Mathematical Modelling and Control

http://www.aimspress.com/journal/mmc

MMC, 2(4): 272–281
DOI:10.3934/mmc.2022026
Received: 14 October 2022
Revised: 25 November 2022
Accepted: 07 December 2022
Published: 27 December 2022

Research article

Stochastic persistence and global attractivity of a two-predator one-prey system with S-type distributed time delays

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Abstract: In this paper, well-posedness and asymptotic behaviors of a stochastic two-predator one-prey system with S-type distributed time delays are studied by using stochastic analytical techniques. First, the existence and uniqueness of global positive solution with positive initial condition is proved. Second, sufficient conditions for persistence in mean and extinction of each species are obtained. Then, sufficient conditions for global attractivity are established. Finally, some numerical simulations are provided to support the analytical results.

Keywords: persistence in mean; extinction; predator-prey system; time delay; global attractivity

1. Introduction

In 1977 and 1984, Freedman and Waltman ([1,2]) studied the following two-predator one-prey system:

$$\begin{cases} \dfrac{\mathrm{d}x_1(t)}{\mathrm{d}t} = x_1(t)\left[r_1 - a_{11}x_1(t) - a_{12}x_2(t) - a_{13}x_3(t)\right], \\ \dfrac{\mathrm{d}x_2(t)}{\mathrm{d}t} = x_2(t)\left[-r_2 + a_{21}x_1(t) - a_{22}x_2(t) - a_{23}x_3(t)\right], \\ \dfrac{\mathrm{d}x_3(t)}{\mathrm{d}t} = x_3(t)\left[-r_3 + a_{31}x_1(t) - a_{32}x_2(t) - a_{33}x_3(t)\right], \end{cases} \tag{1.1}$$

where $x_i(t)$ stands for the size of the ith population and all the parameters are positive constants.

However, the deterministic system has its limitation in mathematical modeling of ecosystems since the parameters involved in the system are unable to capture the influence of environmental noises [3–5]. Therefore, it is of enormous importance to study the effects of environmental noises on the dynamics of population systems. Assume that the parameters r_i are affected by white noises, i.e., $r_1 \hookrightarrow r_1 + \sigma_1\dot{B}_1(t)$, $-r_2 \hookrightarrow -r_2 + \sigma_2\dot{B}_2(t)$, $-r_3 \hookrightarrow -r_3 + \sigma_3\dot{B}_3(t)$, where $B_i(t)$ are mutually independent standard Wiener processes defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Then, the stochastic two-predator one-prey system with white noises can be expressed as follows:

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[r_1 - a_{11}x_1(t) - a_{12}x_2(t) - a_{13}x_3(t)\right]\mathrm{d}t \\ \qquad\qquad + \sigma_1 x_1(t)\mathrm{d}B_1(t), \\ \mathrm{d}x_2(t) = x_2(t)\left[-r_2 + a_{21}x_1(t) - a_{22}x_2(t) - a_{23}x_3(t)\right]\mathrm{d}t \\ \qquad\qquad + \sigma_2 x_2(t)\mathrm{d}B_2(t), \\ \mathrm{d}x_3(t) = x_3(t)\left[-r_3 + a_{31}x_1(t) - a_{32}x_2(t) - a_{33}x_3(t)\right]\mathrm{d}t \\ \qquad\qquad + \sigma_3 x_3(t)\mathrm{d}B_3(t). \end{cases} \tag{1.2}$$

On the other hand, "all species should exhibit time delay" in the real world, and incorporating time delays in biological systems makes the systems much more realistic than those without time delays ([6–10]). Hence, in this paper we concern the dynamics of the following stochastic two-predator one-prey system with S-type distributed time delays:

$$\begin{cases} dx_1(t) = x_1(t)\left[r_1 - \mathcal{D}_{11}(x_1)(t) - \mathcal{D}_{12}(x_2)(t)\right. \\ \qquad \left. -\mathcal{D}_{13}(x_3)(t)\right] dt + \sigma_1 x_1(t) dB_1(t), \\ dx_2(t) = x_2(t)\left[-r_2 + \mathcal{D}_{21}(x_1)(t) - \mathcal{D}_{22}(x_2)(t)\right. \\ \qquad \left. -\mathcal{D}_{23}(x_3)(t)\right] dt + \sigma_2 x_2(t) dB_2(t), \\ dx_3(t) = x_3(t)\left[-r_3 + \mathcal{D}_{31}(x_1)(t) - \mathcal{D}_{32}(x_2)(t)\right. \\ \qquad \left. -\mathcal{D}_{33}(x_3)(t)\right] dt + \sigma_3 x_3(t) dB_3(t), \end{cases} \tag{1.3}$$

where $\mathcal{D}_{ji}(x_i)(t) = a_{ji}x_i(t) + \int_{-\tau_{ji}}^{0} x_i(t+\theta)d\mu_{ji}(\theta)$, $\int_{-\tau_{ji}}^{0} x_i(t+\theta)d\mu_{ji}(\theta)$ are Lebesgue-Stieltjes integrals, $\tau_{ji} > 0$ are time delays, $\mu_{ji}(\theta)$ are nondecreasing bounded variation functions defined on $[-\tau, 0]$, $\tau = \max_{i,j=1,2,3}\left\{\tau_{ji}\right\}$.

2. Persistence in mean and Extinction

Denote $A_{ij} = a_{ij} + \int_{-\tau_{ij}}^{0} d\mu_{ij}(\theta)$, $D_1 = r_1 - \frac{\sigma_1^2}{2}$, $D_i = r_i + \frac{\sigma_i^2}{2}$ $(i = 2, 3)$ and

$$\begin{cases} \Theta = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ -A_{21} & A_{22} & A_{23} \\ -A_{31} & A_{32} & A_{33} \end{vmatrix}, \quad \Theta_1 = \begin{vmatrix} D_1 & A_{12} & A_{13} \\ -D_2 & A_{22} & A_{23} \\ -D_3 & A_{32} & A_{33} \end{vmatrix}, \\ \Theta_2 = \begin{vmatrix} A_{11} & D_1 & A_{13} \\ -A_{21} & -D_2 & A_{23} \\ -A_{31} & -D_3 & A_{33} \end{vmatrix}, \quad \Theta_3 = \begin{vmatrix} A_{11} & A_{12} & D_1 \\ -A_{21} & A_{22} & -D_2 \\ -A_{31} & A_{32} & -D_3 \end{vmatrix}. \end{cases}$$

Assume that $\Theta > 0$. For the matrix corresponding to Θ (respectively, Θ_k), denote by M_{ij}^{Θ} (respectively, $M_{ij}^{\Theta_k}$) the complement minor of the element at the i-th row and the j-th column $(i, j, k = 1, 2, 3)$.

Theorem 2.1. *For any $(\xi_1, \xi_2, \xi_3)^T \in C([-\tau, 0], \mathbb{R}_+^3)$, system (1.3) has a unique global solution $(x_1(t), x_2(t), x_3(t))^T \in \mathbb{R}_+^3$ on $t \in [0, +\infty)$ a.s. Moreover, for any constant $p > 0$, there are $K_i(p) > 0$ such that*

$$\sup_{t \geq 0} \mathbb{E}\left[x_i^p(t)\right] \leq K_i(p) \quad (i = 1, 2, 3). \tag{2.1}$$

Proof. The proof is rather standard and here is omitted (see, e.g., [11] and [12]). □

Lemma 2.1. *([13]) Suppose $Z(t) \in C(\Omega \times [0, +\infty), \mathbb{R}_+)$ and $\lim_{t\to+\infty} \frac{o(t)}{t} = 0$.*

(i) *If there exists constant $\delta_0 > 0$ such that for $t \gg 1$,*

$$\ln Z(t) \leq \delta t - \delta_0 \int_0^t Z(s)ds + o(t), \tag{2.2}$$

then

$$\begin{cases} \limsup_{t\to+\infty} t^{-1} \int_0^t Z(s)ds \leq \frac{\delta}{\delta_0} \ a.s. & (\delta \geq 0); \\ \lim_{t\to+\infty} Z(t) = 0 \ a.s. & (\delta < 0). \end{cases} \tag{2.3}$$

(ii) *If there exist constants $\delta > 0$ and $\delta_0 > 0$ such that for $t \gg 1$,*

$$\ln Z(t) \geq \delta t - \delta_0 \int_0^t Z(s)ds + o(t), \tag{2.4}$$

then

$$\liminf_{t\to+\infty} t^{-1} \int_0^t Z(s)ds \geq \frac{\delta}{\delta_0} \ a.s. \tag{2.5}$$

Lemma 2.2. *Consider the following auxiliary system:*

$$\begin{cases} dX_1(t) = X_1(t)\left[r_1 - \mathcal{D}_{11}(X_1)(t)\right] dt + \sigma_1 X_1(t) dB_1(t), \\ dX_i(t) = X_i(t)\left[-r_i + \mathcal{D}_{i1}(X_1)(t) - \mathcal{D}_{ii}(X_i)(t)\right] dt \\ \qquad + \sigma_i X_i(t) dB_i(t), \ (i = 2, 3). \end{cases} \tag{2.6}$$

(a) *If $D_1 < 0$, then $\lim_{t\to+\infty} X_i(t) = 0$ a.s. $(i = 1, 2, 3)$.*

(b) *If $D_1 \geq 0$, $-D_i + A_{i1}\frac{D_1}{A_{11}} < 0$, then*

$$\lim_{t\to+\infty} t^{-1} \int_0^t X_1(s)ds = \frac{D_1}{A_{11}}, \ \lim_{t\to+\infty} X_i(t) = 0 \ a.s. \ (i = 2, 3).$$

(c) *If $D_1 \geq 0$, $-D_i + A_{i1}\frac{D_1}{A_{11}} \geq 0$, then*

$$\lim_{t\to+\infty} t^{-1} \int_0^t X_1(s)ds = \frac{D_1}{A_{11}},$$
$$\lim_{t\to+\infty} t^{-1} \int_0^t X_i(s)ds = A_{ii}^{-1}\left(-D_i + A_{i1}\frac{D_1}{A_{11}}\right) \ a.s. \ (i = 2, 3).$$

Proof. By Itô's formula, we have

$$\begin{cases} \ln X_1(t) = D_1 t - A_{11}\int_0^t X_1(s)ds - \mathcal{T}_{11}(X_1)(t) + o(t), \\ \ln X_i(t) = -D_i t + A_{i1}\int_0^t X_1(s)ds - A_{ii}\int_0^t X_i(s)ds \\ \qquad + \mathcal{T}_{i1}(X_1)(t) - \mathcal{T}_{ii}(X_i)(t) + o(t), \ (i = 2, 3), \end{cases} \tag{2.7}$$

where

$$\mathcal{T}_{ji}(X_i)(t) = \int_{-\tau_{ji}}^{0}\int_{\theta}^{0} X_i(s)dsd\mu_{ji}(\theta) - \int_{-\tau_{ji}}^{0}\int_{t+\theta}^{t} X_i(s)dsd\mu_{ji}(\theta).$$

Case (i) : $D_1 < 0$. Then $\lim_{t\to+\infty} X_1(t) = 0$ a.s. Hence, for $\forall\epsilon \in \left(0, \frac{D_i}{2}\right)$ and $t \gg 1$,

$$\ln X_i(t) \leq (-D_i + \epsilon)t - a_{ii}\int_0^t X_i(s)ds, \ (i = 2, 3). \tag{2.8}$$

So, $\lim_{t\to+\infty} X_i(t) = 0$ a.s. ($i = 2, 3$).

$\mathfrak{Case}$ (ii) : $D_1 \ge 0$. Then,

$$\lim_{t\to+\infty} t^{-1} \int_0^t X_1(s)\mathrm{d}s = \frac{D_1}{A_{11}} \quad a.s. \tag{2.9}$$

Consider the following SDDE:

$$\mathrm{d}\widetilde{X_i(t)} = \widetilde{X_i(t)}\left(-r_i + \mathcal{D}_{i1}(X_1)(t) - a_{ii}\widetilde{X_i(t)}\right)\mathrm{d}t + \sigma_i\widetilde{X_i(t)}\mathrm{d}B_i(t), \ \ (i = 2, 3).$$

Then, $X_i(t) \le \widetilde{X_i(t)}$ a.s. ($i = 2, 3$). By Itô's formula,

$$\ln\widetilde{X_i(t)} = \left(-D_i + A_{i1}\frac{D_1}{A_{11}}\right)t - a_{ii}\int_0^t \widetilde{X_i(s)}\mathrm{d}s + o(t).$$

In view of Lemma 2.1, we obtain:

$(1)^{\dagger}$ If $D_1 \ge 0$, $-D_i + A_{i1}\frac{D_1}{A_{11}} < 0$, then $\lim_{t\to+\infty}\widetilde{X_i(t)} = 0$ a.s. ($i = 2, 3$).

$(2)^{\dagger}$ If $D_1 \ge 0$, $-D_i + A_{i1}\frac{D_1}{A_{11}} \ge 0$, then

$$\lim_{t\to+\infty} t^{-1} \int_0^t \widetilde{X_i(s)}\mathrm{d}s = a_{ii}^{-1}\left(-D_i + A_{i1}\frac{D_1}{A_{11}}\right) \ a.s. \ (i = 2, 3).$$

Therefore, for arbitrary constant $\gamma > 0$,

$$\lim_{t\to+\infty} t^{-1} \int_{t-\gamma}^t X_i(s)\mathrm{d}s = 0 \quad a.s. \quad (i = 1, 2, 3). \tag{2.10}$$

Based on (2.10) and system (2.7), for $i = 2, 3$,

$$\ln X_i(t) = \left(-D_i + A_{i1}\frac{D_1}{A_{11}}\right)t - A_{ii}\int_0^t X_i(s)\mathrm{d}s + o(t).$$

Thanks to Lemma 2.1, we obtain:

$(1)^{\ddagger}$ If $D_1 \ge 0$, $-D_i + A_{i1}\frac{D_1}{A_{11}} < 0$, then $\lim_{t\to+\infty} X_i(t) = 0$ a.s. ($i = 2, 3$).

$(2)^{\ddagger}$ If $D_1 \ge 0$, $-D_i + A_{i1}\frac{D_1}{A_{11}} \ge 0$, then

$$\lim_{t\to+\infty} t^{-1} \int_0^t X_i(s)\mathrm{d}s = A_{ii}^{-1}\left(-D_i + A_{i1}\frac{D_1}{A_{11}}\right) \ a.s. \ (i = 2, 3).$$

Therefore, the desired assertion ($\mathfrak{b}$) follows from combining (2.9) with $(1)^{\ddagger}$, and ($\mathfrak{c}$) follows from combining (2.9) with $(2)^{\ddagger}$. □

Lemma 2.3. *For system (1.3),* $\limsup_{t\to+\infty} t^{-1}\ln x_i(t) \le 0$ *a.s.* ($i = 1, 2, 3$).

Proof. Thanks to Lemma 2.2 and (2.7), system (2.6) satisfies $\lim_{t\to+\infty} t^{-1}\ln X_i(t) = 0$ a.s. ($i = 1, 2, 3$). From the stochastic comparison theorem, we obtain the desired assertion. □

Theorem 2.2. *For system (1.3):*

($\mathfrak{i}$) *If* $D_1 < 0$, *then* $\lim_{t\to+\infty} x_i(t) = 0$ *a.s.* ($i = 1, 2, 3$).

($\mathfrak{ii}$) *If* $D_1 \ge 0$, $\mathrm{M}_{33}^{\Theta_2} < 0$, $\mathrm{M}_{22}^{\Theta_3} < 0$, *then*

$$\lim_{t\to+\infty} t^{-1} \int_0^t x_1(s)\mathrm{d}s = \frac{D_1}{A_{11}}, \quad \lim_{t\to+\infty} x_i(t) = 0 \quad a.s. \quad (i = 2, 3).$$

($\mathfrak{iii}$) *If* $D_1 \ge 0$, $\mathrm{M}_{13}^{\Theta} \le 0$, $\Theta_3 < 0$, $\mathrm{M}_{33}^{\Theta_2} > 0$, *then*

$$\lim_{t\to+\infty} t^{-1} \int_0^t x_i(s)\mathrm{d}s = \frac{\mathrm{M}_{33}^{\Theta_i}}{\mathrm{M}_{33}^{\Theta}}, \quad \lim_{t\to+\infty} x_3(t) = 0 \quad a.s. \quad (i = 1, 2).$$

($\mathfrak{iv}$) *If* $\Theta_1 > 0$, $\Theta_2 > 0$, $\Theta_3 > 0$, $M_{11}^{\Theta} > 0$, *then*

$$\lim_{t\to+\infty} t^{-1} \int_0^t x_i(s)\mathrm{d}s = \frac{\Theta_i}{\Theta} \quad a.s. \quad (i = 1, 2, 3).$$

($\mathfrak{v}$) *If* $D_1 \ge 0$, $\mathrm{M}_{12}^{\Theta} \ge 0$, $\Theta_2 < 0$, $M_{22}^{\Theta_3} < 0$, *then*

$$\lim_{t\to+\infty} t^{-1} \int_0^t x_1(s)\mathrm{d}s = \frac{D_1}{A_{11}}, \quad \lim_{t\to+\infty} x_i(t) = 0 \quad a.s. \quad (i = 2, 3).$$

($\mathfrak{vi}$) *If* $D_1 \ge 0$, $\mathrm{M}_{12}^{\Theta} \ge 0$, $\Theta_2 < 0$, $M_{22}^{\Theta_3} \ge 0$, *then*

$$\lim_{t\to+\infty} t^{-1} \int_0^t x_i(s)\mathrm{d}s = \frac{\mathrm{M}_{22}^{\Theta_i}}{\mathrm{M}_{22}^{\Theta}}, \quad \lim_{t\to+\infty} x_2(t) = 0 \quad a.s. \quad (i = 1, 3).$$

Proof. According to (2.10), for arbitrary constant $\gamma > 0$,

$$\lim_{t\to+\infty} t^{-1} \int_{t-\gamma}^t x_i(s)\mathrm{d}s = 0 \quad a.s. \quad (i = 1, 2, 3). \tag{2.11}$$

By Itô's formula and (2.11), we derive

$$\begin{cases} \ln x_1(t) = D_1 t - A_{11}\int_0^t x_1(s)\mathrm{d}s - A_{12}\int_0^t x_2(s)\mathrm{d}s \\ \qquad - A_{13}\int_0^t x_3(s)\mathrm{d}s + o(t), \\ \ln x_2(t) = -D_2 t + A_{21}\int_0^t x_1(s)\mathrm{d}s - A_{22}\int_0^t x_2(s)\mathrm{d}s \\ \qquad - A_{23}\int_0^t x_3(s)\mathrm{d}s + o(t), \\ \ln x_3(t) = -D_3 t + A_{31}\int_0^t x_1(s)\mathrm{d}s - A_{32}\int_0^t x_2(s)\mathrm{d}s \\ \qquad - A_{33}\int_0^t x_3(s)\mathrm{d}s + o(t). \end{cases} \tag{2.12}$$

$\mathfrak{Case}$ ($\mathfrak{i}$) : $D_1 < 0$. From Lemma 2.2 ($\mathfrak{a}$), $\lim_{t\to+\infty} x_i(t) = 0$ a.s. ($i = 1, 2, 3$).

$\mathfrak{Case}$ (ii) : $D_1 \geq 0$, $\mathrm{M}_{33}^{\Theta_2} < 0$, $\mathrm{M}_{22}^{\Theta_3} < 0$. Based on system (2.12), for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$\ln x_1(t) \leq (D_1 + \epsilon)\, t - A_{11} \int_0^t x_1(s)\mathrm{d}s. \tag{2.13}$$

By Lemma 2.1 and the arbitrariness of ϵ, we obtain

$$\limsup_{t\to+\infty} t^{-1} \int_0^t x_1(s)\mathrm{d}s \leq \frac{D_1}{A_{11}} \quad a.s. \tag{2.14}$$

Hence, for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$\begin{cases} \ln x_2(t) \leq \left(\dfrac{\mathrm{M}_{33}^{\Theta_2}}{A_{11}} + \epsilon\right) t - A_{22} \displaystyle\int_0^t x_2(s)\mathrm{d}s, \\ \ln x_3(t) \leq \left(\dfrac{\mathrm{M}_{22}^{\Theta_3}}{A_{11}} + \epsilon\right) t - A_{33} \displaystyle\int_0^t x_3(s)\mathrm{d}s. \end{cases} \tag{2.15}$$

According to Lemma 2.1 and the arbitrariness of ϵ, we have

$$\lim_{t\to+\infty} x_i(t) = 0 \quad a.s. \quad (i = 2,3). \tag{2.16}$$

Based on (2.16) and system (2.12), for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$\begin{cases} \ln x_1(t) \leq (D_1 + \epsilon)\, t - A_{11} \displaystyle\int_0^t x_1(s)\mathrm{d}s, \\ \ln x_1(t) \geq (D_1 - \epsilon)\, t - A_{11} \displaystyle\int_0^t x_1(s)\mathrm{d}s. \end{cases} \tag{2.17}$$

In view of Lemma 2.1 and the arbitrariness of ϵ, we obtain

$$\lim_{t\to+\infty} t^{-1} \int_0^t x_1(s)\mathrm{d}s = \frac{D_1}{A_{11}} \quad a.s. \tag{2.18}$$

$\mathfrak{Case}$ (iii) : $D_1 \geq 0$, $\mathrm{M}_{13}^{\Theta} \leq 0$, $\Theta_3 < 0$, $\mathrm{M}_{33}^{\Theta_2} > 0$. Compute

$$\begin{aligned} &\mathrm{M}_{13}^{\Theta} \ln x_1(t) - \mathrm{M}_{23}^{\Theta} \ln x_2(t) + \mathrm{M}_{33}^{\Theta} \ln x_3(t) \\ &= \Theta_3 t - \Theta \int_0^t x_3(s)\mathrm{d}s + o(t). \end{aligned}$$

By Lemma 2.3, for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$\mathrm{M}_{33}^{\Theta} \ln x_3(t) \leq (\Theta_3 + \epsilon)\, t - \Theta \int_0^t x_3(s)\mathrm{d}s. \tag{2.19}$$

From Lemma 2.1 and the arbitrariness of ϵ, we deduce

$$\lim_{t\to+\infty} x_3(t) = 0 \quad a.s. \tag{2.20}$$

By (2.20) and system (2.12), for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$\begin{cases} \ln x_1(t) \leq (D_1 + \epsilon)\, t - A_{11} \displaystyle\int_0^t x_1(s)\mathrm{d}s - A_{12} \displaystyle\int_0^t x_2(s)\mathrm{d}s, \\ \ln x_1(t) \geq (D_1 - \epsilon)\, t - A_{11} \displaystyle\int_0^t x_1(s)\mathrm{d}s - A_{12} \displaystyle\int_0^t x_2(s)\mathrm{d}s, \\ \ln x_2(t) \leq (-D_2 + \epsilon)\, t + A_{21} \displaystyle\int_0^t x_1(s)\mathrm{d}s - A_{22} \displaystyle\int_0^t x_2(s)\mathrm{d}s, \\ \ln x_2(t) \geq (-D_2 - \epsilon)\, t + A_{21} \displaystyle\int_0^t x_1(s)\mathrm{d}s - A_{22} \displaystyle\int_0^t x_2(s)\mathrm{d}s. \end{cases} \tag{2.21}$$

According to (2.21), for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$\begin{aligned} &A_{21} \ln x_1(t) + A_{11} \ln x_2(t) \\ &\geq \left[\mathrm{M}_{33}^{\Theta_2} - (A_{11} + A_{21})\,\epsilon\right] t - \mathrm{M}_{33}^{\Theta} \int_0^t x_2(s)\mathrm{d}s, \\ &A_{22} \ln x_1(t) - A_{12} \ln x_2(t) \\ &\leq \left[\mathrm{M}_{33}^{\Theta_1} + (A_{12} + A_{22})\,\epsilon\right] t - \mathrm{M}_{33}^{\Theta} \int_0^t x_1(s)\mathrm{d}s. \end{aligned}$$

Thanks to Lemma 2.3, for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$\begin{aligned} A_{11} \ln x_2(t) &\geq \left[\mathrm{M}_{33}^{\Theta_2} - (A_{11} + 2A_{21})\,\epsilon\right] t - \mathrm{M}_{33}^{\Theta} \int_0^t x_2(s)\mathrm{d}s, \\ A_{22} \ln x_1(t) &\leq \left[\mathrm{M}_{33}^{\Theta_1} + (2A_{12} + A_{22})\,\epsilon\right] t - \mathrm{M}_{33}^{\Theta} \int_0^t x_1(s)\mathrm{d}s. \end{aligned}$$

In view of Lemma 2.1 and the arbitrariness of ϵ, we obtain

$$\liminf_{t\to+\infty} t^{-1} \int_0^t x_2(s)\mathrm{d}s \geq \frac{\mathrm{M}_{33}^{\Theta_2}}{\mathrm{M}_{33}^{\Theta}} \quad a.s. \tag{2.22-1}$$

$$\limsup_{t\to+\infty} t^{-1} \int_0^t x_1(s)\mathrm{d}s \leq \frac{\mathrm{M}_{33}^{\Theta_1}}{\mathrm{M}_{33}^{\Theta}} \quad a.s. \tag{2.22-2}$$

Thanks to (2.21) and (2.22-2), for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$\ln x_2(t) \leq \left(A_{22} \frac{\mathrm{M}_{33}^{\Theta_2}}{\mathrm{M}_{33}^{\Theta}} + 2\epsilon\right) t - A_{22} \int_0^t x_2(s)\mathrm{d}s. \tag{2.23}$$

Based on Lemma 2.1 and the arbitrariness of ϵ, we obtain

$$\limsup_{t\to+\infty} t^{-1} \int_0^t x_2(s)\mathrm{d}s \leq \frac{\mathrm{M}_{33}^{\Theta_2}}{\mathrm{M}_{33}^{\Theta}} \quad a.s. \tag{2.24}$$

Combining (2.22-1) with (2.24) yields

$$\lim_{t\to+\infty} t^{-1} \int_0^t x_2(s)\mathrm{d}s = \frac{\mathrm{M}_{33}^{\Theta_2}}{\mathrm{M}_{33}^{\Theta}} \quad a.s. \tag{2.25}$$

From (2.21) and (2.25), for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$\ln x_1(t) \geq \left(A_{11} \frac{\mathrm{M}_{33}^{\Theta_1}}{\mathrm{M}_{33}^{\Theta}} - 2\epsilon\right) t - A_{11} \int_0^t x_1(s)\mathrm{d}s. \tag{2.26}$$

Thanks to Lemma 2.1 and the arbitrariness of ϵ, we have

$$\liminf_{t\to+\infty} t^{-1} \int_0^t x_1(s)\mathrm{d}s \geq \frac{\mathrm{M}_{33}^{\Theta_1}}{\mathrm{M}_{33}^{\Theta}} \quad a.s. \tag{2.27}$$

Combining (2.22-2) with (2.27) yields

$$\lim_{t\to+\infty} t^{-1} \int_0^t x_1(s)\mathrm{d}s = \frac{\mathrm{M}_{33}^{\Theta_1}}{\mathrm{M}_{33}^{\Theta}} \quad a.s. \tag{2.28}$$

𝔆𝔞𝔰𝔢 (𝔦𝔳) : $\Theta_1 > 0, \Theta_2 > 0, \Theta_3 > 0, M_{11}^{\Theta} > 0$. According to Lemma 2.1, (2.19) and the arbitrariness of ϵ, we obtain

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x_3(s)\mathrm{d}s \le \frac{\Theta_3}{\Theta} \quad a.s. \tag{2.29}$$

In view of system (2.12), we compute

$$\begin{aligned} &\mathrm{M}_{11}^{\Theta}\ln x_1(t) - \mathrm{M}_{21}^{\Theta}\ln x_2(t) + \mathrm{M}_{31}^{\Theta}\ln x_3(t)\\ =&\ \Theta_1 t - \Theta\int_0^t x_1(s)\mathrm{d}s + o(t).\\ &-\mathrm{M}_{12}^{\Theta}\ln x_1(t) + \mathrm{M}_{22}^{\Theta}\ln x_2(t) - \mathrm{M}_{32}^{\Theta}\ln x_3(t)\\ =&\ \Theta_2 t - \Theta\int_0^t x_2(s)\mathrm{d}s + o(t). \end{aligned}$$

Then, for $\forall\epsilon\in(0,1)$ and $t \gg 1$,

$$\begin{aligned} \mathrm{M}_{11}^{\Theta}\ln x_1(t) &\le (\Theta_1+\epsilon)\,t - \Theta\int_0^t x_1(s)\mathrm{d}s,\\ \mathrm{M}_{22}^{\Theta}\ln x_2(t) &\le (\Theta_2+\epsilon)\,t - \Theta\int_0^t x_2(s)\mathrm{d}s. \end{aligned}$$

According to Lemma 2.1 and the arbitrariness of ϵ, we deduce

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s \le \frac{\Theta_1}{\Theta} \quad a.s. \tag{2.30-1}$$

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x_2(s)\mathrm{d}s \le \frac{\Theta_2}{\Theta} \quad a.s. \tag{2.30-2}$$

Therefore, for $\forall\epsilon\in(0,1)$ and $t \gg 1$,

$$t^{-1}\int_0^t x_i(s)\mathrm{d}s \le \frac{\Theta_i}{\Theta} + \epsilon \quad a.s. \quad (i=1,2,3). \tag{2.31}$$

Based on (2.31) and system (2.12), for $\forall\epsilon\in(0,1)$ and $t \gg 1$,

$$\ln x_1(t) \ge \left(A_{11}\frac{\Theta_1}{\Theta} - \sum_{i=1}^{3} A_{1i}\epsilon\right)t - A_{11}\int_0^t x_1(s)\mathrm{d}s. \tag{2.32}$$

Thanks to Lemma 2.1 and the arbitrariness of ϵ, we obtain

$$\liminf_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s \ge \frac{\Theta_1}{\Theta} \quad a.s. \tag{2.33}$$

Combining (2.30-1) with (2.33) yields

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s = \frac{\Theta_1}{\Theta} \quad a.s. \tag{2.34}$$

According to (2.31), (2.33) and system (2.12), for $\forall\epsilon\in(0,1)$ and $t \gg 1$,

$$\ln x_2(t) \ge \left(A_{22}\frac{\Theta_2}{\Theta} - \sum_{i=1}^{3} A_{2i}\epsilon\right)t - A_{22}\int_0^t x_2(s)\mathrm{d}s. \tag{2.35}$$

From Lemma 2.1 and the arbitrariness of ϵ, we have

$$\liminf_{t\to+\infty} t^{-1}\int_0^t x_2(s)\mathrm{d}s \ge \frac{\Theta_2}{\Theta} \quad a.s. \tag{2.36}$$

Combining (2.30-2) with (2.36) yields

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_2(s)\mathrm{d}s = \frac{\Theta_2}{\Theta} \quad a.s. \tag{2.37}$$

Similarly, for $\forall\epsilon\in(0,1)$ and $t \gg 1$,

$$\ln x_3(t) \ge \left(A_{33}\frac{\Theta_3}{\Theta} - \sum_{i=1}^{3} A_{3i}\epsilon\right)t - A_{33}\int_0^t x_3(s)\mathrm{d}s. \tag{2.38}$$

Thanks to Lemma 2.1 and the arbitrariness of ϵ, we obtain

$$\liminf_{t\to+\infty} t^{-1}\int_0^t x_3(s)\mathrm{d}s \ge \frac{\Theta_3}{\Theta} \quad a.s. \tag{2.39}$$

Combining (2.29) with (2.39) yields

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_3(s)\mathrm{d}s = \frac{\Theta_3}{\Theta} \quad a.s. \tag{2.40}$$

𝔆𝔞𝔰𝔢 (𝔳) : $D_1 \ge 0, \mathrm{M}_{12}^{\Theta} \ge 0, \Theta_2 < 0, \mathrm{M}_{22}^{\Theta_3} < 0$. By Lemma 2.1, (2.30) and the arbitrariness of ϵ, we have

$$\lim_{t\to+\infty} x_2(t) = 0 \quad a.s. \tag{2.41}$$

Hence, for $\forall\epsilon\in(0,1)$ and $t \gg 1$,

$$\begin{cases} \ln x_1(t) \le (D_1+\epsilon)\,t - A_{11}\int_0^t x_1(s)\mathrm{d}s - A_{13}\int_0^t x_3(s)\mathrm{d}s,\\ \ln x_1(t) \ge (D_1-\epsilon)\,t - A_{11}\int_0^t x_1(s)\mathrm{d}s - A_{13}\int_0^t x_3(s)\mathrm{d}s,\\ \ln x_3(t) \le (-D_3+\epsilon)\,t + A_{31}\int_0^t x_1(s)\mathrm{d}s - A_{33}\int_0^t x_3(s)\mathrm{d}s,\\ \ln x_3(t) \ge (-D_3-\epsilon)\,t + A_{31}\int_0^t x_1(s)\mathrm{d}s - A_{33}\int_0^t x_3(s)\mathrm{d}s. \end{cases} \tag{2.42}$$

Based on (2.42), we deduce

$$\begin{aligned} &A_{33}\ln x_1(t) - A_{13}\ln x_3(t)\\ \le&\ \left[\mathrm{M}_{22}^{\Theta_1} + (A_{13}+A_{33})\,\epsilon\right]t - \mathrm{M}_{22}^{\Theta}\int_0^t x_1(s)\mathrm{d}s. \end{aligned} \tag{2.43}$$

Therefore, for $\forall\epsilon\in(0,1)$ and $t \gg 1$,

$$A_{33}\ln x_1(t) \le \left[\mathrm{M}_{22}^{\Theta_1} + (2A_{13}+A_{33})\,\epsilon\right]t - \mathrm{M}_{22}^{\Theta}\int_0^t x_1(s)\mathrm{d}s. \tag{2.44}$$

In view of Lemma 2.1 and the arbitrariness of ϵ, we deduce

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s \le \frac{\mathrm{M}_{22}^{\Theta_1}}{\mathrm{M}_{22}^{\Theta}} \quad a.s. \tag{2.45}$$

According to (2.42) and (2.45), for $\forall\epsilon\in(0,1)$ and $t\gg 1$,

$$\ln x_3(t) \le \left(A_{33}\frac{\mathrm{M}_{22}^{\Theta_3}}{\mathrm{M}_{22}^{\Theta}} + 2\epsilon\right)t - A_{33}\int_0^t x_3(s)\mathrm{d}s. \tag{2.46}$$

Clearly, (2.20) is true. From (2.20) and (2.42), for $\forall\epsilon\in(0,1)$ and $t\gg 1$,

$$\begin{cases} \ln x_1(t) \le (D_1 + 2\epsilon)\, t - A_{11}\displaystyle\int_0^t x_1(s)\mathrm{d}s, \\ \ln x_1(t) \ge (D_1 - 2\epsilon)\, t - A_{11}\displaystyle\int_0^t x_1(s)\mathrm{d}s. \end{cases} \tag{2.47}$$

In view of Lemma 2.1 and the arbitrariness of ϵ, we obtain (2.18).

Case (vi) : $D_1 \ge 0$, $\mathrm{M}_{12}^{\Theta} \ge 0$, $\Theta_2 < 0$, $\mathrm{M}_{22}^{\Theta_3} \ge 0$. Thanks to Lemma 2.1, (2.46) and the arbitrariness of ϵ,

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x_3(s)\mathrm{d}s \le \frac{\mathrm{M}_{22}^{\Theta_3}}{\mathrm{M}_{22}^{\Theta}} \quad a.s. \tag{2.48}$$

According to (2.42) and (2.48), for $\forall\epsilon\in(0,1)$ and $t\gg 1$,

$$\ln x_1(t) \ge \left(A_{11}\frac{\mathrm{M}_{22}^{\Theta_1}}{\mathrm{M}_{22}^{\Theta}} - 2\epsilon\right)t - A_{11}\int_0^t x_1(s)\mathrm{d}s. \tag{2.49}$$

Based on Lemma 2.1 and the arbitrariness of ϵ, we have

$$\liminf_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s \ge \frac{\mathrm{M}_{22}^{\Theta_1}}{\mathrm{M}_{22}^{\Theta}} \quad a.s. \tag{2.50}$$

Combining (2.45) with (2.50) yields

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s = \frac{\mathrm{M}_{22}^{\Theta_1}}{\mathrm{M}_{22}^{\Theta}} \quad a.s. \tag{2.51}$$

By (2.42) and (2.50), for $\forall\epsilon\in(0,1)$ and $t\gg 1$,

$$\ln x_3(t) \ge \left(A_{33}\frac{\mathrm{M}_{22}^{\Theta_3}}{\mathrm{M}_{22}^{\Theta}} - 2\epsilon\right)t - A_{33}\int_0^t x_3(s)\mathrm{d}s. \tag{2.52}$$

Thanks to Lemma 2.1 and the arbitrariness of ϵ, we deduce

$$\liminf_{t\to+\infty} t^{-1}\int_0^t x_3(s)\mathrm{d}s \ge \frac{\mathrm{M}_{22}^{\Theta_3}}{\mathrm{M}_{22}^{\Theta}} \quad a.s. \tag{2.53}$$

Combining (2.48) with (2.53) yields

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_3(s)\mathrm{d}s = \frac{\mathrm{M}_{22}^{\Theta_3}}{\mathrm{M}_{22}^{\Theta}} \quad a.s. \tag{2.54}$$

The proof is complete. □

3. Global attractivity

Theorem 3.1. *Assume that $2a_{jj} > \sum_{i=1}^{3} A_{ij}$ $(j = 1,2,3)$. Let $X(t;\phi) =: (x_1(t;\phi), x_2(t;\phi), x_3(t;\phi))^{\mathrm{T}}$ be the solution to system (1.3) with initial condition $\phi\in C([-\tau,0],\mathbb{R}_+^3)$. Then, for any ϕ and $\phi^*\in C([-\tau,0],\mathbb{R}_+^3)$,*

$$\lim_{t\to+\infty}\mathbb{E}\left[\|X(t;\phi) - X(t;\phi^*)\|\right] = 0. \tag{3.1}$$

Proof. We only need to show

$$\lim_{t\to+\infty}\mathbb{E}\,|x_i(t;\phi) - x_i(t;\phi^*)| = 0 \quad (i = 1,2,3). \tag{3.2}$$

Define

$$W(t;\phi,\phi^*) = \sum_{i=1}^{3}\left|\ln\left(\frac{x_i(t;\phi^*)}{x_i(t;\phi)}\right)\right| + \sum_{i,j=1}^{3}\int_{-\tau_{ji}}^{0}\int_{t+\theta}^{t}|x_i(s;\phi^*) - x_i(s;\phi)|\,\mathrm{d}s\mathrm{d}\mu_{ji}(\theta).$$

From Itô's formula, we derive

$$\mathcal{L}[W(t;\phi,\phi^*)] \le -\sum_{j=1}^{3}\left(2a_{jj} - \sum_{i=1}^{3}A_{ij}\right)\left|x_j(t;\phi^*) - x_j(t;\phi)\right|. \tag{3.3}$$

According to (3.3), we have

$$\begin{aligned}&\mathbb{E}[W(t;\phi,\phi^*)] - \mathbb{E}[W(0;\phi,\phi^*)]\\ &\le -\sum_{j=1}^{3}\left(2a_{jj} - \sum_{i=1}^{3}A_{ij}\right)\int_0^t\mathbb{E}\left[\left|x_j(s;\phi^*) - x_j(s;\phi)\right|\right]\mathrm{d}s,\end{aligned}$$

which implies

$$\int_0^{+\infty}\mathbb{E}\left[|x_i(t;\phi^*) - x_i(t;\phi)|\right]\mathrm{d}t < +\infty \quad (i = 1,2,3). \tag{3.4}$$

Define $G_i(t) = \mathbb{E}\left[|x_i(t;\phi^*) - x_i(t;\phi)|\right]$ $(i = 1,2,3)$. Then,

$$\begin{aligned}&|G_i(t_2) - G_i(t_1)|\\ &\le \mathbb{E}\left[|x_i(t_2;\phi^*) - x_i(t_1;\phi^*)|\right] + \mathbb{E}\left[|x_i(t_2;\phi) - x_i(t_1;\phi)|\right].\end{aligned} \tag{3.5}$$

Based on Hölder's inequality, for $t_2 > t_1$ and $p > 1$,

$$\begin{aligned}&\left(\mathbb{E}\left[|x_j(t_2) - x_j(t_1)|\right]\right)^p \le \mathbb{E}\left[|x_j(t_2) - x_j(t_1)|^p\right]\\ &\le 2^{p-1}\mathbb{E}\left[\left|\left(\int_{t_1}^{t_2} x_j(s)\left(r_j + \sum_{i=1}^{3}a_{ji}x_i(s)\right.\right.\right.\right.\\ &\qquad\left.\left.\left.\left. + \sum_{i=1}^{3}\int_{-\tau_{ji}}^{0}x_i(s+\theta)\mathrm{d}\mu_{ji}(\theta)\right)\mathrm{d}s\right)^p\right|\right]\\ &\qquad + 2^{p-1}\mathbb{E}\left[\left|\int_{t_1}^{t_2}\sigma_j x_j(s)\mathrm{d}B_j(s)\right|^p\right]\\ &\triangleq 2^{p-1}\Upsilon_1 + 2^{p-1}\Upsilon_2 \quad (j = 1,2,3).\end{aligned} \tag{3.6}$$

In view of Theorem 7.1 in [14], for $p \geq 2$, we obtain

$$\Upsilon_2 \leq \left|\sigma_j\right|^p \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} (t_2 - t_1)^{\frac{p-2}{2}} \int_{t_1}^{t_2} \mathbb{E}\left[x_j^p(s)\right] \mathrm{d}s. \quad (3.7)$$

From Hölder's inequality, we derive

$$\begin{aligned}\Upsilon_1 \leq\ & 7^{p-1} r_j^p (t_2 - t_1)^{p-1} \int_{t_1}^{t_2} \mathbb{E}\left[x_j^p(s)\right] \mathrm{d}s \\ &+ \sum_{i=1}^{3} 7^{p-1} a_{ji}^p (t_2 - t_1)^{p-1} \int_{t_1}^{t_2} \mathbb{E}\left[x_i^p(s) x_j^p(s)\right] \mathrm{d}s \\ &+ \sum_{i=1}^{3} 7^{p-1} (t_2 - t_1)^{p-1} \mathbb{E}\left[\int_{t_1}^{t_2} \left(\int_{-\tau_{ji}}^{0} x_i(s+\theta) x_j(s) \mathrm{d}\mu_{ji}(\theta)\right)^p \mathrm{d}s\right].\end{aligned} \quad (3.8)$$

According to Hölder's inequality, we get

$$\begin{aligned}&\mathbb{E}\left[\int_{t_1}^{t_2} \left(\int_{-\tau_{ji}}^{0} x_j(s) x_i(s+\theta) \mathrm{d}\mu_{ji}(\theta)\right)^p \mathrm{d}s\right] \\ &\leq \frac{1}{2}\left(\int_{-\tau_{ji}}^{0} \mathrm{d}\mu_{ji}(\theta)\right)^p \int_{t_1}^{t_2} \mathbb{E}\left[x_j^{2p}(s)\right] \mathrm{d}s \\ &\quad + \frac{1}{2}\left(\int_{-\tau_{ji}}^{0} \mathrm{d}\mu_{ji}(\theta)\right)^{p-1} \int_{t_1}^{t_2} \int_{-\tau_{ji}}^{0} \mathbb{E}\left[x_i^{2p}(s+\theta)\right] \mathrm{d}\mu_{ji}(\theta) \mathrm{d}s.\end{aligned} \quad (3.9)$$

Based on (3.6)-(3.9), for $p \geq 2$ and $|t_2 - t_1| \leq \delta$,

$$\left(\mathbb{E}\left[|x_j(t_2) - x_j(t_1)|\right]\right)^p \leq M_j |t_2 - t_1|^{\frac{p}{2}}, \quad (3.10)$$

where

$$\begin{aligned}M_j =& 14^{p-1} \left\{ r_j^p K_j(p) + \sum_{i=1}^{3} \left[\frac{a_{ji}^p}{2} + \frac{1}{2}\left(\int_{-\tau_{ji}}^{0} \mathrm{d}\mu_{ji}(\theta)\right)^p\right] \right. \\ &\left. \left[K_i(2p) + K_j(2p)\right] \right\} \delta^{\frac{p}{2}} + 2^{p-1} \left|\sigma_j\right|^p \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} K_j(p).\end{aligned}$$

Combining (3.5) with (3.10) yields

$$\left|G_j(t_2) - G_j(t_1)\right| \leq 2 \sqrt[p]{M_j} \sqrt{|t_2 - t_1|}. \quad (3.11)$$

Therefore, (3.2) follows from (3.4), (3.11) and Barbalat's conclusion in [15]. □

4. Numerical simulations

In this section we provide some numerical simulations to show the effectiveness of our main theoretical results by using the Milstein approach mentioned in [16]. Let $\tau_{ji} = \ln 2$, $\mu_{ji}(\theta) = \mu_{ji} e^{\theta}$. Denote

$$\text{Param}(i) = \begin{pmatrix} r_1 & a_{11} & a_{12} & a_{13} & \mu_{11} & \mu_{12} & \mu_{13} \\ r_2 & a_{21} & a_{22} & a_{23} & \mu_{21} & \mu_{22} & \mu_{23} \\ r_3 & a_{31} & a_{32} & a_{33} & \mu_{31} & \mu_{32} & \mu_{33} \end{pmatrix}.$$

4.1. Example 1

Let

$$\text{Param}(1) = \begin{pmatrix} 0.9 & 0.2 & 0.4 & 0.2 & 0.4 & 0.8 & 0.2 \\ 0.2 & 0.6 & 0.3 & 0.1 & 0.6 & 0.6 & 0.2 \\ 0.1 & 0.4 & 0.3 & 0.2 & 0.8 & 0.4 & 0.4 \end{pmatrix},$$

subject to $x_1(\theta) = 0.7e^{\theta}$, $x_2(\theta) = 0.6e^{\theta}$, $x_3(\theta) = 0.5e^{\theta}$, $\theta \in [-\ln 2, 0]$.

Case 1. $\sigma_1 = 1.4$, $\sigma_2 = 0.1$, $\sigma_3 = 0.1$. Then, $D_1 = -0.08$. By Theorem 2.2 (i), all three species are extinctive. See Figure 1(a).

Case 2. $\sigma_1 = 0.1$, $\sigma_2 = 2.0$, $\sigma_3 = 1.9$. Then, $D_1 = 0.895$, $M_{33}^{\Theta_2} = -0.0745$, $M_{22}^{\Theta_3} = -0.046$. From Theorem 2.2 (ii), $x_1(t)$ is persistent in mean, while $x_2(t)$ and $x_3(t)$ are extinctive. See Figure 1(b).

Case 3. $\sigma_1 = 0.1$, $\sigma_2 = 0.1$, $\sigma_3 = 0.1$. Then, $\Theta = 0.225$, $\Theta_1 = 0.16225$, $\Theta_2 = 0.13375$, $\Theta_3 = 0.09825$, $M_{11}^{\Theta} = 0.14$. In view of Theorem 2.2 (iv), all three species are persistent in mean. See Figure 1(c).

4.2. Example 2

Let

$$\text{Param}(2) = \begin{pmatrix} 0.8 & 0.1 & 0.2 & 0.1 & 0.2 & 0.2 & 0.2 \\ 0.3 & 0.2 & 0.4 & 0.2 & 0.4 & 0.4 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.1 & 0.2 & 0.4 & 0.2 \end{pmatrix},$$

subject to $x_1(\theta) = 0.7e^{\theta}$, $x_2(\theta) = 0.6e^{\theta}$, $x_3(\theta) = 0.5e^{\theta}$, $\theta \in [-\ln 2, 0]$.

Case 4. $\sigma_1 = 0.1$, $\sigma_2 = 0.1$, $\sigma_3 = 0.1$. Then, $\Theta = 0.001$, $D_1 = 0.795$, $M_{12}^{\Theta} = 0.01$, $\Theta_2 = -0.00975$, $M_{22}^{\Theta_3} = 0.1975$. According to Theorem 2.2 (vi), both $x_1(t)$ and $x_3(t)$ are persistent in mean, while $x_2(t)$ is extinctive. See Figure 1(d).

Case 5. $\sigma_1 = 0.2$, $\sigma_2 = 1.8$, $\sigma_3 = 1.5$. Then, $\Theta = 0.001$, $D_1 = 0.78$, $M_{12}^{\Theta} = 0.01$, $\Theta_2 = -0.0143$, $M_{22}^{\Theta_3} = -0.031$. Based on Theorem 2.2 (v), $x_1(t)$ is persistent in mean, while both $x_2(t)$ and $x_3(t)$ are extinctive. See Figure 1(e).

4.3. Example 3

Let

$$\text{Param}(3) = \begin{pmatrix} 0.8 & 0.1 & 0.1 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.1 & 0.2 & 0.2 & 0.2 & 0.4 \\ 0.3 & 0.2 & 0.2 & 0.4 & 0.4 & 0.2 & 0.4 \end{pmatrix},$$

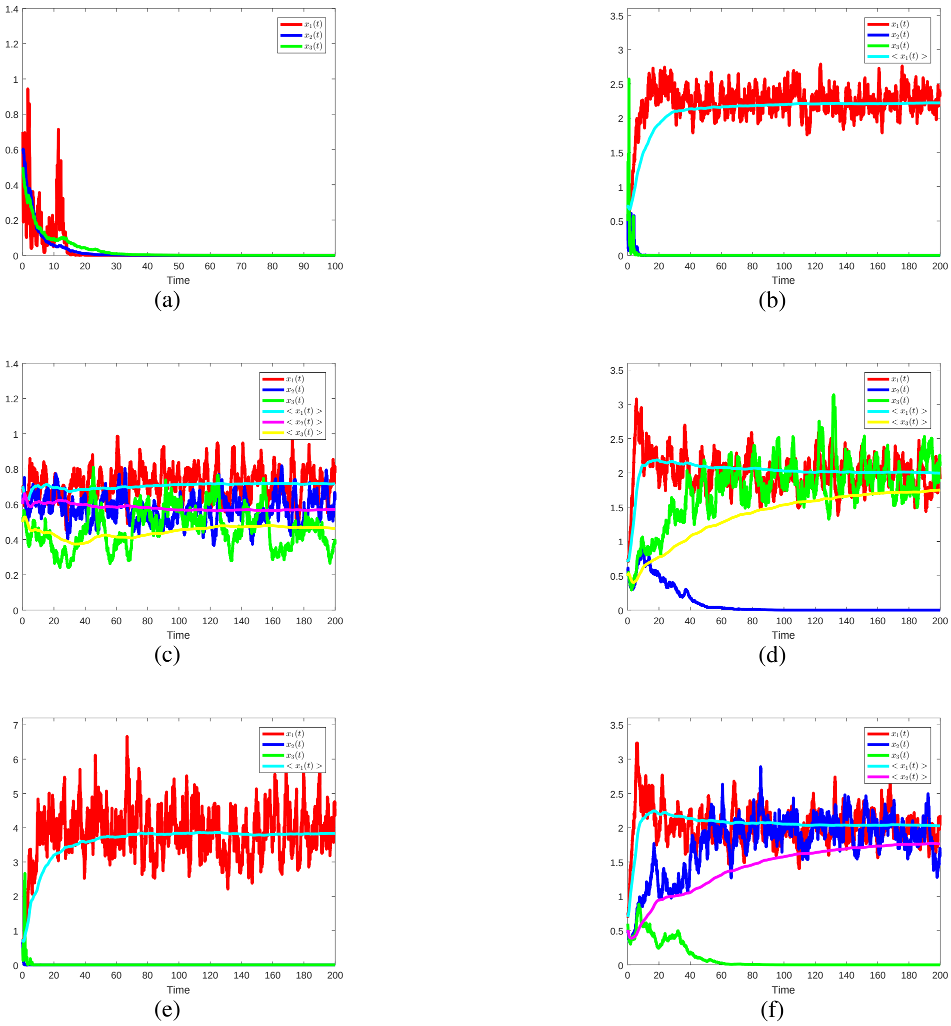

Figure 1. (a) shows the solution to system (1.3) with Param (1) and $\sigma_1 = 1.4$, $\sigma_2 = 0.1$, $\sigma_3 = 0.1$. This subfigure represents that all species in **Case 1** are extinctive; (b) shows the solution to system (1.3) with Param (1) and $\sigma_1 = 0.1$, $\sigma_2 = 2.0$, $\sigma_3 = 1.9$. This subfigure represents that in **Case 2**, $x_1(t)$ is persistent in mean, while $x_2(t)$ and $x_3(t)$ are extinctive; (c) shows the solution to system (1.3) with Param (1) and $\sigma_1 = 0.1$, $\sigma_2 = 0.1$, $\sigma_3 = 0.1$. This subfigure represents that all species in **Case 3** are persistent in mean; (d) shows the solution to system (1.3) with Param (2) and $\sigma_1 = 0.1$, $\sigma_2 = 0.1$, $\sigma_3 = 0.1$. This subfigure represents that in **Case 4**, both $x_1(t)$ and $x_3(t)$ are persistent in mean, while $x_2(t)$ is extinctive; (e) shows the solution to system (1.3) with Param (2) and $\sigma_1 = 0.2$, $\sigma_2 = 1.8$, $\sigma_3 = 1.5$. This subfigure represents that in **Case 5**, $x_1(t)$ is persistent in mean, while both $x_2(t)$ and $x_3(t)$ are extinctive; (f) shows the solution to system (1.3) with Param (3) and $\sigma_1 = 0.1$, $\sigma_2 = 0.1$, $\sigma_3 = 0.1$. This subfigure represents that in **Case 6**, both $x_1(t)$ and $x_2(t)$ are persistent in mean, while $x_3(t)$ is extinctive.

subject to $x_1(\theta) = 0.7e^{\theta}$, $x_2(\theta) = 0.5e^{\theta}$, $x_3(\theta) = 0.6e^{\theta}$, $\theta \in [-\ln 2, 0]$.

Case 6. $\sigma_1 = 0.1$, $\sigma_2 = 0.1$, $\sigma_3 = 0.1$. Then, $\Theta = 0.001$, $D_1 = 0.795$, $M_{13}^{\Theta} = -0.01$, $\Theta_3 = -0.00975$, $M_{33}^{\Theta_2} = 0.1975$. Thanks to Theorem 2.2 (iii), both $x_1(t)$ and $x_2(t)$ are persistent in mean, while $x_3(t)$ is extinctive. See Figure 1(f).

All mentioned above can be confirmed by Figure 1.

Acknowledgement

The work is supported by National Natural Science Foundation of China (No.11901166).

Conflict of interest

The authors declare that there are no conflicts of interest.

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