n e^{(1-x_n)(x_n-m)-(x_n+n)y_n}(t-n), \\ y(t) = y_n e^{b(x_n-c)(x_n+n)}(t-n), \end{cases} \tag{1.5}$$

where $x_n = x(n)$ and $y_n = y(n)$. Letting $t \to (n+1)^-$ in (1.5) produces

$$\begin{cases} x_{n+1} = x_n e^{(1-x_n)(x_n-m)-(x_n+n)y_n}, \\ y_{n+1} = y_n e^{b(x_n-c)(x_n+n)}, \end{cases} \tag{1.6}$$

where $b, c, n > 0$, $1 > m > 0$ are the same as in (1.3). We mainly study the properties of system (1.6) in the sequel.

The rest of the paper is organized as follows: In Section 2, we investigate the existence and stability of fixed points of the system (1.6). In Section 3, we derive the sufficient conditions for the transcritical bifurcation and the Neimark-Sacker bifurcation of the system (1.6) to occur. In Section 4, numerical simulations are performed to verify the above obtained theoretical results and reveal some new dynamical properties.

Before we analyze the fixed points of the system (1.6), we recall the following lemma see [16, 19].

Lemma 1.1. *Let $F(\lambda) = \lambda^2 + P\lambda + Q$, where P and Q are two real constants. Suppose λ_1 and λ_2 are two roots of $F(\lambda) = 0$. Then the following statements hold.*

(i) If $F(1) > 0$, then

$(i.1)$ $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $F(-1) > 0$ and $Q < 1$;

$(i.2)$ $\lambda_1 = -1$ and $\lambda_2 \neq -1$ if and only if $F(-1) = 0$ and $P \neq 2$;

$(i.3)$ $|\lambda_1| < 1$ and $|\lambda_2| > 1$ if and only if $F(-1) < 0$;

$(i.4)$ $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$ *if and only if* $F(-1) > 0$ *and* $Q > 1$*;*

$(i.5)$ λ_1 *and* λ_2 *are a pair of conjugate complex roots with* $|\lambda_1| = |\lambda_2| = 1$ *if and only if* $-2 < P < 2$ *and* $Q = 1$*;*

$(i.6)$ $\lambda_1 = \lambda_2 = -1$ *if and only if* $F(-1) = 0$ *and* $P = 2$*.*

(ii) *If* $F(1) = 0$*, namely,* 1 *is one root of* $F(\lambda) = 0$*, then the another root* λ *satisfies* $|\lambda| = (<, >)1$ *if and only if* $|Q| = (<, >)1$.

(iii) *If* $F(1) < 0$*, then* $F(\lambda) = 0$ *has one root lying in* $(1, \infty)$*. Moreover,*

$(iii.1)$ *the other root* λ *satisfies* $\lambda < (=) - 1$ *if and only if* $F(-1) < (=)0$*;*

$(iii.2)$ *the other root* $-1 < \lambda < 1$ *if and only if* $F(-1) > 0$.

2. Existence and stability of fixed points

In this section, we first consider the existence of fixed points and then analyze the local stability of each fixed point.

The fixed points of the system (1.6) satisfy

$$x = xe^{(1-x)(x-m)-(x+n)y},$$

$$y = ye^{b(x-c)(x+n)}.$$

Considering the biological meanings of the system (1.6), one only takes into account nonnegative fixed points. Thereout, one finds that the system (1.6) has and only has four nonnegative fixed points $E_0 = (0, 0)$, $E_1 = (1, 0)$, $E_2 = (m, 0)$ and $E_3 = (x_0, y_0)$ for $m < c < 1$, where

$$x_0 = c, y_0 = \frac{(1-c)(c-m)}{c+n}.$$

The Jacobian matrix of the system (1.6) at any fixed point $E(x, y)$ takes the following form

$$J(E) = \begin{pmatrix} [1 + x(-2x - y + m + 1)]e^A & -x(x+n)e^A \\ b(2x + n - c)ye^{b(x-c)(x+n)} & e^{b(x-c)(x+n)} \end{pmatrix}.$$

where $A = e^{(1-x)(x-m)-(x+n)y}$.

The characteristic polynomial of Jacobian matrix $J(E)$ reads

$$F(\lambda) = \lambda^2 - p\lambda + q,$$

where

$p = Tr(J(E)) = [1 + x(-2x - y + m + 1)]e^A + e^{b(x-c)(x+n)}$, $q = Det(J(E)) = [1 + x(-2x - y + m + 1)][bx(x + n)(2x + n - c)]e^{A+b(x-c)(x+n)}$.

For the stability of fixed points E_0, E_1, E_2 and E_3, we can easily get the following Theorems 2.1-2.4 respectively.

Theorem 2.1. *The fixed point* $E_0 = (0, 0)$ *of the system* (1.6) *is a sink.*

Proof. The Jacobian matrix $J(E_0)$ of the system (1.6) at the fixed point $E_0 = (0, 0)$ is given by

$$J(E_0) = \begin{pmatrix} e^{-m} & 0 \\ 0 & e^{-bcn} \end{pmatrix}.$$

Obviously, $|\lambda_1| = e^{-m} < 1$ and $|\lambda_2| = e^{-bcn} < 1$, so $E_0 = (0, 0)$ is a sink.

Theorem 2.2. *The following statements about the fixed point* $E_1 = (1, 0)$ *of the system* (1.6) *are true.*

1. *If* $c < 1$*, then* E_1 *is a saddle.*
2. *If* $c = 1$*, then* E_1 *is non-hyperbolic.*
3. *If* $c > 1$*, then* E_1 *is a stable node.*

Proof. The Jacobian matrix of the system (1.6) at $E_1 = (1, 0)$ is

$$J(E_1) = \begin{pmatrix} m & -(1+n) \\ 0 & e^{b(1-c)(1+n)} \end{pmatrix}.$$

Obviously, $\lambda_1 = m$ and $\lambda_2 = e^{b(1-c)(1+n)}$.

Note $|\lambda_1| < 1$ is always true. If $c < 1$, then $|\lambda_2| > 1$, so E_1 is a saddle; if $c = 1$, then $|\lambda_2| = 1$, therefore E_1 is non-hyperbolic; if $c > 1$, implying $|\lambda_2| < 1$, then E_1 is a stable node, namely, a sink. The proof is complete.

Theorem 2.3. *The following statements about the fixed point* $E_2 = (m, 0)$ *of the system* (1.6) *are true.*

1. *If* $c < m$*, then* E_2 *is a source.*
2. *If* $c = m$*, then* E_2 *is non-hyperbolic.*
3. *If* $c > m$*, then* E_2 *is a saddle.*

Proof. The Jacobian matrix of the system (1.6) at $E_2 = (m, 0)$ is

$$J(E_2) = \begin{pmatrix} -m^2 + m + 1 & -m(m+n) \\ 0 & e^{b(m-c)(m+n)} \end{pmatrix}.$$

Obviously, $\lambda_1 = -m^2 + m + 1$ and $\lambda_2 = e^{b(m-c)(m+n)}$.

Note $0 < m < 1$, so $|\lambda_1| > 1$ is always true. If $c < m$, then $|\lambda_2| > 1$, so E_2 is a source; if $c = m$, then $|\lambda_2| = 1$, therefore E_2 is non-hyperbolic; if $c > m$, implying $|\lambda_2| < 1$, then E_2 is a saddle. The proof is finished.

Theorem 2.4. *When* $(1 - c)(c - m) > 0$, *namely,* $0 < m < c < 1$, *the fixed point* $E_3 = (c, \frac{(1-c)(c-m)}{c+n})$ *is a positive fixed point of the system* (1.6). *Let* $b_0 = \frac{c^2+2cn-mn-m-n}{(c+n)^2(1-c)(c-m)}$. *Then the following statements are true about the positive fixed point* E_3.

I. If $m < \frac{c^2+2cn-n}{n+1}$, *then,*

1. *for* $0 < b < b_0$, E_3 *is a stable node;*
2. *for* $b = b_0$, E_3 *is non-hyperbolic;*
3. *for* $b > b_0$, E_3 *is an unstable node.*

II. If $m \geq \frac{c^2+2cn-n}{n+1}$, *then,* E_3 *is an unstable node.*

Proof. The Jacobian matrix of the system (1.6) at E_3 can be simplified as follows

$$J(E_3) = \begin{pmatrix} 1 + \frac{-c(c^2+2cn-mn-m-n)}{c+n} & -c(c+n) \\ b(1-c)(c-m) & 1 \end{pmatrix}.$$

The characteristic polynomial of Jacobian matrix $J(E_3)$ reads as

$$F(\lambda) = \lambda^2 - p\lambda + q, \tag{2.1}$$

where

$p = Tr(J(E_3)) = 2 - b_0c(1 - c)(c - m)(c + n)$,

$q = Det(J(E_3)) = 1 + (b - b_0)c(1 - c)(c - m)(c + n)$. By calculating we get

$F(1) = bc(1 - c)(c - m)(c + n) > 0$,

and

$$F(-1) = 2(2 - c^2) + \frac{2c(n(1 - c) + mn + n)}{c + n} + bc(1 - c)(c - m)(c + n) > 0.$$

I. If $m < \frac{c^2+2cn-n}{n+1}$, then $b_0 > 0$. So, when $0 < b < b_0$, $q < 1$. By Lemma 1.1 (i.1), $|\lambda_1| < 1$ and $|\lambda_2| < 1$, therefore E_3 is a stable node, i.e., a sink. When $b = b_0$, $q = 1$, $-2 < p < 2$. By Lemma 1.1 (i.5), Eq. (2.1) has a pair of conjugate complex roots λ_1 and λ_2 with $|\lambda_1| = |\lambda_2| = 1$, implying E_3 is non-hyperbolic. When $b > b_0 <$, $q > 1$. Lemma 1.1 (i.4) tells us that $|\lambda_1| > 1$ and $|\lambda_2| > 1$, so E_3 is an unstable node, i.e., a source.

II. If $m \geq \frac{c^2+2cn-n}{n+1}$, then $b_0 \leq 0$. So, $b > 0 \geq b_0$. Hence $q > 1$. Lemma 1.1 (i.4) reads that E_3 is an unstable node. The proof is complete.

3. Bifurcation analysis

In this section, we are in a position to use the Center Manifold Theorem and bifurcation theorem to analyze the local bifurcation problems of the fixed points E_1, E_2 and E_3. For related work, refer to [20–25].

3.1. For fixed point $E_1 = (1, 0)$

Theorem 2.2 shows that a bifurcation of E_1 may occur in the space of parameters $(b, c, m, n) \in S_{E_+} = \{(b, c, m, n) \in R^4_+ | b > 0, c > 0, 1 > m > 0, n > 0\}$.

Theorem 3.1. *Set the parameters* $(b, c, m, n) \in S_{E_+} = \{(b, c, m, n) \in R^4_+ | b > 0, c > 0, 1 > m > 0, n > 0\}$. *Let* $c_0 = 1$, *then the system* (1.6) *undergoes a transcritical bifurcation at* E_1 *when the parameter* c *varies in a small neighborhood of* c_0.

Proof. In order to show the detailed process, we proceed according to the following steps.

The first step. Let $u_n = x_n - 1, v_n = y_n - 0$, which transforms the fixed point $E_1 = (1, 0)$ to the origin $O(0, 0)$, and the system (1.6) to

$$\begin{cases} u_{n+1} = (u_n + 1)e^{-u_n(u_n-m+1)-(u_n+n+1)v_n} - 1, \\ v_{n+1} = v_n e^{b(u_n-c+1)(u_n+n+1)}. \end{cases} \tag{3.1}$$

The second step. Giving a small perturbation c^* of the parameter c, i.e., $c^* = c - c_0$, with $0 < |c^*| \ll 1$, the system (3.1) is perturbed into

$$\begin{cases} u_{n+1} = (u_n + 1)e^{-u_n(u_n-m+1)-(u_n+n+1)v_n} - 1, \\ v_{n+1} = v_n e^{b(u_n-c^*)(u_n+n+1)}. \end{cases} \tag{3.2}$$

Letting $c^*_{n+1} = c^*_n = c^*$, the system (3.2) can be written as

$$\begin{cases} u_{n+1} = (u_n + 1)e^{-u_n(u_n-m+1)-(u_n+n+1)v_n} - 1, \\ v_{n+1} = v_n e^{b(u_n-c^*_n)(u_n+n+1)}, \\ c^*_{n+1} = c^*_n. \end{cases} \tag{3.3}$$

The third step. Taylor expanding of the system (3.3) at $(u_n, v_n, c_n^*) = (0, 0, 0)$ takes the form

$$\begin{cases} u_{n+1} = & a_{100}u_n + a_{010}v_n + a_{200}u_n^2 + a_{020}v_n^2 \\ & +a_{110}u_nv_n + a_{300}u_n^3 + a_{030}v_n^3 \\ & +a_{210}u_n^2v_n + a_{120}u_nv_n^2 + o(\rho_1^3), \\ v_{n+1} = & b_{100}u_n + b_{010}v_n + b_{001}c_n^* + b_{200}u_n^2 \\ & +b_{020}v_n^2 + b_{002}c_n^{*2} + b_{110}u_nv_n \\ & +b_{101}u_nc_n^* + b_{011}v_nc_n^* + b_{300}u_n^3 \\ & +b_{030}v_n^3 + b_{003}c_n^{*3} + b_{210}u_n^2v_n \\ & +b_{120}u_nv_n^2 + b_{201}u_n^2c_n^* + b_{102}u_nc_n^{*2} \\ & +b_{021}v_n^2c_n^* + b_{012}v_nc_n^{*2} + b_{111}u_nv_nc_n^* \\ & +o(\rho_1^3), \\ c_{n+1}^* = & c_n^*, \end{cases} \tag{3.4}$$

where $\rho_1 = \sqrt{u_n^2 + v_n^2 + (c_n^*)^2}$,

$$\begin{aligned} &a_{100} = m, a_{010} = -(n+1), \\ &a_{200} = \frac{1}{2}(m-1)^2 + m - 2, a_{020} = \frac{1}{2}(n+1)^2, \\ &a_{110} = -mn - m - 1, \\ &a_{300} = \frac{1}{6}(m-1)^3 + \frac{1}{2}(m-1)^2 - m, \\ &a_{030} = -\frac{1}{6}(n+1)^3, a_{120} = \frac{1}{2}m(n+1)^2 + n + 1, \\ &a_{210} = -\frac{1}{2}(m-1)^2(n+1) - mn - 2m - 2n + 3, \end{aligned}$$

$$\begin{aligned} &b_{100} = b_{001} = b_{200} = b_{020} = b_{002} = b_{101} = b_{300} \\ &\quad = b_{030} = b_{003} = b_{120} = b_{201} = b_{102} = b_{021} = 0, \\ &b_{010} = 1, b_{110} = b(n+1), b_{011} = -b(n+1), \\ &b_{210} = b + \frac{1}{2}b^2(n+1)^2, b_{012} = \frac{1}{2}b^2(n+1)^2, \\ &b_{111} = -b - b^2(n+2)^2. \end{aligned}$$

Let

$$J(E_1) = \begin{pmatrix} a_{100} & a_{010} & 0 \\ b_{100} & b_{010} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$i.e., J(E_1) = \begin{pmatrix} m & -(n+1) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, we derive the three eigenvalues of $J(E_1)$ to be

$$\lambda_1 = m, \qquad \lambda_{2,3} = 1,$$

and the corresponding eigenvectors

$$\begin{aligned} (\xi_1, \eta_1, \varphi_1)^T &= (1, 0, 0)^T, \\ (\xi_2, \eta_2, \varphi_2)^T &= (n+1, m-1, 0)^T, \\ (\xi_3, \eta_3, \varphi_3)^T &= (0, 0, 1)^T. \end{aligned}$$

The fourth step. Let $T = \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 \\ \eta_1 & \eta_2 & \eta_3 \\ \varphi_1 & \varphi_2 & \varphi_3 \end{pmatrix}$, namely,

$$T = \begin{pmatrix} 1 & n+1 & 0 \\ 0 & m-1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\text{then } T^{-1} = \begin{pmatrix} 1 & \frac{1+n}{1-m} & 0 \\ 0 & \frac{1}{m-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Taking the following transformation

$$(u_n, v_n, c_n^*)^T = T(X_n, Y_n, \delta_n)^T,$$

the system (3.4) is changed into the following form

$$\begin{cases} X_{n+1} = mX_n + F(X_n, Y_n, \delta_n) + o(\rho_2^3), \\ Y_{n+1} = Y_n + G(X_n, Y_n, \delta_n) + o(\rho_2^3), \\ \delta_{n+1} = \delta_n, \end{cases} \tag{3.5}$$

where $\rho_2 = \sqrt{X_n^2 + Y_n^2 + \delta_n^2}$,

$$\begin{aligned} F(X_n, Y_n, \delta_n) =& m_{200}X_n^2 + m_{020}Y_n^2 + m_{002}{\delta_n}^2 \\ &+ m_{110}X_nY_n + m_{101}X_n\delta_n \\ &+ m_{011}Y_n\delta_n + m_{300}X_n^3 + m_{030}Y_n^3 \\ &+ m_{003}{\delta_n}^3 + m_{210}X_n^2Y_n \\ &+ m_{120}X_nY_n^2 + m_{201}X_n^2\delta_n \\ &+ m_{102}X_n{\delta_n}^2 + m_{021}Y_n^2\delta_n \\ &+ m_{012}Y_n{\delta_n}^2 + m_{111}X_nY_n\delta_n, \end{aligned}$$

$$\begin{aligned} G(X_n, Y_n, \delta_n) =& l_{200}X_n^2 + l_{020}Y_n^2 + l_{002}{\delta_n}^2 \\ &+ l_{110}X_nY_n + l_{101}X_n\delta_n \\ &+ l_{011}Y_n\delta_n + l_{300}X_n^3 + l_{030}Y_n^3 \end{aligned}$$

$$+ l_{003}{\delta_n}^3 + l_{210}X_n^2Y_n$$
$$+ l_{120}X_nY_n^2 + l_{201}X_n^2\delta_n$$
$$+ l_{102}X_n{\delta_n}^2 + l_{021}Y_n^2\delta_n$$
$$+ l_{012}Y_n{\delta_n}^2 + l_{111}X_nY_n\delta_n,$$

$$m_{200} = a_{200}, m_{300} = a_{300},$$
$$m_{002} = m_{101} = m_{003} = m_{201} = m_{102} = 0,$$
$$m_{020} = (a_{200} - b_{110})(1+n)^2 + a_{020}(m-1)^2 + a_{110}(m-1)(n+1),$$
$$m_{110} = (2a_{200} - b_{110})(1+n) + a_{110}(m-1),$$
$$m_{011} = -b_{011}(1+n),$$
$$m_{030} = (a_{300} - b_{210})(1+n)^3 + a_{030}(m-1)^3 + a_{210}(1+n)^2(m-1) + a_{120}(1+n)(m-1)^2,$$
$$m_{210} = (3a_{300} - b_{210})(1+n) + a_{210}(m-1),$$
$$m_{120} = (3a_{300} - 2b_{210})(1+n)^2 + 2a_{210}(1+n)(m-1) + a_{120}(m-1)^2,$$
$$m_{021} = -b_{111}(1+n)^2, m_{012} = -b_{012}(1+n),$$
$$m_{111} = -b_{111}(1+n),$$
$$l_{200} = l_{002} = 0, l_{020} = b_{110}(1+n), l_{110} = b_{110},$$
$$l_{101} = 0, l_{011} = b_{011}, l_{300} = 0, l_{030} = b_{210}(1+n)^2,$$
$$l_{003} = 0, l_{210} = b_{210}, l_{120} = 2b_{210}(1+n),$$
$$l_{201} = l_{102} = 0, l_{021} = b_{111}(1+n), l_{012} = b_{012},$$
$$l_{111} = b_{111}.$$

The fifth step. Suppose on the center manifold

$$X_n = h(Y_n, \delta_n) = h_{20}Y_n^2 + h_{11}Y_n\delta_n + h_{02}\delta_n^2 + o(\rho_3^2),$$

where $\rho_3 = \sqrt{Y_n^2 + \delta_n^2}$, then, according to

$$X_{n+1} = mh(Y_n, \delta_n) + F(h(Y_n, \delta_n), Y_n, \delta_n) + o(\rho_3^2),$$

$$\begin{aligned} h(Y_{n+1}, \delta_{n+1}) =& h_{20}Y_{n+1}^2 + h_{11}Y_{n+1}\delta_{n+1} + h_{02}\delta_{n+1}^2 \\ &+ o(\rho_3^2) \\ =& h_{20}(Y_n + G(h(Y_n, \delta_n), Y_n, \delta_n))^2 \\ &+ h_{11}(Y_n + G(h(Y_n, \delta_n), Y_n, \delta_n))\delta_n \\ &+ h_{02}\delta_n^2 + o(\rho_3^2). \end{aligned}$$

and $X_{n+1} = h(Y_{n+1}, \delta_{n+1})$, we obtain the center manifold equation to satisfy the following relation

$$mh(Y_n, \delta_n) + F(h(Y_n, \delta_n), Y_n, \delta_n)$$
$$= h_{20}(Y_n + G(h(Y_n, \delta_n), Y_n, \delta_n))^2$$
$$+ h_{11}(Y_n + G(h(Y_n, \delta_n), Y_n, \delta_n))\delta_n + h_{02}\delta_n^2 + o(\rho_3^2).$$

Comparing the corresponding coefficients of terms with the same orders in the above center manifold equation, we get

$$h_{20} = \frac{(m-2)(1+n)^2 - (1+n)(2+n)(m-1) - b(1+n)^3}{1-m},$$
$$h_{11} = \frac{b(1+n)^2}{1-m}, h_{02} = 0.$$

So, the system (3.5) restricted to the center manifold takes as

$$Y_{n+1} = f(Y_n, \delta_n) := Y_n + G(h(Y_n, \delta_n), Y_n, \delta_n) + o(\rho_3^2)$$
$$= Y_n + b(1+n)^2Y_n^2 - b(1+n)Y_n\delta_n$$
$$+ \frac{b(1+n)^2}{1-m}\Big(1 - 2m - n - \frac{1}{2}b(1+n)^2(1+m)\Big)Y_n^3$$
$$+ \Big(\frac{mb^2(1+n)^3}{1-m} - b(1+n)\Big){Y_n}^2\delta_n + \frac{1}{2}b^2(1+n)^2Y_n\delta_n^2$$
$$+ o(\rho_3^3).$$

Therefore one has

$$f(Y_n, \delta_n)|_{(0,0)} = 0, \left.\frac{\partial f}{\partial Y_n}\right|_{(0,0)} = 1, \left.\frac{\partial f}{\partial \delta_n}\right|_{(0,0)} = 0,$$
$$\left.\frac{\partial^2 f}{\partial Y_n \partial \delta_n}\right|_{(0,0)} = -b(1+n) \neq 0,$$
$$\left.\frac{\partial^2 f}{\partial Y_n^2}\right|_{(0,0)} = 2b(1+n)^2 \neq 0.$$

According to (21.1.42)-(21.1.46) in the literature ([26], pp. 507), all the conditions for the occurrence of the transcritical bifurcation are established, hence, it is valid for the occurrence of transcritical bifurcation in the fixed point E_1. The proof is over.

3.2. *For fixed point $E_2 = (m, 0)$*

According to Theorem 2.3, the fixed point $E_2(m, 0)$ is non-hyperbolic, the system (1.6) may undergo a bifurcation (the correspond eigenvalue are $\lambda_1 = -m^2 + m + 1$, $\lambda_2 = 1$). By using the same method as that in Section 3.2, we get the following result.

Theorem 3.2. *Set the parameters $(b, c, m, n) \in S_{E_+} = \{(b, c, m, n) \in R_+^4 | b > 0, c > 0, 1 > m > 0, n > 0\}$. Let*

$c_1 = m$, then the system (1.6) undergoes a transcritical bifurcation at E_2 when the parameter c varies in a small neighborhood of c_1.

3.3. For fixed point $E_3 = (c, \frac{(1-c)(c-m)}{c+n})$

When $m < \frac{c^2+2cn-n}{n+1}$, $b = b_0 = \frac{c^2+2cn-mn-n-m}{(c+n)^2(1-c)(c-m)}$, Theorem 2.4 with Lemma 1.2 ($i.5$) shows that $F(1) > 0$, $F(-1) > 0$, $-2 < p < 2$ and $q = 1$, so λ_1 and λ_2 are a pair of conjugate complex roots with $|\lambda_1| = |\lambda_2| = 1$. At this time we derive that the system (1.6) at the fixed point E_3 can undergo a Neimark-Sacker bifurcation in the space of parameters $(b, c, m, n) \in S_{E_+} = \{(b, c, m, n) \in R^4_+ | b > 0, 1 > c > m, 0 < m < \frac{c^2+2cn-n}{n+1}\}$.

In order to show the process clearly, we carry out the following steps.

The first step. Take the changes of variables $u_n = x_n - x_0, v_n = y_n - y_0$, which transform fixed point $E_3 = (x_0, y_0)$ to the origin $O(0, 0)$, and the system (1.6) into

$$\begin{cases} u_{n+1} = (u_n + x_0)e^{(-u_n - x_0 + 1)(u_n + x_0 - m) - (u_n + x_0 + n)v_n y_n} - x_0, \\ v_{n+1} = (v_n + y_0)e^{b(u_n + x_0 - c)(u_n + x_0 + n)} - y_0. \end{cases} \tag{3.6}$$

The second step. Give a small perturbation b^* of the parameter b, i.e., $b^* = b - b_0$, then the perturbation of the system (3.6) can be regarded as follows

$$\begin{cases} u_{n+1} = (u_n + x_0)e^{(-u_n - x_0 + 1)(u_n + x_0 - m) - (u_n + x_0 + n)v_n y_n} - x_0, \\ v_{n+1} = (v_n + y_0)e^{(b^* + b_0)(u_n + x_0 - c)(u_n + x_0 + n)} - y_0. \end{cases} \tag{3.7}$$

The corresponding characteristic equation of the linearized equation of the system (3.7) at the equilibrium point (0,0) can be expressed as

$$F(\lambda) = \lambda^2 - p(b^*)\lambda + q(b^*) = 0,$$

where

$$p(b^*) = 2 + \frac{c(-c^2 - 2cn + mn + m + n)}{c + n},$$

and

$$q(b^*) = \frac{c[(c^2+2cn-mn-m-n)-(b^*+b_0)(c+n)^2(1-c)(c-m)]}{-(c+n)} + 1.$$

It is easy to derive $p^2(b^*) - 4q(b^*) < 0$ when $b^* = 0$, and $0 < p(b^*) < 2$, then the two roots of $F(\lambda) = 0$ are

$$\lambda_{1,2}(b^*) = \frac{p(b^*) \pm \sqrt{p^2(b^*) - 4q(b^*)}}{2}$$

$$= \frac{p(b^*) \pm i\sqrt{4q(b^*) - p^2(b^*)}}{2},$$

which implies

$$(|\lambda_{1,2}(b^*)|)\Big|_{b^*=0} = \sqrt{q(b^*)}\Big|_{b^*=0} = 1,$$

and

$$\Big(\frac{d|\lambda_{1,2}(b^*)|}{db^*}\Big)\Big|_{b^*=0} = \frac{1}{2}c(c + n)(1 - c)(c - m) > 0.$$

The occurrence of Neimark-Sacker bifurcation requires the following conditions to be satisfied

$$(H.1) \quad \Big(\frac{d|\lambda_{1,2}(b^*)|}{db^*}\Big)\Big|_{b^*=0} \neq 0;$$

$$(H.2) \quad \lambda^i_{1,2}(0) \neq 1, i = 1, 2, 3, 4.$$

Since $p(b^*)\big|_{b^*=0} = 2 + \frac{c(-c^2-2cn+mn+m+n)}{c+n}$ and $q(b^*)\big|_{b^*=0} = 1$, we have $\lambda_{1,2}(0) = \frac{2(c+n)+c(-c^2-2cn+mn+m+n)\pm i\sqrt{4(c+n)^2-[2(c+n)+c(-c^2-2cn+mn+m+n)]^2}}{2(c+n)}$, then it is easy to derive $\lambda^i_{1,2}(0) \neq 1$ for all $i = 1, 2, 3, 4$. According to ([2], pp517-522), they satisfy all of the conditions for Neimark-Sacker bifurcation to occur.

The third step. In order to derive the normal form of the system (3.7), we expand the system (3.7) into power series up to the following third-order form around the origin

$$\begin{cases} u_{n+1} = & c_{10}u_n + c_{01}v_n + c_{20}u_n^2 + c_{11}u_nv_n \\ & +c_{02}v_n^2 + c_{30}u_n^3 + c_{21}u_n^2v_n + c_{12}u_nv_n^2 \\ & +c_{03}v_n^3 + o(\rho_4^3), \\ v_{n+1} = & d_{10}u_n + d_{01}v_n + d_{20}u_n^2 + d_{11}u_nv_n \\ & +d_{02}v_n^2 + d_{30}u_n^3 + d_{21}u_n^2v_n + d_{12}u_nv_n^2 \\ & +d_{03}v_n^3 + o(\rho_4^3), \end{cases} \tag{3.8}$$

where $\rho_4 = \sqrt{u_n^2 + v_n^2}$,

$$c_{10} = 1 + \frac{c(-c^2 - 2cn + mn + m + n)}{c + n},$$

$$c_{01} = -c(c + n),$$

$$c_{20} = -c + \frac{-c^2 - 2cn + mn + m + n}{c + n} + \frac{c(-c^2 - 2cn + mn + m + n)^2}{2(c + n)^2},$$

$$c_{02} = \frac{c(c + n)^2}{2},$$

$$c_{11} = -[2c + n + c(-c^2 - 2cn + mn + m + n)],$$

$$c_{30} = -1 - \frac{c(-c^2 - 2cn + mn + m + n)}{c + n} + \frac{(-c^2 - 2cn + mn + m + n)^2}{c(c + n)^2} + \frac{c(-c^2 - 2cn + mn + m + n)^3}{6(c + n)^3},$$

$$c_{03} = -\frac{c(c + n)^3}{6},$$

$$c_{21} = -1 - (-c^2 - 2cn + mn + m + n) + c[c + n - \frac{-c^2 - 2cn + mn + m + n}{c + n} - \frac{(-c^2 - 2cn + mn + m + n)^2}{2(c + n)}],$$

$$c_{12} = \frac{(c + n)^2}{2} + c(c + n)[1 + \frac{-c^2 - 2cn + mn + m + n}{2}],$$

$$d_{10} = -\frac{-c^2 - 2cn + mn + m + n}{(c + n)^2}, d_{01} = 1,$$

$$d_{20} = \frac{(-c^2 - 2cn + mn + m + n)^2}{2(c + n)^3(1 - c)(c - m)} - \frac{2(-c^2 - 2cn + mn + m + n)(1 - c)(c - m)}{2(c + n)^3(1 - c)(c - m)},$$

$$d_{11} = -\frac{-c^2 - 2cn + mn + m + n}{(c + n)(1 - c)(c - m)},$$

$$d_{02} = d_{03} = d_{12} = 0,$$

$$d_{30} = \frac{(-c^2 - 2cn + mn + m + n)^2}{(c + n)^4(1 - c)(c - m)}[1 - \frac{-c^2 - 2cn + mn + m + n}{6(1 - c)(c - m)}],$$

$$d_{21} = \frac{(-c^2 - 2cn + mn + m + n)}{(c + n)^2(1 - c)(c - m)}[1 - \frac{-c^2 - 2cn + mn + m + n}{2(1 - c)(c - m)}].$$

Let

$$J(E_3) = \begin{pmatrix} c_{10} & c_{01} \\ d_{10} & d_{01} \end{pmatrix}, \text{namely,}$$

$$J(E_3) = \begin{pmatrix} 1 + K & -c(c + n) \\ \frac{-K}{c(c+n)} & 1 \end{pmatrix}.$$

It is easy to derive the two eigenvalues of the matrix $J(E_3)$ are

$$\lambda_{1,2} = (1 + \frac{1}{2}K) \pm \beta i,$$

where $K = \frac{c(-c^2-2cn+mn+m+n)}{c+n}$,
$\beta = \frac{\sqrt{-c(-c^2-2cn+mn+m+n)[4(c+n)+c(-c^2-2cn+mn+m+n)]}}{2(c+n)}$,
with the corresponding eigenvectors

$$v_{1,2} = \begin{pmatrix} -c(c + n) \\ -\frac{1}{2}K \end{pmatrix} \pm i \begin{pmatrix} 0 \\ \beta \end{pmatrix}.$$

Let

$$T = \begin{pmatrix} 0 & -c(c + n) \\ \beta & -\frac{1}{2}K \end{pmatrix}, \text{then,}$$

$$T^{-1} = \begin{pmatrix} -\frac{K}{2c(c+n)\beta} & \frac{1}{\beta} \\ -\frac{1}{c(c+n)} & 0 \end{pmatrix}.$$

Make a change of variables

$$(u, v)^T = T(X, Y)^T,$$

then, the system (3.8) is transformed into the following form

$$\begin{cases} X \to & (1 + \frac{1}{2}K)X - \beta Y + \overline{F}(X, Y) + o(\rho_5^3), \\ Y \to & \beta X + (1 + \frac{1}{2}K)Y + \overline{G}(X, Y) + o(\rho_5^3), \end{cases} \tag{3.9}$$

where $\rho_5 = \sqrt{X^2 + Y^2}$,

$$\overline{F}(X, Y) = e_{20}X^2 + e_{11}XY + e_{02}Y^2 + e_{30}X^3 + e_{21}X^2Y + e_{12}XY^2 + e_{03}Y^3,$$

$$\overline{G}(X, Y) = f_{20}X^2 + f_{11}XY + f_{02}Y^2 + f_{30}X^3 + f_{21}X^2Y + f_{12}XY^2 + f_{03}Y^3,$$

$$e_{20} = \frac{c_{02}\beta K}{2c_{01}}, e_{11} = \frac{c_{01}c_{11}K + 2c_{01}^2 d_{11} - c_{02}K^2}{2c_{01}},$$

$$e_{02} = \frac{4c_{01}^2(c_{20}K + 2c_{01}d_{20} - d_{11}K)}{8c_{01}\beta} + \frac{K^2(c_{02}K - 2c_{01}c_{11})}{8c_{01}\beta},$$

$$e_{30} = \frac{c_{03}\beta^2 K}{2c_{01}}, e_{21} = \frac{(2c_{01}c_{12} - 3c_{03}K)\beta K}{4c_{01}},$$

$$e_{12} = \frac{c_{01}c_{21}K - c_{12}K^2 + 2d_{21}c_{01}^2}{2} + \frac{3c_{03}K^3}{8c_{01}},$$

$$e_{03} = \frac{8c_{01}^3(c_{30}K + 2c_{01}d_{30} - d_{21}K)}{16c_{01}\beta} - \frac{K^2(4c_{01}^2 c_{21} + c_{03}K^2 - 2c_{01}c_{12}K)}{16c_{01}\beta},$$

$$f_{20} = \frac{c_{02}}{c_{01}}\beta^2, f_{11} = c_{11}\beta - \frac{c_{02}}{c_{01}}\beta K,$$

$$f_{02} = c_{01}c_{20} - \frac{1}{2}c_{11}K + \frac{c_{02}}{4c_{01}}K^2,$$
$$f_{30} = \frac{c_{03}}{c_{01}}\beta^3, f_{21} = c_{12}\beta^2 - \frac{3c_{03}}{2c_{01}}\beta^2 K,$$
$$f_{12} = c_{01}c_{21}\beta - c_{12}\beta K + \frac{3c_{03}}{4c_{01}}\beta K^2,$$
$$f_{03} = c_{30}c_{01}^2 - \frac{1}{2}c_{01}c_{21}K + \frac{1}{4}c_{12}K^2 - \frac{c_{03}}{8c_{01}}K^3.$$

Furthermore,

$$\overline{F}_{XX} = \frac{c_{02}\beta K}{c_{01}},$$
$$\overline{F}_{XY} = \frac{c_{01}c_{11}K + 2c_{01}^2 d_{11} - c_{02}K^2}{2c_{01}},$$
$$\overline{F}_{XXX} = \frac{3c_{03}\beta^2 K}{c_{01}},$$
$$\overline{F}_{YY} = \frac{4c_{01}^2(c_{20}K + 2c_{01}d_{20} - d_{11}K)}{4c_{01}\beta} + \frac{K^2(c_{02}K - 2c_{01}c_{11})}{4c_{01}\beta},$$
$$\overline{F}_{XXY} = c_{12}\beta K - \frac{3c_{03}\beta K^2}{2c_{01}},$$
$$\overline{F}_{XYY} = c_{01}c_{21}K - c_{12}K^2 + 2d_{21}c_{01}^2 + \frac{3c_{03}K^3}{4c_{01}},$$
$$\overline{F}_{YYY} = \frac{3c_{01}^3(c_{30}K + 2c_{01}d_{30}) - d_{21}K}{c_{01}\beta} - \frac{3K^2(4c_{01}^2 c_{21} + c_{03}K^2 - 2c_{01}c_{12}K)}{8c_{01}\beta},$$
$$\overline{G}_{XX} = \frac{2c_{02}\beta^2}{c_{01}}, \overline{G}_{XY} = c_{11}\beta - \frac{c_{02}\beta K}{c_{01}},$$
$$\overline{G}_{YY} = 2c_{01}c_{20} - c_{11}K + \frac{c_{02}K^2}{2c_{01}},$$
$$\overline{G}_{XXX} = \frac{6c_{03}\beta^3}{c_{01}}, \overline{G}_{XXY} = 2c_{12}\beta^2 - \frac{3c_{03}\beta^2 K}{c_{01}},$$
$$\overline{G}_{XYY} = 2c_{01}c_{21}\beta - 2c_{12}\beta K + \frac{3c_{03}\beta K^2}{2c_{01}},$$
$$\overline{G}_{YYY} = 6c_{30}c_{01}^2 - 3c_{01}c_{21}K + \frac{3}{2}c_{12}K^2 - \frac{3c_{03}K^3}{4c_{01}}.$$

The fourth step. In order to ensure that the system (3.9) has a Neimark-Sacker bifurcation occurring, we need to calculate the discriminating quantity

$$L = -Re\Big(\frac{(1 - 2\lambda_1)\lambda_2^2}{1 - \lambda_1}\zeta_{20}\zeta_{11}\Big) - \frac{1}{2}|\zeta_{11}|^2 - |\zeta_{02}|^2 + Re(\lambda_2\zeta_{21}), \tag{3.10}$$

and L is required not to be zero, where

$$\zeta_{20} = \frac{1}{8}[\overline{F}_{XX} - \overline{F}_{YY} + 2\overline{G}_{XY} + i(\overline{G}_{XX} - \overline{G}_{YY} - 2\overline{F}_{XY})],$$
$$\zeta_{11} = \frac{1}{4}[\overline{F}_{XX} + \overline{F}_{YY} + i(\overline{G}_{XX} + \overline{G}_{YY})],$$
$$\zeta_{02} = \frac{1}{8}[\overline{F}_{XX} - \overline{F}_{YY} - 2\overline{G}_{XY} + i(\overline{G}_{XX} - \overline{G}_{YY} + 2\overline{F}_{XY})],$$
$$\zeta_{21} = \frac{1}{16}[\overline{F}_{XXX} + \overline{F}_{XYY} + \overline{G}_{XXY} + \overline{G}_{YYY} + i(\overline{G}_{XXX} + \overline{G}_{XYY} - \overline{F}_{XXY} - \overline{F}_{YYY})].$$

By calculation we get

$$\zeta_{20} = \frac{1}{8}\Big(- \frac{4c_{01}^2(c_{20}K + 2c_{01}d_{20} - d_{11}K)}{4c_{01}\beta} - \frac{K^2(c_{02}K - 2c_{01}c_{11})}{4c_{01}\beta} + \frac{(2c_{01}c_{11} - c_{02}K)\beta}{c_{01}}\Big) + \frac{1}{8}\Big(\frac{c_{02}(K^2 + 4\beta^2)}{2c_{01}} - 2c_{01}(c_{20} + d_{11})\Big)i,$$

$$\zeta_{11} = \frac{1}{4}\Big(\frac{c_{02}\beta K}{c_{01}} + \frac{4c_{01}^2(c_{20}K + 2c_{01}d_{20} - d_{11}K)}{4c_{01}\beta} + \frac{K^2(c_{02}K - 2c_{01}c_{11})}{4c_{01}\beta}\Big) + \frac{1}{4}\Big(\frac{c_{02}(4\beta^2 + K^2)}{2c_{01}} + 2c_{01}c_{20} - c_{11}K\Big)i,$$

$$\zeta_{02} = \frac{1}{8}\Big(- \frac{4c_{01}^2(c_{20}K + 2c_{01}d_{20} - d_{11}K)}{4c_{01}\beta} - \frac{K^2(c_{02}K - 2c_{01}c_{11})}{4c_{01}\beta} + \frac{(3c_{02}K - 2c_{01}c_{11})\beta}{c_{01}}\Big) + \frac{1}{4}\Big(\frac{c_{02}(4\beta^2 - 3K^2)}{4c_{01}} + c_{11}K + c_{01}(d_{11} - c_{20})\Big)i,$$

$$\zeta_{21} = \frac{1}{16}\Big(2c_{01}(3c_{30}c_{01} - c_{21}K + c_{01}d_{21}) + c_{12}(\frac{1}{2}K^2 + 2\beta^2)\Big) + \frac{1}{16}\Big(\frac{3c_{03}\beta(K^2 + 2\beta^2)}{c_{01}} + \frac{3K^2(4c_{01}^2 c_{21} + c_{03}K^2 - 2c_{01}c_{12}K)}{8c_{01}\beta} + \beta(2c_{01}c_{21} - 3c_{12}K) - \frac{3c_{01}^3(c_{30}K + 2c_{01}d_{30}) - d_{21}K}{c_{01}\beta}\Big)i.$$

Theorem 3.3. *Assume the parameters b, c, m, n in the space* $S_{E_+} = \{(b, c, m, n) \in R_+^4 | b > 0, 1 > c > m, 0 < m < \frac{c^2+2cn-n}{n+1}\}$. *Let* $b_0 = \frac{c^2+2cn-mn-m-n}{(c+n)^2(1-c)(c-m)}$ *and L be defined as above* (3.10). *If* $L \neq 0$ *holds and the parameter a varies in the small neighborhood of* b_0, *then the system* (1.6) *at the fixed point* E_3 *undergoes a Neimark-Sacker bifurcation. In addition, if* $L < (or >)0$, *then an attracting (or repelling) invariant closed curve bifurcates from the fixed point* E_3 *for* $b < (or >)b_0$.

4. Numerical simulation

In this section, we use the bifurcation diagrams, phase portraits and Lyapunov exponents of the system (1.6) to verify our theoretical results and further reveal some new dynamical behaviors to occur as the parameters vary by Matlab software.

Fix the parameter values $c = 0.8, m = 0.3, n = 0.8$, let $b \in (1.5, 3.0)$ and take the initial values $(x_0, y_0) = (0.55, 0.25), (0.80, 0.05)$ in Fig.2 and Fig.3 respectively. Figure 1(a) shows the bifurcation diagram of (b, x)-plane, from which the fixed point E_3 is stable when $b < b_0 = 2.266$. Moreover, the fixed point E_3 is unstable when $b > b_0$. Hence, the Neimark-Sacker bifurcation occurs at the fixed point $E_3 = (0.800, 0.625)$ when $b = b_0$, whose multipliers are $\lambda_{1,2} = 0.855 \pm 0.519i$ with $|\lambda_{1,2}| = 1$.

The corresponding maximum Lyapunov exponent diagram of the system (1.6) is plotted in Figure 1(b). Figures 2(a)-2(f) and Figures 3(a)-3(d) show that the dynamical properties of the fixed point E_3 change from stable to unstable as the value of the parameter b decreases and there is an occurrence of invariant closed curve around E_3 when $b = b_0$, which agrees to the result of Theorem 3.3.

From the phase portraits in Figs 2 and 3, we infer the stability of E_3. Figures 2(d)-2(f) show that the closed curve is stable outside, while Figures 3(a)-3(d) indicate that the closed curve is stable inside for the fixed point E_3 as long as the assumptions of Theorem 3.2 hold.

5. Discussion and conclusion

In this paper, we discuss the dynamical behaviors of a predator-prey model (1.6) of Gause-type with double Allee effect affecting the prey population. Under the given parametric conditions, we completely show the existence and stability of four nonnegative equilibria $E_0 = (0, 0)$, $E_1 = (1, 0)$, $E_2 = (m, 0)$ and $E_3 = (c, \frac{(1-c)(c-m)}{c+n})$. Then we derive the sufficient conditions for its transcritical bifurcation and Neimark-Sacker bifurcation to occur. Meanwhile, it is clear that the positive equilibrium $E_3 = (x_0, y_0)$ is asymptotically stable when $b < b_0 = \frac{c^2+2cn-mn-m-n}{(c+n)^2(1-c)(c-m)}$ and unstable when $b > b_0$ under the condition $m < \frac{c^2+2cn-n}{n+1}$. Hence, the system (1.6) undergoes a bifurcation which has been shown to be a Neimark-Sacker bifurcation when the parameter b goes through the critical value b_0. Finally, numerical simulations illustrate the theoretical analysis results of the system (1.6).

The perturbations of different parameters in this system may lead to different bifurcations. This demonstrates that this system is sensitive to its parameters. Especially, the occurrence of Neimark-Sacker bifurcation implies that the predator and the prey can coexist under such parametric conditions.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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Conflict of interest

The authors declare that they have no competing interests.

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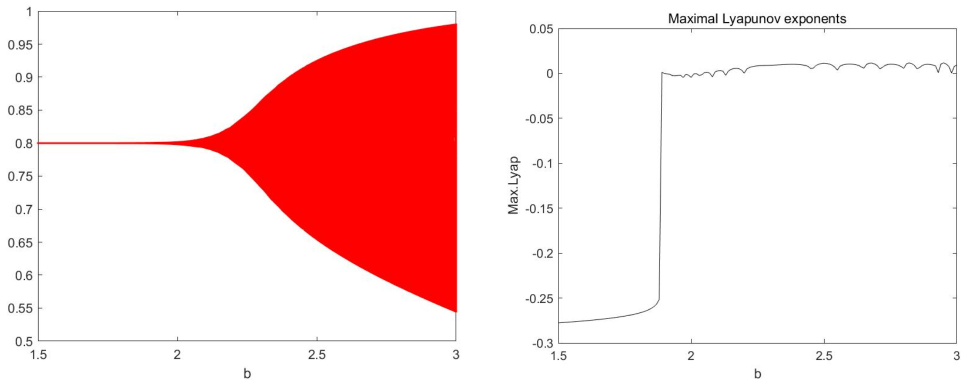
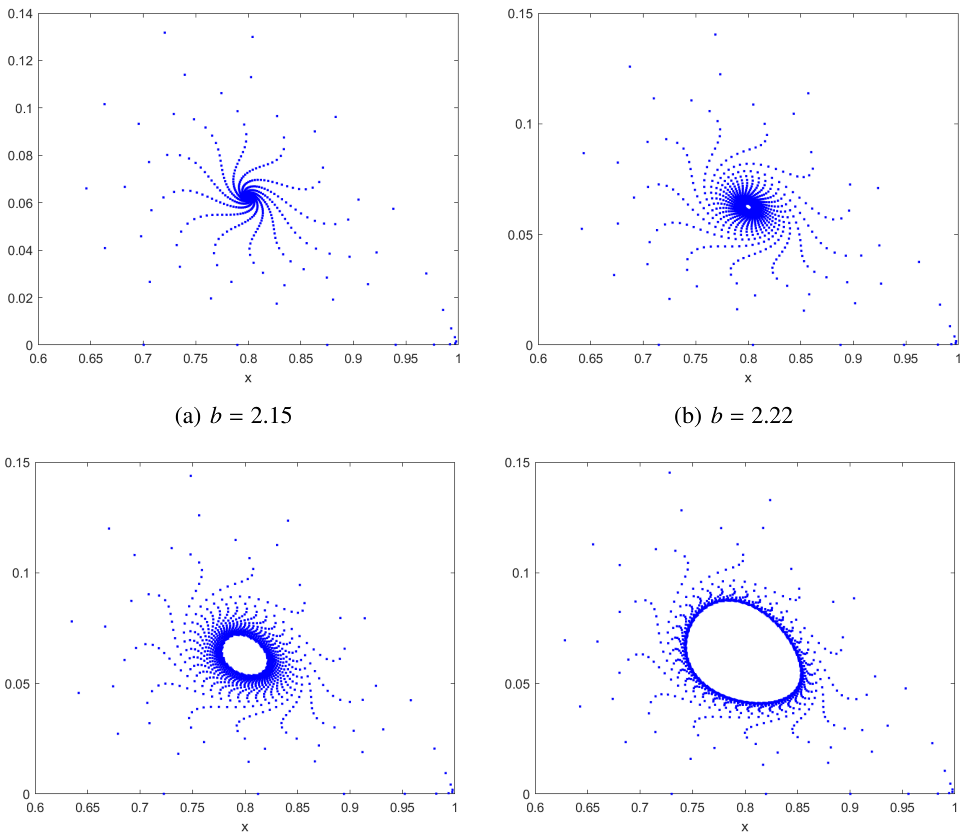


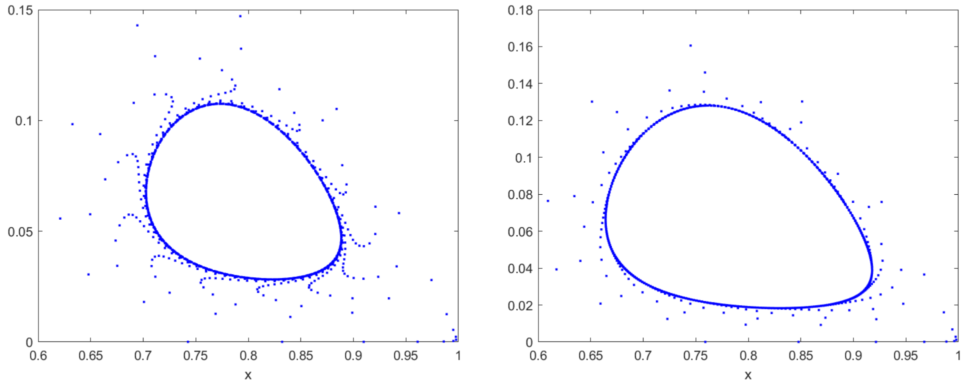
(a) $b \in (1.5, 3.0)$ (b) $b \in (1.5, 3.0)$

Figure 1. Bifurcation of the system (1.6) in (b, x)-plane and Maximal Lyapunov exponent .



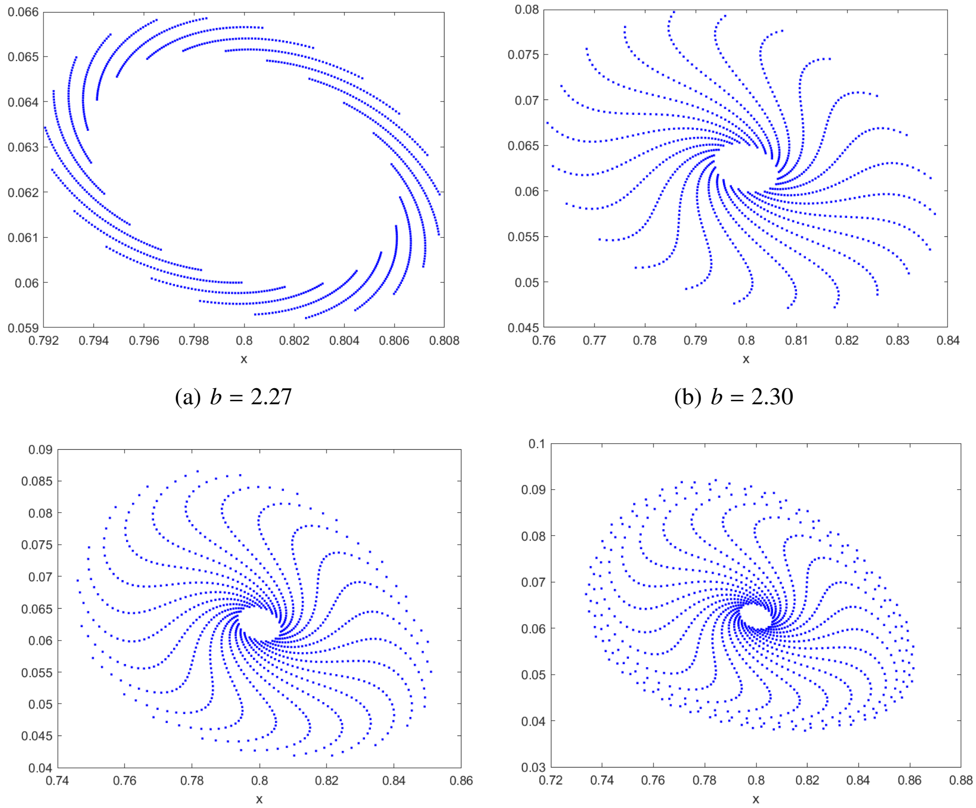
(a) $b = 2.15$ (b) $b = 2.22$

(c) $b = 2.26$ (d) $b = 2.30$



(e) $b = 2.37$ (f) $b = 2.47$

Figure 2. Phase portraits for the system (1.6) with $c = 0.8, m = 0.3, n = 0.8$ and different b when the initial value $(x_0, y_0) = (0.55, 0.25)$.



(a) $b = 2.27$ (b) $b = 2.30$

(c) $b = 2.31$ (d) $b = 2.32$

Figure 3. Phase portraits for the system (1.6) with $c = 0.8, m = 0.3, n = 0.8$ and different b when the initial value $(x_0, y_0) = (0.80, 0.05)$.

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