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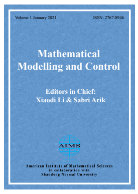

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Research article

Comparing the number of ideals in quadratic number fields

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Abstract: Denote by $a_K(n)$ the number of integral ideals in K with norm n, where K is a algebraic number field of degree m over the rational field Q. Let p be a prime number. In this paper, we prove that, for two distinct quadratic number fields $K_i = Q(\sqrt{d_i})$, $i = 1, 2$, the sets both

$$\{p \mid a_{K_1}(p) < a_{K_2}(p)\} \text{ and } \{p \mid a_{K_1}(p^2) < a_{K_2}(p^2)\}$$

have analytic density $1/4$, respectively.

Keywords: quadratic number fields; prime; Dedekind zeta function

1. Introduction

Suppose that K is an algebraic number field of degree m over the rational field Q. The Dedekind zeta function $\zeta_K(s)$ is defined as, for Re $s > 1$,

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{\mathfrak{N}(\mathfrak{a})^s},$$

where $\mathfrak{a}$ varies over the integral ideals of K and $\mathfrak{N}(\mathfrak{a})$ denotes the norm of $\mathfrak{a}$. Denote by $a_K(n)$ the number of integral ideals in K with norm n, then we can rewrite $\zeta_K(s)$ as

$$\zeta_K(s) = \sum_{n=1}^{\infty} \frac{a_K(n)}{n^s}, \quad \text{Re } s > 1.$$

The arithmetic function $a_K(n)$ is one of the research hotspots in algebraic number theory, since its behavior is not regular. It is known from Chandraseknaran and Good [1] that $a_K(n)$ is multiplicative and satisfies the upper bound

$$a_K(n) \le d(n)^m, \tag{1.1}$$

where $d(n)$ is the divisor function and $m = [K : Q]$.

Let $S_k(N)^{\text{new}}$ be the space of all cuspidal newforms of even integral weight k for the congruence subgroup $\Gamma_0(N) \subseteq SL_2(\mathbb{Z})$, with trivial nebentypus. Let $f \in S_k(N)^{\text{new}}$ be a Hecke eigenform, and denote by $\lambda_f(n)$ the corresponding normalized Hecke eigenvalues, which are studied by many scholars (see [2–8] etc.). In particular, Chiriac [9] proposed an interesting problem: Is it possible that for two distinct newforms f and g the eigenvalues $\lambda_f(p)$ are not less than $\lambda_g(p)$, for almost all primes p? To state Chiriac's result, we say that a set Ξ of primes has analytic density (or Dirichlet density) $\delta > 0$ if and only if

$$\sum_{p\in\Xi} \frac{1}{p^s} \sim \delta \sum_{p} \frac{1}{p^s}, \quad \text{as } s \to 1^+. \tag{1.2}$$

Chiriac [9] showed that the problem he proposed cannot occur by proving that for two distinct cusp forms, the set

$$\{p \mid \lambda_f(p) < \lambda_g(p)\}$$

has analytic density at least $1/16$. In the same paper, assuming that f and g do not have complex multiplication, and that neither is a quadratic twist of the other, Chiriac [9] also proved that the set

$$\{p \mid \lambda_f^2(p) < \lambda_g^2(p)\}$$

has analytic density at least 1/16. Later, some more results were established in this direction, and we refer to the references [10–12] for details.

Motivated by the above works, naturally we draw our attention to the following question: Is it possible that for two distinct quadratic number fields K_1 and K_2 the number of integral ideals $a_{K_1}(p)$ are not less than $a_{K_2}(p)$, for almost all primes p? We are able to show that the answer to above question is negative by proving the following result.

Theorem 1.1. Let $K_i = \mathbb{Q}(\sqrt{d_i})$, $i = 1, 2$ be two quadratic number fields, where $d_1, d_2 \neq 0, 1$ are two distinct square-free integers. Then the sets both

$$\{p \mid a_{K_1}(p) < a_{K_2}(p)\} \text{ and } \{p \mid a_{K_1}(p^2) < a_{K_2}(p^2)\}$$

have analytic density 1/4, respectively.

Since the structure of the quadratic number field is more detailed, we can get a precise density result in Theorem 1.1. One key point of the proof is the famous Čebotarev Density theorem.

2. Preliminaries

The following lemma is the famous Čebotarev Density theorem, which can be found in [13, Theorem 31].

Lemma 2.1. Let K, k be algebraic number fields such that K is Galois over k, let σ be an element of Gal(K/k) and denote by $\langle\sigma\rangle$ the conjugacy class of σ. Let $\mathcal{S}$ be the set of prime ideals $\mathfrak{p}$ of k such that for every $\mathfrak{P}$ above $\mathfrak{p}$ the Frobenius element $[\frac{K/k}{\mathfrak{p}}]$ lies in $\langle\sigma\rangle$. Then $\mathcal{S}$ has Dirichlet density card$(\langle\sigma\rangle)/$card(Gal(K/k)).

For convenience, we write

$$\Xi_1 = \{p \mid a_{K_1}(p) < a_{K_2}(p)\} \text{ and } \Xi_2 = \{p \mid a_{K_1}(p^2) < a_{K_2}(p^2)\}.$$

Let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic number field with a square-free integer $d \neq 0, 1$. It is known that its discriminant is

$$D = \begin{cases} d, & \text{if } d \equiv 1 \pmod 4, \\ 4d, & \text{if } d \equiv 2, 3 \pmod 4. \end{cases}$$

Then we have the following proposition.

Proposition 2.2. Let $K_i = \mathbb{Q}(\sqrt{d_i})$, $i = 1, 2$ be two quadratic number fields, where $d_1, d_2 \neq 0, 1$ are two distinct square-free integers. D_1 and D_2 are the discriminants of K_1 and K_2, respectively. Then for $p \nmid D_1D_2$, both $p \in \Xi_1$ and $p \in \Xi_2$ are equivalent to that p is inert in K_1 and splits in K_2, respectively.

Proof. We first consider the quadratic number field $K = \mathbb{Q}(\sqrt{d})$. Since $p \nmid D$, the prime p does not ramify. Thus the prime p either splits or is inert in K.

When p splits in K, then $pO_K = \mathfrak{p}_1\mathfrak{p}_2$ with $\mathfrak{p}_1 \neq \mathfrak{p}_2$. Then we have that the ideals with norm p are $\mathfrak{p}_1, \mathfrak{p}_2$ and the ideals with norm p^2 are $\mathfrak{p}_1^2, \mathfrak{p}_1\mathfrak{p}_2, \mathfrak{p}_2^2$. When p is inert in K, the pO_K is the only prime ideal with norm p^2 of K above p. There are no ideals with norm p.

Thus we have

$$a_K(p) = \begin{cases} 2, & \text{if } p \text{ splits in } K, \\ 0, & \text{if } p \text{ is inert in } K, \end{cases}$$

$$a_K(p^2) = \begin{cases} 3, & \text{if } p \text{ splits in } K, \\ 1, & \text{if } p \text{ is inert in } K. \end{cases}$$

Then from the above two formulas we can get this proposition. □

With the help of Čebotarev Density theorem and Proposition 2.2, we can complete the proof of Theorem 1.1.

3. Proof of Theorem 1.1

Note that in the quadratic number field K, p ramifies if and only if $p \mid D$. Thus the number of p which ramifies in K is limited. We just need to focus on the case that p does not ramifies. From Proposition 2.2, it is sufficient to prove that the density of primes which are inert in K_1 and split in K_2 is 1/4.

Let $S = K_1K_2 = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$. Due to the fact that $d_1, d_2 \neq 0, 1$ are two distinct square-free integers, we have

$$K_1 \cap K_2 = \mathbb{Q}, \quad \text{Gal}(S/\mathbb{Q}) \cong \text{Gal}(K_1/\mathbb{Q}) \times \text{Gal}(K_2/\mathbb{Q}).$$

We know that S is the splitting field of the polynomial $(x^2 - d_1)(x^2 - d_2)$ with roots $\omega_1 = \sqrt{d_1}, \omega_2 = -\sqrt{d_1}, \omega_3 = \sqrt{d_2}, \omega_4 = -\sqrt{d_2}$. By the ordering of roots we can identify Gal$(S/\mathbb{Q})$ with the permutation group

$$V_4 = \{\text{id}, (1, 2), (3, 4), (1, 2)(3, 4)\}.$$

For a prime p, we write Frob_p as Frobenius element of $\text{Gal}(S/Q) \cong V_4$ corresponding to p. Then p is inert in K_1 and splits in K_2 if and only if $\text{Frob}_p = (1, 2)$. From Lemma 2.1, we know that density of primes which are inert in K_1 and split in K_2 is $1/4$. Therefore, we complete the proof of Theorem 1.1.

4. Conclusions

Let K be a algebraic number field and suppose that $a_K(n)$ denotes the number of integral ideals in K with norm n. In this paper, for the question: Is it possible that for two distinct quadratic number fields K_1 and K_2 the number of integral ideals $a_{K_1}(p)$ are not less than $a_{K_2}(p)$, for almost all primes p? We give a negative answer and further show that the sets both

$$\{p \mid a_{K_1}(p) < a_{K_2}(p)\} \text{ and } \{p \mid a_{K_1}(p^2) < a_{K_2}(p^2)\}$$

have analytic density $1/4$, respectively.

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Conflict of interest

The authors declare there is no conflict of interest in this paper.

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